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Synchronization in Networks of Nonlinear, Non-Introspective, Minimum-Phase Agents Without Exchange of Controller States

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***Abstract*— We consider the synchronization problem for homogeneous networks of nonlinear SISO agents connected via diffusive partial-state coupling. The agents are non-introspective (i.e., they do not have access to their own state or output), and thus unable to manipulate their own dynamics in order to present themselves differently to the network. Moreover, the agents are not allowed to exchange additional information, such as internal controller states, over the network. Unlike many other designs for nonlinear synchronization, we do not require the agents to be passive; we rely instead on a canonical form that requires the nonlinearities to have a certain lower-triangular structure.**

I. INTRODUCTION

In recent years, a large body of work has emerged on the topic of *synchronization*, where the goal is to secure agreement among networked agents on a common state or output trajectory. Most of this work has focused on on synchronization of *linear* agents, based on diffusive *state coupling* [1]–[5] or *partial-state coupling* [6]. In the context of the latter, Li, Duan, Chen, and Huang [7] introduced a distributed observer that makes additional use of the network by allowing the agents to exchange information with their neighbors about their internal estimates, effectively requiring another layer of communication. On the other hand, Seo, Shim, and Back [8] presented a *low-gain* control design that does not require the exchange of internal states, provided the poles of the agent dynamics are located in the closed left-half complex plane. Much of the synchronization literature is rooted in the seminal work of Wu and Chua [9], [10].

The works cited above are concerned with *homogeneous* networks; others have considered *heterogeneous* networks, where the agents are governed by non-identical dynamical models [11]–[15]. Design methodologies for heterogeneous networks typically assume that the agents are *introspective*, meaning that they have access to information about their own state or output in addition to the information received from the network. The authors have recently considered the more challenging case of *non-introspective* agents, and developed a methodology based on a distributed high-gain observer [16]. Like for several other designs for heterogeneous networks [12], [14], it was assumed that the agents can exchange internal controller states with neighboring agents in the network; this issue was recently addressed by introducing a design that combines elements of high-gain and low-gain design [17].

Some authors have also studied synchronization in networks with *nonlinear* agent dynamics [18]–[25], sometimes in combination with network heterogeneity. Explicit control designs for nonlinear networks have to a large degree centered on the relatively strict assumption of *passivity*. Passivity can in some cases be ensured by first applying local pre-feedbacks to the system; however, this requires the system to be introspective.

A. Topics of This Paper

In this paper we consider networks of agents connected via partial state coupling, with nonlinear and time-varying dynamics that go beyond the assumption of passivity. Moreover, these agents are non-introspective (hence, they cannot make themselves passive by employing local pre-feedbacks), and they are unable to exchange any additional information, such as controller states, via the network.

Roughly speaking, we require the agent dynamics to be transformable to a canonical form in which the nonlinearities appear in a lower-triangular form. Unlike a passive system, we place no requirements on the open-loop stability of the agent dynamics, and no requirements on the relative degree of the system. We do, however, assume that the linear part of the system is minimum-phase. We furthermore consider only SISO systems; although it is straightforward to extend the results to certain classes of MIMO systems, the more general problem is difficult (see, e.g., [26] for an indication of the complexity).

We start in Section II by presenting our precise problem formulation and our assumptions on the network topology and the agent dynamics. The latter involves the introduction of a nonlinear canonical form that is an extension of the *special coordinate basis* (SCB) [27] for linear SISO systems. In Section III, we present our main result, which is a constructive design that ensures synchronization under the given assumptions. The design is based on a combination of low-gain and high-gain techniques, similar to the authors' previous design for heterogeneous networks [17]. In Section IV we consider the question of when and how a given nonlinear system can be transformed to the required canonical form via linear state and input transformations. In Section V we present an example that illustrates our design technique.

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B. Notation and Definitions

For a matrix A, A' denotes its transpose and A^* denotes its conjugate transpose. The Kronecker product between A and B is denoted by $A \otimes B$. We denote by $[X_1;\ldots;X_n]$ the vector or matrix obtained by stacking $X_1,\ldots,X_n$.

II. Problem Formulation

The networks that will be considered in this paper consist of N nonlinear SISO agents, with the state and output of agent $i \in \{1,\ldots,N\}$ denoted by x_i and y_i, respectively. Our goal is to achieve state synchronization among the agents, meaning that $\lim_{t\to\infty}(x_i - x_j) = 0$ for all $i,j \in \{1,\ldots,N\}$.

A. Network Communication

The agents are non-introspective; hence, agent i does not have access to its own state x_i or output y_i. The information available to each agent comes from the network, in the form of a linear combination of its own output relative to that of the other agents. In particular, agent i has access to the quantity

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j),$$

where $a_{ij} \geq 0$ and $a_{ii} := 0$.

The communication topology of the network can be described by a directed graph (digraph) $\mathscr{G}$ with nodes corresponding to the agents in the network and edges given by the coefficients a_{ij}. In particular, $a_{ij} > 0$ implies that an edge exists from agent j to i, in which case j is called a *parent* of agent i and agent i is called a *child* of agent j. The weight of the edge equals the magnitude of a_{ij}. We say that there exists a *directed path* from node i to node j if $\mathscr{G}$ contains a sequence of edges originating at node i and terminating at node j.

