

FUNCTIONAL CALCULUS FOR C_0 -SEMIGROUPS USING INFINITE-DIMENSIONAL SYSTEMS THEORY

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Dedicated to Charles Batty on the occasion of his sixtieth birthday.

ABSTRACT. In this short note we use ideas from systems theory to define a functional calculus for infinitesimal generators of strongly continuous semigroups on a Hilbert space. Among others, we show how this leads to new proofs of (known) results in functional calculus.

1. INTRODUCTION

Let A be a linear operator on the linear space X . In essence, a functional calculus provides for every (scalar) function f in the algebra \mathcal{A} a linear operator $f(A)$ from (a subspace of) X to X such that

- $f \mapsto f(A)$ is linear;
- $f(s) \equiv 1$ is mapped on the identity I ;
- If $f(s) = (s - r)^{-1}$, then $f(A) = (A - rI)^{-1}$;
- For $f = f_1 \cdot f_2$ we have $f(A) = f_1(A)f_2(A)$.

As the domains of the operators $f(A)$ might differ, the above properties have to be seen formally, and, in general, need to be made rigorous. It is well-known that self-adjoint (or unitary operators) on a Hilbert space have a functional calculus with \mathcal{A} being the set of continuous functions from \mathbb{R} (or the torus \mathbb{T} respectively) to \mathbb{C} , (von Neumann [10]). This theory has been further extended to different operators and algebra's, see e.g. [7], [3], and [2]. For an excellent overview, in particular on the \mathcal{H}^∞ -calculus, we refer to the book by Markus Haase, [5].

For the algebra of bounded analytic functions on the left half-plane and A the infinitesimal generator of a strongly continuous semigroup, we show how to build a functional calculus using infinite-dimensional systems theory.

2. FUNCTIONAL CALCULUS FOR \mathcal{H}_∞^-

We choose our class of functions to be \mathcal{H}_∞^- , i.e., the algebra of bounded analytic functions on the left half-plane. For A we choose the generator of an exponentially stable strongly continuous semigroup on the Hilbert space X . This semigroup will be denoted by $(e^{At})_{t \geq 0}$. We refer to [4] for a detailed overview on C_0 -semigroups.

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In the following all semigroups are assumed to be strongly continuous. To explain our choice/set-up we start with the following observation.

Let h be an integrable function from \mathbb{R} to \mathbb{C} which is zero on $(0, \infty)$ and let $t \mapsto \mathbb{1}(t)$ denote the indicator function of $[0, \infty)$, i.e., $\mathbb{1}(t) = 1$ for $t \geq 0$ and $\mathbb{1}(t) = 0$ for $t < 0$. Then for $t > 0$

$$\begin{aligned} (h * e^{A \cdot} x_0 \mathbb{1}(\cdot))(t) &= \int_{-\infty}^{\infty} h(\tau) e^{A(t-\tau)} x_0 \mathbb{1}(t-\tau) d\tau \\ &= \left[\int_{-\infty}^t h(\tau) e^{-A\tau} d\tau \right] e^{At} x_0 \\ &= \left[\int_{-\infty}^0 h(\tau) e^{-A\tau} d\tau \right] e^{At} x_0. \end{aligned}$$

Hence the convolution of h with the semigroup gives an operator times the semigroup. We denote this operator by $g(A)$, with g the Laplace transform of h .

Now we want to extend the mapping $g \mapsto g(A)$. Therefore we need the Hardy space $H^2(X) = H^2(\mathbb{C}_+; X)$, i.e., the set of X -valued functions, analytic on the right half-plane which are uniformly square integrable along every line parallel to the imaginary axis. By the (vector-valued) Paley-Wiener Theorem, this space is isomorphic to $L^2((0, \infty); X)$ under the Laplace transform, see [1, Theorem 1.8.3].

