H^∞ CONTROL OF SYSTEMS WITH MULTIPLE I/O DELAYS*

Gjerrit Meinsma[†] **and** **Leonid Mirkin**[‡]

[†]*Department of Applied Mathematics, University of Twente, 7500 AE Enschede, The Netherlands*
E-mail: g.meinsma@utwente.nl

[‡]*Faculty of Mechanical Engineering, Technion — IIT, Haifa 32000, Israel*
E-mail: mirkin@technion.ac.il

Abstract: In this paper the standard (four-block) H^∞ control problem for systems with multiple i/o delays in the feedback loop is studied. The central idea is to see the multiple delay operator as a special series connection of elementary delay operators, called the *adobe delay* operators. The adobe delay case is solved and thereby the general case is solved as a nested set of solutions to adobe delay problems.

Keywords: Time-delay systems, dead-time compensation, H^∞ control

1 INTRODUCTION

Input/output time delays arise naturally in numerous control application, both from physical delays in processes and control interfaces and from the use of delays to model complicated high-frequency dynamics. Optimal control of time-delay systems has been an active research area since the late 60's, first in the H^2 (LQG) setting (Kleinman, 1969; Soliman and Ray, 1972) and then in the H^∞ setting (Foias *et al.*, 1996; Mirkin and Tadmor, 2002).

Time-delay systems can in principle be treated in the framework of the general theory of infinite-dimensional systems, both in the time (van Keulen, 1993) and in the frequency (Foias *et al.*, 1996) domains. These approaches, however, result in rather abstract results (i.e., in terms of operator Riccati equations), from which it might not be clear what the structures of solvability conditions and controllers are and how (if) they can be computed and implemented. This motivated researchers to seek for more problem-oriented approaches that exploit the special structure of the delay operator, see the review paper (Mirkin and Tadmor, 2002) and the references therein.

Although substantial progress has been made in this direction during the last two decades, the vast majority of the results (in both H^2 and H^∞ settings) is still limited to systems with a *single delay*. On the other hand, in MIMO systems different input/output channels might have different delays, so that multiple delay results are of great importance. Earlier treatments of multiple-delay systems are either produced quite complicated solutions (Soliman and Ray, 1972; Foias *et al.*, 1996) or were heavily based on the simplifying assumption that the delay operator commutes with the plant (Grimble and Hearns, 1998) which limits the scope of their applicability. An exception to this is a recent work by Kojima and Ishijima (Kojima and Ishijima, 2001), who derive explicit H^∞ solution for the case when the disturbance and/or control inputs are delayed. Yet in (Kojima and Ishijima, 2001) only input delays are considered and it is assumed that the controller has access to the full plant state.

In this paper the H^∞ control problem of systems with input/output delays is studied. The setup that we shall address is depicted in Fig. 1, where P is a given finite-dimensional plant, K_Λ is a controller to be determined, and Λ_u and Λ_y are given delay operators. When $\Lambda_u = \mathrm{e}^{-sh_u} I$ and $\Lambda_y = \mathrm{e}^{-sh_y} I$, such a setup corresponds to the single-delay problem. In our case the delay operators are more general block-diagonal matrices (see Section 2 for details). This enables to deal with different delays in different control and measurement channels.

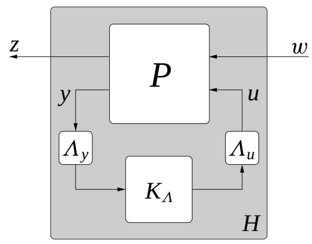


Fig. 1: Original 4-block problem formulation

The central idea of this paper is to split the multiple-delay problem to a nested sequence of simpler problems which we call *adobe problems*. The adobe problem is a problem with a single delay in *a part of* input or output channels. We sometimes distinguish *adobe input delay* and *adobe output delay* problems. These are apparently the simplest nontrivial generalizations of the single delay case. We show that both input and output adobe delay problems can be solved in an unified fashion using the approach developed in (Meinsma *et*

*This research was supported by THE ISRAEL SCIENCE FOUNDATION (grant No. 106/01).

al., 2002) (though with some nontrivial modifications). The solutions to the adobe problems are then tailored to constitute the solution to the original problem.

The advantage of the proposed approach is twofold. First, the split of the problem to elementary adobe problems (apart from the fact that this allows us to find the solution) clarifies how additional delays in certain channels affect the performance. Second, the approach results in a transparent controller structure. The controller consists of a finite-dimensional system with a feedback/feedforward part that, though infinite dimensional, can be easily implemented owing to the fact that its components may be chosen to be FIR. This structure is reminiscent of that of the single-delay H^∞ dead-time compensators (DTC) in (Meinsma and Zwart, 2000; Mirkin, 2003), though the presence of feedforward interchannel interconnections is unique to the multiple delay case.

It is worth stressing in this respect that there appears to be no natural generalization of single-delay Smith predictor (dead-time compensator) schemes to the case of multiple delays, see, e.g., the discussion in (Jerome and Ray, 1986). We believe that a byproduct of our solution might be a suggestion of a possible form of the multiple delay DTC.

