# Comparing Böhm-Like Trees*

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**Abstract.** Extending the infinitary rewriting definition of Böhm-like trees to infinitary Combinatory Reduction Systems (iCRSs), we show that each Böhm-like tree defined by means of infinitary rewriting can also be defined by means of a direct approximant function. In addition, we show that counterexamples exists to the reverse implication.

## 1 Introduction

In $\lambda$-calculus, a Böhm tree defines a denotational semantics based on syntax. Essentially, a Böhm tree of a term can been seen as an infinite normal form of the term, omitting subterms that do not 'compute' anything.

What constitutes a term that 'computes' something is not universally determined. Within $\lambda$-calculus three alternatives exist: the head normal forms [2, 3], the weak head normal forms [4, 5], and the root-stable terms [6]. These define, besides the Böhm trees, the Lévy-Longo trees and the Berarducci trees. As a result, abstract definitions have appeared that are parameterized over the set of terms 'computing' something. These are the so-called Böhm-like trees [1].

The abstract definitions can be divided into two classes, based on the concrete definition taken as a starting point. One is based on infinitary rewriting [7, 8]; the other is based on so-called direct approximants [1, 9, 10].

*Infinitary Rewriting.* Within this class, a Böhm-like tree *is* a normal form, albeit not in the original (finite) system but in an infinite system. The infinite system extends the finite one with infinite terms and infinite reductions. Rules are added rewriting terms *not* 'computing' anything — the meaningless terms — to a fresh nullary symbol $\bot$. Pivotal are a number of conditions on the set of meaningless terms guaranteeing that each term has a unique normal form.

*Direct Approximants.* Within this class, terms are partially ordered by adding a fresh nullary function symbol $\bot$. The direct approximant function — the parameterized component — replaces by $\bot$ any subterm that either reduces to a redex or does *not* 'compute' anything. This yields a normal form that approximates the Böhm-like tree. The tree is obtained by gathering the direct approximants of all the reducts of a term and taking the least upper bound. Pivotal are a number of conditions on the direct approximant function guaranteeing that the least upper bound exists.

* This paper extends earlier unpublished work from the author's Ph.D. thesis [1].

We show that the direct approximant approach is more expressive than the infinitary rewriting one in the context of Combinatory Reduction Systems (CRSs): Each Böhm-like tree defined by means of infinitary rewriting can also be defined by means of a direct approximant function. The reverse, however, does not hold.

**Overview.** In Sect. 2 we give some preliminaries, mostly regarding infinitary Combinatory Reduction Systems (iCRSs). In Sect. 3 we extend to iCRSs the infinitary rewriting approach to Böhm-like trees. In Sect. 4 we compare, after shortly reviewing direct approximants. Finally, in Sect. 5 we conclude.

## 2 Preliminaries

We outline some basic facts concerning iCRSs; see [11–13] for more detailed accounts. Throughout, we denote the first infinite ordinal by $\omega$, and arbitrary ordinals by $\alpha$, $\beta$, $\gamma$, etc. By $\mathbb{N}$ we denote the natural numbers including zero.

**Terms and Substitutions.** Let $\Sigma$ be a signature with each element of finite arity. Moreover, assume a countably infinite set of variables and, for each finite arity, a countably infinite set of meta-variables — countably infinite sets suffice given 'Hilbert hotel'-style renaming.

Infinite terms are usually defined by metric completion [11]. Here, we give the shorter, but equivalent, definition from [12]:

**Definition 2.1.** *The set of* meta-terms *is defined by interpreting the following rules coinductively, where $s$ and $s_1$, ..., $s_n$ are again meta-terms:*

1. *each variable $x$ is a meta-term,*
2. *if $x$ is a variable, then $[x]s$ is a meta-term,*
3. *if $Z$ is an $n$-ary meta-variable, then $Z(s_1, \ldots, s_n)$ is a meta-term, and*
4. *if $f \in \Sigma$ is $n$-ary, then $f(s_1, \ldots, s_n)$ is a meta-term.*

*The set of* finite *meta-terms, a subset of the set of meta-terms, is the set inductively defined by the above rules. A* term *is a meta-term without meta-variables. A* context *is a meta-term over $\Sigma \cup \{\Box\}$ and a* partial meta-term *is a meta-term over $\Sigma_\bot = \Sigma \cup \{\bot\}$, with $\Box$ and $\bot$ fresh nullary function symbols.*

We consider (meta-)terms modulo $\alpha$-equivalence. A meta-term of the form $[x]s$ is called an *abstraction*; a variable $x$ in $s$ is called *bound* in $[x]s$. Meta-terms with meta-variables only occur in rewrite rules; rewriting itself is defined over terms.

Partial meta-terms are partially ordered where $\bot \preccurlyeq s$ for each partial meta-term $s$ and such that term formation is monotonic modulo $\alpha$-equivalence [1, 7].

The set of *positions* [11] of a meta-term $s$, denoted $\mathcal{P}os(s)$, is a set of *finite* strings over $\mathbb{N}$, with each string denoting the 'location' of a subterm in $s$. If $p$ is a position of $s$, then $s|_p$ is the *subterm of $s$ at* position $p$. The length of $p$ is denoted by $|p|$. There exists a well-founded order $<$ on positions: $p < q$ iff $p$ is a proper prefix of $q$. The concatenation of positions $p$ and $q$ is denoted by $p \cdot q$.

A *valuation* [14], denoted $\bar{\sigma}$, substitutes terms for meta-variables in meta-terms and is defined by coinductively interpreting the rules of valuations for

CRSs [11]. In CRSs, applying a valuation to a meta-term yields a unique term. This is not the case for iCRSs [11]. To alleviate this problem, the set of meta-terms satisfying the so-called 'finite chains property' is defined in [11]:

**Definition 2.2.** *Let $s$ be a meta-term. A* chain *in $s$ is a sequence of (context, position)-pairs $(C_i[\Box], p_i)_{i<\alpha}$, with $\alpha \leq \omega$, such that for each $(C_i[\Box], p_i)$ there exists a term $t_i$ with $C_i[t_i] = s|_{p_i}$ and $p_{i+1} = p_i \cdot q$ where $q$ is the position of the hole in $C_i[\Box]$. A* chain of meta-variables *in $s$ is such that for each $i < \alpha$ it holds that $C_i[\Box] = Z(t_1, \ldots, t_n)$ with $t_j = \Box$ for exactly one $1 \leq j \leq n$.*

*The meta-term $s$ is said to satisfy the* finite chains property *if no infinite chain of meta-variables occurs in $s$.*

Remark that $\Box$ only occurs in $C_i[\Box]$ if $i + 1 < \alpha$, otherwise $C_i[\Box] = s|_{p_i}$. The meta-term $[x_1]Z_1([x_2]Z_2(\ldots[x_n]Z_n(\ldots)))$ e.g. satisfies the finite chains property, while $Z(Z(\ldots Z(\ldots)))$ does not. Finite meta-terms always satisfy the finite chains property. The following is shown in [11]:

**Proposition 2.3.** *Let $s$ be a meta-term satisfying the finite chains property and let $\bar{\sigma}$ be a valuation. There is a unique term that is the result of applying $\bar{\sigma}$ to $s$.*

**Rewriting.** Recall that a *pattern* is a finite meta-term each meta-variable of which has distinct bound variables as arguments and that a meta-term is *closed* if all variables occur bound [14].

**Definition 2.4.** *A* rewrite rule *is a pair of closed meta-terms $(l, r)$, denoted $l \to r$, with $l$ a finite pattern of the form $f(s_1, \ldots, s_n)$ and $r$ satisfying the finite chains property such that all meta-variables that occur in $r$ also occur in $l$.*

*An* infinitary Combinatory Reduction System (iCRS) *is a pair $\mathcal{C} = (\Sigma, R)$ with $\Sigma$ a signature and $R$ a set of rewrite rules.*

Left-linearity and orthogonality are defined as for CRSs [14] (left-hand sides of rewrite rules are finite). A rewrite rule is *collapsing* if the root of its right-hand side is a meta-variable. Moreover, a pattern is *fully-extended*, if, for each meta-variable $Z$ and abstraction $[x]s$ with an occurrence of $Z$ in its scope, $x$ is an argument of that occurrence of $Z$; a rewrite rule is *fully-extended* if its left-hand side is and an iCRS is *fully-extended* if all its rewrite rules are.

**Definition 2.5.** *A* rewrite step *is a pair of terms $(s, t)$ denoted $s \to t$ and adorned with a context $C[\Box]$, a rewrite rule $l \to r$, and a valuation $\bar{\sigma}$ such that $s = C[\bar{\sigma}(l)]$ and $t = C[\bar{\sigma}(r)]$. The term $\bar{\sigma}(l)$ is called an $l \to r$-*redex *and* occurs *at position $p$ and depth $|p|$ in $s$, where $p$ is the position of the hole in $C[\Box]$.*

*A position $q$ of $s$ occurs in the* redex pattern *of the redex at position $p$ if $q \geq p$ and if there does not exist a position $q'$ with $q \geq p \cdot q'$ such that $q'$ is the position of a meta-variable in $l$.*

Above, $\bar{\sigma}(l)$ and $\bar{\sigma}(r)$ are well-defined, as both left- and right-hand sides of rewrite rules satisfy the finite chains property (left-hand sides as they are finite).

