Centrum voor Wiskunde en Informatica

REPORT*RAPPORT*

From exponential coordinates to bicovariant differential calculi on matrix quantum groups

N. van den Hijligenberg and R. Martini

Department of Analysis, Algebra and Geometry

AM-R9518 1995

Report AM-R9518
ISSN 0924-2953

CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

From Exponential Coordinates to Bicovariant Differential Calculi on Matrix Quantum Groups

N. van den Hijligenberg
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands
nico@cwi.nl

R. Martini
University of Twente
P.O. Box 217, 7500 AE Enschede, The Netherlands
martini@math.utwente.nl

Abstract

A procedure to obtain bicovariant differential calculi on matrix quantum groups is presented. The construction is based on the description of the matrix quantum group as a quantized universal enveloping algebra by the use of exponential coordinates. The procedure is illustrated by applying it to the two-dimensional solvable quantum group and the Heisenberg quantum group.

AMS Subject Classification (1991): 16S30, 16W30, 17B35, 17B37, 81R50.
Keywords & Phrases: Universal enveloping algebras, Lie algebras, Hopf algebras, Quantum theory, Quantum groups, Colour Lie superalgebras, non-commutative geometry.
Note: The first author is supported by NWO, Grant N. 611-307-100.

1 Introduction

In the past few years a substantial number of papers have been published dealing with examples of bicovariant differential calculi (see [1]) on special classes of quantum groups, especially on simple quantum groups (see e.g. [2], [3], [4], [5]). However, to our best knowledge not much is known concerning differential calculi on nonsemisimple and inhomogeneous quantum groups. In this paper we present a method for constructing 'natural' bicovariant differential calculi on matrix quantum groups which can effectively be applied to these types of quantum groups. The adjective natural refers to the two main aspects of the calculus. The first is that we demand the bicovariance to be induced by the slightly richer structure of a differential Hopf algebra. The second denotes the fact that the calculus in some sense arises from an extension of the duality between the Hopf algebra of functions on the group and the universal enveloping algebra of its corresponding Lie algebra. We will explain this in more detail in the sequel.

We consider a matrix quantum group, i.e. a function algebra which is quantized by means of the well known RTT-relations for a given solution of the quantum Yang Baxter equation R. The transfer matrix T can classically be represented as $T = e^{\sum x^i l_i}$ where the dynamical variables x^i can be interpreted as exponential coordinates on the group generated by the matrices l_i. In [6] the authors show that these exponential coordinates can be fruitfully utilized in order to present the quantized function algebra as a deformation $U_h(\mathfrak{g}_x)$ of the enveloping algebra of $\mathfrak{g}_x$. Here $\mathfrak{g}_x$ denotes the Lie bialgebra which is dual to the Lie bialgebra generated by the matrices l_i equipped with the coproduct that arises from the Poisson bracket. Hence, the exponential coordinates enable us to apply certain techniques, that are developped for quantized universal enveloping (QUE) algebras, to quantized function algebras.

In particular, the method to construct differential calculi on QUE algebras of Poincaré-Birkhoff-Witt type presented in [7], can be used to construct differential calculi on matrix quantum groups. The resulting De Rham complexes can elegantly be described in terms of quantizations of the enveloping algebras of colour Lie superalgebraic extensions of the Lie algebras involved. Since these De Rham complexes possess a differential Hopf algebra structure that extends the usual Hopf algebra structure of the enveloping algebra, the obtained differential calculi are intrinsically bicovariant. More importantly, the construction of those De Rham complexes can be computed on (colour) Lie superalgebraic level. For more details on this we simply refer to the examples presented in [7]. We will illustrate this procedure for the matrix quantum groups in detail by applying it to the two-dimensional solvable quantum group and to the Heisenberg quantum group.

2 The two-dimensional solvable Quantum Group

We apply the method described in the introduction to construct a bicovariant differential calculus on the two-dimensional solvable quantum group.

2.1 The quantum group

To construct the quantum group we begin by considering the classical algebraic group of 2×2-matrices of the form

$$T = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \tag{2.1}$$

The algebra of regular functions on this group is isomorphic to the free commutative algebra on the alphabet $\{a, a^{-1}, b\}$. It is usually equipped with a bialgebra structure by defining the comultiplication $\Delta(T) = T \odot T$ and the counit $\epsilon(T) = I$, i.e.

$$\Delta(a) = a \otimes a \quad \epsilon(a) = 1 \quad \Delta(b) = a \otimes b + b \otimes 1 \quad \epsilon(b) = 0 \tag{2.2}$$

This bialgebra becomes a Hopf algebra by defining the antipode S by $S(T) = T^{-1}$, which boils down to

$$S(a) = a^{-1} \qquad S(b) = -a^{-1}b \tag{2.3}$$

A corresponding quantum group can be introduced by imposing the commutation relations $RT_1T_2 = T_2T_1R$ with the R-matrix given by

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1-q^{-1} & q^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.4}$$

In terms of the coordinates a and b in (2.1) this is equivalent to $ab = qba$. In order to define a differential calculus on this quantum group one needs to introduce certain commutation relations between the 1-forms da and db and the coordinate functions a and b. These commutation relations need to be chosen in such a way that they are compatible in some sense with the relation $ab = qba$.

