Improved Upper Bounds for λ-Backbone Colorings Along Matchings and Stars

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Abstract. We continue the study on backbone colorings, a variation on classical vertex colorings that was introduced at WG2003. Given a graph $G = (V, E)$ and a spanning subgraph H of G (the backbone of G), a λ-backbone coloring for G and H is a proper vertex coloring $V \rightarrow \{1, 2, \dots\}$ of G in which the colors assigned to adjacent vertices in H differ by at least λ. The main outcome of earlier studies is that the minimum number ℓ of colors for which such colorings $V \rightarrow \{1, 2, \dots, \ell\}$ exist in the worst case is a factor times the chromatic number (for all studied types of backbones). We show here that for split graphs and matching or star backbones, ℓ is at most a small additive constant (depending on λ) higher than the chromatic number. Despite the fact that split graphs have a nice structure, these results are difficult to prove. Our proofs combine algorithmic and combinatorial arguments. We also indicate other graph classes for which our results imply better upper bounds on ℓ than the previously known bounds.

1 Introduction and Related Research

Coloring has been a central area in Graph Theory for more than 150 years. Some reasons for this are its appealingly simple definition, its large variety of open problems, and its many application areas. Whenever conflicting situations between pairs of objects can be modeled by graphs, and one is looking for a partition of the set of objects in subsets of mutually non-conflicting objects, this can be viewed as a graph coloring problem. This holds for classical settings such as neighboring countries (map coloring) or interfering jobs on machines (job scheduling), as well as for more recent settings like colliding data streams in optical networks (wavelength assignment) or interfering transmitters and receivers for broadcasting, mobile phones and sensors (frequency assignment), to name just a few. Except perhaps for the notorious map coloring problem, all of the

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above settings play an important role in Computer Science as well, e.g., in areas like parallel and distributed computing, embedded systems, optical networks, sensor networks and mobile networks. Apart from these applications areas, graph coloring has been a central theme within Theoretical Computer Science, especially within Complexity Theory and the currently very popular area of Exact Algorithms.

In [7] backbone colorings are introduced, motivated and put into a general framework of coloring problems related to frequency assignment. Graphs are used to model the topology and interference between transmitters (receivers, base stations): the vertices represent the transmitters; two vertices are adjacent if the corresponding transmitters are so close (or so strong) that they are likely to interfere if they broadcast on the same or 'similar' frequency channels. The problem is to assign the frequency channels in an economical way to the transmitters in such a way that interference is kept at an 'acceptable level'. This has led to various different types of coloring problems in graphs, depending on different ways to model the level of interference, the notion of similar frequency channels, and the definition of acceptable level of interference (See, e.g., [15],[20]). We refer to [6] and [7] for an overview of related research, but we repeat the general framework and some of the related research here for convenience and background.

> Given two graphs G_1 and G_2 with the property that G_1 is a spanning subgraph of G_2, one considers the following type of coloring problems: Determine a coloring of (G_1 and) G_2 that satisfies certain restrictions of type 1 in G_1, and restrictions of type 2 in G_2.

Many known coloring problems fit into this general framework. We mention some of them here explicitly, without giving details. The first variant is known as the distance-2 coloring problem. Much of the research has been concentrated on the case that G_1 is a planar graph. We refer to [1], [4], [5], [18], and [21] for more details. A closely related variant is known as the radio coloring problem and has been studied (under various names) in [2], [8], [9], [10], [11], [12], and [19]. A third variant is known as the radio labeling problem. We refer to [14] and [17] for more particulars.

In the WG2003 paper [7], a situation is modeled in which the transmitters form a network in which a certain substructure of adjacent transmitters (called the backbone) is more crucial for the communication than the rest of the network. This means more restrictions are put on the assignment of frequency channels along the backbone than on the assignment of frequency channels to other adjacent transmitters.

Postponing the relevant definitions to the next subsections, we consider the problem of coloring the graph G_2 (that models the whole network) with a proper vertex coloring such that the colors on adjacent vertices in G_1 (that models the backbone) differ by at least $\lambda \geq 2$. This is a continuation of the study in [7] and [22]. Throughout the paper we consider two types of backbones: matchings and disjoint unions of stars. We give many details on the matching case (for which the proofs are the most involved), but due to page limits refrain from details for the other case (that is simpler).