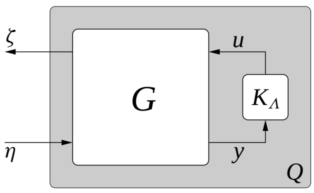


Fig. 2: A scattering representation

Notation Throughout the paper we frequently use scattering representations such as in Fig. 2. The arrows here can be confusing: what is meant in this figure is that $\left[\begin{smallmatrix}\zeta\\ \eta\end{smallmatrix}\right] = G\left[\begin{smallmatrix}u\\ y\end{smallmatrix}\right]$ and $u = Ky$. If the dimensions of η and y are the same then generically there corresponds to each K a unique mapping Q from η to ζ, denoted as $Q = \mathcal{C}_r(G, K)$. It is easy to verify that

$$\mathcal{C}_r(G, K) = (G_{11}K + G_{12})(G_{21}K + G_{22})^{-1}.$$

Once in a while we use the conventional lower linear fractional transformations (LFT's). For example the LFT $\mathcal{F}_l(P, \Lambda_u K_\Lambda \Lambda_y)$ means by definition the mapping from w to z in the system in Fig. 1.

We say that $K(s)$ is *proper* if $\sup_{\mathrm{Re}\, s>\rho} \|K(s)\| < \infty$ for some large enough $\rho \in \mathbb{R}$. As shown in (Weiss, 1994), an LTI system has a causal implementation iff its transfer matrix is proper. If $G(\infty) = I$ then properness of K implies properness of $Q \doteq \mathcal{C}_r(G, K)$, and since $K = \mathcal{C}_r(G^{-1}, Q)$ we in fact have that then the mapping is causally invertible.

Borrowing from (Mirkin, 2003) we define the *completion* operator $\boldsymbol{\pi}_h$, which "analytically completes" the impulse response of an h-delay system to a delay-free system. The "analytic completion" for delayed systems of the form $\mathrm{e}^{-sh}P = \mathrm{e}^{-sh}C(sI - A)^{-1}B$ is defined formally as

$$\boldsymbol{\pi}_h(\mathrm{e}^{-sh}P) = \left[\begin{array}{c|c} A & B \\ \hline C\mathrm{e}^{-Ah} & 0 \end{array}\right] - \mathrm{e}^{-sh}\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

($h > 0$). For finite dimensional P, the sum of $\mathrm{e}^{-sh}P$ an its completion $\boldsymbol{\pi}_h(\mathrm{e}^{-sh}P)$ is again finite dimensional.

A mapping $Q \in H^\infty$ is γ*-contractive* if $\|Q\|_\infty < \gamma$ (when $\gamma = 1$ we simply say "contractive"). A transfer matrix Q is *bistable* if $Q, Q^{-1} \in H^\infty$. The number of entries of a vector-valued signal w is denoted as n_w, for example $u(t) \in \mathbb{R}^{n_u}$.

2 PROBLEM FORMULATION

As mentioned in the Introduction, we study the feedback setup in Fig. 1. We assume that that the plant P there has the realization

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \qquad (1)$$

and that the following standard assumptions hold:

$\mathcal{A}_1$: (C_2, A, B_2) is stabilizable and detectable;

$\mathcal{A}_2$: $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank $\forall\omega \in \mathbb{R} \cup \infty$;

$\mathcal{A}_3$: $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank $\forall\omega \in \mathbb{R} \cup \infty$.

Note that assumptions $\mathcal{A}_2$ and $\mathcal{A}_3$ imply that $D_{12}'D_{12} > 0$ and $D_{21}D_{21}' > 0$, respectively. Note also that we do not assume that D_{11} and D_{22} are zero as these assumptions hardly simplify the results to come and, moreover, in delay systems nonzero D_{22} may appear naturally.

The delay elements are assumed to be of the diagonal form

$$\Lambda_u(s) = \mathrm{diag}\{\mathrm{e}^{-h_{u,q}s}I_{m_q}, \cdots, \mathrm{e}^{-h_{u,1}s}I_{m_1}, I_{m_0}\} \qquad (2a)$$

with $0 < h_{u,1} < \cdots < h_{u,q}$ ($\sum m_i = n_u$) and

$$\Lambda_y(s) = \mathrm{diag}\{I_{p_0}, \mathrm{e}^{-h_{y,1}s}I_{p_1}, \cdots, \mathrm{e}^{-h_{y,r}s}I_{p_r}\} \qquad (2b)$$

with $0 < h_{y,1} < \cdots < h_{y,r}$ ($\sum p_i = n_y$). In other words, we assume that there are q different input delay channels, r different output delay channels, and, possibly, two delay-free channels; $m_0 = 0$ ($p_0 = 0$) implies that there is no delay-free input (output) channel. Moreover, all delay channels are ordered (from large to small in Λ_u and from small to large in Λ_y). The latter assumption can be made without loss of generality; otherwise, a simple channel permutation is to be applied.

The problem studied in this paper is formulated as follows:

SHP: Given the system in Fig. 1 with the generalized plant P as in (1) satisfying $\mathcal{A}_{1\text{–}3}$ and the delays Λ_u and Λ_y as in (2). Determine whether there exists a proper K_Λ so that $K \doteq \Lambda_u K_\Lambda \Lambda_y$ internally stabilizes the system and guarantees that

$$\|\mathcal{F}_l(P, \Lambda_u K_\Lambda \Lambda_y)\|_\infty < \gamma \qquad (3)$$

for a given $\gamma > 0$, and then characterize all such K_Λ if one exists.

This problem is a nontrivial generalization of the single-delay H^∞ problem extensively studied in the control literature for the last two decades (Mirkin and Tadmor, 2002).