2.2 The duality

From [6] we derive the following factorization of the matrix T.

$$T = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\alpha} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = e^{\alpha x} e^{\beta y} \tag{2.5}$$

where

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.6}$$

Classically, this factorization expresses the duality between the group and its Lie algebra by means of the exponential mapping. This expression of T in terms of exponentials combined with the discussions presented in [6] and [8] motivated us to study the function algebra in terms of the coordinates $\alpha = log(a)$ and $\beta = a^{-1}b$. We will construct the differential calculus in these coordinates and afterwards transform the results to the original coordinates a and b. We write $q = e^h$, hence the commutation relation for α and β is $[\alpha, \beta] = h\beta$. The coalgebra structure in terms of α and β is described by

$$\begin{array}{lll} \Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha & \epsilon(\alpha) = 0 & S(\alpha) = -\alpha \\ \Delta(\beta) = 1 \otimes \beta + \beta \otimes e^{-\alpha} & \epsilon(\beta) = 0 & S(\beta) = -e^{-h}e^{\alpha}\beta \end{array} \tag{2.7}$$

By the substitution $\alpha' = \frac{\alpha}{h}$, the Hopf algebra deformation of the function algebra is described by $[\alpha', \beta] = \beta$ and

$$\begin{array}{lll} \Delta(\alpha') = \alpha' \otimes 1 + 1 \otimes \alpha' & \epsilon(\alpha') = 0 & S(\alpha') = -\alpha' \\ \Delta(\beta) = 1 \otimes \beta + \beta \otimes e^{-h\alpha'} & \epsilon(\beta) = 0 & S(\beta) = -e^{-h}e^{h\alpha'}\beta \end{array} \tag{2.8}$$

Hence, this Hopf algebra can be seen as a QUE algebra on the two-dimensional Lie algebra with basis $\{\alpha', \beta\}$ and commutator $[\alpha', \beta] = \beta$. The corresponding classical limit defines a Lie bialgebra with cocommutator given by $\delta(\alpha) = 0$ and $\delta(\beta) = \alpha \otimes \beta - \beta \otimes \alpha$. This Lie bialgebra is isomorphic to the dual of the Lie bialgebra with basis $\{x, y\}$, commutator $[x, y] = y$ and cocommutator $\delta(x) = 0$, $\delta(y) = x \otimes y - y \otimes x$ which arises from the Poisson Lie Group structure that originated the quantization. It is this form of duality that will enable us to construct a bicovariant differential calculus on the quantum group.

2.3 The bicovariant differential calculus

In [7] the authors present a method to construct differential calculi of PBW type on QUE algebras. It is shown that the differential Hopf algebra structure of the resulting De Rham complexes can be described as QUE algebras of certain colour Lie superalgebraic extensions of the original Lie algebra. Fundamental in their approach is a compatibility condition between the differential operator and the cocommutator of the Lie bialgebra that is defined as the classical limit of the quantization. We have applied this method in order to construct a De Rham complex on the Hopf algebra defined by (2.8). By resubstituting $\alpha' = h\alpha$ we obtained the result in terms of the coordinates $\{\alpha, \beta\}$.

The resulting differential Hopf algebra Ω can be described as follows. It is an $\mathbb{N}$-graded algebra generated by the elements α, β of degree zero and the elements $d\alpha, d\beta$ of degree one. The defining relation $[\alpha, \beta] = h\beta$ is extended with the following set of relations:

$$\begin{array}{llll} [\alpha, d\alpha] = -hd\alpha & [\alpha, d\beta] = 0 & [\beta, d\alpha] = -hd\beta & \beta d\beta - e^h d\beta.\beta = 0 \\ [d\alpha, d\alpha] = 0 & [d\alpha, d\beta] = 0 & [d\beta, d\beta] = 0 & \end{array} \tag{2.9}$$

We note that [,] denotes the 'colour' commutator which is defined by

$$[x, y] = xy - (-1)^{pq}yx \qquad x \in \Omega^p, y \in \Omega^q. \tag{2.10}$$

The comultiplication Δ and the counit ϵ are defined by (2.7) and

$$\begin{array}{ll} \Delta(d\alpha) = d\alpha \otimes 1 + 1 \otimes d\alpha & \epsilon(d\alpha) = 0 \\ \Delta(d\beta) = 1 \otimes d\beta + d\beta \otimes e^{-\alpha} + \beta \otimes d\alpha.e^{-\alpha}(\frac{1 - e^h}{h}) & \epsilon(d\beta) = 0. \end{array} \tag{2.11}$$

These extensions are a direct consequence of the fact that Δ and ϵ are differential algebra morphisms. The antipode S is an anti differential algebra morphism, in particular we have $S \circ d = d \circ S$ yielding

$$S(d\alpha) = -d\alpha \qquad S(d\beta) = -(d\beta - \frac{e^h - 1}{h}d\alpha)e^{\alpha} \tag{2.12}$$

The differential operator $d : \Omega \to \Omega$ is the unique graded derivation of degree one that satisfies

$$d(\alpha) = d\alpha \quad d(\beta) = d\beta \quad d(d\alpha) = 0 \quad d(d\beta) = 0 \tag{2.13}$$

From this one easily sees that d has the characteristic property of an exterior derivative: $d^2 = d \circ d = 0$. By performing the inverse transformation of coordinates given by $a = e^{\alpha}, b = e^{\alpha}\beta$, the previous results yield the following theorem.