We shall make use of the matrix $G = [g_{ij}]$, where $g_{ii} = \sum_{j=1}^{N} a_{ij}$ and $g_{ij} = -a_{ij}$ for $j \neq i$. This matrix is known as the *Laplacian* of $\mathscr{G}$ and has the property that all the row sums are zero. In terms of the coefficients of G, ζ_i can be rewritten as

$$\zeta_i = \sum_{j=1}^{N} g_{ij} y_j.$$

We make the following assumption about the network's communication topology.

Assumption 1: The graph $\mathscr{G}$ contains a directed spanning tree.

Remark 1: A directed tree is a subgraph of $\mathscr{G}$ in which every node has exactly one parent, except a single root node with no parents. Moreover, there must be a directed path from the root node to every other node in the tree. A *directed spanning tree* is a directed tree containing all the nodes of the graph.

Assumption 1 implies that the Laplacian G has a single eigenvalue at the origin and that all the other eigenvalues are located in the open right-half complex plane [28]. In our design, we do not assume knowledge of the network topology; however, we do assume knowledge of a lower bound $\tau > 0$ on the real parts of the non-zero eigenvalues.

B. Nonlinear Agent Dynamics

We shall assume that the dynamics of each agent complies with a certain nonlinear canonical form. In order to introduce this canonical form, consider first a *linear* SISO system

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u}, \qquad \bar{x} \in \mathbb{R}^n, \quad \bar{u} \in \mathbb{R}, \tag{1a}$$
$$y = \bar{C}\bar{x}, \qquad y \in \mathbb{R}, \tag{1b}$$

of relative degree $\rho \geq 1$. For this linear system there always exist nonsingular state and input transformations $\Gamma_x \in \mathbb{R}^{n\times n}$ and $\Gamma_u \in \mathbb{R}$ such that, by defining $\bar{x} = \Gamma_x x$ and $\bar{u} = \Gamma_u u$, the dynamics of the system with state x, input u, and output y is in the SCB [27]. The state vector x in the SCB can be partitioned as $x = [x_a; x_d]$, where $x_a \in \mathbb{R}^{n-\rho}$ and $x_d \in \mathbb{R}^{\rho}$, and the state equations are given by

$$\dot{x}_a = A_a x_a + L_{ad} y, \tag{2a}$$
$$\dot{x}_d = A_d x_d + B_d(u + E_{da}x_a + E_{dd}x_d), \tag{2b}$$
$$y = C_d x_d, \tag{2c}$$

where the matrices $A_d \in \mathbb{R}^{\rho\times\rho}$, $B_d \in \mathbb{R}^{\rho\times 1}$, and $C_d \in \mathbb{R}^{1\times\rho}$ have the special form

$$A_d = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C_d = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{3}$$

Furthermore, the eigenvalues of A_a are the invariant zeros of $(\bar{A},\bar{B},\bar{C})$, meaning that, if the system (1) is minimum-phase, then A_a is Hurwitz. The transformations Γ_x and Γ_u can be calculated using available software, either numerically [29] or symbolically [30].

The nonlinear canonical form assumed in this paper is an extension of (2), differing only in the presence of a time-varying nonlinearity. The precise definition of this canonical form is given in the following assumption.

Assumption 2:

1) The dynamics of agent i is given by

$$\dot{x}_{ia} = A_a x_{ia} + L_{ad} y_i, \tag{4a}$$
$$\dot{x}_{id} = A_d x_{id} + \phi_d(t, x_{ia}, x_{id}) + B_d(u_i + E_{da}x_{ia} + E_{dd}x_{id}), \tag{4b}$$
$$y_i = C_d x_{id}, \tag{4c}$$

where $x_{ia} \in \mathbb{R}^{n-\rho}$, $x_{id} \in \mathbb{R}^{\rho}$, $u_i \in \mathbb{R}$, and $y_i \in \mathbb{R}$. Moreover, A_a is Hurwitz and the matrices A_d, B_d, and C_d have the special form shown in (3).

2) The function $\phi_d(t, x_{ia}, x_{id})$ is continuously differentiable and Lipschitz continuous with respect to (x_{ia}, x_{id}), uniformly in t, and piecewise continuous with respect to t. Moreover, the nonlinearity satisfies the following lower-triangular structure:

$$\frac{\partial \phi_{dj}(t, x_{ia}, x_{id})}{\partial x_{idk}} = 0, \quad \forall k > j, \tag{5}$$

where $\phi_{dj}(t, x_{ia}, x_{id})$ denotes the j'th element of $\phi_d(t, x_{ia}, x_{id})$ and x_{idk} denotes the k'th element of x_{id}.

The canonical form in (4) is similar to various types of chained, lower-triangular canonical forms common in the context of high-gain observer design and output feedback control (see, e.g., [31]). Among the practically relevant types of systems encompassed by this canonical form are mechanical systems with nonlinearities occurring at the acceleration level. In Section IV we show how one can easily determine whether a given nonlinear system can be transformed to the canonical form by a linear transformation, and how to find the appropriate transformation.