Definition 2.1. Let X be a Hilbert space. For $g \in \mathcal{H}_{\infty}^-$ and $f \in L^2((0, \infty); X)$ we define the Toeplitz operator

$$(1) \quad M_g(f) = \mathfrak{L}^{-1} [\Pi(g(\mathfrak{L}(f)))],$$

where \mathfrak{L} and \mathfrak{L}^{-1} denotes the Laplace transform and its inverse, respectively, and Π is the projection from $L^2(i\mathbb{R}, X)$ onto $H^2(X)$.

Remark 2.2. If we take $f(t) = e^{At} x_0$, $t \geq 0$, and “ $g = \mathfrak{L}(h)$ ”, then this extends the previous convolution.

The following norm estimate is easy to see.

Lemma 2.3. Under the conditions of Definition 2.1 we have that M_g is a bounded linear operator from $L^2((0, \infty); X)$ to itself with norm satisfying

$$(2) \quad \|M_g\| \leq \|g\|_{\infty}.$$

To show that Definition 2.1 leads to a functional calculus, we need the following concept from infinite dimensional systems theory, see e.g. [16].

Definition 2.4. Let Y be a Hilbert space, and C a linear operator bounded from $D(A)$, the domain of A , to Y . C is an admissible output operator if the mapping $x_0 \mapsto C e^{A \cdot} x_0$ can be extended to a bounded mapping from X to $L^2([0, \infty); Y)$.

Since in this paper only admissible output operators appear, we shall sometimes omit “output”. In [17] the following was proved.

Theorem 2.5. Let A be the generator of an exponentially stable semigroup on the Hilbert space X . For every $g \in \mathcal{H}_{\infty}^-$ there exists a linear mapping $g(A) : D(A) \mapsto X$ such that

$$((M_g(e^{A \cdot} x_0))(t) = g(A) e^{At} x_0, \quad x_0 \in D(A).$$

Furthermore,

- $g(A)$ is an admissible operator;

- $g(A)e^{At}$ extends to a bounded operator for $t > 0$;
- $g(A)$ commutes with the semigroup;
- $g(A)$ can be extended to a closed operator $g_{\Gamma}(A)$ such that $g \mapsto g_{\Gamma}(A)$ has the properties of an (unbounded) functional calculus;
- This (unbounded) calculus extends the Hille-Phillips calculus.

Hence in general the functional calculus constructed in this way will contain unbounded operators. However, they may not be “too unbounded”, as the product with any admissible operator is again admissible.

Theorem 2.6 (Lemma 2.1 in [17]). *Let A be the generator of an exponentially stable semigroup on the Hilbert space X and let C be an admissible operator, then*

$$(M_g(Ce^{A \cdot} x_0))(t) = Cg(A)e^{At}x_0, \quad x_0 \in D(A^2).$$

Moreover, $Cg(A)$ extends to an admissible output operator.

3. ANALYTIC SEMIGROUPS

From Theorem 2.5 we know that $g(A)e^{At}$ is a bounded operator for $t > 0$. In this section we show that for analytic semigroups the norm of $g(A)e^{At}$ behaves like $|\log(t)|$ for t close to zero. Let A generate an exponentially stable, analytic semigroup on the Hilbert space X . Then there exists a $M, \omega > 0$ such that, see [11, Theorem 2.6.13],

$$(3) \quad \|(-A)^{\frac{1}{2}}e^{At}\| \leq M \frac{1}{\sqrt{t}}e^{-\omega t}, \quad t > 0.$$

Using this inequality, we prove the following estimate.

Theorem 3.1. *Let A generate an exponentially stable, analytic semigroup on the Hilbert space X . There exists $m, \varepsilon_0 > 0$ such that for every $g \in \mathcal{H}_{\infty}^{-}$, $\varepsilon \in (0, \varepsilon_0)$*

$$(4) \quad \|g(A)e^{A\varepsilon}\| \leq m\|g\|_{\infty}|\log(\varepsilon)|.$$

If we assume that $(-A^)^{\frac{1}{2}}$ or $(-A)^{\frac{1}{2}}$ is admissible, then*

$$(5) \quad \|g(A)e^{A\varepsilon}\| \leq m\|g\|_{\infty}\sqrt{|\log(\varepsilon)|} \quad \text{for } \varepsilon \in (0, \varepsilon_0).$$