Theorem 1 *The differential Hopf algebra Ω described hereafter is a De Rham complex of Poincaré-Birkhoff-Witt type on the two-dimensional solvable quantum group. In particular it defines a bicovariant differential calculus. The algebra structure of Ω is defined as the $\mathbb{N}$-graded algebra generated by the set $\{a, a^{-1}, b\}$ of degree zero and the set $\{da, db\}$ of degree one, imposed with the following defining relations:*

$$aa^{-1} = a^{-1}a = 1 \quad ab = qba \tag{2.14}$$

$$\begin{array}{ll} a.da = q^{-1}da.a & a.db = db.a \\ b.da = q^{-1}da.b + (q^{-1}-1)db.a & b.db = q^{-1}db.b \end{array} \tag{2.15}$$

$$(da)^2 = 0 \quad dbda = -dadb \quad (db)^2 = 0 \tag{2.16}$$

It is equipped with a differential operator $d : \Omega \to \Omega$ which is defined as the unique graded derivation of degree one that satisfies

$$d(a) = da \quad d(a^{-1}) = -qda.a^{-2} \quad d(b) = db \quad d(da) = 0 \quad d(db) = 0 \tag{2.17}$$

Its comultiplication Δ, counit ϵ and antipode S are defined as the unique (anti) differential algebra morphisms satisfying

$$\begin{array}{lll} \Delta(a) = a \otimes a & \epsilon(a) = 1 & S(a) = a^{-1} \\ \Delta(b) = a \otimes b + b \otimes 1 & \epsilon(b) = 0 & S(b) = -a^{-1}b \end{array} \tag{2.18}$$

which implies

$$\begin{array}{lll} \Delta(a^{-1}) = a^{-1} \otimes a^{-1} & \epsilon(a^{-1}) = 1 & S(a^{-1}) = a \\ \Delta(da) = da \otimes a + a \otimes da & \epsilon(da) = 0 & S(da) = -qda.a^{-2} \\ \Delta(db) = da \otimes b + a \otimes db + db \otimes 1 & \epsilon(db) = 0 & S(db) = qda.a^{-2}b - db.a^{-1} \end{array} \tag{2.19}$$

We remark that this differential Hopf algebra is an extension to a Hopf algebra of a differential calculus defined on the quantum plane. More specifically, it corresponds to the calculus III for $s = q$ described in [9].

The only result of the preceding theorem we did not yet prove is the statement that the differential calculus is bicovariant. This is a direct consequence of the differential Hopf algebra structure of Ω. The associativity of the multiplication $\mu : \Omega \otimes \Omega \to \Omega$ implies that the restrictions $\mu_l : \Omega^0 \otimes \Omega^1 \to \Omega^1$ and $\mu_r : \Omega^1 \otimes \Omega^0 \to \Omega^1$ turn Ω^1 into a bimodule over Ω^0. Doe to the PBW property Ω^1 is generated by $d\Omega^0$ as right (and left) Ω^0-module. The unique decomposition of the restriction of Δ to Ω^1 defined by

$$\Delta = \Delta_L + \Delta_R \text{ with } \Delta_L : \Omega^1 \to \Omega^0 \otimes \Omega^1 \text{ and } \Delta_R : \Omega^1 \to \Omega^1 \otimes \Omega^0 \tag{2.20}$$

gives, due to the coassociativity property, Ω^1 the structure of a bicomodule over Ω^0. From the fact that Δ is a differential algebra morphism one easily sees that

$$\begin{array}{l} \Delta_L(df.g) = (id \otimes d) \circ \Delta(f).\Delta(g) \\ \Delta_R(df.g) = (d \otimes id) \circ \Delta(f).\Delta(g) \end{array} \tag{2.21}$$

This shows that the differential calculus described in the theorem is bicovariant, and in particular Ω^1 is a bicovariant Ω^0-bimodule. We remark that, although in most papers one considers Ω^1 as a left Ω^0-module, we will mainly work with Ω^1 as a right Ω^0-module.

2.4 Maurer-Cartan forms

A 1-form ω is called left-coinvariant if it satisfies

$$\Delta_L(\omega) = 1 \otimes \omega \tag{2.22}$$

where Δ_L denotes the left corepresentation of Ω^0 on Ω^1 as defined in (2.20). From the expression of T in terms of exponentials one can obtain a set of left Maurer-Cartan 1-forms as follows.

$$d(T) = d(e^{\alpha x}e^{\beta y}) = d(e^{\alpha x})e^{\beta y} + e^{\alpha x}d(e^{\beta y}) =$$
$$(\frac{e^h - 1}{h})e^{\alpha x}xd\alpha.e^{\beta y} + e^{\alpha x}e^{\beta y}yd\beta = e^{\alpha x}e^{\beta y}((\frac{e^h-1}{h})(xd\alpha + yd\alpha.\beta) + yd\beta)$$

We can write this result as $d(T) = T\Theta$ with

$$\Theta = \begin{pmatrix} \frac{e^h-1}{h}d\alpha & \frac{e^h-1}{h}d\alpha.\beta + d\beta \\ 0 & 0 \end{pmatrix} \tag{2.23}$$

