PATHS AND CYCLES IN COLORED GRAPHS *

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Abstract

Let G be an (edge-)colored graph. A path (cycle) is called monochromatic if all the edges of it have the same color, and is called heterochromatic if all the edges of it have different colors. In this note, some sufficient conditions for the existence of monochromatic and heterochromatic paths and cycles are obtained. We also propose a conjecture on the existence of paths and cycles with many colors.

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1 Introduction

We use [1] for terminology and notation not defined here and consider simple graphs only.

Let $G = (V(G), E(G))$ be a graph. By an *edge-coloring* of G we will mean a function $C : E(G) \to \mathbb{N}$. If G is assigned such a coloring, then we say that G is a *colored graph*, and we call $C(e)$ the color of the edge $e \in E(G)$. We note that C is not necessarily a proper edge-coloring, i.e., two adjacent edges may have the same color.

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For a subgraph H of G, we let $C(H) = \cup_{e \in E(H)} C(e)$ and $c(H) = |C(H)|$. For a vertex v of G, the *colored neighbor* $CN(v)$ of v is defined as the set $\{C(e) | e \text{ is incident to } v\}$ and the *colored degree* $d^c(v) = |CN(v)|$. A path (cycle) is called *monochromatic*, if all the edges of it have the same color; and is called *heterochromatic*, if all the edges of it have different colors. For a vertex z, a *z-path* is one with z as an end-vertex of it. We shall use $\lceil x \rceil$ for the smallest integer not smaller than x.

A usual graph can be regarded as a colored graph in which all edges have different colors. The number of the colors of a subgraph is just the number of the edges of it. The colored degree of a vertex is the degree of it.

Our aim in this note is to investigate sufficient conditions for the existence of monochromatic and heterochromatic paths and cycles. We also propose a conjecture on the existence of paths and cycles with many colors.

2 Monochromatic Paths and Cycles

The *arboricity* $a(G)$ of a graph G is defined as the minimum number of edge-disjoint forests into which G can be decomposed. It is also the minimum number of colors necessary to color the edges of G so that no cycle has edges all of the same color. So, we have

Theorem 1
Let G be colored graph. If $c(G) \le a(G)$, then G contains at least one monochromatic cycle.

The arboricity $a(G)$ could be found by the matroid partitioning algorithms of Edmonds [4]. Picard and Queyranne [5] showed that this number can be determined in at most $O(n^4)$ operations, by using network flow method.

Furthermore, we have the following obvious result.

Theorem 2
Let G be a colored graph with color classes $E_1, E_2, \cdots, E_c$. Then, G has a monochromatic path (cycle) of length at least l if and only if some $G[E_i]$ has a path (cycle) of length at least l.

3 Heterochromatic Paths and Cycles

The following results on the existence of long paths and cycles are well known.

Theorem A (Erdös and Gallai [3])
Let G be a graph of order n and size m. Then, G contains a path of length at least $2m/n$.

Theorem B (Erdös and Gallai [3])

Let G be a 2-edge-connected graph of order n and size m. Then, G contains a cycle of length at least $2m/(n-1)$.

For a colored graph G, by selecting an edge in each color class, we can get a colored graph G', such that all the edges of G' have different colors, and $c(G') = c(G)$. From Theorems A and B, we have the following two results.

Theorem 3
Let G be a colored graph of order n. Then, G contains a heterochromatic path of length at least $2c(G)/n$.

Theorem 4
Let G be a 2-edge-connected colored graph of order n. Then, G contains a heterochromatic cycle of length at least $2c(G)/(n-1)$.

The following two theorems on the existence of long heterochromatic paths are not difficult to be proved.

Theorem 5
Let G be a colored graph and k an integer. Suppose that $d^c(v) \geq k$ for any vertex v of G, then for any vertex z of G there exists a heterochromatic z-path of length at least $\lceil (k+1)/2 \rceil$.

Theorem 6
Let G be a colored graph and s an integer. Suppose that $|CN(u) \cup CN(v)| \geq s$ for any pair of vertices u and v of G, then G contains a heterochromatic path of length at least $\lceil s/3 \rceil + 1$.