Matching backbones reflect the necessity to assign considerably different frequencies to pairwise very close (or most likely interfering) transmitters. This occurs in real world applications such as military scenarios, where soldiers or military vehicles carry two (or sometimes more) radios for reliable communication.

For star backbones one could think of applications to sensor networks. If sensors have low battery capacities, the tasks of transmitting data are often assigned to specific sensors, called cluster heads, that represent pairwise disjoint clusters of sensors. Within the clusters there should be a considerable difference between the frequencies assigned to the cluster head and to the other sensors within the same cluster, whereas the differences between the frequencies assigned to the other sensors within the cluster, or between different clusters, is of a secondary importance. This situation is well reflected by a backbone consisting of disjoint stars.

We concentrate on the case that G_2 is a split graph, but will indicate how our results could be used in case G_2 is a general graph, and for which type of graphs this could be useful. The motivation for looking at split graphs is twofold. First of all, split graphs have nice structural properties, which lead to substantially better upper bounds on the number of colors in this context of backbone colorings. Secondly, every graph can be turned into a split graph by considering any (e.g., a maximum or maximal) independent set and turning the remaining vertices into a clique. The number of colors needed to color the resulting split graph is an upper bound for the number of colors one needs to color the original graph. We will indicate classes of non-split graphs for which our results also imply better upper bounds.

Although split graphs have a very special structure, they are not completely artificial in the context of, e.g., sensor networks. As an example, consider a sensor network within a restricted area (like a lab) with two distinct types of nodes: weak sensors with a very low battery capacity, like heat sensors, smoke sensors, body tags, etc., and PCs, laptops, etc., with much stronger power properties. The weak sensors are very unlikely to interfere with one another (especially if they are put with a certain purpose on fixed locations), while the other equipment is likely to interfere (within this restricted area). Weak sensors interfere with pieces of the other equipment within their vicinity. In such cases, the situation can be modeled as a split graph.

1.1 Terminology and Previous Results

For undefined terminology we refer to [3]. Let $G = (V, E)$ be a graph, where $V = V_G$ is a finite set of vertices and $E = E_G$ is a set of unordered pairs of two different vertices, called edges. A function $f : V \to \{1, 2, 3, \ldots\}$ is a *vertex coloring* of V if $|f(u) - f(v)| \geq 1$ holds for all edges $uv \in E$. A vertex coloring $f : V \to \{1, \ldots, k\}$ is called a *k-coloring*, and the *chromatic number* $\chi(G)$ is the smallest integer k for which there exists a k-coloring. A set $V' \subseteq V$ is *independent* if its vertices are mutually nonadjacent; it is a *clique* if its vertices are mutually adjacent. By definition, a k-coloring partitions V into k independent sets $V_1, \ldots, V_k$. Let H be a *spanning subgraph* of G, i.e., $H = (V_G, E_H)$ with $E_H \subseteq E_G$. Given an integer $\lambda \geq 2$, a vertex coloring f of G is a λ-*backbone coloring* of (G, H), if $|f(u) - f(v)| \geq \lambda$ holds for all edges $uv \in E_H$. The λ-*backbone coloring number* $\textsc{bbc}_\lambda(G, H)$ of (G, H) is the smallest integer ℓ for which there exists a λ-backbone coloring $f : V \to \{1, \ldots, \ell\}$. A *star* S_q is a complete 2-partite graph with independent sets $V_1 = \{r\}$ and V_2 with $|V_2| = q$; the vertex r is called the *root* and the vertices in V_2 are called the *leaves* of the star S_q. In our context a *matching* M is a collection of pairwise disjoint stars that are all copies of S_1. We call a spanning subgraph H of a graph G a *star backbone* of G if H is a collection of pairwise disjoint stars, and a *matching backbone* if H is a (perfect) matching.

Obviously, $\textsc{bbc}_\lambda(G, H) \geq \chi(G)$ holds for any backbone H of a graph G. We are interested in tight upper bounds for $\textsc{bbc}_\lambda(G, H)$ in terms of $\chi(G)$. In [22], it has been shown that the upper bounds in case of star and matching backbones roughly grow like $(2 - \frac{1}{\lambda})\chi(G)$ and $(2 - \frac{2}{\lambda+1})\chi(G)$, respectively. In all worst cases the backbone coloring numbers grow proportionally to a multiplicative factor times the chromatic number. Although these upper bounds in [22] are tight, they are probably only reached for very special graphs. To analyze this further, we turned to study the special case of split graphs. This was motivated by the observation in [7] that for split graphs and tree backbones the 2-backbone coloring number differs at most 2 from the chromatic number. We show a similar behavior for the general case with $\lambda \geq 2$ and matching and star backbones in split graphs. This can have nice implications for upper bounds on the λ-backbone coloring numbers for matching and star backbones in other graphs, if they satisfy certain conditions.