Hence, $\Theta = T^{-1}d(T)$, so

$$\Delta_L(\Theta) = \Delta(T^{-1})\Delta_L(d(T)) = \Delta(T)^{-1}(1 \otimes d) \circ \Delta(T) =$$
$$(T \odot T)^{-1}(T \odot d(T)) = (T \odot T)^{-1}(T \odot T\Theta) = 1 \otimes \Theta$$

which implies

$$\Delta_L(\Theta) = 1 \otimes \Theta. \tag{2.24}$$

So the entries of Θ are left Maurer-Cartan forms. Therefore we define

$$\theta_1 = \frac{e^h - 1}{h}d\alpha \qquad \theta_2 = \frac{e^h - 1}{h}d\alpha.\beta + d\beta. \tag{2.25}$$

In terms of the coordinates a and b one obtains

$$\theta_1 = qda.a^{-1} = a^{-1}da \qquad \theta_2 = db.a^{-1} = a^{-1}db \tag{2.26}$$

so these are like the classical left Maurer-Cartan forms. The corresponding structure matrix M which is defined by the property (see e.g. [1] and [10])

$$\Delta_R(\theta_i) = \theta_j \otimes M_i^j \tag{2.27}$$

is equal to

$$M = \begin{pmatrix} 1 & a^{-1}b \\ 0 & a^{-1} \end{pmatrix} \tag{2.28}$$

Associated to the Maurer-Cartan forms one can define the commutation matrix ω which describes the commutation between the functions and these 1-forms. This matrix is defined by

$$f\theta_k = \theta_l\omega(f)_k^l \quad f \in \Omega^0. \tag{2.29}$$

One can easily see that ω is multiplicative, i.e. $\omega(fg) = \omega(f)\omega(g)$ for all functions f and g. For that reason it suffices to compute $\omega(f)$ for the generators $f = a$ and $f = b$. By direct computation we find

$$\omega(a) = \begin{pmatrix} q^{-1}a & 0 \\ 0 & a \end{pmatrix} \qquad \omega(b) = \begin{pmatrix} b & 0 \\ (1-q)a & b \end{pmatrix} \tag{2.30}$$

and hence

$$\omega(a^m b^n) = \begin{pmatrix} q^{-m}a^m b^n & 0 \\ [n]_{q^{-1}}(1-q)ab^{n-1} & a^m b^n \end{pmatrix} \quad m \in \mathbb{Z}, n \in \mathbb{N} \tag{2.31}$$

Remark that $[n]_q$ denotes the q-analogue of the natural number n defined by

$$[n]_q = \frac{q^n - 1}{q - 1} \tag{2.32}$$

It is possible to introduce a functional matrix F such that (see e.g. [1] and [10])

$$g\theta_k = \theta_l(F_k^l \star g) = \theta_l(id \otimes F_k^l) \circ \Delta(g) \quad g \in \Omega^0. \tag{2.33}$$

This functional matrix has the obvious multiplicative properties:

$$F_j^i(g_1 g_2) = F_k^i(g_1)F_j^k(g_2) \quad F_j^i(1) = \delta_j^i \tag{2.34}$$

One easily verifies that for the generators a and b one has

$$F(a) = \begin{pmatrix} q^{-1} & 0 \\ 0 & 1 \end{pmatrix} \qquad F(b) = \begin{pmatrix} 0 & 0 \\ (1-q) & 0 \end{pmatrix} \tag{2.35}$$

and hence

$$F(a^m) = \begin{pmatrix} q^{-m} & 0 \\ 0 & 1 \end{pmatrix} F(a^m b) = \begin{pmatrix} 0 & 0 \\ (1-q) & 0 \end{pmatrix} F(a^m b^n) = 0 \quad (n \geq 2) \tag{2.36}$$

2.5 Left-invariant vector fields

We define the operators $\nabla^k : \Omega^0 \to \Omega^0$ by

$$d(f) = \theta_k \nabla^k(f) \qquad f \in \Omega^0. \tag{2.37}$$

and we call these operators left-invariant vector fields. One can easily show that they have the property $\Delta \circ \nabla^k = (1 \otimes \nabla^k) \circ \Delta$ (see e.g. [11]) which expresses the left-coinvariance. From

$$\theta_k \nabla^k(fg) = d(fg) = d(f)g + fd(g) = \theta_k \nabla^k(f)g + f\theta_l \nabla^l(g) = \theta_k(\nabla^k(f)g + \omega(f)^k_l \nabla^l(g))$$

we derive the so-called quantum Leibnitz rule

$$\nabla^k(fg) = \nabla^k(f)g + \omega(f)^k_l \nabla^l(g). \tag{2.38}$$

We compute explicitly the action of the left-invariant vector fields on ordered monomials $a^p b^m$. Similar to the vector fields one can define the partial derivatives with respect to the coordinates a and b by

$$d(f) = da.\partial_a f + db.\partial_b f \quad f \in \Omega^0 \tag{2.39}$$

and by a direct computation one finds that

$$\partial_a(a^p b^m) = [p]_{q^{-1}} a^{p-1} b^m \quad \partial_b(a^p b^m) = [m]_{q^{-1}} a^p b^{m-1} \tag{2.40}$$

One can interpret this result in terms of the differential q-operator on functions which is defined by

$$\frac{df}{d_q x} = \frac{f(qx) - f(x)}{x(q-1)} \tag{2.41}$$

With this definition one can write

$$\partial_a = \frac{\partial}{\partial_{q^{-1}} a} \qquad \partial_b = \frac{\partial}{\partial_{q^{-1}} b} \tag{2.42}$$