The results in Theorem 5 and 6 are best possible.

In the following, we give an sufficient condition for the existence of heterochromatic triangles or quadrilaterals and heterochromatic Hamilton cycles.

Theorem 7
Let G be a colored graph of order $n(\geq 4)$, such that $|CN(u) \cup CN(v)| \geq n-1$ for any pair of vertices u and v of G. Then, G contains at least one heterochromatic triangle or one heterochromatic quadrilateral.

Theorem 8
Let G be a colored graph of order n, such that $|CN(u) \cup CN(v)| \geq 2n-3$ for any pair of vertices u and v of G. Then, G contains a heterochromatic Hamilton cycle.

We don't believe that the results in Theorems 7 and 8 are best possible. But we can't provide examples to show this.

4 Paths and Cycles with Many Colors

As we mentioned in Section 1, if we regard a (usual) graph as a colored graph in which all edges have different colors, then the number of the colors of a subgraph is just the number of the edges of it. It is well known that the problem of finding the longest paths and the longest cycles in a graph is NP-complete. Therefore, the problem of finding paths and cycles with as many colors as possible in a colored graph is also NP-complete.

In the past decades, many sufficient conditions for the existence of long paths and cycles has been derived . The following result is due to Dirac.

Theorem C (Dirac [2])
Let G be graph and d an integer. If $d(v) \geq d$ for every vertex v of G, then G contains (1) a path of length at least d, and (2) a cycle of length at least $d+1$ if $d > 1$.

It is an interesting problem that whether Theorem C admits a generalization to colored graphs. We have the following conjecture.

Conjecture 1
Let G be a colored graph and d an integer. If $d^c(v) \geq d$ for every vertex v of G, then G contains (1) a path with at least $d-1$ colors, and (2) a cycle with at least d colors if $d > 1$.

If the above conjecture is true, then it would be best possible. This can be shown by considering a special coloring of complete graphs. Let K_n be a complete graph of order n. The γ-coloring of K_n when n is even is defined as a proper $(n-1)$-coloring of K_n. In the case that n is odd, the γ-coloring of K_n is defined as follows: first assign a γ-coloring to $K_n - v$ for some vertex v, then assign the edges incident to v with $n-1$ colors which are all different from the colors of the γ-coloring of $K_n - v$. It is not difficult to verify that, for the complete graph K_n with γ-coloring, $d^\gamma(v) \geq n-1$ for any vertex v of it, but there is no path with more than $n-2$ colors and no cycle with more than $n-1$ colors.

By improving the connectivity of graphs, Dirac give the following

Theorem D (Dirac [2])
Let G be a 2-connected graph and d an integer. If $d(v) \geq d$ for every vertex v of G, then G contains either a Hamilton cycle or a cycle of length at least $2d$.

Let $K_{n,n+1}$ be the complete bipartite graph with bipartition (X, Y) such that $|X| = n$ and $|Y| = n+1$. Assign an edge-coloring to $K_{n,n+1}$ as follows: first color the graph $K_{n,n+1} - y$ for some vertex $y \in Y$ properly, then assign the colors of $K_{n,n+1}$ to the n edges incident to y respectively. It is easy to know that $d^c(v) \geq n$ for each vertex v of $K_{n,n+1}$ but $K_{n,n+1}$ contains neither a Hamilton cycle nor a cycle with more than n colors. This shows that, different from Theorem D, improving the connectivity of the graphs in Conjecture 1 can not guarantee cycles with more colors.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, Macmillan-Elsevier, London-New York (1976).

[2] G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc., **2** (1952), 69-81.

[3] P. Erdös and T.Gallai, On maximal paths and circuits in graphs, Acta Math. Acad. Sci. Hung., **10** (1959) 337-356.

[4] E.L. Lawler, Combinatorial Optimization: Networks and Matroids, Holt, Rinehart and Winston, New York (1976).

[5] J.C. Picard and M. Queyranne, A network flow solution to some non-linear 0-1 programming problems, with applications to graph theory, Networks, **12** (1982) 141-159.