III. Synchronization

In this section we present our control design for networks satisfying Assumptions 1 and 2.

Let $\delta \in (0,1]$ and $\varepsilon \in (0,1]$ denote a low-gain and a high-gain parameter, respectively. It is easy to see that (A_d, B_d, C_d) is controllable and observable. Let therefore K be chosen such that $A_d - KC_d$ is Hurwitz. Recalling the definition of $\tau > 0$ from the end of Section II-A, let $P_\delta = P_\delta' > 0$ be the solution of the algebraic Riccati equation

$$P_\delta A_d + A_d' P_\delta - \tau P_\delta B_d B_d' P_\delta + \delta I_\rho = 0, \tag{6}$$

and define $F_\delta = -B_d' P_\delta$.

Next, let $S_\varepsilon := \mathrm{diag}(1, \ldots, \varepsilon^{\rho-1})$, and define $K_\varepsilon = \varepsilon^{-1} S_\varepsilon^{-1} K$ and $F_{\delta\varepsilon} = \varepsilon^{-\rho} F_\delta S_\varepsilon$. Now, for each $i \in \{1, \ldots, N\}$, define the following dynamic controller:

$$\dot{\hat{x}}_{ia} = A_a \hat{x}_{ia} + L_{ad} C_d \hat{x}_{id}, \tag{7a}$$

$$\begin{aligned} \dot{\hat{x}}_{id} &= A_d \hat{x}_{id} + \phi_d(t, \hat{x}_{ia}, \hat{x}_{id}) + K_\varepsilon(\zeta_i - C_d \hat{x}_{id}) \\ &\quad + B_d(E_{da}\hat{x}_{ia} + E_{dd}\hat{x}_{id}), \end{aligned} \tag{7b}$$

$$u_i = F_{\delta\varepsilon} \hat{x}_{id}. \tag{7c}$$

Theorem 1: Consider the network with agents described by (4) and the dynamic controller described by (7). Under Assumptions 1 and 2 there exists a $\delta^* \in (0,1]$ such that, for each $\delta \in (0, \delta^*]$, there exists an $\varepsilon^*(\delta) \in (0,1]$ such that, for all $\varepsilon \in (0, \varepsilon^*(\delta)]$, $\lim_{t\to\infty}(x_i - x_j) = 0$ for all $i, j \in \{1, \ldots, N\}$.

Proof: For each $i \in \{1, \ldots, N-1\}$, let $\bar{x}_i = [\bar{x}_{ia}; \bar{x}_{id}] := x_N - x_i$ and $\hat{\bar{x}}_i = [\hat{\bar{x}}_{ia}; \hat{\bar{x}}_{id}] := \hat{x}_N - \hat{x}_i$, where $\hat{x}_i = [\hat{x}_{ia}; \hat{x}_{id}]$. The synchronization objective is achieved if $\bar{x}_i \to 0$ for all $i \in \{1, \ldots, N-1\}$. By Taylor's theorem [32, Theorem 11.1], we can write $\phi_d(t, x_{Na}, x_{Nd}) - \phi_d(t, x_{ia}, x_{id}) = \Phi_{ia}(t)\bar{x}_{ia} + \Phi_{id}(t)\bar{x}_{id}$, where $\Phi_{ia}(t)$ and $\Phi_{id}(t)$ are given by

$$\Phi_{ia}(t) = \int_0^1 \frac{\partial \phi_d}{\partial x_{ia}}(t, x_{ia} + p\bar{x}_{ia}, x_{id} + p\bar{x}_{id})\,\mathrm{d}p,$$
$$\Phi_{id}(t) = \int_0^1 \frac{\partial \phi_d}{\partial x_{id}}(t, x_{ia} + p\bar{x}_{ia}, x_{id} + p\bar{x}_{id})\,\mathrm{d}p.$$

Note that, due to the Lipschitz property of the nonlinearity, the elements of $\Phi_{ia}(t)$ and $\Phi_{id}(t)$ are uniformly bounded, and the lower-triangular structure of the nonlinearity implies that $\Phi_{id}(t)$ is lower-triangular. Similarly, we have $\phi_d(t, \hat{x}_{Na}, \hat{x}_{Nd}) - \phi_d(t, \hat{x}_{ia}, \hat{x}_{id}) = \hat{\Phi}_{ia}(t)\hat{\bar{x}}_{ia} + \hat{\Phi}_{id}(t)\hat{\bar{x}}_{id}$, for matrices $\hat{\Phi}_{ia}(t)$ and $\hat{\Phi}_{id}(t)$ with the same properties.