1.2 New Results

A *split graph* is a graph whose vertex set can be partitioned into a clique and an independent set, with possibly edges in between. The size of a largest clique in G is denoted by $\omega(G)$. Split graphs were introduced by Hammer & Földes [16]; see also the book [13] by Golumbic. They form an interesting subclass of the class of perfect graphs. Hence, split graphs satisfy $\chi(G) = \omega(G)$, and many NP-hard problems are polynomially solvable when restricted to split graphs.

In Section 2 we present sharp upper bounds for the λ-backbone coloring numbers of split graphs with matching or star backbones. We apply them to certain other graphs, too. All upper bounds are only a small additive constant (depending on λ and for non-split graphs also on $\alpha(G)$) higher than $\chi(G)$, in contrast to earlier results, which show a multiplicative factor times $\chi(G)$.

2 Matching and Star Backbones

In this section we present sharp upper bounds on the λ-backbone coloring numbers of split graphs along matching and star backbones. Our result on matching backbones is summarized in the next theorem which will be proved in Section 3.

Theorem 1. *Let $\lambda \geq 2$ and let $G = (V, E)$ be a split graph with $\chi(G) = k \geq 2$. For every matching backbone $M = (V, E_M)$ of G,*

$$\mathrm{BBC}_\lambda(G, M) \leq \begin{cases} \lambda + 1 & \text{if } k = 2 & (i) \\ k + 1 & \text{if } k \geq 4 \text{ and } \lambda \leq \min\{\frac{k}{2}, \frac{k+5}{3}\} & (ii) \\ k + 2 & \text{if } k = 9 \text{ or } k \geq 11 \text{ and } \frac{k+6}{3} \leq \lambda \leq \lceil \frac{k}{2} \rceil & (iii) \\ \lceil \frac{k}{2} \rceil + \lambda & \text{if } k = 3, 5, 7 \text{ and } \lambda \geq \lceil \frac{k}{2} \rceil & (iv) \\ \lceil \frac{k}{2} \rceil + \lambda + 1 & \text{if } k = 4, 6 \text{ or } k \geq 8 \text{ and } \lambda \geq \lceil \frac{k}{2} \rceil + 1. & (v) \end{cases}$$

All the bounds are tight.

We will now show how these results can yield upper bounds for non-split graphs. For this purpose we first implicitly define a function f by the upper bounds $\mathrm{BBC}_\lambda(G, M) \leq f(\lambda, \chi(G))$ from the above theorem. Note that f is a nondecreasing function in λ and $\chi(G)$. Let $G = (V, E)$ be a graph and $V_1 \subseteq V$ be an independent set with $|V_1| = \alpha(G)$ and let $V_2 = V \setminus V_1$. Let W be the subset of V_1 consisting of vertices that are adjacent to all vertices in V_2. If W is non-empty, then we choose one $v \in W$ and move it to V_2, i.e., $V_2 := V_2 \cup \{v\}$. The meaning of this choice will become clear after the next sentence. Let $S(G)$ be the split graph with clique V_2 and independent set V_1. Since we moved one vertex from W to V_2 in case $W \neq \emptyset$, we guarantee that no vertex of V_1 is adjacent to all vertices of V_2. So $\chi(S(G)) = \omega(S(G)) = |V(G)| - \alpha(G)$ or $\chi(S(G)) = |V(G)| - \alpha(G) + 1$. Let the edges between V_1 and V_2 be defined according to E. Then we obtain: $\mathrm{BBC}_\lambda(G, M) \leq \mathrm{BBC}_\lambda(S(G), M) \leq f(\lambda, \chi(S(G))) \leq f(\lambda, |V(G)| - \alpha(G) + 1)$.

These upper bounds are almost sharp in the sense that we have examples for sharpness for most values of λ and $\alpha(G)$, but we (still) have a discrepancy of 1 in some cases. We will present the tedious details in a full journal version of this paper.