3 EQUIVALENT ONE-BLOCK REFORMULATION

It is clear that the problem is solvable only if so is its delay-free counterpart (delays just impose additional constraints on the controller). Following (Mirkin, 2003), we exploit this fact to reduce the *four-block* H^∞ problem with multiple delays to an equivalent *one-block* H^∞ problem with multiple delays. To this end we first need the standard solution, i.e., the solution for the situation that there are no delays.

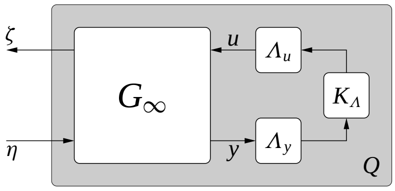


Fig. 3: An equivalent one-block problem

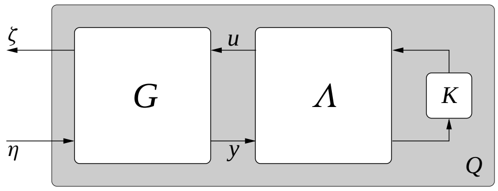


Fig. 4: Input and output delays combined into one block

3.1 Review of the standard delay-free solution

The solution to the standard delay-free H^∞ problem is currently well understood (Green and Limebeer, 1995; Kimura, 1996), so we only present here its features that are relevant for our development. Also, hereafter we assume without loss of generality that γ is such that the delay-free version of **SHP** is solvable.

We start with some nomenclature related to the H^∞ solution. Let $X \geq 0$ and $Y \geq 0$ be the stabilizing solutions to the standard H^∞ Riccati equations; $Z \doteq (I - \gamma^{-2}YX)^{-1}$ (well-defined by the solvability assumption); F_1 and F_2 be the H^∞ gains associated with the state-feedback problem whereas L_1 and L_2 be the corresponding filtering gains; $A_F \doteq A + B_1F_1 + B_2F_2$ and $A_L \doteq A + L_1C_1 + L_2C_2$ be the (stable) "closed-loop" matrices associated with state-feedback and filtering, respectively. Introduce also the following transfer matrix,

$$G_\infty(s) \doteq D_\infty \left[\begin{array}{c|c} A_L & B_\infty \\ \hline C_\infty Z^{-1} & I \end{array}\right], \tag{4}$$

where $B_\infty \doteq \begin{bmatrix} B_2 + L_1D_{12} + L_2D_{22} & -L_2 \end{bmatrix}$ and

$$C_\infty \doteq \begin{bmatrix} F_2 \\ C_2 + D_{21}F_1 + D_{22}F_2 \end{bmatrix},$$

and D_∞ is a nonsingular matrix obtained by the J-factorization of a matrix constructed from the feedthrough term of $P(s)$. It can be shown (Kimura, 1996) that

$$G_\infty(s)^{-1} = \left[\begin{array}{c|c} A_F & -ZB_\infty \\ \hline C_\infty & I \end{array}\right] D_\infty^{-1},$$

so that G_∞ is bistable. With these definitions, the standard solution then goes to show that the **SHP** is equivalent to finding a (proper) K for which

$$Q \doteq \mathcal{C}_r(G_\infty, K) \tag{5}$$

is contractive. In the delay-free case this settles the problem completely because the mapping $K \to Q$ is invertible,

$$K = \mathcal{C}_r(G_\infty^{-1}, Q), \tag{6}$$

and K is proper for almost every contractive Q. This yields the well known parameterization of all solutions K to the **SHP**: simply take any contractive Q and the resulting K does the job.

3.2 Including the delays

In our situation $K = \Lambda_u K_\Lambda \Lambda_y$ and we cannot invert the mapping (5) because the resulting K in (6) would simply cancel the delays in $\Lambda_u K_\Lambda \Lambda_y$ and hence the resulting K_Λ would not be proper.

Still we may begin the analysis with the simplified problem of finding contractive Q as in (5) so that the **SHP** is recast as the (one-block) problem of finding a proper K_Λ guaranteeing that the mapping $\eta \to \zeta$ in Fig. 3 is contractive, i.e., that $\| \mathcal{C}_r(G_\infty, \Lambda_u K_\Lambda \Lambda_y)\|_\infty < 1$. This reduction has a couple of advantages over a direct treatment of the **SHP**. First, it separates the delay-free problem from the delay problem thereby clarifying what part of the problem may be contributed purely to the delays. Moreover, it is useful to adopt chain-scattering representations rather than the more common LFT's since it reveals some extra structure. For example, the fact that G_∞ is bistable simplifies the further analysis considerably. Furthermore it allows us to consider the input and output delays on an equal footing. To see this, let us define the joint delay operator $\Lambda = \mathrm{diag}\{\Lambda_u, \Lambda_y^{-1}\}$ (mind the *inverse* Λ_y^{-1}). Then

$$\mathcal{C}_r(G_\infty, \Lambda_u K_\Lambda \Lambda_y) = \mathcal{C}_r(G_\infty \Lambda, K_\Lambda),$$

see Fig. 4. The so defined joint delay operator generally has advance elements (negative delays). Yet this is not an obstacle as Λ may be multiplied by a scalar operator α without affecting the mapping $\mathcal{C}_r(G_\infty \Lambda\alpha, K)$. We choose α to be the maximal delay term $\mathrm{e}^{-h_{y,r}s}$ in Λ_y that results in

$$\begin{aligned} \Lambda &\doteq \mathrm{e}^{-h_{y,r}s}\,\mathrm{diag}\big\{\Lambda_u, \Lambda_y^{-1}\big\} \\ &= \mathrm{diag}\{\mathrm{e}^{-h_{q+r}s} I_{n_{q+r}}, \cdots, \mathrm{e}^{-h_1 s} I_{n_1}, I_{n_0}\} \end{aligned} \tag{7}$$

(note that the input and output delay-free channels are united). Here $h_{q+r} > \cdots > h_1 > 0$ so that $h_{q+r} = h_{u,q} + h_{y,r}$ is the maximal delay between any two channels u_i and y_j in the system in Fig. 1. Note also that that $n_0 \neq 0$.