We say that a redex $s$ *overlaps* a term $t$ at position $p$, if $p$ occurs in the redex pattern of $s$ and $s|_p = t$ [7]. Moreover a redex and a rewrite step are *collapsing* if the employed rewrite rule is. Using rewrite steps, we define reductions:

**Definition 2.6.** *A* transfinite reduction *with domain* $\alpha > 0$ *is a sequence of terms* $(s_\beta)_{\beta<\alpha}$ *such that* $s_\beta \to s_{\beta+1}$ *for all* $\beta + 1 < \alpha$. *For each* $s_\beta \to s_{\beta+1}$, *let* $d_\beta$ *denote the depth of the contracted redex. The reduction is* strongly convergent *if* $\alpha$ *is a successor ordinal and if for every limit ordinal* $\gamma \leq \alpha$ *it holds that* $s_\beta$ *converges to* $s_\gamma$ *and* $d_\beta$ *tends to infinity in case* $\beta$ *approaches* $\gamma$ *from below.*

Consider the rules $a \to a$ and $f(Z) \to g(f(Z))$. The reduction

$$f(a) \to g(f(a)) \to \cdots \to g^n(f(a)) \to \cdots g^\omega ,$$

with $g^\omega$ denoting $g(g(\ldots g(\ldots)))$, is strongly convergent. The reduction

$$f(a) \to f(a) \to \cdots \to f(a) \to \cdots$$

is not strongly convergent, as each contracted redex occurs at depth 1.

By $s \twoheadrightarrow^\alpha t$, resp. $s \twoheadrightarrow^{\leq\alpha} t$, we denote a strongly convergent reduction of length $\alpha$, resp. of length at most $\alpha$. By $s \twoheadrightarrow t$, resp. $s \to^* t$, we denote a strongly convergent reduction of arbitrary length, resp. of finite length.

Across strongly convergent reductions we assume that a position that occurs in the redex pattern of a contracted redex does not have any descendants; likewise for residuals [11]. We write $P/(s \twoheadrightarrow t)$ for the descendants of a set of positions $P \subseteq \mathcal{P}os(s)$ across a strongly convergent reduction $s \twoheadrightarrow t$ and $\mathcal{U}/(s \twoheadrightarrow t)$ for the residuals of a set $\mathcal{U}$ of subterms of $s$ across $s \twoheadrightarrow t$.

Below, we appeal to a number of properties of iCRSs. The first is compression:

**Theorem 2.7 (Compression [11]).** *For every fully-extended, left-linear iCRS, if* $s \twoheadrightarrow^\alpha t$, *then* $s \twoheadrightarrow^{\leq\omega} t$.

A term $s$ is *hypercollapsing*, resp. *root-active*, if for all $s \twoheadrightarrow t$ there exists a $t \twoheadrightarrow t'$ such that $t'$ is a collapsing redex, resp. a redex. We write $s \sim_{hc} t$ if $t$ can be obtained from $s$ by replacing hypercollapsing subterms in $s$ by other hypercollapsing subterms.

Let $\sim$ be an equivalence relation. Confluence modulo $\sim$ means that if $s \twoheadrightarrow s'$ and $t \twoheadrightarrow t'$ with $s \sim t$, then $s' \twoheadrightarrow s''$ and $t' \twoheadrightarrow t''$ with $s'' \sim t''$. For $\sim_{hc}$ we have:

**Theorem 2.8.** *Given a fully-extended, orthogonal iCRSs, the relation* $\sim_{hc}$ *is an equivalence relation and the system is confluent modulo* $\sim_{hc}$.

The above is shown in [12] under assumption that rewrite rules have finite right-hand sides; in [13] the result is extended to allow for infinite right-hand sides.

## 3 Infinitary Rewriting

We extend the infinitary rewriting approach to Böhm-like trees from [7, 8] to fully-extended, orthogonal iCRSs. Given an iCRS and a set of so-called meaningless terms, this means we define a confluent and normalising rewrite system.

Following the pattern laid down in [7, 8], we start in Sect. 3.1 by stating a number of axioms for sets of meaningless terms. Assuming some of the axioms, we consider 'meaningful' terms in Sect. 3.2 and we define Böhm-like trees in Sect. 3.3. In Sect. 3.4, we construct a set of partial terms given a set of terms and show the axioms are preserved. Finally, in Sect. 3.5, we consider some examples, some of which employ the construction from the Sect. 3.4.

### 3.1 Axioms

To state our axioms, assume $\mathcal{U}$ is a set of terms. We call the terms in this set *meaningless*; intuitively they are not supposed to 'compute' anything.

Let $s$ and $t$ be terms with $P \subseteq \mathcal{P}os(s)$ such that $s|_p \in \mathcal{U}$ for each $p \in P$. We write $s \to_P^{\mathcal{U}} t$, resp. $s \leftrightarrow_P^{\mathcal{U}} t$, if $t$ can be obtained from $s$ by replacing the subterms at positions in $P$ by arbitrary terms, resp. by terms from $\mathcal{U}$. Remark that $\leftrightarrow^{\mathcal{U}}$ is reflexive and symmetric, i.e. $s \leftrightarrow_P^{\mathcal{U}} s$ and $s \leftrightarrow_P^{\mathcal{U}} t$ iff $t \leftrightarrow_P^{\mathcal{U}} s$. We write $s \to^{\mathcal{U}} t$ and $s \leftrightarrow^{\mathcal{U}} t$ if the set of positions is irrelevant or clear from the context.

The considered axioms stem from [8] and are as follows:

**Residuals** If $s \twoheadrightarrow t$ and $s|_p \in \mathcal{U}$, then $t|_q \in \mathcal{U}$ for all $q \in p/(s \twoheadrightarrow t)$.
**Overlap** If a redex $s$ overlaps a term in $\mathcal{U}$, then $s \in \mathcal{U}$.
**Root-activeness** If $s$ is root-active, then $s \in \mathcal{U}$.
**Hypercollapsingness** If $s$ is hypercollapsing, then $s \in \mathcal{U}$.
**Indiscernability** If $s \leftrightarrow^{\mathcal{U}} t$, then $s \in \mathcal{U}$ iff $t \in \mathcal{U}$.

Intuitively, residuals and overlap state, resp., that no information can be obtained about meaningless terms by reducing them or by placing them in a context. All root-active terms, which includes all hypercollapsing terms, reduce indefinitely at the root and do not become stable. Hence, it is reasonable to consider these terms to be meaningless. This will also guarantee the existence of normal forms later on. Indiscernability states that the identities of the meaningless subterms of a meaningless term are irrelevant.

Indiscernability coincides with transitivity, as shown in [8, Lemma 12.9.17]:

**Lemma 3.1.** *A set $\mathcal{U}$ satisfies indiscernability iff $\leftrightarrow^{\mathcal{U}}$ is transitive.*

Hence, in case $\mathcal{U}$ satisfies indiscernability, $\leftrightarrow^{\mathcal{U}}$ is an equivalence relation.

The next lemma introduces two derived axioms describing the simulation of one reduction by another. These axioms are used extensively in the remainder.

**Lemma 3.2.** *In a fully-extended, left-linear iCRS, if $\mathcal{U}$ satisfies residuals and overlap, then for $s \twoheadrightarrow s'$:*

**Simulation** *if $s \to^{\mathcal{U}} t$, there exists a term $t'$ such that $t \twoheadrightarrow t'$ and $s' \to^{\mathcal{U}} t'$, and*
**Bisimulation** *if $s \leftrightarrow^{\mathcal{U}} t$, there exists a term $t'$ such that $t \twoheadrightarrow t'$ and $s' \leftrightarrow^{\mathcal{U}} t'$.*

*Proof (Sketch).* By ordinal induction on the length of $s \twoheadrightarrow s'$, using fully-extendedness and left-linearity. Employ the fact that each subterm in $\mathcal{U}$ has a residual — a redex pattern either occurs fully inside or fully outside a subterm in $\mathcal{U}$ by overlap — and the fact that each residual of a subterm in $\mathcal{U}$ is in $\mathcal{U}$ — by residuals. □

### 3.2 Meaningful Terms

Meaningless subterms can occur in the reducts of a term $s$ — even without $s$ having meaningless subterms itself. In such a case, $s$ cannot be called completely meaningful. Contrary, any term not possessing this property can be considered meaningful. Following [7] and assuming an iCRS $\mathcal{C}$ and set of terms $\mathcal{U}$, we define:

**Definition 3.3.** *A term $s$ is* totally meaningful *if no subterm of any reduct of $s$ occurs in $\mathcal{U}$.*

With the help of totally meaningful terms, we can express the intuition that meaningless terms should be "computational irrelevant" [7]:

**Definition 3.4.** *The set $\mathcal{U}$ is called* generic*, if for every $s \in \mathcal{U}$ and context $C[\Box]$ reduction of $C[s]$ to a totally meaningful term implies reduction of $C[t]$ to a totally meaningful term for every term $t$.*

Simulation is a sufficient criterion for genericity to hold:

**Theorem 3.5 (Genericity).** *If $\mathcal{C}$ is fully-extended and left-linear and $\mathcal{U}$ satisfies simulation, then $\mathcal{U}$ is generic.*

*Proof.* Let $C[s] \twoheadrightarrow s'$ with $s \in \mathcal{U}$ and $s'$ totally meaningful. If $t$ is arbitrary, then $C[s] \to^{\mathcal{U}} C[t]$. Hence, by simulation a term $t'$ exists such that $s' \to^{\mathcal{U}} t'$. Since $s'$ is totally meaningful, $s' = t'$ and genericity follows. □

Above, "computational irrelevancy" is expressed employing reduction. Alternatively, it can be expressed employing conversion; in which case we define [7]:

**Definition 3.6.** *The iCRS $\mathcal{C}$ is* relative consistent *given $\mathcal{U}$, if $s\ (\twoheadrightarrow \cdot \leftrightarrow^{\mathcal{U}} \cdot \twoheadleftarrow)^*\ t$ implies $s\ (\twoheadrightarrow \cdot \twoheadleftarrow)^*\ t$ for all totally meaningful terms $s$ and $t$.*

To show that relative consistency holds under the assumption of certain axioms, we first state a confluence theorem:

**Theorem 3.7 (Confluence).** *If $\mathcal{C}$ is fully-extended and orthogonal and $\mathcal{U}$ satisfies bisimulation, hypercollapsingness, and indiscernability, then $\mathcal{C}$ is confluent modulo $\mathcal{U}$.*

The proof is similar to that of [7, Lemma 23], observing Lemma 3.1 and using bisimulation instead of [7, Lemma 21]. Lemma 14 in [7] is Theorem 2.8.