When can these bounds be useful for other (non-split) graphs? To answer this question, we should compare the new bound $f(\lambda, |V(G)| - \alpha(G) + 1)$ with the bound $(2 - \frac{2}{\lambda+1})\chi(G)$ from [22]. To get some insight into situations for which this gives an improvement, we apply a very rough calculation in which we use that the first bound is roughly of order $|V(G)| - \alpha(G)$ (disregarding some additive constant depending on λ), and the second one is roughly of order $2\chi(G)$ (disregarding the factor $\frac{2}{\lambda+1}$). Adopting these rough estimates, the first bound is better than the second one whenever $|V(G)| - \alpha(G) \leq 2\chi(G)$. This is, of course, the case when G is a split graph, since then $|V(G)| - \alpha(G) \leq \omega(G) = \chi(G)$. Now suppose we have a graph G with the following structure: An independent set I of G with cardinality $\alpha(G)$ shares at most one vertex with a clique C of G with cardinality $\omega(G)$, and $r = |V(G) \setminus (I \cup C)| \leq \omega(G)$. Then clearly $|V(G)| - \alpha(G) \leq 2\omega(G) \leq 2\chi(G)$. This gives large classes of non-split graphs for which the new bounds are better than the old bounds. Also if we apply a more careful analysis: If r is small compared to $(1 - \frac{2}{\lambda+1})\omega(G) + \lambda$, we get an improvement. We omit the details.

For split graphs with star backbones we obtained the following result.

Theorem 2. *Let $\lambda \geq 2$ and let $G = (V, E)$ be a split graph with $\chi(G) = k \geq 2$. For every star backbone $S = (V, E_S)$ of G,*

$$\mathrm{BBC}_\lambda(G, S) \leq \begin{cases} k + \lambda & \text{if either } k = 3 \text{ and } \lambda \geq 2 \text{ or } k \geq 4 \text{ and } \lambda = 2 \\ k + \lambda - 1 & \text{in the other cases.} \end{cases}$$

The bounds are tight.

The proof of Theorem 2 has been postponed to the journal version of our paper. We can apply the results to obtain upper bounds for certain non-split graphs that improve bounds in [22], in a similar way as we did in the case of matching backbones, using a function $g(\lambda, \chi(G))$ which is implicitly defined by the upper bounds from Theorem 2. We omit the details.

3 Proof of Theorem 1

Given a graph $G = (V, E)$ with a matching backbone $M = (V, E_M)$, $u \in V$ is called a *matching neighbor* of $v \in V$ if $(u, v) \in E_M$, denoted by $u = mn(v)$.

Throughout this section, $G = (V, E)$ denotes a split graph, and V is assumed to be partitioned in a largest clique C and an independent set I. Moreover, $|V|$ is assumed to be even to allow for a perfect matching in the graph G. The set of nonneighbors of a vertex u will be denoted by $NN(u)$. Note that in G, every vertex of I has at least one nonneighbor in C (otherwise C would not be a largest clique). However, for a vertex $u \in C$, the set $NN(u)$ may be empty.

For some $p \leq \alpha(G)$ a *splitting set of cardinality* p, named an s-set for short, is a subset $\{v_1, \ldots, v_p\} \subseteq I$ such that

$$\left\{ \bigcup_{i=1\ldots p} NN(v_i) \right\} \bigcap \left\{ \bigcup_{i=1\ldots p} \{mn(v_i)\} \right\} = \emptyset.$$

Note that if (G, M) has an s-set of cardinality p, then it also has an s-set of cardinality q, for all $q \leq p$.

We need the following technical lemmas on the existence of certain s-sets for our proof. The proof of the second lemma is postponed to the journal version of our paper.

Lemma 1. *Given (G, M), let $k' = |C'|$ for a clique C' in G and let $i' = |I'|$ for an independent set I' in G. If $i' = k'$ and every vertex in I' has at most one nonneighbor in C' and every vertex in I' has exactly one matching neighbor in C' and $\lceil \frac{k'}{3} \rceil \geq p$, then (G, M) has an s-set of cardinality p.*

Proof. Below we partition the disjoint sets C' and I' in the sets C'_1, C'_2, I'_1 and I'_2 with cardinalities c'_1, c'_2, i'_1 and i'_2, respectively. Then we show that one can pick at least $\lceil \frac{i'_1}{3} \rceil$ vertices from I'_1 and at least $\lceil \frac{i'_2}{3} \rceil$ vertices from I'_2 to form an s-set with cardinality $q \geq \lceil \frac{i'_1}{3} \rceil + \lceil \frac{i'_2}{3} \rceil \geq \lceil \frac{k'}{3} \rceil$, which will prove the lemma.