Noting that the row sums of G are zero, we have $\zeta_N - \zeta_i = -\sum_{j=1}^N (g_{ij} - g_{Nj}) y_j = \sum_{j=1}^N (g_{ij} - g_{Nj})(y_N - y_j) = \sum_{j=1}^{N-1} \bar{g}_{ij} C_d \bar{x}_{jd}$, where $\bar{g}_{ij} = g_{ij} - g_{Nj}$, $i, j \in \{1, \ldots, N-1\}$. It follows that we can write

$$\begin{aligned}
\dot{\bar{x}}_{ia} &= A_a \bar{x}_{ia} + L_{ad} C_d \bar{x}_{id}, \\
\dot{\hat{\bar{x}}}_{ia} &= A_a \hat{\bar{x}}_{ia} + L_{ad} C_d \hat{\bar{x}}_{id}, \\
\dot{\bar{x}}_{id} &= A_d \bar{x}_{id} + \Phi_{ia}(t)\bar{x}_a + \Phi_{id}(t)\bar{x}_d \\
&\quad + B_d(F_{\delta\varepsilon}\hat{\bar{x}}_{id} + E_{da}\bar{x}_{ia} + E_{dd}\bar{x}_{id}), \\
\dot{\hat{\bar{x}}}_{id} &= A_d \hat{\bar{x}}_{id} + \hat{\Phi}_{ia}(t)\hat{\bar{x}}_a + \hat{\Phi}_{id}(t)\hat{\bar{x}}_d \\
&\quad + B_d(E_{da}\hat{\bar{x}}_{ia} + E_{dd}\hat{\bar{x}}_{id}) \\
&\quad + \sum_{j=1}^{N-1} \bar{g}_{ij} K_\varepsilon C_d \bar{x}_{jd} - K_\varepsilon C_d \hat{\bar{x}}_{id}.
\end{aligned}$$

Next, define $\xi_{ia} = \bar{x}_{ia}$, $\hat{\xi}_{ia} = \hat{\bar{x}}_{ia}$, $\xi_{id} = S_\varepsilon \bar{x}_{id}$, and $\hat{\xi}_{id} = S_\varepsilon \hat{\bar{x}}_{id}$. Then, using the identities $S_\varepsilon A_d S_\varepsilon^{-1} = \varepsilon^{-1} A_d$, $S_\varepsilon B_d = \varepsilon^{\rho-1} B_d$, and $C_d S_\varepsilon^{-1} = C_d$, we have

$$\dot{\xi}_{ia} = A_a \xi_{ia} + V_{iad} \xi_{id}, \tag{8a}$$
$$\dot{\hat{\xi}}_{ia} = A_a \hat{\xi}_{ia} + \hat{V}_{iad} \hat{\xi}_{id}, \tag{8b}$$
$$\varepsilon \dot{\xi}_{id} = A_d \xi_{id} + B_d F_\delta \hat{\xi}_{id} + V_{ida}^\varepsilon(t)\xi_{ia} + V_{idd}^\varepsilon(t)\xi_{id}, \tag{8c}$$
$$\begin{aligned} \varepsilon \dot{\hat{\xi}}_{id} &= A_d \hat{\xi}_{id} + \hat{V}_{ida}^\varepsilon(t)\hat{\xi}_a + \hat{V}_{idd}^\varepsilon(t)\hat{\xi}_{id} \\ &\quad + \sum_{j=1}^{N-1} \bar{g}_{ij} K C_d \xi_{jd} - K C_d \hat{\xi}_{id}, \end{aligned} \tag{8d}$$

where $V_{iad} = \hat{V}_{iad} = L_{ad} C_d$, $V_{ida}^\varepsilon(t) := \varepsilon^\rho B_d E_{da} + \varepsilon S_\varepsilon \Phi_{ia}(t)$, $\hat{V}_{ida}^\varepsilon(t) := \varepsilon^\rho B_d E_{da} + \varepsilon S_\varepsilon \hat{\Phi}_{ia}(t)$, $V_{idd}^\varepsilon(t) := \varepsilon^\rho B_d E_{dd} S_\varepsilon^{-1} + \varepsilon S_\varepsilon \Phi_{id}(t) S_\varepsilon^{-1}$, and $\hat{V}_{idd}^\varepsilon(t) := \varepsilon^\rho B_d E_{dd} S_\varepsilon^{-1} + \varepsilon S_\varepsilon \hat{\Phi}_{id}(t) S_\varepsilon^{-1}$. Clearly $\|V_{iad}\|$ and $\|\hat{V}_{iad}\|$ are ε-independent, whereas $\|V_{ida}^\varepsilon\|$ and $\|\hat{V}_{ida}^\varepsilon\|$ are $O(\varepsilon)$. Furthermore, due to the lower-triangular structure of Φ_{id} and $\hat{\Phi}_{id}$, $\|V_{idd}^\varepsilon\|$ and $\|\hat{V}_{idd}^\varepsilon\|$ are also $O(\varepsilon)$ (see [26]).

Define $\bar{G} = [\bar{g}_{ij}]$, $i, j \in \{1, \ldots, N-1\}$. It follows from the proof of Lemma 1 of Zhang and Tian [33] that the eigenvalues of $\bar{G}$ are the nonzero eigenvalues of G. Let $\xi_a = [\xi_{1a}; \ldots; \xi_{(N-1)a}]$, $\hat{\xi}_a = [\hat{\xi}_{1a}; \ldots; \hat{\xi}_{(N-1)a}]$, $\xi_d = [\xi_{1d}; \ldots; \xi_{(N-1)d}]$, and $\hat{\xi}_d = [\hat{\xi}_{1d}; \ldots; \hat{\xi}_{(N-1)d}]$. Then