We can now show relative consistency. Remark that the assumed axioms are much stronger than in the case of genericity.

**Theorem 3.8 (Relative Consistency).** *If $\mathcal{C}$ is fully-extended and orthogonal and $\mathcal{U}$ satisfies bisimulation, hypercollapsingness, and indiscernability, then $\mathcal{C}$ is relatively consistent given $\mathcal{U}$.*

*Proof.* Let $s\ (\twoheadrightarrow \cdot \leftrightarrow^{\mathcal{U}} \cdot \twoheadleftarrow)^*\ t$, with $s$ and $t$ totally meaningful. By induction on the number of changes in the direction of the rewrite relation in $s\ (\twoheadrightarrow \cdot \leftrightarrow^{\mathcal{U}} \cdot \twoheadleftarrow)^*\ t$ and Theorem 3.7 there exist terms $s'$ and $t'$ such that $s \twoheadrightarrow s' \leftrightarrow^{\mathcal{U}} t' \twoheadleftarrow t$. Hence, since $s$ and $t$ are totally meaningful, $s' = t'$ and the result follows. □

### 3.3 Böhm-Like Trees

In this section, we define Böhm-like trees by means of infinitary rewriting. The definition proceeds in two steps. In the first, we define an iCRS that extends the iCRS whose Böhm-like trees we want to define. In the second step, we give sufficient criteria — in the form of our axioms — implying that the defined iCRS is confluent and normalising. Confluence and normalisation imply that each term has a unique normal form, the Böhm-like tree of that term.

We assume that our set of meaningless terms is a set of *partial* terms, we denote this set by $\mathcal{U}_\perp$. In the next section, we show how to obtain such a set of partial terms from a set of (non-partial) terms.

Our iCRS and Böhm-like tree are defined as follows:

**Definition 3.9.** *The* Böhm-like iCRS *of an iCRS* $\mathcal{C} = (\Sigma, R)$ *and a set of partial terms* $\mathcal{U}_\perp$ *is a pair* $\mathcal{B} = (\Sigma_\perp, R \cup B)$ *with* $B = \{b \to_\perp \perp \mid b \in \mathcal{U}_\perp,\ b \neq \perp\}$.

*A* rewrite step *in* $\mathcal{B}$ *is a pair of partial terms* $(s,t)$ *denoted* $s \to t$ *and adorned with a context* $C[\Box]$ *and a rule* $l \to r \in R$ *or a rule* $b \to_\perp \perp \in B$ *such that:*

- $s = C[\bar{\sigma}(l)]$ *and* $t = C[\bar{\sigma}(r)]$ *with* $\bar{\sigma}$ *a valuation, or*
- $s = C[b]$ *and* $t = C[\perp]$.

*A* Böhm-like tree *of a partial term* $s$ *is a normal form of* $s$ *with respect to* $\mathcal{B}$.

Remark that the definition of rewrite steps deviates slightly from the usual one; no valuation is employed in case the rule originates from $B$. Reduction-wise nothing changes; we employ strongly convergent reductions.

Writing $s \twoheadrightarrow_R t$ for a reduction in case all rewrite rules originate from the set $R$, we have the following:

**Lemma 3.10.** *Given a fully-extended, left-linear iCRS and a set* $\mathcal{U}_\perp$*:*

1. *if* $\mathcal{U}_\perp$ *satisfies root-activeness, then every term has a Böhm-like tree, and*
2. *if* $\mathcal{U}_\perp$ *satisfies residuals, then* $s \twoheadrightarrow t$ *implies* $s \twoheadrightarrow_R \cdot \twoheadrightarrow_\perp t$.

The proof of the first part, resp. of the second part, is identical to that of [7, Theorem 1], resp. [7, Lemma 27].

The following now suffices to ensure that each (partial) term has a unique Böhm-like tree.

**Theorem 3.11.** *Given a fully-extended, orthogonal iCRS, if* $\mathcal{U}_\perp$ *satisfies residuals, overlap, root-activeness, and indiscernability and if* $\perp \in \mathcal{U}_\perp$*, then* $\mathcal{B}$ *is confluent and every term has a unique Böhm-like tree.*

The proof is similar to that of [7, Theorem 2], using Lemma 3.10 instead of Theorem 1 and Lemma 27 in [7], Theorem 2.8 instead of Lemma 14 in [7], and Lemma 3.1 instead of Lemma 15 in [7].

In case the Böhm-like tree of a term $s$ is uniquely defined by the set $\mathcal{U}_\perp$, we denote it by $\mathrm{BLT}^\infty(s)$. The following is immediate by the previous theorem:

**Corollary 3.12 (Congruence).** *Given a fully-extended, orthogonal iCRS, if* $\mathcal{U}_\perp$ *satisfies residuals, overlap, root-activeness, and indiscernability and if* $\perp \in \mathcal{U}_\perp$*, then for all terms* $s$ *and* $t$ *and each context* $C[\Box]$ *it holds that* $\mathrm{BLT}^\infty(s) = \mathrm{BLT}^\infty(t)$ *implies* $\mathrm{BLT}^\infty(C[s]) = \mathrm{BLT}^\infty(C[t])$.

*Remark 3.13.* Overlap can be replaced by bisimulation in the above theorem. Doing so, we can prove uniqueness of Böhm-like trees for certain iCRSs and sets $\mathcal{U}_\perp$ where overlap *does* occur.

Consider for example the rule:

$$f(g(Z)) \to f(Z)$$

and the set

$$\mathcal{U}_\perp = \{g^n(\perp),\ f(g^\omega) \mid n \in \mathbb{N}\}.$$

Residuals, root-activeness, and indiscernability follow easily. Concerning bisimulation, the only interesting case is $f(g^{n+1}(\bot)) \to f(g^n(\bot))$ with $f(g^{n+1}(\bot)) \leftrightarrow^{\mathcal{U}} f(g^m(\bot))$. As $g^n(\bot) \in \mathcal{U}$ for all $n \in \mathbb{N}$, we have the following diagram:

$$\begin{array}{ccc} f(g^{n+1}(\bot)) & \overset{\mathcal{U}}{\longleftrightarrow} & f(g^m(\bot)) \\ \downarrow & & \parallel \\ f(g^n(\bot)) & \overset{\mathcal{U}}{\longleftrightarrow} & f(g^m(\bot)) \end{array}$$

Thus, bisimulation holds. As $0 \in \mathbb{N}$, we also have $\bot \in \mathcal{U}_\bot$. Hence, we find that every term has a unique Böhm-like tree although $\mathcal{U}_\bot$ does not satisfy overlap.

### 3.4 Extending $\mathcal{U}$ with $\bot$

Assume we have at our disposal an iCRS $\mathcal{C} = (\Sigma, R)$ and a set of (non-partial) terms $\mathcal{U}$. We next define a set of partial terms $\mathcal{U}_\bot \supseteq \mathcal{U}$ [7]. The set is defined in such a way that each of the axioms satisfied by $\mathcal{U}$ is also satisfied by $\mathcal{U}_\bot$. The construction slightly simplifies some of our examples in the next section.

**Definition 3.14.** *A* $\bot$-instance *of a partial term $s$ is a term $t$ obtained by replacing every $\bot$ in $s$ by a term in $\mathcal{U}$, i.e. $s \leftarrow^{\mathcal{U}}_{P} t$, where $P = \{p \in \mathcal{P}os(s) \mid s|_p = \bot\}$.*

*The set $\mathcal{U}_\bot$ is the union of $\{\bot\}$ and the set of partial terms each of which has a $\bot$-instance in $\mathcal{U}$.*

Note that if $t$ is a $\bot$-instance of $s$, then $s \preccurlyeq t$; the reverse does not necessarily hold. Explicit inclusion of $\bot$ in $\mathcal{U}_\bot$ only makes difference in case $\mathcal{U}$ is empty, we then have $\mathcal{U}_\bot = \{\bot\}$. Otherwise, $\bot$ is included automatically as each term in $\mathcal{U}$ is a $\bot$-instance of $\bot$. Inclusion of $\{\bot\}$ is needed in light of Theorem 3.11.

As promised, we have the following:

**Lemma 3.15.** *For each of residuals, overlap, root-activeness, hypercollapsingness, and indiscernability, if $\mathcal{U}$ satisfies the property, then so does $\mathcal{U}_\bot$.*

Each property in the lemma follows easily; see [7, Lemma 25]. Roughly, we are required to show that each considered partial term has a $\bot$-instance in $\mathcal{U}$.