From

$$d(f) = da.\partial_a f + db.\partial_b f = \theta_1 \nabla^1(f) + \theta_2 \nabla^2(f) \tag{2.43}$$

one finds that

$$\nabla^1 = q^{-1} a \partial_a = q^{-1} a \frac{\partial}{\partial_{q^{-1}} a} \qquad \nabla^2 = a \partial_b = a \frac{\partial}{\partial_{q^{-1}} b} \tag{2.44}$$

2.6 The Quantum Lie algebra

In the classical situation the Lie algebra corresponding to the solvable group is defined to be the Lie algebra of left-invariant vector fields. The classical left-invariant vector fields ∇^1, ∇^2 satisfy the commutation relation $[\nabla^1, \nabla^2] = \nabla^2$. In order to find the quantum analogue of this commutation relation we consider

$$\begin{aligned} d^2(f) = d\theta_1 \nabla^1 f - \theta_1\theta_1(\nabla^1)^2 f - \theta_1\theta_2 \nabla^2\nabla^1 f + \\ d\theta_2 \nabla^2 f - \theta_2\theta_1 \nabla^1\nabla^2 f - \theta_2\theta_2(\nabla^2)^2 f = 0. \end{aligned} \tag{2.45}$$

From the commutation relations (2.14), (2.15) and (2.16) we find the following for the products of Maurer-Cartan forms

$$\theta_1\theta_1 = 0 \quad \theta_2\theta_1 = -q\theta_1\theta_2 \quad \theta_2\theta_2 = 0 \tag{2.46}$$

From the Maurer-Cartan equation $d(\Theta) = -\Theta^2$ it follows that

$$d(\theta_1) = 0 \qquad d(\theta_2) = -\theta_1\theta_2. \tag{2.47}$$

By substituting (2.46) and (2.47) in (2.45) one obtains the following quantum commutator

$$\nabla^1\nabla^2 - q^{-1}\nabla^2\nabla^1 = q^{-1}\nabla^2 \tag{2.48}$$

2.7 The Laplacian

In formula (2.43) we defined the partial derivatives with respect to the coordinates $\{a, b\}$. Since this definition considers the forms as right-module over the functions, these partial derivatives can be considered as right-partial derivatives. Analogously, one can define left-partial derivatives by

$$d(f) = \overleftarrow{\partial_a}(f).da + \overleftarrow{\partial_b}(f).db \quad f \in \Omega^0 \tag{2.49}$$

By direct computation one finds that

$$\overleftarrow{\partial_a}(a^p b^m) = [p]_q q^m a^{p-1} b^m \qquad \overleftarrow{\partial_b}(a^p b^m) = [m]_q q^p a^p b^{m-1} \tag{2.50}$$

We introduce a Laplacian on the quantum group by

$$\Delta_q = \overleftarrow{\partial_a} \circ \overrightarrow{\partial_a} + \overleftarrow{\partial_b} \circ \overrightarrow{\partial_b}\,. \tag{2.51}$$

By substituting (2.40) and (2.50) in (2.51) we obtain the following expression for the Laplacian

$$\Delta_q = \lambda_q^b \circ \frac{\partial}{\partial_q a} \circ \frac{\partial}{\partial_{q^{-1}} a} + \lambda_q^a \circ \frac{\partial}{\partial_q b} \circ \frac{\partial}{\partial_{q^{-1}} b} \tag{2.52}$$

The symbol λ_q denotes the q-scalar multiplication operator which is defined by

$$\lambda_q f(x) = f(qx). \tag{2.53}$$

The upper index indicates with respect to which coordinate this operator acts.

3 The Quantum Heisenberg Group

We apply the method described in the introduction to construct a bicovariant differential calculus on the quantum Heisenberg group.

3.1 The quantum Heisenberg group

At first we consider the classical Heisenberg group, i.e. the algebraic group of 3×3-matrices of the form

$$T = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

The algebra of regular functions on this group is isomorphic to the free commutative algebra on the alphabet $\{a, b, c\}$. Similar to the solvable group its Hopf algebra structure is defined by $\Delta(T) = T \odot T$, $\epsilon(T) = I$ and $S(T) = T^{-1}$ yielding

$$\begin{array}{lll} \Delta(a) = a \otimes 1 + 1 \otimes a & \epsilon(a) = 0 & S(a) = -a \\ \Delta(b) = b \otimes 1 + 1 \otimes b + a \otimes c & \epsilon(b) = 0 & S(b) = -b + ac \\ \Delta(c) = c \otimes 1 + 1 \otimes c & \epsilon(c) = 0 & S(c) = -c \end{array} \tag{3.2}$$