$$\begin{aligned}
\dot{\xi}_a &= (I_{N-1} \otimes A_a)\xi_a + V_{ad}\xi_d, \\
\dot{\hat{\xi}}_a &= (I_{N-1} \otimes A_a)\hat{\xi}_a + \hat{V}_{ad}\hat{\xi}_d, \\
\varepsilon \dot{\xi}_d &= (I_{N-1} \otimes A_d)\xi_d + (I_{N-1} \otimes B_d F_\delta)\hat{\xi}_d \\
&\quad + V_{da}^\varepsilon(t)\xi_a + V_{dd}^\varepsilon(t)\xi_d, \\
\varepsilon \dot{\hat{\xi}}_d &= (I_{N-1} \otimes A_d)\hat{\xi}_d + \hat{V}_{da}^\varepsilon(t)\hat{\xi}_a + \hat{V}_{dd}^\varepsilon(t)\hat{\xi}_d \\
&\quad + (\bar{G} \otimes K C_d)\xi_d - (I_{N-1} \otimes K C_d)\hat{\xi}_d,
\end{aligned}$$

where $V_{ad} = \mathrm{diag}(V_{1ad}, \ldots, V_{(N-1)ad})$, and $\hat{V}_{ad}$, $V_{da}^\varepsilon(t)$, $\hat{V}_{da}^\varepsilon(t)$, $V_{dd}^\varepsilon(t)$, and $\hat{V}_{dd}^\varepsilon(t)$ are similarly defined. Define U such that $U^{-1}\bar{G}U = J$, where J is the Jordan form of $\bar{G}$, and let $\nu_a = (JU^{-1} \otimes I_{n-\rho})\xi_a$, $\tilde{\nu}_a = \nu_a - (JU^{-1} \otimes I_{n-\rho})\hat{\xi}_a$, $\nu_d = (JU^{-1} \otimes$

$I_\rho)\xi_d$, and $\tilde{\nu}_d = \nu_d - (U^{-1}\otimes I_\rho)\hat{\xi}_d$. Then

$$\begin{aligned}
\dot{\nu}_a &= (I_{N-1}\otimes A_a)\nu_a + W_{ad}\nu_d,\\
\dot{\tilde{\nu}}_a &= (I_{N-1}\otimes A_a)\tilde{\nu}_a + W_{ad}\nu_d - \hat{W}_{ad}(\nu_d - \tilde{\nu}_d),\\
\varepsilon\dot{\nu}_d &= (I_{N-1}\otimes A_d)\nu_d + (J\otimes B_dF_\delta)(\nu_d - \tilde{\nu}_d)\\
&\quad + W^\varepsilon_{da}(t)\nu_a + W^\varepsilon_{dd}(t)\nu_d,\\
\varepsilon\dot{\tilde{\nu}}_d &= (I_{N-1}\otimes A_d)\tilde{\nu}_d + (J\otimes B_dF_\delta)(\nu_d - \tilde{\nu}_d)\\
&\quad + W^\varepsilon_{da}(t)\nu_a - \hat{W}^\varepsilon_{da}(t)(\nu_a - \tilde{\nu}_a)\\
&\quad + W^\varepsilon_{dd}(t)\nu_d - \hat{W}^\varepsilon_{dd}(t)(\nu_d - \tilde{\nu}_d) - (I_{N-1}\otimes KC_d)\tilde{\nu}_d,
\end{aligned}$$

where $W_{ad} = (JU^{-1}\otimes I_{n-\rho})V_{ad}(UJ^{-1}\otimes I_\rho)$, $\hat{W}_{ad} = (JU^{-1}\otimes I_{n-\rho})\hat{V}_{ad}(U\otimes I_\rho)$, $W^\varepsilon_{da}(t) = (JU^{-1}\otimes I_\rho)V^\varepsilon_{da}(t)(UJ^{-1}\otimes I_{n-\rho})$, $W^\varepsilon_{dd}(t) = (JU^{-1}\otimes I_\rho)V^\varepsilon_{dd}(t)(UJ^{-1}\otimes I_\rho)$, $\hat{W}^\varepsilon_{da}(t) = (U^{-1}\otimes I_\rho)\hat{V}^\varepsilon_{da}(t)(UJ^{-1}\otimes I_{n-\rho})$, and $\hat{W}^\varepsilon_{dd}(t) = (U^{-1}\otimes I_\rho)\hat{V}^\varepsilon_{dd}(t)(U\otimes I_\rho)$. Finally, let N_a and N_d be defined such that $\eta_a := N_a[\nu_a;\tilde{\nu}_a] = [\nu_{1a};\tilde{\nu}_{1a};\ldots;\nu_{(N-1)a};\tilde{\nu}_{(N-1)a}]$, and $\eta_d := N_d[\nu_d;\tilde{\nu}_d] = [\nu_{1d};\tilde{\nu}_{1d};\ldots;\nu_{(N-1)d};\tilde{\nu}_{(N-1)d}]$. Then

$$\dot{\eta}_a = \tilde{A}_a\eta_a + \tilde{W}_{ad}\eta_d, \tag{9a}$$

$$\varepsilon\dot{\eta}_d = \tilde{A}_\delta\eta_d + \tilde{W}^\varepsilon_{da}(t)\eta_a + \tilde{W}^\varepsilon_{dd}(t)\eta_d, \tag{9b}$$