C' and I' are split up in the following way: C'_1 consists of all the vertices in C' that either have zero nonneighbors in I' or have at least two nonneighbors in I' or have exactly one nonneighbor in I', whose matching neighbor in C' has no nonneighbors in I'; C'_2 consists of all other vertices in C'. Obviously, they all have exactly one nonneighbor in I'; I'_1 consists of the matching neighbors of the vertices in C'_1; I'_2 consists of the matching neighbors of the vertices in C'_2.

Clearly, $i'_1 = c'_1$ and $i'_2 = c'_2$. Now assume that there are ℓ_1 vertices in C'_1 that have no nonneighbors in I' and put them in L_1. Also assume that there are ℓ_2 vertices in C'_1 that have at least two nonneighbors in I' and put them in L_2. Finally, assume that there are ℓ_3 vertices in C'_1 that have exactly one nonneighbor in I', whose matching neighbor has no nonneighbors in I' and put them in L_3. Then $\ell_1 \geq \ell_2$ and $\ell_1 \geq \ell_3$ and $c'_1 = \ell_1 + \ell_2 + \ell_3$, so $c'_1 \leq 3\ell_1$.
Let L'_1, L'_2 and L'_3 be the sets of matching neighbors of the vertices in L_1, L_2 and L_3, respectively. Now we pick from I'_1 the ℓ_1 vertices in L'_1 and put them in the s-set. Notice that these vertices do not violate the definition of an s-set, because the set of their nonneighbors and the set of their matching neighbors are two disjoint sets. The matching neighbors of the nonneighbors of the ℓ_1 vertices in the s-set are either in L'_2 or in L'_3, so we exclude the vertices in these two sets for use in the s-set. On the other hand, the matching neighbors of the ℓ_1 vertices in the s-set do not have nonneighbors, so we do not have to worry about that. From the observations above it is clear that we can pick $\ell_1 \geq \lceil \frac{c'_1}{3} \rceil = \lceil \frac{i'_1}{3} \rceil$ vertices from I'_1 that can be used in the s-set. Moreover, any vertices from I'_2 that we will put in the s-set do not conflict with the vertices from L'_1 that are in the s-set already. So the only thing we have to do now is to pick at least $\lceil \frac{i'_2}{3} \rceil$ vertices from I'_2 that can be used in the s-set. Simply pick an arbitrary vertex from I'_2 and put it in the s-set. Now delete from I'_2 the matching neighbor of its nonneighbor and the unique nonneighbor of its matching neighbor if they happen to be in I'_2. Continuing this way, we 'throw away' at most two vertices of I'_2 for every vertex of I'_2 that we put in the s-set. It is easy to see that we can pick at least $\lceil \frac{i'_2}{3} \rceil$ vertices from I'_2 that we can put in the s-set. Therefore, the cardinality of the s-set will be at least $\lceil \frac{i'_1}{3} \rceil + \lceil \frac{i'_2}{3} \rceil \geq \lceil \frac{i'}{3} \rceil = \lceil \frac{k'}{3} \rceil$, which proves the lemma. □

Lemma 2. *Given (G, M), let $k = \omega(G) = |C|$ and let $i = |I|$. If $i \leq k$ and every vertex in I has exactly one nonneighbor in C and $\lceil \frac{k}{3} \rceil \geq p$, then (G, M) has an s-set S with $|S| = p - \frac{k-i}{2}$ such that there are no matching edges between elements of the set of nonneighbors of vertices of S.*

Proof of the bounds in Theorem 1. First of all, note that for technical reasons we split up the proof in more and different subcases than there appear in the formulation of the theorem. The exact relation between the subcases in the theorem and those in the following proof is given as follows: Subcase **i** of the theorem is proven in **a**. The proof of subcase **ii** can be found in **b**. For even k the proof of subcase **iii** is given in **c**, for odd k in **d**. The three cases with $k = 3$ and $\lambda = 2$, $k = 5$ and $\lambda = 3$ and $k = 7$ and $\lambda = 4$ from subcase **iv** are treated in **b**, the others in **e**. Finally, subcase **v** is proven in **f** for even k and in **g** for odd k.