We define a quantum Heisenberg group by means of the R-matrix from [12]. This R-matrix is given by

$$R = \begin{pmatrix} I_3 & 2he_3^2 & -he_2^2 \\ 0 & I_3 - he_3^1 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{3.3}$$

where e_j^i denotes the 3×3-matrix with 1 on the i-th row and j-th column and 0 in all other positions. In terms of the coordinates a, b and c this yields the following commutation relations

$$[a, b] = ha \quad [a, c] = 0 \quad [b, c] = -hc. \tag{3.4}$$

3.2 The duality

We computed the following factorization

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{cy} e^{bz} e^{ax} \tag{3.5}$$

where

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.6}$$

This indicates that $\{a, b, c\}$ are exponential coordinates. By the substitution $b' = \frac{b}{h}$ the deformation of the function algebra on the Heisenberg group is described by

$$[a, b'] = a \quad [a, c] = 0 \quad [b', c] = -c \tag{3.7}$$

and

$$\begin{array}{lll} \Delta(a) = a \otimes 1 + 1 \otimes a & \epsilon(a) = 0 & S(a) = -a \\ \Delta(b') = b' \otimes 1 + 1 \otimes b' + h^{-1} a \otimes c & \epsilon(b') = 0 & S(b') = -b' + h^{-1} ac \\ \Delta(c) = c \otimes 1 + 1 \otimes c & \epsilon(c) = 0 & S(c) = -c \end{array} \tag{3.8}$$

This Hopf algebra can be seen as a QUE algebra with deformation parameter h^{-1} on the Lie bialgebra with basis $\{a, b', c\}$, commutator given by (3.7) and cocommutator described by

$$\delta(a) = 0 \quad \delta(b') = a \otimes c - c \otimes a \quad \delta(c) = 0. \tag{3.9}$$

This Lie bialgebra is dual to the standard three-dimensional Heisenberg Lie bialgebra which is described by the commutator

$$[x, y] = z \quad [x, z] = 0 \quad [y, z] = 0, \tag{3.10}$$

and cocommutator

$$\delta(x) = x \otimes z - z \otimes x \quad \delta(y) = y \otimes z - z \otimes y \quad \delta(z) = 0. \tag{3.11}$$

Note that the Heisenberg Lie bialgebra is coboundary (see e.g. [13]) with R-matrix given by $r = x \otimes y - y \otimes x$. The corresponding Poisson Lie group structure which originates from the classical limit of the quantization, is therefore given by the well known Sklyanin-bracket (see [14]) $\{T \overset{\otimes}{,} T\} = [T \otimes T, r]$, or explicitly

$$\{a, b\} = a \quad \{a, c\} = 0 \quad \{b, c\} = -c \tag{3.12}$$

In the next section we will use the description of the Heisenberg quantum group by (3.7) and (3.8) to construct a bicovariant differential calculus.

3.3 The bicovariant differential calculus

The description of the function algebra as a QUE algebra in (3.7) and (3.8) enables us to apply the method of quantization described in [7]. We computed a differential operator which is compatible with the cocommutator given by (3.9) and constructed a De Rham complex on the given QUE algebra. The result is as follows.

Theorem 2 *The differential Hopf algebra Ω described hereafter is a De Rham complex of Poincaré-Birkhoff-Witt type, and hence a bicovariant differential calculus, on the quantum Heisenberg group with R-matrix (3.3). The algebra structure of Ω is defined as the $\mathbb{N}$-graded algebra generated by the set $\{a, b, c\}$ of degree zero and the set $\{da, db, dc\}$ of degree one, imposed with the relations (3.4) extended with*

$$\begin{array}{lll} [a, da] = 0 & [a, db] = (1 - \lambda) h da & [a, dc] = 0 \\ [b, da] = -\lambda h da & [b, db] = -h(\mu da + 2\lambda db + \tau dc) & [b, dc] = -\lambda h dc \\ [c, da] = 0 & [c, db] = (1 - \lambda) h dc & [c, dc] = 0 \end{array} \tag{3.13}$$

and

$$dbda = -dadb \quad dcda = -dadc \quad dcdb = -dbdc \quad (da)^2 = (db)^2 = (dc)^2 = 0 \tag{3.14}$$

The parameters μ and τ can be taken arbitrarily, the parameter λ needs to be equal to ± 1. The differential operator $d : \Omega \to \Omega$ is defined as the unique graded derivation of degree one that satisfies

(3.15) $$d(a) = da \quad d(b) = db \quad d(c) = dc \quad d(da) = 0 \quad d(db) = 0 \quad d(dc) = 0$$

The comultiplication Δ, counit ϵ and antipode S are defined as the unique (anti) differential algebra morphisms satisfying (3.2). This implies

(3.16) $$\begin{array}{ll} \Delta(da) = da \otimes 1 + 1 \otimes da & \epsilon(da) = 0 \\ \Delta(db) = db \otimes 1 + 1 \otimes db + da \otimes c + a \otimes dc & \epsilon(db) = 0 \\ \Delta(dc) = dc \otimes 1 + 1 \otimes dc & \epsilon(dc) = 0 \end{array}$$

and

(3.17) $$S(da) = -da \quad S(db) = -db + da.c + da.c \quad S(dc) = -dc$$

The fact that the parameter λ must be equal to ± 1 can easily be seen by considering the overlap ambiguity $b.a.db$. By direct computation and using relations (3.4) and (3.13) one finds $(ba)db - b(adb) = h^2(\lambda^2 - 1)da$. So due to the PBW basis it follows that $\lambda^2 = 1$.