where $\tilde{A}_a = (I_{2(N-1)}\otimes A_a)$,

$$\tilde{A}_\delta = I_{N-1}\otimes\begin{bmatrix} A_d & 0\\ 0 & A_d - KC_d\end{bmatrix} + J\otimes\begin{bmatrix} B_dF_\delta & -B_dF_\delta\\ B_dF_\delta & -B_dF_\delta\end{bmatrix},$$

and

$$\begin{aligned}
\tilde{W}_{ad} &= N_a\begin{bmatrix} W_{ad} & 0\\ W_{ad}-\hat{W}_{ad} & \hat{W}_{ad}\end{bmatrix}N_d^{-1},\\
\tilde{W}^\varepsilon_{da}(t) &= N_d\begin{bmatrix} W^\varepsilon_{da}(t) & 0\\ W^\varepsilon_{da}(t)-\hat{W}^\varepsilon_{da}(t) & \hat{W}^\varepsilon_{da}(t)\end{bmatrix}N_a^{-1},\\
\tilde{W}^\varepsilon_{dd}(t) &= N_d\begin{bmatrix} W^\varepsilon_{dd}(t) & 0\\ W^\varepsilon_{dd}(t)-\hat{W}^\varepsilon_{dd}(t) & \hat{W}^\varepsilon_{dd}(t)\end{bmatrix}N_d^{-1}.
\end{aligned}$$

Note that $\tilde{W}^\varepsilon_{da}$ and $\tilde{W}^\varepsilon_{dd}$ are $O(\varepsilon)$.

Due to its upper block-triangular structure, the eigenvalues of $\tilde{A}_\delta$ are the eigenvalues of the matrices

$$\bar{A}_\delta := \begin{bmatrix} A_d + \lambda B_dF_\delta & -\lambda B_dF_\delta\\ \lambda B_dF_\delta & A_d - KC_d - \lambda B_dF_\delta\end{bmatrix}, \tag{10}$$

for each eigenvalue λ of $\bar{G}$ along the diagonal of J.

Noting that A_d has all its poles in the closed left-half complex plane, the matrix $\bar{A}_\delta$ corresponds to the system matrix from Seo et al. [8, Eq. (19)] except for the appearance of λ instead of $\lambda - 1$ in the second row. The proof of [8, Theorem 4] can now be followed to prove that $\bar{A}_\delta$ is Hurwitz for all δ less than some sufficiently small $\delta^* > 0$. Note that δ^* is independent of the high-gain parameter ε.

Let $\tilde{P}_\delta = \tilde{P}^*_\delta > 0$ be the solution of the Lyapunov equation $\tilde{P}_\delta\tilde{A}_\delta + \tilde{A}^*_\delta\tilde{P}_\delta = -I_{2(N-1)\rho}$, and let $\tilde{P}_a = \tilde{P}'_a > 0$ be the solution of the Lyapunov equation $\tilde{P}_a\tilde{A}_a + \tilde{A}'_a\tilde{P}_a = -I_{2(N-1)(n-\rho)}$, which exists because $\tilde{A}_a$ is Hurwitz. Consider the Lyapunov function $V = \varepsilon\eta^*_d\tilde{P}_\delta\eta_d + \varepsilon\eta^*_a\tilde{P}_a\eta_a$, for which we have

$$\begin{aligned}
\dot{V} &= -\|\eta_d\|^2 + 2\mathrm{Re}(\eta^*_d\tilde{P}_\delta\tilde{W}^\varepsilon_{da}(t)\eta_a)\\
&\quad + 2\mathrm{Re}(\eta^*_d\tilde{P}_\delta\tilde{W}^\varepsilon_{dd}(t)\eta_d) - \varepsilon\|\eta_a\|^2\\
&\quad + 2\varepsilon\mathrm{Re}(\eta^*_a\tilde{P}_a\tilde{W}_{ad}\eta_d)\\
&\le -(1-2\varepsilon\gamma_1)\|\eta_d\|^2 - \varepsilon\|\eta_a\|^2 + 2\varepsilon\gamma_2\|\eta_d\|\|\eta_a\|,
\end{aligned}$$

where $\varepsilon\gamma_1 \ge \|\tilde{P}_\delta\tilde{W}^\varepsilon_{dd}\|$ and $\varepsilon\gamma_2 \ge \|\tilde{P}_\delta\tilde{W}^\varepsilon_{da}\| + \varepsilon\|\tilde{P}_a\tilde{W}_{ad}\|$. Let ε be chosen small enough that $1 - 2\varepsilon\gamma_1 \ge \frac{1}{2}$. Then

$$\dot{V} \le -\begin{bmatrix}\|\eta_d\| & \|\eta_a\|\end{bmatrix}\begin{bmatrix}\frac{1}{2} & -\varepsilon\gamma_2\\ -\varepsilon\gamma_2 & \varepsilon\end{bmatrix}\begin{bmatrix}\|\eta_d\|\\ \|\eta_a\|\end{bmatrix}.$$