It will be useful to perform another simplification at this stage: to replace G_∞ with a transfer matrix having the identity feedthrough term. The given direct feedthrough term in (4) is D_∞. A special D_∞ can be constructed as follows. Note that **SHP** is solvable only if so is its finite-horizon version at any interval $[0, \tau]$. Therefore, **SHP** must also be solvable at $[0, \tau]$ for $\tau \to 0$. In the delay-free case the latter is equivalent to the existence of a matrix D_K so that $\mathcal{F}_l(P(\infty), D_K)$ is γ-contractive. Yet delayed loops do not participate in such finite-horizon problem (they are open on $[0, \tau]$ whenever τ is small enough). Hence, **SHP** is solvable only if there exists a matrix $D_0 \in \mathbb{R}^{m_0 \times p_0}$ so that

$$\| \mathcal{F}_l(P(\infty), E_u D_0 E_y')\| < \gamma, \tag{8}$$

where the matrices

$$E_u \doteq \begin{bmatrix} 0 \\ I_{m_0} \end{bmatrix} \in \mathbb{R}^{n_u \times m_0} \text{ and } E_y \doteq \begin{bmatrix} I_{p_0} \\ 0 \end{bmatrix} \in \mathbb{R}^{n_y \times p_0}$$

are the directions of the delay-free input and output channels, respectively. Using D_0 it is now possible to construct a special D_∞ of the form

$$D_\infty = V \begin{bmatrix} I & -E_u D_0 E_y' \\ 0 & I \end{bmatrix}, \quad V \text{ is lower triangular.}$$

(Proof omitted due to lack of space.) This D_∞ has the property that $\Lambda^{-1} D_\infty \Lambda$ is bi-causal, bi-stable and the mapping mapping $K_\Lambda \to K = \mathcal{C}_r(\Lambda^{-1} D_\infty \Lambda, K_\Lambda)$ is causally invertible. If we define

$$G(s) \doteq G_\infty(s) D_\infty^{-1} = \left[\begin{array}{c|c} A_L & B_\infty D_\infty^{-1} \\ \hline D_\infty C_\infty Z^{-1} & I \end{array}\right] \tag{9}$$

then G has a identity direct feedthrough term and $\mathcal{C}_r(G_\infty \Lambda, K_\Lambda) = \mathcal{C}_r(G\Lambda, K)$ with $K = \mathcal{C}_r(\Lambda^{-1} D_\infty \Lambda, K_\Lambda)$. We thus end up with the following one-block H^∞ problem:

OBP: Given the system in Fig. 4 with G and Λ as in (9) and (7), respectively, determine whether there exists a proper K which guarantees that

$$\| \mathcal{C}_r(G\Lambda, K)\|_\infty < 1, \tag{10}$$

and then characterize all such K if one exists.

The following lemma, which was actually proved above, establishes that **SHP** can be solved in terms of the simpler problem **OBP**:

Lemma 1. ***SHP*** *is solvable only if so is its delay-free counterpart and there exists a matrix* D_0 *such that* (8) *holds. If these conditions hold, then* ***SHP*** *is solvable iff* ***OBP*** *is solvable. Moreover, a proper* K *solves the latter problem iff*

$$K_\Lambda \doteq \mathcal{C}_r(\Lambda^{-1} D_\infty^{-1} \Lambda, K)$$

solves the former.

4 ADOBE DELAY PROBLEM

By *adobe* delay we mean the case that the joint delay operator is of the form

$$\Lambda = \begin{bmatrix} e^{-sh} I_\mu & 0 \\ 0 & I_\rho \end{bmatrix} \tag{11}$$

for some $\mu < n_u + n_y$ and $\rho = n_u + n_y - \mu$. These adobe problems serve as building blocks from which the general **OBP** will be solved later.

Note that the dimensions (μ, ρ) do not necessarily match the dimensions of the input and output signals. In fact, the case of $\mu = n_u$ (and, consequently, $\rho = n_y$) corresponds to the single-delay problem treated in (Meinsma *et al.*, 2002). Indeed, for the single-delay problem

$$\Lambda_u = e^{-h_u s} I_{n_u} \quad \text{and} \quad \Lambda_y = e^{-h_y s} I_{n_y},$$

that results in $\Lambda = \text{diag}\{e^{-hs} I_{n_u}, I_{n_y}\}$ with $h = h_y + h_u$. The case $\mu > n_u$ can then be thought of as resulting from

$$\Lambda_u = I_{n_u} \quad \text{and} \quad \Lambda_y = \text{diag}\{I_{n_y - \rho}, e^{-h_y s} I_\rho\}. \tag{12}$$

We thus call the corresponding adobe problem *the adobe plant output delay* problem. Similarly, $\mu < n_u$ may correspond to

$$\Lambda_u = \text{diag}\{e^{-h_u s} I_\mu, I_{n_u - \mu}\} \quad \text{and} \quad \Lambda_y = I_{n_y}, \tag{13}$$

so we call it *the adobe plant input delay* problem. It is worth stressing that in the last two cases controller structures and interpretations are quite different (see below). On the other hand, the *formulae* in all cases above are in a sense the same.