3.4 Maurer-Cartan forms

From the universal T-matrix expression (3.5) we deduce that

(3.18) $$d(T) = T(ydc + zdb + xda - zdc.a) = T\Theta$$

and hence left-coinvariant Maurer-Cartan forms are given by

(3.19) $$\Theta = \begin{pmatrix} 0 & \theta_1 & \theta_2 \\ 0 & 0 & \theta_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & da & db - dc.a \\ 0 & 0 & dc \\ 0 & 0 & 0 \end{pmatrix}$$

The corresponding structure matrix M defined by (2.27) is equal to

(3.20) $$M = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & -a & 1 \end{pmatrix}$$

The corresponding commutation matrices $\omega(f)$ (see (2.29)) for the generators $f = a$, $f = b$ and $f = c$ are

(3.21) $$\omega(a) = \begin{pmatrix} a & (1-\lambda)h & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \quad \omega(c) = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & (1-\lambda)h & c \end{pmatrix}$$

and

(3.22) $$\omega(b) = \begin{pmatrix} b - \lambda h & -h\mu & 0 \\ 0 & b - 2\lambda h & 0 \\ 0 & h(1-\lambda)a - h\tau & b - \lambda h \end{pmatrix}$$

By induction one can prove that

(3.23) $$\omega(a^p) = \begin{pmatrix} a^p & p(1-\lambda)ha^{p-1} & 0 \\ 0 & a^p & 0 \\ 0 & 0 & a^p \end{pmatrix} \omega(c^p) = \begin{pmatrix} c^p & 0 & 0 \\ 0 & c^p & 0 \\ 0 & p(1-\lambda)hc^{p-1} & c^p \end{pmatrix}$$

and

(3.24) $$\omega(b^p) = \begin{pmatrix} (b-\lambda h)^p & \lambda\mu((b-2\lambda h)^p - (b-\lambda h)^p) & 0 \\ 0 & (b-2\lambda h)^p & 0 \\ 0 & \lambda\tau((b-2\lambda h)^p - (b-\lambda h)^p) + \frac{1-\lambda}{2}a((b+2h)^p - b^p) & (b-\lambda h)^p \end{pmatrix}$$

from which the matrix $\omega(a^p b^q c^r)$ can easily be ontained. The functional matrix F defined by property (2.33) is for the generators a, b and c given by

$$F(a) = \begin{pmatrix} 0 & (1-\lambda)h & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} F(b) = \begin{pmatrix} -\lambda h & -\mu h & 0 \\ 0 & -2\lambda h & 0 \\ 0 & -\tau h & -\lambda h \end{pmatrix} F(c) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & (1-\lambda)h & 0 \end{pmatrix}$$

3.5 Left-invariant vector fields

By formula (2.37) we define left-invariant vector fields on the quantum Heisenberg group. Due to the equality

$$d(f) = da.\, \vec{\partial}_a\, f + db.\, \vec{\partial}_b\, f + dc.\, \vec{\partial}_c\, f = \theta_1 \nabla^1(f) + \theta_2 \nabla^2(f) + \theta_3 \nabla^3(f)$$

one has

(3.25) $$\nabla^1 = \vec{\partial}_a \quad \nabla^2 = \vec{\partial}_b \quad \nabla^3 = \vec{\partial}_c + a\, \vec{\partial}_b$$

Hence, in order to find expressions for the left-invariant vector fields we compute the right-partial derivatives with respect to the coordinates $\{a, b, c\}$. We will make use of the following special operators:

1. The symbol σ_α denotes the shifting operator over a distance α, i.e.

   (3.26) $$\sigma_\alpha f(x) = f(x + \alpha)$$

2. The symbol ∇_α denotes the difference operator defined by

   (3.27) $$\nabla_\alpha f(x) = \frac{f(x+\alpha) - f(x)}{\alpha}$$

3. The composition $D_\alpha = \nabla_\alpha \circ \nabla_{-\alpha} = \nabla_{-\alpha} \circ \nabla_\alpha$ equals

   (3.28) $$D_\alpha f(x) = \frac{f(x+\alpha) - 2f(x) + f(x-\alpha)}{\alpha^2}$$

   We remark that D_α is the well known second order central difference operator.

In the following we consider these operators to act partially with respect to the coordinate b. It turns out that the right-partial derivatives can be written as follows

(3.29) $$\vec{\partial}_a = \sigma_{-\lambda h} \circ (\sigma_h \circ \frac{\partial}{\partial a} - \frac{1}{2} h\mu D_h) \quad \vec{\partial}_b = \nabla_{-2\lambda h} \quad \vec{\partial}_c = \sigma_{-\lambda h} \circ (\frac{\partial}{\partial c} - \frac{1}{2} h\tau D_h)$$

3.6 The Quantum Lie algebra

The classical left-invariant vector fields $\nabla^1, \nabla^2, \nabla^3$ satisfy the commutation relations

(3.30) $$[\nabla^1, \nabla^2] = 0 \quad [\nabla^1, \nabla^3] = \nabla^2 \quad [\nabla^2, \nabla^3] = 0$$

In order to find the quantum analogue of these commutation relations we consider the equation $d^2(f) = 0$. From the commutation relations we find the following for the products of Maurer-Cartan forms

(3.31) $$\begin{array}{lll} \theta_2\theta_1 = -\theta_1\theta_2 & \theta_3\theta_1 = -\theta_1\theta_3 & \theta_2\theta_3 = -\theta_3\theta_2 \\ \theta_1\theta_1 = 0 & \theta_2\theta_2 = (1-\lambda)h\theta_1\theta_3 & \theta_3\theta_3 = 0 \end{array}$$