The first-order principal minor of the above matrix is $\frac{1}{2} > 0$. The second-order principal minor is $\frac{1}{2}\varepsilon - \varepsilon^2\gamma_2^2$, which is positive for all $\varepsilon < 1/(2\gamma_2^2)$. It follows that $\eta_a \to 0$ and $\eta_d \to 0$, which implies $\bar{x}_i \to 0$ for all $i \in \{1,\ldots,N-1\}$. ■

IV. Transforming Nonlinear Time-Varying Systems to the Canonical Form

Our design for nonlinear time-varying agents requires the system to be given in the particular canonical form (4). Given an arbitrary nonlinear time-varying system, one would therefore like to know (i) whether it is possible to transform it to this canonical form; and (ii) how the appropriate transformation can be constructed. If we limit ourselves to *linear* state and input transformations, then both questions are simultaneously answered by the following theorem.

Theorem 2: Consider the nonlinear time-varying system

$$\dot{\bar{x}}_i = \bar{A}\bar{x}_i + \bar{B}\bar{u}_i + \bar{\phi}(t,\bar{x}_i), \qquad \bar{x}_i \in \mathbb{R}^n,\ \bar{u}_i \in \mathbb{R}, \tag{11a}$$

$$y_i = \bar{C}\bar{x}_i, \qquad y_i \in \mathbb{R}, \tag{11b}$$

where $(\bar{A},\bar{B},\bar{C})$ is minimum-phase and of relative degree $\rho \ge 1$; and where $\bar{\phi}(t,\bar{x}_i)$ is continuously differentiable and Lipschitz continuous with respect to $\bar{x}_i$, uniformly in t, and piecewise continuous with respect to t. Let $\Gamma_x \in \mathbb{R}^{n\times n}$ and $\Gamma_u \in \mathbb{R}$ be nonsingular state and input transformations such that the triple $(A,B,C) = (\Gamma_x^{-1}\bar{A}\Gamma_x, \Gamma_x^{-1}\bar{B}\Gamma_u, \bar{C}\Gamma_x)$ is in the SCB, and define $\bar{x}_i = \Gamma_x x_i$ and $\bar{u}_i = \Gamma_u u_i$. Then either

- the system with state x_i, input u_i, and output y_i satisfies the canonical form of Assumption 2; or
- there exists no set of linear, non-singular state and input transformations that take the system to the canonical form of Assumption 2.

Proof: First, note that the linear portion of (4) is in the SCB. Thus, all we have to show is that all transformations that take the linear portion of the system to the SCB are equivalent with respect to satisfying point 2 of Assumption 2. Consider therefore the system (4) satisfying point 2 of Assumption 2, and let (A,B,C) denote the corresponding linear triple. Let $\tilde{\Gamma}_x$ and $\tilde{\Gamma}_u$ denote state and input transformations such that $(\tilde{A},\tilde{B},\tilde{C}) = (\tilde{\Gamma}_x^{-1}A\tilde{\Gamma}_x, \tilde{\Gamma}_x^{-1}B\tilde{\Gamma}_u, C\tilde{\Gamma}_x)$ is also in the SCB. Define $x_i = \tilde{\Gamma}_x\tilde{x}_i$, and $u_i = \tilde{\Gamma}_u\tilde{u}_i$, and partition $\tilde{x}_i$ as $\tilde{x}_i = [\tilde{x}_{ia};\tilde{x}_{id}]$, where $\tilde{x}_{ia} \in \mathbb{R}^{n-\rho}$ and $\tilde{x}_{id} \in \mathbb{R}^\rho$. Then we can write

$$\begin{aligned}
\dot{\tilde{x}}_{ia} &= \tilde{A}_a\tilde{x}_{ia} + \tilde{L}_{ad}y_i + \tilde{\phi}_a(t,\tilde{x}_{ia},\tilde{x}_{id}),\\
\dot{\tilde{x}}_{id} &= A_d\tilde{x}_{id} + \tilde{\phi}_d(t,\tilde{x}_{ia},\tilde{x}_{id})\\
&\quad + B_d(\tilde{u}_i + \tilde{E}_{da}\tilde{x}_{ia} + \tilde{E}_{dd}\tilde{x}_{id}),\\
y_i &= C_d\tilde{x}_{id},
\end{aligned}$$

and we need to show that $\tilde{\phi}_a(t,\tilde{x}_{ia},\tilde{x}_{id}) = 0$ and that $\tilde{\phi}_d(t,\tilde{x}_{ia},\tilde{x}_{id})$ satisfies (5).