### 3.5 Examples

We consider three interesting sets of meaningless terms from [7] defining Böhm-like trees. We show that in the higher-order case these sets also define Böhm-like trees. The sets are those of the root-active, opaque, and Huet-Lévy undefined terms.

**Root-Active.** As argued above, root-active terms are essentially meaningless. Hence, it is interesting to consider the set solely consisting of these terms. This set defines a Böhm-like tree, given a fully-extended, orthogonal iCRS.

Recall from [15] that a term is root-active iff a perpetual reduction starts from it, i.e. a reduction with an infinite number of root-steps. We have:

**Proposition 3.16.** *Let $s$ and $t$ be terms. If $s$ is root-active and $s \leftrightarrow^{\mathcal{U}} t$, then $t$ is root-active.*

*Proof (Sketch).* Since a term is root-active iff it has a perpetual reduction starting from it, consider a perpetual reduction $S$ starting from $s$ and define a perpetual reduction starting from $t$. To do so, omit those steps from $S$ that occur inside subterms that are residuals of subterms replaced in $s \leftrightarrow^{\mathcal{U}} t$. □

Employing the above, we have the following:

**Proposition 3.17.** *The root-active terms satisfy residuals, overlap, root-activeness, and indiscernability.*

*Proof.* Residuals and overlap follow by orthogonality. Root-activeness is immediate by definition. Indiscernability follows by Proposition 3.16. □

The root-active terms also satisfy hypercollapsingness, as every hypercollapsing term is root-active. Simulation and bisimulation follow by Lemma 3.2. Hence, genericity and relative consistency also follow. By Lemma 3.15, each term has a Böhm-like tree with respect to the set each partial terms each of which has a $\bot$-instance that is root-active.

**Opaque.** Similar to root-activeness, opaqueness takes an axiom as its starting point, in this case overlap. We again assume a fully-extended, orthogonal iCRS.

**Definition 3.18.** *A closed term $s$ is* opaque *iff no term to which $s$ reduces is overlapped by a redex at a non-root position. A term is* opaque *iff every closed substitution instance is.*

The above definition stems from [7]. The definition in [16], i.e. that a term $s$ is opaque iff no term reachable from $s$ is overlapped by a redex at a non-root position, cannot be the intended one, as it is not closed under substitution in case of $\lambda$-calculus. For example, the open term $x$ would then be opaque, while the substitution instance $\lambda y.y$ is not, as it is a subterm of $(\lambda y.y)z$.

We have the following; the proof is identical to the one in Sect. 8.1.3 of [7]:

**Proposition 3.19.** *The set of opaque terms satisfies residuals, overlap, root-activeness, and indiscernability.*

The opaque terms also satisfy hypercollapsingness, as every hypercollapsing term is root-active. Simulation and bisimulation follow by Lemma 3.2. Hence, genericity and relative consistency also follow. By Lemma 3.15, each term has a Böhm-like tree with respect to the set each partial terms each of which has a $\bot$-instance that is opaque.

**Huet-Lévy Undefined.** As shown in [7, Sect. 8.1.4], the Huet-Lévy TRS — a starting point for the direct approximant approach [1, 10] — can be used to define a set of meaningless terms. This approach extends to CRSs, assuming a fully-extended, orthogonal CRS $\mathcal{C}$, i.e. only allowing finite terms and reductions.

**Definition 3.20.** *The* Huet-Lévy CRS *of $\mathcal{C}$ is defined as $\mathcal{HL} = (\Sigma_\bot, HL)$, with:*

$$HL = \{d \to \bot \mid d \textit{ a partial pattern}\} \cup \{l \to \bot \mid l \to r \in R\},$$

*where a partial pattern $d$ is any pattern $\bot \neq d \preccurlyeq l$ with $l \to r \in R$ such that no valuation $\bar{\sigma}$ exists with $\bar{\sigma}(d) = \bar{\sigma}(l)$.*

By the definition, Huet-Lévy CRSs are orthogonal, because $\mathcal{C}$ is orthogonal, and without collapsing rules. We easily obtain the following:

**Proposition 3.21.** *The Huet-Lévy CRS $\mathcal{HL}$ of $\mathcal{C}$ is confluent. Any finite partial term $s$ has a unique normal form $\omega_{\mathrm{HL}}(s)$ and for all finite partial terms $s$ and $t$:*

1. $\omega_{\mathrm{HL}}(s) \preccurlyeq s$,
2. *if a redex occurs at position $p$ in $s$, then* $\omega_{\mathrm{HL}}(s) \preccurlyeq s[\bot]_p$, *and*
3. *if $s \to t$, then* $\omega_{\mathrm{HL}}(s) \preccurlyeq \omega_{\mathrm{HL}}(t)$,

We can now define the following two sets:

$$\mathcal{U}^{\mathrm{f}}_{\mathrm{HL}} = \{s \text{ a finite partial term} \mid \forall s \to^* t : \omega_{\mathrm{HL}}(t) = \bot\}$$
$$\mathcal{U}_{\mathrm{HL}} = \{s \mid \forall s \twoheadrightarrow t \text{ and } u \preccurlyeq t : u \in \mathcal{U}^{\mathrm{f}}_{\mathrm{HL}}\}$$

**Proposition 3.22.** *The terms in $\mathcal{U}_{\mathrm{HL}}$ satisfies residuals, overlap, root-activeness, and indiscernability.*

*Proof.* Overlap, resp. residuals, follows by orthogonality of the CRS, resp. of the Huet-Lévy CRS. Root-activeness follows by Proposition 3.21(2).

In the case of indiscernability, consider $s \leftrightarrow^{\mathcal{U}_{\mathrm{HL}}} t$ with $s \in \mathcal{U}_{\mathrm{HL}}$ and let $t \twoheadrightarrow t'$. By bisimulation, which follows from Lemma 3.2, there exists a reduction $s \twoheadrightarrow s'$ such that $s' \leftrightarrow^{\mathcal{U}_{\mathrm{HL}}} t'$. Consider any finite partial term $u_t \preccurlyeq t'$. As $s' \leftrightarrow^{\mathcal{U}_{\mathrm{HL}}} t'$, we have a finite partial term $u_s \preccurlyeq s'$ such that $u_s \leftrightarrow^{\mathcal{U}^{\mathrm{f}}_{\mathrm{HL}}} u_t$. Since $u_s$ and $u_t$ are finite and $u_s \leftrightarrow^{\mathcal{U}^{\mathrm{f}}_{\mathrm{HL}}} u_t$, there exists a finite partial term $u$ such that $u_s \to^* u \; {}^*\!\leftarrow u_t$, employing the reduction rules of the Huet-Lévy CRS. Hence, as $s \in \mathcal{U}_{\mathrm{HL}}$ implies $u_s \to^* \bot$, we have $u_t \to^* \bot$ and indiscernability follows. □

The set $\mathcal{U}_{\mathrm{HL}}$ also satisfies hypercollapsingness, as every hypercollapsing term is root-active. Simulation and bisimulation follow by Lemma 3.2. Hence, genericity and relative consistency also follow. Moreover, as $\bot \in \mathcal{U}_{\mathrm{HL}}$, each term has a Böhm-like tree with respect to $\mathcal{U}_{\mathrm{HL}}$.

*Remark 3.23.* The definition of the set $\mathcal{U}_{\mathrm{HL}}$ differs from the one in [7], which requires an additional nullary function symbol. It is easily shown that $\mathcal{U}_{\mathrm{HL}}$ and the set defined in [7] yield exactly the same Böhm-like tree for each term.

*Remark 3.24.* Consider the CRS encoding of the $\beta$-rule from $\lambda$-calculus:

$$\mathtt{app}(\mathtt{lam}([x]Z(x)), Z') \to Z(Z')$$

This rule yields the following Huet-Lévy CRS:

$$\mathtt{app}(\mathtt{lam}([x]Z(x)), Z') \to \bot$$
$$\mathtt{app}(\mathtt{lam}(\bot), Z') \to \bot$$
$$\mathtt{app}(\bot, Z') \to \bot$$

Any term that is the encoding of a term from $\lambda$-calculus is in $\mathcal{U}_{\mathrm{HL}}$ iff the term does not have a weak head normal form, i.e. the $\lambda$-term does not reduce to a term of the form $\lambda x.s$ or $x s_1 s_2 \ldots s_n$. Hence, $\mathcal{U}_{\mathrm{HL}}$ defines the Lévy-Longo tree [4,5,17].

This means that not only the opaque terms define an iTRS analogue of Lévy-Longo trees, as stated in [16], but so does $\mathcal{U}_{\mathrm{HL}}$. The set of opaque terms and

$\mathcal{U}_{\mathrm{HL}}$ do not need to coincide: Consider a ruleless CRS. All terms are opaque, while $\mathcal{U}_{\mathrm{HL}} = \{\bot\}$. Thus, the question whether an analogue of the Lévy-Longo tree exists for TRSs [10] does not have a unique answer.

## 4 Comparison

Having defined Böhm-like trees by means of infinitary rewriting, we can now compare this approach with the direct approximant approach. To do so, we first recall the direct approximant definition for CRSs from [9].

**Definition 4.1.** *Let $\mathcal{C} = (\Sigma, R)$ be an orthogonal CRS. A* direct approximant function *is a map $\omega$ on finite partial terms, such that for all terms $s$ and $t$:*

1. *$\omega(s) \preccurlyeq s$,*
2. *if a redex occurs at position $p$ in $s$, then $\omega(s) \preccurlyeq s[\bot]_p$, and*
3. *if $s \to t$, then $\omega(s) \preccurlyeq \omega(t)$,*

*where $\omega(s)$ is called the* direct approximant *of $s$.*

Hence, $\omega_{\mathrm{HL}}$, as defined in Sect. 3.5, is a direct approximant function.