Furthermore, from the Maurer-Cartan equation one finds

(3.32) $$d(\theta_1) = 0 \quad d(\theta_2) = -\theta_1\theta_3 \quad d(\theta_3) = 0$$

By substituting (3.31) and (3.32) in the expression $d^2(f) = 0$ one obtains the following quantum commutator

(3.33) $$\begin{array}{l} \nabla^1\nabla^2 - \nabla^2\nabla^1 = 0 \\ \nabla^1\nabla^3 - \nabla^3\nabla^1 = \nabla^2 + (1-\lambda)h(\nabla^2)^2 \\ \nabla^2\nabla^3 - \nabla^3\nabla^2 = 0 \end{array}$$

Note that in the case $\lambda = 1$ the commutation of Maurer-Cartan forms (3.31) coincides with the classical commutation relations and therefore the quantum Lie algebra is isomorphic to the classical Lie algebra.

3.7 The Laplacian

The left-partial derivatives are equal to

$$\overleftarrow{\partial}_a = \sigma_{\lambda h} \circ (\frac{\partial}{\partial a} + \frac{1}{2} h \mu D_h) \quad \overleftarrow{\partial}_b = \nabla_{2\lambda h} \quad \overleftarrow{\partial}_c = \sigma_{\lambda h} \circ (\sigma_{-h} \circ \frac{\partial}{\partial c} + \frac{1}{2} h \tau D_h) \tag{3.34}$$

This yields the following expression for the Laplacian

$$\Delta = \overleftarrow{\partial}_a \circ \overrightarrow{\partial}_a + \overleftarrow{\partial}_b \circ \overrightarrow{\partial}_b + \overleftarrow{\partial}_c \circ \overrightarrow{\partial}_c = \sigma_h \circ \frac{\partial^2}{\partial a^2} + \frac{1}{2} h \mu (\sigma_h - 1) \circ \frac{\partial}{\partial a} \circ D_h - \tag{3.35}$$

$$\frac{1}{4} \mu^2 h^2 (D_h)^2 + D_{2h} + \sigma_{-h} \circ \frac{\partial^2}{\partial c^2} + \frac{1}{2} h \tau (1 - \sigma_{-h}) \circ \frac{\partial}{\partial c} \circ D_h - \frac{1}{4} \tau^2 h^2 (D_h)^2 =$$

$$\sigma_h \circ \frac{\partial^2}{\partial a^2} + D_{2h} + \sigma_{-h} \circ \frac{\partial^2}{\partial c^2} + \frac{1}{2} h^2 (\mu \frac{\partial}{\partial a} \circ \nabla_h + \tau \frac{\partial}{\partial c} \circ \nabla_{-h} - \frac{1}{2} (\mu^2 + \tau^2) D_h) \circ D_h$$

We remark that, although there are differences between the differential calculi on the quantum Heisenberg group corresponding to the values 1 and -1 for the parameter λ, the Laplacian is independent of the parameter λ.

References

[1] S.L. Woronowicz (1989). Differential calculus on compact matrix pseudogroups (quantum groups). *Comm. Math. Phys.* **122**, 125-170.

[2] B. Jurco (1991). Differential calculus on quantized simple Lie groups. *Lett. Math. Phys.* **22**, 177.

[3] G. Maltsiniotis (1990). Groupes quantiques et structures differentielles (Quantum groups and differential structures). *C.R. Acad. Sci. Paris* Sér I. **311**, 831-834.

[4] K. Schmüdgen and A. Schüler (1995). Classification of Bicovariant Differential Calculi on Quantum Groups of Type A, B, C and D. *Comm. Math. Phys.* **167**, 635-670.

[5] K. Bresser (1995). *All bicovariant differential calculi on* $GL_q(3,\mathbb{C})$ *and* $SL_q(3,\mathbb{C})$, Preprint.

[6] C. Fronsdal and A. Galindo (1993). The Dual of a Quantum Group. *Lett. Math. Phys.* **27**, 59-71.

[7] N. van den Hijligenberg, R. Martini and G.F. Post (1995). *Quantization of Differential Calculi on Universal Enveloping Algebras*, Preprint.

[8] F. Bonechi, E. Celeghini, R. Giachetti, C.M. Perena, E. Sorace and M. Tarlini (1994). Exponential mapping for non-semisimple quantum groups. *J. Phys. A:Math. Gen.* **27**, 1307-1315.

[9] T. Brzezinski, H. Dabrowski and J. Rembielinski (1992). On the quantum differential calculus and the quantum holomorphicity. *J. Math. Phys.* **33**, 19.

[10] P. Aschieri and L. Castellani (1993). An introduction to non-commutative differential geometry on quantum groups. *Int. J. Mod. Phys. A.* **8**, 1667.

[11] F. Muller-Hoissen (1992). Differential calculi on the quantum group $GL_{p,q}(2)$. *J. Phys. A:Math. Gen.* **25**, 1703-1734.

[12] E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini (1991). The quantum Heisenberg group $H(1)_q$. *J. Math. Phys.* **32** 5, 1155-1158.

[13] V. Chari and A. Pressley (1994). *Quantum Groups.* Cambridge University Press, Cambridge. p. 50-59.

[14] E.K. Sklyanin (1982). Quantum variant of the method of inverse scattering problem. *J. Soviet. Math.* **19**, 1546-1596.