4.1 The main result

Let us rewrite the realization of G from (9) as follows:

$$G(s) = \left[\begin{array}{c|cc} A_L & B_\mu & B_\rho \\ \hline C_\mu & I_\mu & 0 \\ C_\rho & 0 & I_\rho \end{array}\right], \tag{14}$$

where the partitioning is compatible with (11). Throughout this section we denote $J \doteq \text{diag}\{I_{n_u}, -I_{n_y}\}$ and also introduce the following two signature matrices:

$$J_\mu \doteq \begin{bmatrix} I_\mu & 0 \end{bmatrix} J \begin{bmatrix} I_\mu \\ 0 \end{bmatrix} \quad \text{and} \quad J_\rho \doteq \begin{bmatrix} 0 & I_\rho \end{bmatrix} J \begin{bmatrix} 0 \\ I_\rho \end{bmatrix}$$

Denote now the symplectic matrix function

$$\Sigma(t) = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{bmatrix} \doteq e^{Ht}, \tag{15}$$

where H is the following Hamiltonian matrix,

$$H \doteq \begin{bmatrix} A_L - B_\rho C_\rho & -B_\rho J_\rho B_\rho' \\ -C_\mu' J_\mu C_\mu & -A_L' + C_\rho' B_\rho' \end{bmatrix} \tag{16}$$

(note that H does not depend on B_μ). To simplify the notations, we write Σ instead of $\Sigma(h)$. Then the main result of this section is as follows:

Theorem 1. ***OBP*** *with joint delay operator* (11) *is solvable iff* $\Sigma_{22}(t)$ *is nonsingular* $\forall t \in [0, h]$. *In that case* K *solves* ***OBP*** *iff*

$$K = \mathcal{C}_r\left(\left[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\right] \tilde{G}^{-1}, \tilde{Q}\right)$$

where

$$\tilde{G} = \left[\begin{array}{c|cc} A_L & \Sigma_{22}' B_\mu + \Sigma_{12}' C_\mu' J_\mu & B_\rho \\ \hline J_\mu C_\mu \Sigma_{22}^{-\prime} & I_\mu & 0 \\ C_\rho - J_\rho B_\rho' \Sigma_{22}^{-1} \Sigma_{21} & 0 & I_\rho \end{array}\right]$$

is bistable,

$$\Pi = \pi_h \left\{ e^{-sh} \left[\begin{array}{cc|c} H_{11} & H_{12} & B_\mu \\ H_{21} & H_{22} & -C_\mu' J_\mu \\ \hline C_\rho & J_\rho B_\rho' & 0 \end{array}\right] \right\}$$

is FIR, and $\|\tilde{Q}\|_\infty < 1$ *but otherwise arbitrary.*

The following corollary of Theorem 1 will be used in the sequel:

Corollary 1. *Let the condition of Theorem 1 hold. Then* K *solves the adobe* ***OBP*** *iff*

$$\mathcal{C}_r\left(\tilde{G}, \mathcal{C}_r\left(\left[\begin{smallmatrix} I & 0 \\ -\Pi & I \end{smallmatrix}\right], K\right)\right),$$

is a contraction.

The proof of Theorem 1 follows the steps in (Meinsma *et al.*, 2002), though the proofs of some of these steps are nontrivially different.

4.2 Necessity: finite-horizon problem

Consider the finite-horizon version of **OBP**. If Λ_u, Λ_y are given by (13), then then delayed channels of u are zero $\forall t \in [0, h]$. Hence, these channels can be safely eliminated on this finite horizon. The system then can be equivalently described as

$$\begin{bmatrix} \zeta \\ \eta \end{bmatrix} = G_\rho \begin{bmatrix} u_\rho \\ y_\rho \end{bmatrix}, \qquad u_\rho = K_\rho y_\rho. \tag{17}$$

Here

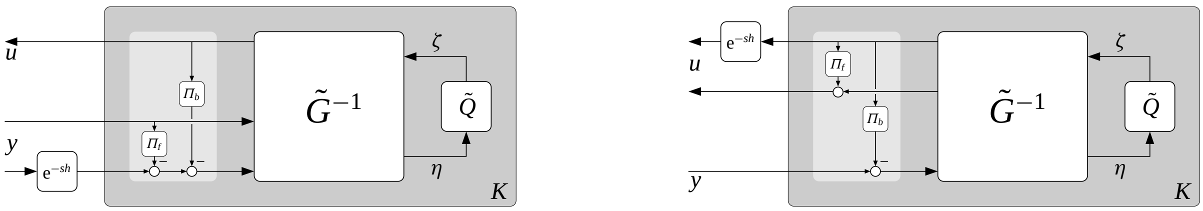
$$G_\rho(s) \doteq \left[\begin{array}{c|c} A_L & B_\rho \\ \hline C_\mu & 0 \\ C_\rho & I_\rho \end{array}\right] \doteq \left[\begin{array}{c|c} A_L & B_\rho \\ \hline C_g & D_\rho \end{array}\right],$$

$u_\rho : [0, h] \mapsto \mathbb{R}^{\mu - n_u}$ and $y_\rho : [0, h] \mapsto \mathbb{R}^{n_y}$. Thus, the finite-horizon version of **OBP** with delays as in (13) is solvable only if there exists a causal K_ρ such that

$$\sup \frac{\|\zeta\|_{L^2[0,h]}}{\|\eta\|_{L^2[0,h]}} < 1, \tag{18}$$

where the supremum is taken over all η and ζ satisfying (17).