The definition only concerns CRSs and not iCRSs. As such, our comparison only concerns the Böhm-like trees of *finite* terms. Since each pair that defines a CRS also defines an iCRS, with the reductions of the CRS forming a subset of the reductions of the iCRS, this does not pose any obstacle in our comparison.

In the current context, Böhm-like trees are defined as follows:

**Definition 4.2.** *Let $s$ be a finite partial term. The* Böhm-like tree *of $s$ with respect to $\omega$, denoted $\mathrm{BLT}(s)$, is defined as:*

$$\mathrm{BLT}(s) = \bigsqcup \{\omega(t) \mid s \to^* t\} .$$

The set $\{\omega(t) \mid s \to^* t\}$ is directed by confluence and the third clause of the direct approximant definition. Hence, the least upper bound exists.

Usually, $\mathrm{BLT}(s)$ is defined by means of downward closure instead of the least upper bound [1, 9, 10], with the (infinite) terms being defined by means of ideal completion. However, downward closure and the least upper bound coincide in case of ideals. Replacing downward closure by the least upper bound allows us to avoid the introduction of (infinite) terms by means of ideal completion, using the isomorphic definition of terms given in Sect. 2 [1].

Obviously, each finite partial term has a unique Böhm-like tree. Moreover, Böhm-like trees are preserved under rewriting:

**Theorem 4.3.** *If $s \to^* t$, with $s$ and $t$ finite, then* $\mathrm{BLT}(s) = \mathrm{BLT}(t)$.

*Proof.* Let $s \to^* t$. By confluence of $\mathcal{C}$ there exists for every $s \to^* s'$ and $t \to^* t'$ a partial term $u$ such that $s' \to^* u \;{}^*\!\!\leftarrow t'$. Hence, by the third clause of Definition 4.1 and the definition of Böhm-like trees we have $\mathrm{BLT}(s) = \mathrm{BLT}(t)$. □

### 4.1 From Infinitary Rewriting to Direct Approximants

Assume $\mathcal{C} = (\Sigma, R)$ is a fully-extended, orthogonal CRS and $\mathcal{U}$ is a set of meaningless terms satisfying residuals, overlap, root-activeness, and indiscernability such that $\bot \in \mathcal{U}$. We show that we can define a direct approximant function such that for each finite term we have that the Böhm-like tree it defines is identical to the Böhm-like tree we would obtain by means of infinitary rewriting.

We first define a map:

**Definition 4.4.** *The map $\omega_{\mathcal{U}}$ on finite partial terms is defined for each term $s$ as the largest term $t$, with respect to the prefix order, such that $t \preccurlyeq s[\bot]_p$ for all $p \in \mathcal{P}os(s)$ with $s|_p$ either transfinitely reducible to a redex or to term in $\mathcal{U}$.*

We now show:

**Lemma 4.5.** *The map $\omega_{\mathcal{U}}$ defines a direct approximant function.*

*Proof.* We consider each of the clauses of Definition 4.1 in turn:

1. That $\omega_{\mathcal{U}}(s) \preccurlyeq s$ is immediate by the definition of $\omega_{\mathcal{U}}$.
2. That $\omega_{\mathcal{U}}(s) \preccurlyeq s[\bot]_p$ for all $p \in \mathcal{P}os(s)$ if redex occurs at $p$ in $s$, follows by the fact that $\omega_{\mathcal{U}}(s) \preccurlyeq s[\bot]_p$ if $s|_p$ transfinitely reduces to a redex.
3. That $s \to t$ implies $\omega_{\mathcal{U}}(s) \preccurlyeq \omega_{\mathcal{U}}(t)$, follows, as for each position $p$ parallel or above the contracted redex (in both $s$ and $t$), we have that $s|_p$ transfinitely reduces to a redex or to term in $\mathcal{U}$ if $t|_p$ does. □

Write $\mathrm{BLT}^{\infty}_{\mathcal{U}}$ for the Böhm-like tree defined by the Böhm-like iCRS $\mathcal{B}$ of $\mathcal{C}$ and $\mathcal{U}$ and write $\mathrm{BLT}_{\mathcal{U}}$ for the tree defined by $\omega_{\mathcal{U}}$. We show our main result, i.e. coincidence of $\mathrm{BLT}^{\infty}_{\mathcal{U}}$ and $\mathrm{BLT}_{\mathcal{U}}$. The proof effectively defines a bisimulation.

**Theorem 4.6.** *If $s$ is a finite partial term, then $\mathrm{BLT}_{\mathcal{U}}(s) = \mathrm{BLT}^{\infty}_{\mathcal{U}}(s)$.*

*Proof.* Given a finite partial term $s$, we show by induction on positions $p$ that $p \in \mathcal{P}os(\mathrm{BLT}_{\mathcal{U}}(s))$ iff $p \in \mathcal{P}os(\mathrm{BLT}^{\infty}_{\mathcal{U}}(s))$ and $root(\mathrm{BLT}_{\mathcal{U}}(s)|_p) = root(\mathrm{BLT}^{\infty}_{\mathcal{U}}(s)|_p)$.

Obviously, if $p$ is the root position, it is a position of both Böhm-like trees. Moreover, if $p = q \cdot i$, then $p$ is a position of both Böhm-like trees given that $q$ is such a position, with $root(\mathrm{BLT}_{\mathcal{U}}(s)|_q) = root(\mathrm{BLT}^{\infty}_{\mathcal{U}}(s)|_q)$ of arity $n$ and $0 \leq i \leq n$, considering $[x]$ to be a unary function symbol for every variable $x$. This leaves to show for each position $p$ that $root(\mathrm{BLT}_{\mathcal{U}}(s)|_p) = root(\mathrm{BLT}^{\infty}_{\mathcal{U}}(s)|_p)$.

Suppose $root(\mathrm{BLT}_{\mathcal{U}}(s)|_p) = f$. Either $f = \bot$ or $f \neq \bot$. If $f = \bot$, we have by definition of $\omega_{\mathcal{U}}$ for every $s \to^* t$ with $p \in \mathcal{P}os(\omega_{\mathcal{U}}(t))$ that $t|_p$ transfinitely reduces to a redex or term in $\mathcal{U}$. The first implies $t|_p$ is root-active and, whence, in $\mathcal{U}$. Thus, $root(\mathrm{BLT}^{\infty}_{\mathcal{U}}(s)|_p) = \bot$, as $t|_p$ transfinitely reduces to a term in $\mathcal{U}$ and $p \in \mathcal{P}os(\omega_{\mathcal{U}}(t))$. In case $f \neq \bot$, $s \to^* t$ with $p \in \mathcal{P}os(\omega_{\mathcal{U}}(t))$ and $root(\omega_{\mathcal{U}}(t)|_p) = f$ by definition of $\omega_{\mathcal{U}}$ . Hence, again by definition of $\omega_{\mathcal{U}}$, $t|_p$ neither transfinitely reduces to a redex nor to term in $\mathcal{U}$, implying $root(\mathrm{BLT}^{\infty}_{\mathcal{U}}(s)|_p) = f$.

Now suppose $root(\mathrm{BLT}^{\infty}_{\mathcal{U}}(s)|_p) = f$. As before, either $f = \bot$ or $f \neq \bot$. In case $f = \bot$, there exists by Lemma 3.10(2) and compression a reduction $s \to^* t$ such that $p \in \mathcal{P}os(t)$, all $t|_q$ with $q < p$ not reducible to a redex of $\mathcal{B}$, and $t|_p$ transfinitely reducible to a term in $\mathcal{U}$. Hence, by definition of $\omega_{\mathcal{U}}$, we have $p \in \mathcal{P}os(\omega_{\mathcal{U}}(t))$ and $\omega_{\mathcal{U}}(t)|_p = \bot$, which implies $root(\mathrm{BLT}_{\mathcal{U}}(s)|_p) = \bot$. In case $f \neq \bot$, there exists by Lemma 3.10(2) and compression a finite partial term $t$ such that $t|_q$ with $q \leq p$ not reducible to a redex of $\mathcal{B}$. Hence, $root(\omega_{\mathcal{U}}(t)) = f$, which implies $root(\mathrm{BLT}_{\mathcal{U}}(s)|_p) = f$.

Hence, $root(\mathrm{BLT}_{\mathcal{U}}(s)|_p) = f$ iff $root(\mathrm{BLT}^{\infty}_{\mathcal{U}}(s)|_p) = f$, as required. □

### 4.2 From Direct Approximants to Infinitary Rewriting

Although a Böhm-like tree defined by a direct approximant function exists for every Böhm-like tree defined by a set of meaningless terms, the reverse does not hold. To see this, recall congruence holds for every Böhm-like tree defined by a set of meaningless terms (see Corollary 3.12). Congruence does not necessarily hold for Böhm-like trees defined by direct approximant functions. Consider e.g. the fully-extended, orthogonal CRS consisting of the following two rewrite rules:

$$\texttt{IsEmpty}(\texttt{nil}) \to \texttt{true}$$
$$\texttt{IsEmpty}(x : xs) \to \texttt{false}$$

Moreover, consider the following rules, forming a confluent and terminating CRS:

$$\texttt{IsEmpty}(xs) \to \bot$$
$$\texttt{nil} \to \bot$$

The map $\omega$ assigning to each term its normal form with respect to the last two rules defines a direct approximant function for the CRS consisting of the first two rules. However, the Böhm-like tree defined by $\omega$ is not congruent:

$$\mathrm{BLT}(\bot) = \bot = \mathrm{BLT}(\texttt{nil}),$$

but placed in the context $\texttt{IsEmpty}(\Box)$:

$$\mathrm{BLT}(\texttt{IsEmpty}(\bot)) = \texttt{IsEmpty}(\bot) \neq \texttt{true} = \mathrm{BLT}(\texttt{IsEmpty}(\texttt{nil})).$$

Hence, a class of Böhm-like trees exists that can be defined by means of direct approximant functions, but not by means of a set of meaningless terms.