If both $(-A^)^{\frac{1}{2}}$ and $(-A)^{\frac{1}{2}}$ are admissible, then $g(A)$ is bounded.*

Proof. For $y \in D(A^*)$, $x \in D(A^2)$ we have

$$\begin{aligned} \frac{1}{2}\langle y, g(A)e^{A2\varepsilon}x \rangle &= \int_0^{\infty} \langle y, (-A)e^{A2t}g(A)e^{A2\varepsilon}x \rangle dt \\ &= \int_0^{\infty} \langle (-A^*)^{\frac{1}{2}}e^{A^*\varepsilon}e^{A^*t}y, g(A)(-A)^{\frac{1}{2}}e^{A\varepsilon}e^{At}x \rangle dt, \end{aligned}$$

where we used that $g(A)$ commutes with the semigroup. Using Cauchy-Schwarz's inequality, we find

$$\begin{aligned} (6) \quad \frac{1}{2}|\langle y, g(A)e^{A2\varepsilon}x \rangle| &\leq \|(-A^*)^{\frac{1}{2}}e^{A^*\varepsilon}e^{A^* \cdot}y\|_{L^2} \|g(A)(-A)^{\frac{1}{2}}e^{A\varepsilon}e^{A \cdot}x\|_{L^2} \\ &= \|(-A^*)^{\frac{1}{2}}e^{A^*\varepsilon}e^{A^* \cdot}y\|_{L^2} \cdot \|M_g\left((-A)^{\frac{1}{2}}e^{A\varepsilon}e^{A \cdot}x\right)\|_{L^2} \\ &\leq \|(-A^*)^{\frac{1}{2}}e^{A^*\varepsilon}e^{A^* \cdot}y\|_{L^2} \cdot \|g\|_{\infty} \cdot \|(-A)^{\frac{1}{2}}e^{A\varepsilon}e^{A \cdot}x\|_{L^2}, \end{aligned}$$

where we used Lemma 2.3. Hence it remains to estimate the two L^2 -norms. Since X is a Hilbert space $(e^{A^*t})_{t \geq 0}$ is an analytic semigroup as well. Hence both L^2 -norms behave similarly. We do the estimate for e^{At} . For $\omega\varepsilon < 1/4$,

$$\begin{aligned} \|(-A)^{\frac{1}{2}}e^{A\varepsilon}e^{A\cdot}x\|_{L^2}^2 &= \int_0^\infty \|(-A)^{\frac{1}{2}}e^{A\varepsilon}e^{At}x\|^2 dt \\ &= \int_\varepsilon^\infty \|(-A)^{\frac{1}{2}}e^{At}x\|^2 dt \\ &\leq M^2 \int_\varepsilon^\infty \frac{e^{-2\omega t}}{t} \|x\|^2 dt \\ &= M^2 \|x\|^2 \int_1^\infty \frac{e^{-2\varepsilon\omega t}}{t} dt \\ &\leq M^2 \|x\|^2 m_1 |\log(\varepsilon\omega)|, \end{aligned}$$

where we used (3) and m_1 is an absolute constant.

Combining the estimates and using the fact that ω is fixed, we find that there exists a constant $m_3 > 0$ such that for all $x \in D(A^2)$ and $y \in D(A^*)$ there holds

$$|\langle y, g(A)e^{A2\varepsilon}x \rangle| \leq m_3 |\log(\varepsilon)| \|g\|_\infty \|x\| \|y\|.$$

Since $D(A^2)$ and $D(A^*)$ are dense in X , we have proved the estimate (4).