Subcase a. If $k = 2$ then G is bipartite, and we use colors 1 and $\lambda + 1$.

For $k \geq 3$, let $G = (V, E)$ be a split graph with $\chi(G) = k$ and with a perfect matching backbone $M = (V, E_M)$. Let C and I be a partition of V such that C with $|C| = k$ is a clique of maximum size, and such that I with $|I| = i$ is an independent set. Without loss of generality, we assume that every vertex in I has exactly one nonneighbor in C.

Subcase b. Here we consider the cases with $k \geq 4$ and $\lambda \leq \min\{\frac{k}{2}, \frac{k+5}{3}\}$ together with the three separate cases with $k = 3$ and $\lambda = 2$, $k = 5$ and $\lambda = 3$ and $k = 7$ and $\lambda = 4$. The reason for this is that these are exactly the cases for which we obtain $k \geq 2\lambda - 1$ and $\lceil \frac{k}{3} \rceil \geq \lambda - 1$ and for which we need show the existence of a λ-backbone coloring using at most $k + 1$ colors. By Lemma 2, we find that (G, M) has an s-set of cardinality $y = \lambda - 1 - \frac{k-i}{2}$ such that there are no matching edges between the nonneighbors of vertices in the s-set. We make a partition of C into six disjoint sets $C_1, \ldots, C_6$, with cardinalities $c_1, \ldots, c_6$, respectively, as follows: C_1 consists of those vertices in C that have a matching neighbor in C and a nonneighbor in the s-set. Notice that by definition of the s-set, there are no matching edges between vertices in C_1; C_2 consists of those vertices in C that have a matching neighbor in I and a nonneighbor in the s-set; C_3 contains one end vertex of each matching edge in C that has no end vertex in C_1; C_4 consists of those vertices in C whose matching neighbor is in I and that are neither matching neighbor nor nonneighbor of any vertex in the s-set; C_5 consists of those vertices in C that have a matching neighbor in the s-set; C_6 consists of those vertices in C that have a matching neighbor in C and that are not already in C_1 or C_3. It is easily verified that

$$
\begin{array}{lll}
c_1 + c_2 \leq y, & c_3 = \frac{k-i}{2} - c_1, & c_4 = i - y - c_2, \\
c_5 = y, & c_6 = \frac{k-i}{2}, & \sum_{i=1}^{6} c_i = k.
\end{array}
$$

An algorithm that constructs a feasible λ-backbone coloring of (G, M) with at most $k + 1$ colors is given below. In this algorithm I'' denotes the set of vertices of I that are not in the s-set.

Coloring Algorithm 1

1 Color the vertices in C_1 with colors from the set $\{1, \ldots, c_1\}$.
2 Color the vertices in C_2 with colors from the set $\{c_1 + 1, \ldots, c_1 + c_2\}$.
3 Color the vertices in the s-set by assigning to them the same colors as their nonneighbors in C_1 or C_2. Note that different vertices in the s-set can have the same nonneighbor in C_1 or C_2, so a color may occur more than once in the s-set.
4 Color the vertices in C_3 with colors from the set $\{c_1 + c_2 + 1, \ldots, c_1 + c_2 + c_3\}$.
5 Color the vertices in C_4 with colors from the set $\{c_1 + c_2 + c_3 + 1, \ldots, c_1 + c_2 + c_3 + c_4\}$.

6 Color the vertices in C_5 with colors from the set $\{c_1+c_2+c_3+c_4+1,\ldots,c_1+c_2+c_3+c_4+c_5\}$; start with assigning the lowest color from this set to the matching neighbor of the vertex in the s-set with the lowest color and continue this way.

7 Color the vertices in C_6 with colors from the set $\{c_1+c_2+c_3+c_4+c_5+1,\ldots,c_1+c_2+c_3+c_4+c_5+c_6\}$; start with assigning the lowest color from this set to the matching neighbor with the lowest color in $C_1 \cup C_3$ and continue this way.

8 Finally, color the vertices of I'' with color $k+1$.

We postpone the correctness proof of this algorithm to the journal version.