In the remainder we consider two Böhm-like trees defined by direct approximant functions which we have sets of meaningless terms that do define the same Böhm-like trees: the Berarducci-like trees and the Huet-Lévy trees.

**Berarducci-Like Trees.** Define $\omega_{\mathrm{BeL}}(s)$ as the largest term $t$ with respect to $\preccurlyeq$ such that $t \preccurlyeq s[\bot]_p$ iff the subterm at position $p$ in $s$ reduces to a redex. Given a fully-extended, orthogonal CRS, it is easily shown that $\omega_{\mathrm{BeL}}$ defines a direct approximant function; the one associated with Berarducci-like trees.

We show for every fully-extended, orthogonal CRS $\mathcal{C} = (\Sigma, R)$ that its Berarducci-like tree and the Böhm-like tree defined by the set of root-active terms (see Sect. 3.5) coincide for every finite partial term.

Denote the set of terms each of which has a $\bot$-instance that is a root-active term by $\mathcal{U}_{\mathrm{BeL}}$. Moreover, denote by $\mathrm{BLT}^{\infty}_{\mathrm{BeL}}$ the Böhm-like tree defined by $\mathcal{U}_{\mathrm{BeL}}$ and denote by $\mathrm{BLT}_{\mathrm{BeL}}$ the Berarducci-like tree. We show that $\mathrm{BLT}^{\infty}_{\mathrm{BeL}}$ and $\mathrm{BLT}_{\mathrm{BeL}}$ are identical as maps on the finite partial terms. We start with a lemma:

**Lemma 4.7.** *Let $\mathcal{U}$ be defined as:*

$$\mathcal{U} = \{s \text{ is a partial term} \mid s \text{ either root-active or } s \twoheadrightarrow \bot\}.$$

*It holds that $\mathcal{U} = \mathcal{U}_{\mathrm{BeL}}$.*

*Proof.* We show $\mathcal{U}_{\mathrm{BeL}} \subseteq \mathcal{U}$ and $\mathcal{U} \subseteq \mathcal{U}_{\mathrm{BeL}}$. Thus, suppose $s \in \mathcal{U}_{\mathrm{BeL}}$. By definition of $\mathcal{U}_{\mathrm{BeL}}$, there exists for $s$ a $\bot$-instance $t$ that is root-active. The subterms replaced by $\bot$ either contribute or do not contribute to $t$ being root-active. In case the subterms contribute, we have by orthogonality that $s \twoheadrightarrow \bot$. In case they do not contribute, we have by orthogonality that $s$ is root-active. Hence, $s \in \mathcal{U}$.

That $\mathcal{U} \subseteq \mathcal{U}_{\mathrm{BeL}}$ follows by orthogonality when we replace each $\bot$ in every term of $\mathcal{U}$ by a closed root-active term. In case no root-active term exists, $\mathcal{U} = \{\bot\}$ and we are done immediately. □

We can now prove:

**Theorem 4.8.** *If $s$ is a finite partial term, then* $\mathrm{BLT}^{\infty}_{\mathrm{BeL}}(s) = \mathrm{BLT}_{\mathrm{BeL}}(s)$.

*Proof.* Let $\omega_{\mathcal{U}_{\mathrm{BeL}}}$ be defined according to Definition 4.4, with $\mathcal{U}_{\mathrm{BeL}}$ assuming the rôle of $\mathcal{U}$. By Lemma 4.7, compression, the observation that $\mathcal{U}_{\mathrm{BeL}}$ is closed under transfinite expansion, and Definition 4.4, we have that $\omega_{\mathcal{U}_{\mathrm{BeL}}}$ replaces by $\bot$ precisely every maximal subterm that reduces to a redex — note that $s \twoheadrightarrow \bot$ either has a redex at the root or $s = \bot$. Hence, $\omega_{\mathcal{U}_{\mathrm{BeL}}} = \omega_{\mathrm{BeL}}$ and, by Theorem 4.6, we have for each finite partial term $s$ that $\mathrm{BLT}^{\infty}_{\mathrm{BeL}}(s) = \mathrm{BLT}_{\mathrm{BeL}}(s)$. □

**Huet-Lévy Trees.** By Proposition 3.21, the Huet-Lévy CRS of a fully-extended, orthogonal CRS $\mathcal{C}$ defines a direct approximant function and, hence, a Böhm-like tree, the Huet-Lévy tree.

Denote by $\mathrm{BLT}^{\infty}_{\mathrm{HL}}$ the Böhm-like tree defined by $\mathcal{U}_{\mathrm{HL}}$ and by $\mathrm{BLT}_{\mathrm{HL}}$ the Huet-Lévy tree. We show that $\mathrm{BLT}^{\infty}_{\mathrm{HL}}$ and $\mathrm{BLT}_{\mathrm{HL}}$ are identical as maps on the finite partial terms.

**Theorem 4.9.** *If $s$ is a finite partial term, then* $\mathrm{BLT}^{\infty}_{\mathrm{HL}}(s) = \mathrm{BLT}_{\mathrm{HL}}(s)$.

*Proof.* Suppose $s$ is a finite partial term and let $\omega_{\mathcal{U}_{\mathrm{HL}}}$ be defined according to Definition 4.4, with $\mathcal{U}_{\mathrm{HL}}$ assuming the rôle of $\mathcal{U}$. By definition of $\omega_{\mathcal{U}_{\mathrm{HL}}}$, the subterms of $s$ that either transfinitely reduce to a redex or term in $\mathcal{U}_{\mathrm{HL}}$ are replaced by $\bot$. Hence, by definition of $\mathcal{U}_{\mathrm{HL}}$, all replaced subterms of $s$ have $\bot$ as their Huet-Lévy direct approximant.

If $\omega_{\mathcal{U}_{\mathrm{HL}}}(s)$ does not replace a certain subterm by $\bot$, then the subterm does not reduce to a redex. Moreover, by definition of $\mathcal{U}_{\mathrm{HL}}$ the subterm reduces in a finite number of steps to a term with a Huet-Lévy direct approximant unequal to $\bot$. Hence, by orthogonality there exists a term $t$ and a reduction $s \to^{*} t$ such that $\omega_{\mathcal{U}_{\mathrm{HL}}}(s) \preccurlyeq \omega_{\mathrm{HL}}(t)$.

By the facts from the first paragraph and by orthogonality of the Huet-Lévy CRS, we also have $\omega_{\mathrm{HL}}(s) \preccurlyeq \omega_{\mathcal{U}_{\mathrm{HL}}}(s)$. Hence, $\mathrm{BLT}_{\mathcal{U}_{\mathrm{HL}}}(s) = \mathrm{BLT}_{\mathrm{HL}}(s)$ and by Theorem 4.6 we obtain $\mathrm{BLT}^{\infty}_{\mathrm{HL}}(s) = \mathrm{BLT}_{\mathrm{HL}}(s)$. □

## 5 Conclusion

Somewhat remarkably, there is a difference between the infinitary rewriting approach to Böhm-like trees and the direct approximant approach: Each Böhm-like tree defined by infinitary rewriting coincides with a Böhm-like tree defined by a direct approximant function but the reverse is not the case. The difference seems to be due to the infinitary rewriting approach yielding congruent Böhm-like trees (see Corollary 3.12).

To enable our comparison, we extended to iCRSs the infinitary rewriting approach to Böhm-like trees. Contrary to most of the previous theory developed for iCRSs, no serious complications arise due to iCRSs being higher-order. However, as noted by Van Oostrom (private communication), a number of reasonable Böhm-like trees cannot be defined due to the overlap axiom (see Remark 3.13).

At least two questions remain: First, can either the infinitary rewriting approach be extended or the direct approximant approach be restricted as to obtain coincidence between the two approaches? Second, can the overlap axiom be replaced by some new axiom as to allow certain forms of overlap?