We continue with the proof of inequality (5). If $(-A^*)^{\frac{1}{2}}$ is admissible, then (6) implies that

$$\begin{aligned} \frac{1}{2} |\langle y, g(A)e^{A2\varepsilon}x \rangle| &\leq \|(-A^*)^{\frac{1}{2}}e^{A^*\varepsilon}e^{A^*\cdot}y\|_{L^2} \|g(A)(-A)^{\frac{1}{2}}e^{A\varepsilon}e^{A\cdot}x\|_{L^2} \\ &\leq m_2 \|y\| \cdot \|M_g \left((-A)^{\frac{1}{2}}e^{A\varepsilon}e^{A\cdot}x \right)\|_{L^2}. \end{aligned}$$

The estimate follows as shown previously. Let us now assume that $(-A)^{\frac{1}{2}}$ is admissible. Then by Theorem 2.6 there holds

$$\begin{aligned} \|g(A)(-A)^{\frac{1}{2}}e^{A\varepsilon}e^{A\cdot}x\|_{L^2} &\leq \|g(A)(-A)^{\frac{1}{2}}e^{A\cdot}x\|_{L^2} \\ &= \|M_g \left((-A)^{\frac{1}{2}}e^{A\cdot}x \right)\|_{L^2} \\ &\leq \|g\|_\infty \|(-A)^{\frac{1}{2}}e^{A\cdot}x\|_{L^2} \\ &\leq \|g\|_\infty m \|x\|, \end{aligned}$$

where we have used Lemma 2.3 and the admissibility of $(-A)^{\frac{1}{2}}$. Now the proof of (5) follows similarly as in the first part.

If $(-A)^{\frac{1}{2}}$ and $(-A^*)^{\frac{1}{2}}$ are both admissible, then we see from the above that the epsilon disappears from the estimate, and since the semigroup is strongly continuous, $g(A)$ extends to a bounded operator. \square

In [13], it is shown that for any $\delta \in (0, 1)$ there exists an analytic, exponentially stable semigroup on a Hilbert space, and $g \in \mathcal{H}_\infty^-$ such that $(-A)^{\frac{1}{2}}$ is admissible and $\|g(A)e^{A\varepsilon}\| \sim (\sqrt{|\log(\varepsilon)|})^{1-\delta}$. Similarly, the sharpness of (4) is shown.

In the next section we relate the above theorem to results in the literature.

4. CLOSING REMARKS

A natural question is whether the calculus above coincides with other definitions of the \mathcal{H}_∞^- -calculus. As the construction extends the Hille-Phillips calculus, the answer is “yes”, see [14].

In [15], Vitse showed a similar estimate as in (4) for analytic semigroups on general Banach spaces by using the Hille-Phillips calculus. The setting there is slightly different since bounded analytic semigroups and functions $g \in \mathcal{H}_\infty^-$ with bounded Fourier spectrum are considered. In [13], the authors improve Vitse’s result with a more direct technique. In the course of that work, the approach to Theorem 3.1 via the calculus construction used here was obtained. Moreover, the techniques here and in Vitse’s work [15] require that the functions f are bounded, analytic on a half-plane. In [13] it is shown that the corresponding result is even true for functions f that are only bounded, analytic on sectors which are larger than the sectorality sector of the generator A .

Furthermore, Haase and Rozendaal proved that (4) holds for general (exponentially stable) semigroups on Hilbert spaces, see [6]. Their key tool is a *transference principle*. More general, they show that on general Banach spaces one has to consider the analytic multiplier algebra $\mathcal{AM}_2(X)$, as the function space to obtain a corresponding result. Note that $\mathcal{AM}_2(X)$ is continuously embedded in \mathcal{H}_∞^- with equality if X is a Hilbert space.

The difference in the transference principle and the approach followed here is that in the transference principle, estimates are first proved for “nice” functions and then extended to the whole space \mathcal{H}_∞^- . Whereas we prove the result first for “nice” elements in X , and then extend the operators $g(A)$.

The fact that the calculus is bounded for analytic semigroups when both $(-A)^{\frac{1}{2}}$ and $(-A^*)^{\frac{1}{2}}$ are admissible, can already be found in [8]. However, as the admissibility of $(-A)^{\frac{1}{2}}$ is equivalent to A satisfying *square function estimates*, the result is much older and goes back to McIntosh, [9].

The construction of the \mathcal{H}_∞^- -calculus followed here can be adapted to general Banach spaces, see [12, 14].

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