Subcase c. Here we consider the case $k = 2m$, $m \geq 6$ and $\frac{k+6}{3} \leq \lambda \leq \frac{k}{2}$. We obtain $k \geq 2\lambda$. We color the k vertices in C with colors from the sets $\{2,\ldots,\frac{k}{2}+1\}$ and $\{\frac{k}{2}+2,\ldots,k+1\}$. If there are matching edges in C, then we color them such that the first colors from both sets are assigned to the end vertices of one matching edge, the second colors from both sets are assigned to the end vertices of another matching edge, and so on. For later reference we call this a greedy coloring. We can color up the two end vertices of $\frac{k}{2}$ matching edges in C this way, which suffices. Vertices in I get color $k+2$ if their matching neighbor in C is colored by a color from the first set, and vertices in I get color 1 if their matching neighbor in C is colored by a color from the second set. This yields a λ-backbone coloring of (G,M) with at most $k+2$ colors.

Subcase d. We now consider the case $k = 2m+1$, $m \geq 4$ and $\frac{k+6}{3} \leq \lambda \leq \frac{k+1}{2}$. We obtain $k \geq 2\lambda-1$. For this case i is odd, otherwise there is no perfect matching in G. If $i = 1$, then there are $\frac{k-1}{2}$ matching edges in C. We can color their end vertices with colors from the two sets $\{1,\ldots,\frac{k-1}{2}\}$ and $\{\frac{k-1}{2}+3,\ldots,k+1\}$ by a greedy coloring. The distance between the colors of the end vertices of a matching edge in C is then $\frac{k-1}{2}+2 \geq \frac{2\lambda-2}{2}+2 = \lambda+1$. For the other vertex in C we use color $\frac{k-1}{2}+1$ and its matching neighbor in I gets color $k+2$. Note that $k+2-\frac{k-1}{2}-1 = \frac{k+3}{2} \geq \frac{2\lambda+2}{2} = \lambda+1$. If $3 \leq i \leq k$, there are $\frac{k-i}{2}$ matching edges in C. We color their end vertices with colors from the two sets $\{2,\ldots,\frac{k-i}{2}+1\}$ and $\{\frac{k+i}{2}+2,\ldots,k+1\}$ by a greedy coloring. The distance between the colors of the end vertices in a matching edge in C is then $\frac{k+i}{2} \geq \frac{2\lambda-1+i}{2} \geq \frac{2\lambda+2}{2} = \lambda+1$. The other i vertices in C are colored with colors from the sets $\{\frac{k-i}{2}+2,\ldots,\frac{k+3}{2}\}$ and $\{\frac{k+3}{2}+1,\ldots,\frac{k+i}{2}+1\}$. The cardinality of the first set is $\frac{i+1}{2}$ and of the second set $\frac{i-1}{2}$, adding up to exactly i. Vertices in I get color $k+2$ if their matching neighbor in C is colored by a color from the first set, or get color 1 if their matching neighbor in C is colored by a color from the second set. Notice that $k+2-\frac{k+3}{2} = \frac{2k+4-k-3}{2} = \frac{k+1}{2} \geq \frac{2\lambda}{2} = \lambda$ and $\frac{k+3}{2}+1-1 = \frac{k+3}{2} \geq \frac{2\lambda+2}{2} = \lambda+1$, so this yields a λ-backbone coloring of (G,M) with at most $k+2$ colors.

Subcase e. Next, we consider the case $k = 3,5,7$ and $\lambda \geq \frac{k+6}{3}$. We obtain $\lambda > \frac{k+1}{2}$ and $\lceil \frac{k}{3} \rceil = \frac{k-1}{2}$. By Lemma 2, we find that (G,M) has an s-set of cardinality $z = \frac{k-1}{2} - \frac{k-i}{2} = \frac{i-1}{2}$ such that there are no matching edges between the nonneighbors of vertices in the s-set. We have to construct a λ-backbone coloring of (G, M) using at most $\frac{k+1}{2} + \lambda$ colors. Obviously, colors from the set $\{\frac{k+1}{2} + 1, \ldots, \lambda\}$ can not be used at all, so we must find a λ-backbone coloring with colors from the sets $\{1, \ldots, \frac{k+1}{2}\}$ and $\{\lambda + 1, \ldots, \frac{k+1}{2} + \lambda\}$. We partition C in six disjoint sets exactly like we did in (b). For the cardinalities of the sets, we now find the following relations:

$$\begin{array}{lll} c_1 + c_2 \leq \frac{i-1}{2}, & c_3 = \frac{k-i}{2} - c_1, & c_4 = i - z - c_2, \\ c_5 = z, & c_6 = \frac{k-i}{2}, & \sum_{i=1}^{6} c_i = k. \end{array}$$

The following variation on Coloring Algorithm 1 constructs a feasible λ-backbone coloring of (G, M).