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## References

1. Ketema, J.: Böhm-Like Trees for Rewriting. PhD thesis, Vrije Universiteit, Amsterdam (2001)
2. Lévy, J.J.: Réductions correctes et optimales dans le lambda-calcul. PhD thesis, Université de Paris VII (1978)
3. Barendregt, H.P.: The Lambda Calculus: Its Syntax and Semantics. Revised edn. Elsevier Science (1985)
4. Lévy, J.J.: An algebraic interpretation of the $\lambda\beta$K-calculus and the labelled $\lambda$-calculus. In: Proceedings of the Symposium on $\lambda$-calculus and Computer Science Theory. Volume 37 of LNCS. (1975) 147–165
5. Longo, G.: Set-theoretical models of $\lambda$-calculus: Theories, expansions, isomorphisms. Annals of Pure and Applied Logic **24** (1983) 153–188
6. Berarducci, A.: Infinite $\lambda$-calculus and non-sensible models. In: Logic and Algebra. Volume 180 of Lect. Notes Pure Appl. Math. (1996) 339–378
7. Kennaway, R., van Oostrom, V., de Vries, F.J.: Meaningless terms in rewriting. JFLP **1** (1999)
8. Kennaway, R., de Vries, F.J.: Infinitary rewriting. [18] Chapter 12
9. Blom, S.C.C.: Term Graph Rewriting: syntax and semantics. PhD thesis, Vrije Universiteit, Amsterdam (2001)
10. Ketema, J.: Böhm-like trees for term rewriting systems. In: RTA 2004. Volume 3091 of LNCS. (2004) 233–248
11. Ketema, J., Simonsen, J.G.: Infinitary combinatory reduction systems. In: RTA 2005. Volume 3467 of LNCS. (2005) 438–452
12. Ketema, J., Simonsen, J.G.: On confluence of infinitary combinatory reduction systems. In: LPAR 2005. Volume 3835 of LNAI. (2005) 199–214
13. Ketema, J., Simonsen, J.G.: Infinitary combinatory reduction systems. Technical Report D-558, Department of Computer Science, University of Copenhagen (2006)
14. Klop, J.W., van Oostrom, V., van Raamsdonk, F.: Combinatory reduction systems: introduction and survey. TCS **121**(1 & 2) (1993) 279–308
15. Ketema, J.: On normalisation of infinitary combinatory reduction systems. In: RTA 2008. Volume 5117 of LNCS. (2008) 172–186
16. Kennaway, R., et al.: Infinitary rewriting: From syntax to semantics. In: Processes, Terms and Cycles. Volume 3838 of LNCS. (2005) 148–172
17. Kennaway, J.R., et al.: Infinitary lambda calculus. TCS **175**(1) (1997) 93–125
18. Terese, ed.: Term Rewriting Systems. Cambridge University Press (2003)

## A Omitted Proofs

*Proof (Lemma 3.1).* Let $\mathcal{U}$ satisfy indiscernability and let $s_1 \leftrightarrow^{\mathcal{U}}_{P_1} s_2 \leftrightarrow^{\mathcal{U}}_{P_2} s_3$. Define $Q$ as the set of minimal positions in $P_1 \cup P_2$ with respect to the prefix order on positions. The set $Q$ is a set of positions of each of $s_1$, $s_2$ and $s_3$. Since $\mathcal{U}$ satisfies indiscernability, we have for all positions in $Q$ that the subterms of $s_1$, $s_2$, and $s_3$ are in $\mathcal{U}$. Hence, $s_1 \leftrightarrow^{\mathcal{U}} s_3$ follows, as required.

To see that transitivity implies indiscernability, assume there exist terms $s \in \mathcal{U}$ and $t \notin \mathcal{U}$ such that $s \leftrightarrow^{\mathcal{U}}_{P} t$. Let $p$ be a position of minimal depth in $P$ and observe that $s$ and $t$ are identical up to depth $|p|$. Trivially, $s|_p \leftrightarrow^{\mathcal{U}} s \leftrightarrow^{\mathcal{U}} t|_p$, since all these terms are in $\mathcal{U}$, and by transitivity of $\leftrightarrow^{\mathcal{U}}$ we have $s|_p \leftrightarrow^{\mathcal{U}}_{Q_1} t \leftrightarrow^{\mathcal{U}}_{Q_2} t|_p$. Since $t \notin \mathcal{U}$, neither $Q_1$ nor $Q_2$ can be empty or be equal to $\{\epsilon\}$. Thus, there exists a non-empty context $C[\Box, \ldots, \Box]$ with $n$ holes such that $s|_p = C[s_1, \ldots s_n]$ and $t|_p = C[t_1, \ldots, t_n]$ with $s_i \leftrightarrow^{\mathcal{U}}_{R_i} t_i$ for all $1 \leq i \leq n$ and we can define:

$$P' = (P - \{p\}) \cup \{p \cdot q_i \cdot r \mid 1 \leq i \leq n,\ q_i \text{ the position } i\text{th hole, and } r \in R_i\}\,.$$

By definition, $s \leftrightarrow^{\mathcal{U}}_{P'} t$. Moreover, $|p| < |p \cdot q_i \cdot r|$ for any $q$, as $C[\Box, \ldots, \Box]$ is non-empty, and $p$ is of minimal depth in $P$. Hence, repeating the argument yields that $s$ and $t$ are identical up to any arbitrary depth. Thus, $s = t$ and $t \in \mathcal{U}$, contradiction. □

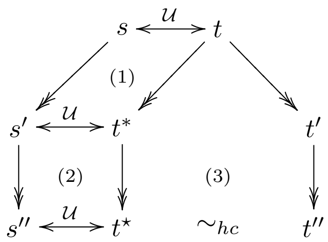
*Proof (Lemma 3.2).* Let $s \twoheadrightarrow^{\alpha} s'$ and $s \rightarrow^{\mathcal{U}} t$ (resp. $s \leftrightarrow^{\mathcal{U}} t$). We prove the result by ordinal induction on $\alpha$. If $\alpha = 0$, then the result is immediate.

If $\alpha = \beta + 1$, assume $s \twoheadrightarrow^{\alpha} s' = s \twoheadrightarrow^{\beta} s_\beta \rightarrow s'$. By the induction hypothesis there exists a term $t_\beta$ such that $t \twoheadrightarrow t_\beta$ and $s_\beta \rightarrow^{\mathcal{U}}_{P} t_\beta$ (resp. $s_\beta \leftrightarrow^{\mathcal{U}}_{P} t_\beta$). There are two possibilities for $s_\beta \rightarrow s'$ since $\mathcal{U}$ satisfies overlap: the redex pattern of the contracted redex occurs either fully outside all subterms at positions in $P$ or not.

- If the redex pattern of the contracted redex occurs outside all subterms at positions in $P$, then $s_\beta \rightarrow^{\mathcal{U}} t_\beta$ (resp. $s_\beta \leftrightarrow^{\mathcal{U}} t_\beta$) together with left-linearity and fully-extendedness implies that a redex employing the same rewrite rule as the redex contracted in $s_\beta \rightarrow s'$ occurs at the same position in $t_\beta$. Contracting the redex in $t_\beta$ yields a step $t_\beta \rightarrow t'$. That $s' \rightarrow^{\mathcal{U}} t'$ (resp. $s' \leftrightarrow^{\mathcal{U}} t'$) follows by $s_\beta \rightarrow^{\mathcal{U}} t_\beta$ (resp. $s_\beta \leftrightarrow^{\mathcal{U}} t_\beta$) and the fact that the same rewrite rule is employed in both $s_\beta \rightarrow s'$ and $t_\beta \rightarrow t'$: Clearly, $s'$ and $t'$ are identical at all positions $p$ that descend from positions whose subterms are not replaced in $s_\beta \rightarrow^{\mathcal{U}} t_\beta$ (or $s_\beta \leftrightarrow^{\mathcal{U}} t_\beta$). Moreover, all residuals of subterms replaced in $s_\beta \rightarrow^{\mathcal{U}} t_\beta$ (resp. $s_\beta \leftrightarrow^{\mathcal{U}} t_\beta$) are in $\mathcal{U}$ by the residuals axiom.
- If the redex occurs inside a subterm at a position in $P$, it follows by residuals that $s' \rightarrow^{\mathcal{U}} t_\beta$ (resp. $s' \leftrightarrow^{\mathcal{U}} t_\beta$). Hence, we can define $t' = t_\beta$.

If $\alpha$ is a limit ordinal, the result is immediate by strong convergence, the induction hypothesis, and the residuals axiom; note in the successor ordinal case that any reduction step of $s \twoheadrightarrow s'$ 'simulated' by $t \twoheadrightarrow t'$ occurs at the same depth. □

*Proof (Theorem 3.7).* Let $s \leftrightarrow^{\mathcal{U}} t$ and assume $s \twoheadrightarrow s'$ and $t \twoheadrightarrow t'$. Consider the following diagram:




In the diagram, (1) and (2) exist by bisimulation and (3) exists by Theorem 2.8. The result now follows by the diagram, the fact that all hypercollapsing subterms are included in $\mathcal{U}$, and transitivity of $\leftrightarrow^{\mathcal{U}}$. $\square$

*Proof (Lemma 3.10).*

1. Let $s$ be a term. If $s$ is not root-active, then $s$ reduces to a head normal form. Recursively apply this argument to all non-root-active subterms of the obtained head normal form. This yields a strongly convergent reduction to a term $t$ all whose redexes employing rules from $R$ occur in root-active subterms. Iteratively, reduce all maximal subterms of $t$ in $\mathcal{U}_\perp$ to $\perp$. This is possible by a strongly convergent reduction, as only finitely many subterms occur at each depth and as every depth is finite. Hence, as replacement of subterms by $\perp$ cannot create any redexes employing rules from $R$, by fully-extendedness and left-linearity, and as all root-active terms are in $\mathcal{U}_\perp$, the result follows.
2. Let $s \twoheadrightarrow t$. By fully-extendedness and left-linearity a strongly convergent reduction $s \twoheadrightarrow_R t'$ — omitting from $s \twoheadrightarrow t$ all steps employing rules from $B$ — is readily defined by ordinal induction.
Define $t_\alpha$ as $t'$ with each subterm a position in $t'$ replaced $\perp$ if that position descents from one of the positions of the first $\alpha$ redexes in $s \twoheadrightarrow t$ omitted from $s \twoheadrightarrow t'$. By the residuals axiom a redex occurs in $t_\alpha$ at every position descending from the $\alpha + 1$th step in $s \twoheadrightarrow t$ omitted from $s \twoheadrightarrow t'$. Contracting the occurring redexes in a depth-wise fashion yields a strongly convergent reduction $t_\alpha \twoheadrightarrow_\perp t_{\alpha+1}$, as only a finite number of subterms occur at each depth. Since $s \twoheadrightarrow t$ is strongly convergent, concatenating all these reductions yields a strongly convergent reduction $t' \twoheadrightarrow_\perp t$, as required. $\square$

*Proof (Theorem 3.11).* Let $s$ be a term. Obviously, $s$ has a Böhm-like tree by Lemma 3.10(1).