The above is a finite horizon closed loop argument. Now if the delays Λ_u, Λ_y are given by (12) then dually a finite-horizon open-loop argument applies. In this case the last ρ



(a) Adobe plant output delay

(b) Adobe plant input delay

Fig. 5: Controller structure

channels of y are delayed. Hence for y of the form $y = \left[\begin{smallmatrix} 0 \\ y_\rho \end{smallmatrix}\right]$ we do not get any response u on $[0, h]$ whatever K is (a long as it is causal). Therefore for any such y Eqn. (17) holds with u_ρ and K_ρ void. It may be verified that every y of the form $y = \left[\begin{smallmatrix} 0 \\ y_\rho \end{smallmatrix}\right] : [0, h] \mapsto \left[\begin{smallmatrix} \mathbb{R}^{n_y - \rho} \\ \mathbb{R}^{\rho} \end{smallmatrix}\right]$ is possible by proper choice of input η. Hence also in this case the finite-horizon version of the **OBP** is solvable only if (18) holds over all possible ζ, η of the form (17). The two finite-horizon necessary requirements (closed-loop and open-loop) have a joint characterization.

We start with the following technical result:

Lemma 2. *The operator* $G_\rho^* J G_\rho : L^2[0, t] \mapsto L^2[0, t]$ *is singular iff* $\det \Sigma_{22}(t) = 0$.

Proof. Can be deduced from (Gohberg and Kaashoek, 1984). Details are omitted because of space limitations. □

Now we are in the position to formulate our main result:

Lemma 3. *Let* Λ *be as in* (11). *There exists a causal* K *such that* (18) *holds only if* $\det \Sigma_{22}(t) \neq 0$ *for all* $t \in [0, h]$.

Proof. Assume to the contrary that $\Sigma_{22}(t)$ is singular for some $t \in [0, h]$. By Lemma 2 this means that

$$\xi^\circ \doteq \begin{bmatrix} u_\rho^\circ \\ y_\rho^\circ \end{bmatrix} \neq 0$$

exists such that $G_\rho^* J G_\rho \xi^\circ = 0$. Now for any such ξ° define the “worst” signals

$$\begin{bmatrix} \zeta^\circ \\ \eta^\circ \end{bmatrix} \doteq G_\rho \xi^\circ$$

(notice that $(\zeta^\circ, \eta^\circ) \neq 0$ because D_ρ has full column rank, and that by construction, $\|\zeta^\circ\|_{L^2[0,t]} = \|\eta^\circ\|_{L^2[0,t]}$). In what follows all mappings and inner products are over $[0, t]$. Take $\eta = \eta^\circ$ as input to the system of Fig. 4. Then given any causal K the resulting closed loop signals $\left[\begin{smallmatrix} u_\rho \\ y_\rho \end{smallmatrix}\right] : [0, h] \mapsto \mathbb{R}^\rho$ are unique and they are such that

$$\begin{bmatrix} H\eta^\circ \\ \eta^\circ \end{bmatrix} = G_\rho \begin{bmatrix} u_\rho \\ y_\rho \end{bmatrix}, \qquad H \doteq \mathcal{C}_r(G, K).$$

Hence

$$\begin{aligned} \langle H\eta^\circ, \zeta^\circ \rangle - \langle \eta^\circ, \eta^\circ \rangle &= \left\langle \left[\begin{smallmatrix} H \\ I \end{smallmatrix}\right] \eta^\circ, J \left[\begin{smallmatrix} \zeta^\circ \\ \eta^\circ \end{smallmatrix}\right] \right\rangle \\ &= \left\langle G_\rho \left[\begin{smallmatrix} u_\rho \\ y_\rho \end{smallmatrix}\right], J G_\rho \xi^\circ \right\rangle = 0. \end{aligned}$$

This together with the fact that $\langle \zeta^\circ, \zeta^\circ \rangle = \langle \eta^\circ, \eta^\circ \rangle$ shows that

$$\langle H\eta^\circ, \zeta^\circ \rangle = \langle \eta^\circ, \eta^\circ \rangle = \langle \zeta^\circ, \zeta^\circ \rangle.$$

Cauchy-Schwartz inequality yields then that $\|H\|_{L^2[0,t]} \geq 1$ (and equality holds only if $H\eta^\circ = \zeta^\circ$, in which case $\|H\eta^\circ\|_2 = \|\eta^\circ\|_2$, hence the name “worst disturbance” for η°). The proof is complete on noting that $\|H\|_{L^2[0,t]} \leq \|H\|_{L^2[0,h]}$, $\forall t \leq h$. □

It is worth noting that the condition of Lemma 3 is actually also sufficient (in fact it is a byproduct of Theorem 1). We, however, do not need this fact in the proof of Theorem 1.

4.3 Controller structure

For implementation of the controller in Theorem 1 it is convenient to repartition $\left[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\right]$ compatibly with the dimensions of u and y in Fig. 3.