Coloring Algorithm 2

1 - 5 are the same as in Coloring Algorithm 1.
6 Color the vertices in C_5 with colors from the set $\{\lambda + 1, \ldots, \lambda + c_5\}$; start with assigning the lowest color from this set to the matching neighbor of the vertex in the s-set with the lowest color and continue this way.
7 Color the vertices in C_6 with colors from the set $\{\lambda + c_5 + 1, \ldots, \lambda + c_5 + c_6\}$; start with assigning the lowest color from this set to the matching neighbor with the lowest color in $C_1 \cup C_3$ and continue this way.
8 Finally, color the vertices in I'' with color $\frac{k+1}{2} + \lambda$.

We postpone the correctness proof of this algorithm to the journal version.

Subcase f. We consider the case $k = 2m$, $m \geq 2$ and $\lambda \geq \frac{k}{2} + 1$. For this case we find that i is even, otherwise there is no perfect matching of G. If $i = 0$, then there are $\frac{k}{2}$ matching edges in C. We can use color pairs $\{1, \lambda + 1\}, \{2, \lambda + 2\}, \ldots, \{\frac{k}{2}, \frac{k}{2} + \lambda\}$ for their end vertices, because $\lambda + 1 > \frac{k}{2}$. If $i \geq 2$, then there are $\frac{k-i}{2}$ matching edges in C. We can color their end vertices with colors from the two sets $\{2, \ldots, \frac{k-i}{2} + 1\}$ and $\{\frac{i}{2} + \lambda + 1, \ldots, \frac{k}{2} + \lambda\}$, using greedy coloring. The distance between the two colors on every matching edge in C is then $\frac{i}{2} + \lambda - 1 \geq \lambda$. The other i vertices in C are colored with colors from the sets $\{\frac{k-i}{2} + 2, \ldots, \frac{k}{2} + 1\}$ and $\{\lambda + 1, \ldots, \frac{i}{2} + \lambda\}$, which are exactly i colors. The colors in the first set have distance at least λ to color $\frac{k}{2} + \lambda + 1$, so we color the matching neighbors in I of the vertices in C that are colored with colors from this set with color $\frac{k}{2} + \lambda + 1$. The colors in the second set have distance at least λ to color 1, so we color the matching neighbors in I of the vertices in C that are colored with colors from this set with color 1. This yields a feasible λ-backbone coloring of (G, M) with at most $\frac{k}{2} + \lambda + 1$ colors.

Subcase g. Finally, we consider the case $k = 2m + 1$, $m \geq 4$ and $\lambda \geq \frac{k+1}{2} + 1$. For this case we find that i is odd, otherwise there is no perfect matching of G. There are $\frac{k-i}{2}$ matching edges in C. We can color their end vertices with colors from the two sets $\{2, \ldots, \frac{k-i}{2} + 1\}$ and $\{\frac{i+3}{2} + \lambda, \ldots, \frac{k+1}{2} + \lambda\}$ by a greedy coloring.

Notice that $\frac{i+3}{2} + \lambda - \frac{k-i}{2} - 1 = \frac{i+3+2\lambda-k+i-2}{2} = \frac{2i+1-k+2\lambda}{2} \geq \frac{2i+1-k+k+2}{2} > 0$, so that these sets are disjoint. The distance between the two colors on every matching edge in C is $\frac{i-1}{2} + \lambda \geq \lambda$. The other i vertices in C are colored with colors from the sets $\{\frac{k-i}{2}+2, \ldots, \frac{k+1}{2}\}$ and $\{\lambda+1, \ldots, \frac{i+1}{2}+\lambda\}$, which are exactly i colors that have not been used so far. Vertices in I get color $\frac{k+1}{2}+\lambda+1$ if their matching neighbor in C is colored by a color from the first set, and get color 1 otherwise. This yields a λ-backbone coloring of (G, M) with at most $\frac{k+1}{2}+\lambda+1$ colors. □

Proof of the tightness of the bounds in Theorem 1. We postpone the proof to the journal version.

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