Suppose that $t_0 \twoheadleftarrow s \twoheadrightarrow t_1$. Again by Theorem 3.10(1), the terms $t_0$ and $t_1$ each have a Böhm-like tree, write these resp. $t_0^*$ and $t_1^*$. By Lemma 3.10(2) there exist terms $s_0'$ and $s_1'$ such that $s \twoheadrightarrow_R s_0' \twoheadrightarrow_\perp t_0^*$ and $s \twoheadrightarrow_R s_1' \twoheadrightarrow_\perp t_1^*$. Moreover, by Lemma 2.8 we have $s_0' \twoheadrightarrow_R s_0^*$ and $s_1' \twoheadrightarrow_R s_1^*$ with $s_0^* \sim_{hc} s_1^*$ (see Fig. 1).

By the indiscernability axiom and the fact that $\perp \in \mathcal{U}_\perp$, we may consider the reductions $s_0' \twoheadrightarrow_\perp t_0^*$ and $s_1' \twoheadrightarrow_\perp t_1^*$ to be replacements by $\perp$ of the maximal subterms of $s_0'$ and $s_1'$ that are in $\mathcal{U}_\perp$. Thus, $s_0' \leftrightarrow^{\mathcal{U}_\perp} t_0^*$ and $s_1' \leftrightarrow^{\mathcal{U}_\perp} t_1^*$. Moreover, by Lemma 3.2 and the fact that $t_0^*$ and $t_1^*$ are normal forms, we have that $s_0^* \leftrightarrow^{\mathcal{U}_\perp} t_0^*$ and $s_1^* \leftrightarrow^{\mathcal{U}_\perp} t_1^*$. As all hypercollapsing terms are root-active, it follows by transitivity of $\leftrightarrow^{\mathcal{U}_\perp}$ that $t_0^* \leftrightarrow^{\mathcal{U}_\perp} t_1^*$. Hence, since $t_0^*$ and $t_1^*$ are normal forms of $\mathcal{B}$, we have $t_0^* = t_1^*$, as required. □

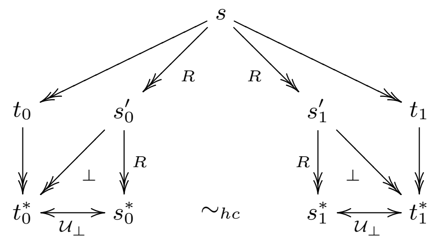



**Fig. 1.** The proof of Theorem 3.11

*Proof (Lemma 3.15).* We give a proof for each of the axioms in turn:

**Residuals** Let $s$ be a partial term and $s \twoheadrightarrow t$. By definition of $\mathcal{U}_\perp$ we can obtain a partial term $s^*$ by replacing every maximal subterm of $s$ that is in $\mathcal{U}_\perp$ by a term in $\mathcal{U}$ such that the subterm becomes a $\perp$-instance — $s^*$ may be partial, as $\mathcal{U}_\perp$ may be $\{\perp\}$. Replace each $\perp$ in $s^*$ by a fresh variable to obtain a term $s'$. By fully-extendedness and orthogonality $s' \twoheadrightarrow t'$, where the redex pattern and position of the redex contracted in the $\alpha$th step of $s \twoheadrightarrow t$ and $s' \twoheadrightarrow t'$ are identical. By construction, we have for every position $p \in \mathcal{P}os(s)$ that $p \in \mathcal{P}os(s')$ and $p/(s \twoheadrightarrow t) \subseteq p/(s' \twoheadrightarrow t')$. Hence, the result follows as $\mathcal{U}$ satisfies residuals.

**Overlap** Suppose $s$ is a redex overlapping a term $t$ in $\mathcal{U}_\perp$ at position $p \in \mathcal{P}os(s)$. By definition of $\mathcal{U}_\perp$ there exists a term $t' \in \mathcal{U}$ that is a $\perp$-instance of $t$. Note that $\mathcal{U}_\perp \neq \{\perp\}$, as $t' \in \mathcal{U}$. Hence, there exists a term $s'$, a $\perp$-instance of $s$, with $s'|_p = t'$. The term $s'$ is a redex by fully-extendedness and orthogonality. Thus, as $\mathcal{U}$ satisfies overlap, we have $s' \in \mathcal{U}$ and, as $s'$ is a $\perp$-instance of $s$, it follows that $s \in \mathcal{U}_\perp$, as required.

**Hypercollapsingness (resp. root-activeness)** Let $s$ be a partial term that is hypercollapsing (resp. root-active). Recursively replace each $\perp$ in $s$ by $s$ to obtain a term $t$, avoiding the capture of free variables. By fully-extendedness and orthogonality $t$ is hypercollapsing (resp. root-active). Moreover, as $\mathcal{U}$ contains all hypercollapsing (resp. root-active) terms, $t \in \mathcal{U}$ and, by construction, $s$ has $t$ as a $\perp$-instance. Hence, $s \in \mathcal{U}_\perp$, as required.

**Indiscernability** Let $s \leftrightarrow_P^{\mathcal{U}_\perp} t$ and $s \in \mathcal{U}_\perp$. By definition of $\mathcal{U}_\perp$ there exists a term $s' \in \mathcal{U}$ that is a $\perp$-instance of $s$. Moreover, for each $p \in P$ there exist terms $s'_p, t'_p \in \mathcal{U}$ such that $s'_p$ is a $\perp$-instance of $s|_p$ and $t'_p$ is a $\perp$-instance of

$t|_p$. By indiscernability of $\mathcal{U}$ and the fact that $s'_p \in \mathcal{U}$, we have $s'|_p \in \mathcal{U}$ for each $p \in P$. Define $t'$ as

$$t'|_p = \begin{cases} t'_q|_r & \text{if } p = q \cdot r \text{ for } q \in P \\ s'|_p & \text{otherwise} \end{cases}$$

By definition $t'$ is a $\bot$-instance of $t$ and $s' \leftrightarrow_P^{\mathcal{U}} t'$, as $s'|_p, t'|_p \in \mathcal{U}$ for all $p \in P$. Hence, by indiscernability of $\mathcal{U}$, it follows that $t' \in \mathcal{U}$ and, thus, $t \in \mathcal{U}_\bot$, as required. □

*Proof (Proposition 3.16).* Let $P$ be the set of positions of root-active subterms in $s$ that are replaced to obtain $t$. By definition of $s$ there exists a perpetual reduction $S$ starting from it. The redex patterns employed in the steps of $S$ either occur completely outside or completely inside the subterms at positions that descent form those in $P$, by root-activeness and orthogonality. It is irrelevant that any terms are substituted along $S$ into the subterms at positions that descent from those in $P$ by orthogonality and the fact that free variables cannot get bound when substituted in the subterms.

Omit from $S$ all steps that occur inside the subterms that occur at positions that descent from those in $P$ to obtain a reduction $S'$ of length $\alpha$. By definition of $S'$, together with orthogonality and fully-extendedness, there exists a reduction $T$ of length $\alpha$ starting in $t$ such that for all $\beta \leq \alpha$ we have that the redex pattern and position of the redex contracted in the $\beta$th step of both $S'$ and $T$ are identical. Hence, if $S'$ is perpetual then so is $T$ and the result follows since perpetuality implies root-activeness. If $S'$ is not perpetual, then $s$ reduces to a subterm at a position $p \in P$ and the same holds for $T$. As the subterm at position $p$ in $t$ is root-active, there exist a perpetual reduction starting from it. As earlier, it is irrelevant that any terms are substituted in the subterm by orthogonality and the fact that free variables cannot get bound when substituted. Hence, $T$ can be prolonged to obtain a perpetual reduction and the result follows again as perpetuality implies root-activeness. □

*Proof (Proposition 3.19).* Observe that each term has a closed substitution instance: replace each free variable by $[x]x$. Residuals and overlap follow immediately by definition of opaqueness. As each root-active term reduces to a redex, it follows by orthogonality that no redex can overlap a root-active term at a non-root position. Hence, opaqueness follows for the root-active terms.

In case of indiscernability, assume that $s$ is not opaque and that $s \leftrightarrow^{\mathcal{U}} t$. By definition of opaqueness there is a closed substitution instance $s'$ of $s$ that reduces to a term $s''$ overlapped by a redex at a non-root position. For $s'$ there exists a term $t'$ such that $s' \leftrightarrow^{\mathcal{U}} t'$ and $t'$ a closed substitution instance of $t$. By bisimulation, which follows from residuals and overlap, there exists a reduction $t' \twoheadrightarrow t''$ such that $s'' \leftrightarrow^{\mathcal{U}} t''$. By opaqueness, the subterms replaced in $s'' \leftrightarrow^{\mathcal{U}} t''$ are irrelevant — they cannot overlap any redex — and it follows that $t''$ is overlapped by a redex at a non-root position. Hence, $t$ is not opaque and we obtain indiscernability. □