In the adobe plant output delay case ($\mu \geq n_u$) we have:

$$\begin{bmatrix} I_\mu & 0 \\ \Pi & I_\rho \end{bmatrix} = \left[\begin{array}{c:cc} I_{n_u} & 0 & 0 \\ \hdashline 0 & I_{\mu - n_u} & 0 \\ \Pi_b & \Pi_f & I_\rho \end{array}\right].$$

The structure of the controller K from Theorem 1 hence is as shown in Fig. 5(a). It consists of the rational (bistable) part $\tilde{G}$, a free contractive parameter $\tilde{Q}$, and two irrational stable (FIR) blocks: Π_b and Π_f. The former FIR block is in fact the internal feedback in the controller reminiscent the classical dead-time compensators (DTC) or Smith predictors. The only difference from the DTC that appears in the single-delay H^∞ control is that Π_b acts only on a part of the measurement channels, namely, on the delayed channel. On the other hand, Π_f acts as an interchannel feedforward part of the controller and has no direct counterpart in the Smith predictor literature.

In the adobe plant input delay case ($\mu \leq n_u$) we have:

$$\begin{bmatrix} I_\mu & 0 \\ \Pi & I_\rho \end{bmatrix} = \left[\begin{array}{cc:c} I_{n_u} & 0 & 0 \\ \Pi_f & I_{\rho - n_y} & 0 \\ \hdashline \Pi_b & 0 & I_{n_y} \end{array}\right].$$

The structure of this controller is shown in Fig. 5(b). As in the output delay case, the DTC part of the controller contains two different FIR blocks. The first one, Π_b, acts as an internal feedback from the *delayed* control channel to the measured signal, while the second one, Π_f, acts as an interchannel feedforward from the delayed control channel to the delay-free one.

5 DECOMPOSITION

Now we are in a position to address the decomposition of **OBP** to a series of adobe problems. We return to the general joint delay operator Λ in (7), which contains $q + r$ descendant ordered delay blocks and for that reason we refer to it as a *$(q + r)$-delay operator*. In the future references, we denote **OBP** with the data G and Λ as **OBP(**G, Λ**)**. Also, given two equally dimensioned joint delay operators Λ_α and Λ_β of the form (7), we write $\Lambda_\alpha \succ \Lambda_\beta$ (or, equivalently, $\Lambda_\beta \prec \Lambda_\alpha$) if the last (delay-free) block of Λ_α has *strictly larger* dimension than that of Λ_β.

It is readily verified that the $(q + r)$-delay operator can be decomposed as follows:

$$\Lambda = \Lambda_1 \tilde{\Lambda}, \tag{19}$$

where

$$\Lambda_1 \doteq \begin{bmatrix} e^{-h_1 s} I_{\mu_1} & 0 \\ 0 & I_{\rho_1} \end{bmatrix} \quad (\text{with } \rho_1 = n_0)$$

is the joint delay operator of the adobe problem, cf. (11), and $\tilde{\Lambda}$ is actually a $(q+r-1)$-delay operator with the (n_0+n_1)-dimensional delay-free channel (i.e., $\tilde{\Lambda} \succ \Lambda$) and the smallest delay $h_2 - h_1$. Thus,

$$\mathcal{C}_r(G\Lambda, K) = \mathcal{C}_r\big(G\Lambda_1, \mathcal{C}_r(\tilde{\Lambda}, K)\big).$$

As the delay block $\tilde{\Lambda}$ above just imposes additional constraints on K, **OBP(**G, Λ**)** is solvable *only if* so is the adobe delay problem **OBP(**G, Λ_1**)**. According to Corollary 1, the latter problem is solvable iff the condition of Theorem 1 holds and

$$\tilde{Q} \doteq \mathcal{C}_r\big(\tilde{G}\big[\begin{smallmatrix} I & 0 \\ -\Pi & I \end{smallmatrix}\big], \mathcal{C}_r(\tilde{\Lambda}, K)\big)$$

is a contraction in H^∞, where $\tilde{G}$ and Π are defined in Theorem 1. It is important to note now that any nonsingular lower triangular matrix "causally commutes" with joint delay operators of form (7) in the sense that $\Lambda^{-1} O_L \Lambda$ is bi-causal. Thus, K is proper iff

$$\tilde{K} \doteq \mathcal{C}_r\big(\tilde{\Lambda}^{-1}\big[\begin{smallmatrix} I & 0 \\ -\Pi & I \end{smallmatrix}\big]\tilde{\Lambda}, K\big)$$

is proper and, consequently, $\tilde{Q}$ is a contraction iff there exists a proper $\tilde{K}$ so that

$$\| \mathcal{C}_r(\tilde{G}\tilde{\Lambda}, \tilde{K}) \| < 1.$$

Yet this is just another one-block problem, **OBP(**$\tilde{G}, \tilde{\Lambda}$**)**. Moreover, since $\tilde{\Lambda} \succ \Lambda$, the latter problem has reduced complexity comparing with the original problem **OBP(**G, Λ**)**. We thus just proved the following result:

Lemma 4. *Let G be as in* (14) *and Λ as in* (19). *Then **OBP(**G, Λ**)** is solvable iff the adobe problem **OBP(**G, Λ_1**)** and the reduced complexity **OBP(**$\tilde{G}, \tilde{\Lambda}$**)** are both solvable. Furthermore, in that case a proper K solves **OBP(**G, Λ**)** iff*

$$K = \mathcal{C}_r\big(\tilde{\Lambda}^{-1}\big[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\big]\tilde{\Lambda}, \tilde{K}\big)$$

*with $\tilde{K}$ a solution of **OBP(**$\tilde{G}, \tilde{\Lambda}$**)** (here $\tilde{G}$ and Π are as defined in Theorem 1).*