Let

$$\tilde{\Gamma}_x = \begin{bmatrix}\Gamma_{xaa} & \Gamma_{xad}\\ \Gamma_{xda} & \Gamma_{xdd}\end{bmatrix}$$

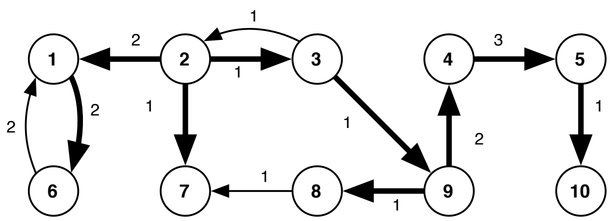

Fig. 1. Network graph

be partitioned according to the dimensions of x_{ia} and x_{id}. Note that

$$\begin{bmatrix} \tilde{C} \\ \vdots \\ \tilde{C}\tilde{A}^{\rho-1} \end{bmatrix} = \begin{bmatrix} C \\ \vdots \\ CA^{\rho-1} \end{bmatrix} = \begin{bmatrix} 0 & C_d \\ \vdots & \vdots \\ 0 & C_d A_d^{\rho-1} \end{bmatrix} = \begin{bmatrix} 0 & I_\rho \end{bmatrix}.$$

On the other hand,

$$\begin{bmatrix} \tilde{C} \\ \vdots \\ \tilde{C}\tilde{A}^{\rho-1} \end{bmatrix} = \begin{bmatrix} C \\ \vdots \\ CA^{\rho-1} \end{bmatrix} \tilde{\Gamma}_x = \begin{bmatrix} 0 & I_\rho \end{bmatrix} \tilde{\Gamma}_x = \begin{bmatrix} \Gamma_{xda} & \Gamma_{xdd} \end{bmatrix}.$$

It follows that $\Gamma_{xda} = 0$ and $\Gamma_{xdd} = I_\rho$, which implies that $\tilde{x}_{id} = x_{id}$, and hence $\tilde{\phi}_d(t, \tilde{x}_{ia}, \tilde{x}_{id})$ satisfies (5).

Next, we have $\tilde{\Gamma}_x \tilde{B} = B\tilde{\Gamma}_u$, which implies $\Gamma_{xad} B_d = 0$, meaning that column ρ of Γ_{xad} is zero. Furthermore, we have $\tilde{\Gamma}_x \tilde{A} = A\tilde{\Gamma}_x$, which implies $\Gamma_{xaa}\tilde{L}_{ad}C_d + \Gamma_{xad}(A_d + B_d\tilde{E}_{dd}) = A_a\Gamma_{xad} + L_{ad}C_d$. It follows that $(\Gamma_{xaa}\tilde{L}_{ad} - L_{ad})C_d = A_a\Gamma_{xad} - \Gamma_{xad}A_d$. Let $1 < k \le \rho$ and note that column k on the left-hand side of the last equation is zero. Suppose that column k of Γ_{xad} is also zero (note that this holds for $k = \rho$) which implies that column k of $A_a\Gamma_{xad}$ is zero. Since column k of $\Gamma_{xad}A_d$ is equal to column $k-1$ of Γ_{xad}, it follows that this column is also zero. By induction, $\Gamma_{xad} = 0$, and hence $x_{ia} = \Gamma_{xaa}\tilde{x}_{ia}$. It now follows that $\tilde{\phi}_a(t, \tilde{x}_{ia}, \tilde{x}_{id}) = 0$. ■

V. Example

Consider a network of $N = 10$ agents, illustrated in Figure 1. This network has a directed spanning tree rooted at node 2, and thus it satisfies Assumption 1. The real part of the non-zero eigenvalues are bounded below by approximately 0.76. For design purposes, we assume that a lower bound $\tau = 0.6$ is known.

The agent model is given by

$$\dot{x}_{ia} = -2x_{ia} + y_i,$$
$$\dot{x}_{id} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{id} + \phi_d(t, x_{id}) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i,$$
$$y_i = \begin{bmatrix} 1 & 0 \end{bmatrix} x_{id},$$

where

$$\phi_d(t, x_{id}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0.3\sin(t)x_{ia} + \sin(0.1x_{id1})).$$

It is easy to see that this model is in the canonical form of Assumption 2. We start the design by selecting $K = [3; 2]$, to place the poles of $A_d - KC_d$ at -1 and -2. Next, we find the solution of the algebraic Riccati equation (6) for a given $\delta \in (0, 1]$, and we compute $K_\varepsilon = \varepsilon^{-1}S_\varepsilon^{-1}K$ and $F_{\delta\varepsilon} = \varepsilon^{-2}F_\delta S_\varepsilon$ for a given $\varepsilon \in (0, 1]$. Finally, we implement the controller (7) with the computed values. After some trial and error, we find that $\delta = 10^{-5}$ and $\varepsilon = 1$ ensures synchronization, which yields $K_\varepsilon = [3; 2]$ and $F_{\delta\varepsilon} \approx [-0.0041, -0.1167]$. Figure 2 shows the agent outputs $y_1, \ldots, y_{10}$, which clearly synchronize.

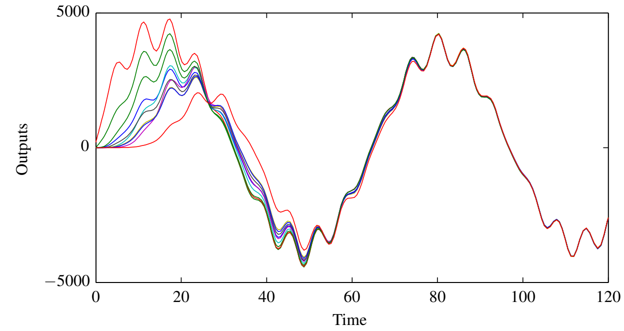

Fig. 2. Agent outputs for nonlinear example

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