Now, we can proceed with the $(q+r-1)$-delay operator exactly the same way as with the $(q+r)$-delay operator before. More precisely, let us substitute $\tilde{G} \to G$, $\tilde{\Lambda} \to \Lambda$, and $\tilde{K} \to K$. Then, repeating arguments from the beginning of this section, the solvability of the one-block problem with the $(q+r-1)$-delay operator can be shown to be equivalent to the solvability of a adobe problem with

$$\Lambda_2 \doteq \begin{bmatrix} e^{-(h_2-h_1)s} I_{\mu_2} & 0 \\ 0 & I_{\rho_2} \end{bmatrix} \quad (\rho_2 = n_0 + n_1)$$

and a one-block problem with a $(q+r-2)$-delay operator. This procedure can obviously be repeated $q+r$ times, each time resulting to an **OBP** with a "smaller" delay operator, until we end up with a one-block problem with (0)-delay operator, the solution for which consists simply on the inversion of its "G" transfer function.

OBP (and therefore **SHP**) can thus be solved iteratively, in $q+r$ iterations. The ith iteration consists on solving the adobe delay problem **OBP(**G_i, Λ_i**)**, where

$$\Lambda_i \doteq \begin{bmatrix} e^{-(h_i-h_{i-1})s} I_{\mu_i} & 0 \\ 0 & I_{\rho_i} \end{bmatrix} \quad (\rho_i = \textstyle\sum_{j=0}^{i-1} n_j) \qquad (20a)$$

and (bistable) G_i is generated by the following sequence:

$$G_i = \tilde{G}_{i-1} \quad [\text{with } G_1 = G \text{ as defined by (9)}], \qquad (20b)$$

where $\tilde{G}_{i-1}$ implies the "$\tilde{G}$" matrix appearing in the solution of the adobe problem **OBP(**G_{i-1}, Λ_{i-1}**)**. The solutions of all iterations are then tailored to constitute the solution to the original multiple delay problem. The following theorem, which is the main result of this paper, summarizes the reasoning above.

Theorem 2. ***OBP** is solvable iff so are all **OBP(**G_i, Λ_i**)**, $i = 1, \ldots, q+r$. In this case, all solutions to the former are parameterized as*

$$K = \mathcal{C}_r\big(\Pi_\Lambda G_\Lambda^{-1}, Q_\Lambda\big),$$

where $G_\Lambda \doteq \tilde{G}_{q+r}$ is bistable and finite dimensional,

$$\Pi_\Lambda \doteq \Lambda^{-1} \prod_{i=1}^{q+r} \Lambda_i \big[\begin{smallmatrix} I & 0 \\ \Pi_i & I \end{smallmatrix}\big]$$

is bistable, and Q_Λ is an arbitrary contraction.

REFERENCES

Curtain, R. F. and H. Zwart (1995). *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer-Verlag. NY.

Foias, C., H. Özbay and A. Tannenbaum (1996). *Robust Control of Infinite Dimensional Systems: Frequency Domain Methods*. Vol. 209 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag. London.

Gohberg, I. and M. A. Kaashoek (1984). Time varying linear systems with boundary conditions and integral operators, I. The transfer operator and its properties. *Integral Equations and Operator Theory* **7**, 325–391.

Green, M. and D. J. N. Limebeer (1995). *Linear Robust Control*. Prentice-Hall. Englewood Cliffs, NJ.

Grimble, M. J. and G. Hearns (1998). LQG controllers for state-space systems with pure transport delays: Application to hot strip mills. *Automatica* **34**(10), 1169–1184.

Jerome, N. F. and W. H. Ray (1986). High-performance multivariable control strategies for systems having time delays. *AIChE Journal* **32**(6), 914–931.

Kimura, H. (1996). *Chain-Scattering Approach to H^∞ Control*. Birkhäuser. Boston.

Kleinman, D. L. (1969). Optimal control of linear systems with time-delay and observation noise. *IEEE Trans. Automat. Control* **14**(5), 524–527.

Kojima, A. and S. Ishijima (2001). H^∞ control for preview and delayed strategies. In: *Proc. 40th IEEE Conf. Decision and Control*. Orlando, FL. pp. 991–996.

Meinsma, G. and H. Zwart (2000). On H^∞ control for dead-time systems. *IEEE Trans. Automat. Control* **45**(2), 272–285.

Meinsma, G., L. Mirkin and Q.-C. Zhong (2002). Control of systems with I/O delay via reduction to a one-block problem. *IEEE Trans. Automat. Control* **47**(11), 1890–1895.

Mirkin, L. (2003). On the extraction of dead-time controllers and estimators from delay-free parametrizations. *IEEE Trans. Automat. Control* **48**(4), 543–553.

Mirkin, L. and G. Tadmor (2002). H^∞ control of system with I/O delay: A review of some problem-oriented methods. *IMA J. Math. Control & Information* **19**, 185–199.

Soliman, M. A. and W. H. Ray (1972). Optimal feedback control for linear-quadratic systems having time delays. *Int. J. Control* **15**(4), 609–627.

van Keulen, B. (1993). *H^∞ Control for Distributed Parameter Systems: A State Space Approach*. Birkhäuser. Boston.

Weiss, G. (1994). Transfer functions of regular linear systems. Part I: Characterizations of regularity. *Trans. Amer. Math. Soc.* **342**(2), 827–854.

Zhou, K., J. C. Doyle and K. Glover (1995). *Robust and Optimal Control*. Prentice-Hall. Englewood Cliffs, NJ.