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# Spare parts inventory pooling: how to share the benefits?

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**Abstract**

We consider a stock point for expensive, low-usage items that is operated by multiple decision makers. Each faces a Poisson demand process, and the joint stock point is controlled by a continuous-review base stock policy with full backordering. We consider penalty costs for backorders and holding costs for stock on hand. For this model, we derive structural properties of the resulting cost function. We use these to prove not only that it is cost effective to share one stock point with all parties involved, but also that collaboration (inventory pooling) can be supported by a stable cost allocation, i.e., the core of the associated cooperative game is non-empty. These results hold under optimized and under exogenously given base stock levels. For the former case, we further identify a stable cost allocation that would be easy to implement in practice and that induces players to reveal their private information truthfully.

## 1 Introduction

Users of capital goods such as trams, manufacturing plants, and airplanes are often confronted with the difficult task of guaranteeing high availabilities of their expensive, technologically advanced systems. A commonly used strategy to prevent costly downtimes is

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to immediately replace any failed component with a functioning spare part. Obviously, this strategy functions only if there are enough spare parts available. This leads to a challenging problem from an inventory management perspective, especially for expensive components that fail infrequently. Consider, for instance, the problem faced by a tram operator in a city such as Amsterdam. Even with a fleet of hundreds of trams, the number of failures observed for some tram component is subject to a lot of uncertainty: one year a given component might not fail at all, while the next year it may suddenly fail in four different trams. Aggravating this demand uncertainty with long replenishment lead times of sometimes more than a year, it becomes clear that having adequate stocks of spare parts on hand is essential. But at the same time, these stocks tie up a lot of capital. The sale of spare parts and after-sales services in the United States has been pegged at 8% of gross domestic product, which translates to $1 trillion every year in the United States alone (see Cohen et al., 2006, and references therein).

Intuitively, it makes sense for companies in the same area to pool common spare parts. Indeed, as stated by Cohen et al. (2006): "The best way for companies to realize economies of scale is to pool spare parts". Tram operators in the Netherlands are a good example. In the Netherlands, the local public transport in the three largest cities (Amsterdam, Rotterdam, and The Hague, all of which are within an hour's driving distance of each other) is operated by a separate company per city. Although the operators use trams of different models, there is still a lot of commonality on the component level, enabling an excellent opportunity for inventory pooling. Another example setting is that of independently managed plants of a large energy company, as described in Guajardo and Ronnqvist (2011): the plants currently hold their inventory separately, but annual savings of 44% are obtainable if pooling is taken into account. Kukreja et al. (2001) describe a similar case of pooling possibilities between independently operating power-generating plants; for this case, substantial savings of 68% can be achieved by pooling of common parts. Spare parts pools for multiple companies already exist in the airline industry (e.g., the Abacus component exchange pool of Fokker Services) and they have motivated researchers such as Wong et al. (2005) to develop quantitative spare parts models with pooled inventories. As a final example, a number of European air forces and navies are already pooling their spare parts, and other countries have recently shown interest in joining the pool (see, e.g., Hale, 2011).

Although the successful collaborations in aerospace and defense are encouraging, cooperative pooling of spare parts between multiple decision makers is not a common practice yet. One major obstacle appears to be the identification of a fair cost sharing mechanism. This has been recognized as an important research direction by other researchers as well (Wong et al., 2005; Guajardo and Ronnqvist, 2011). In our contacts with the capital goods industry, we find that practitioners are mainly hesitant to pool spare parts because they are not sure it will lead to a cost saving for themselves. Some of the commonly stated fears are that some group of firms may end up paying to subsidize the others, that the other companies may not disclose their private information truthfully, and that new members might take more benefit out of the pool than they bring in. To tackle these issues, we will apply concepts from cooperative game theory to study an inventory model that is appropriate for pooled spare parts.

Pooling is especially interesting for expensive, low-demand spare parts with long lead times and no emergency supply flexibility, since parts with such characteristics have the largest economic impact. For such parts, a continuous-review base stock policy with one-for-one replenishments is appropriate. In this paper, we will analyze the resulting single-echelon inventory model, taking into account holding costs incurred while spare parts are in stock and penalty costs incurred while spare parts are backordered. Backlogging is common in practice; moreover, in the stream of literature on spare parts models, as reviewed in Section 2, it is generally assumed that unfulfilled demands are backordered. Previous analytical investigations of cooperative games that arise from spare parts pooling between multiple players (Karsten et al., 2009, 2011a) have focused on a model with an emergency supply option (see Section 2 for more details). But especially for very specific components that are manufactured (or repaired) by one original equipment manufacturer only, there is no alternative emergency supplier with a negligible lead time — backordering is the only available option for these components.

The spare parts inventory model with backordering that we will consider has been studied extensively in the literature, due to its high practical relevance. As a result, the steady-state distributions of the number of components on hand and on backorder are well-known; the same holds for the average long-term costs and the behavior of these costs as a function of the base stock level. What has not been analyzed thus far is the behavior of these costs for varying demand rates and the effects of pooling. We contribute in this respect by deriving new, interesting properties of the costs as a function of the demand rate, and by showing that pooling the demand streams and inventory of a number of given stock points leads to reduced backorders, inventory, and costs.

We use these structural properties to prove that the associated cooperative games have a non-empty core, i.e., there exists a stable allocation of expected costs that gives no group of players an incentive to split off and form a separate pool. We further strengthen this by showing that, under the innocuous assumption that strictly positive base stock levels are used, it is possible to allocate costs in such a way that each group of players is *strictly* better off. These results hold for both settings that we will study: optimized and exogenously given base stock levels. In the former case, any coalition is assumed to pick an optimal base stock level for the aggregate demand rate of its members. In the latter case, adjustment of base stock levels is not possible, and players can only pool their given stocks of (repairable) spare parts.

Our main interest, however, lies in the setting with optimization. For that setting, we aim to select and implement a cost allocation rule; preferably one with appealing properties from a practical perspective. Four relevant properties that an allocation rule might satisfy are that: (1) it gives a stable and fair allocation of the costs to the various players, (2) it stimulates growth, i.e., it makes it interesting for participating players to allow more players to join, (3) it is easy to understand and implement, (4) it gives players an incentive to disclose all relevant information truthfully. Identification of an allocation rule satisfying all these requirements may seem to be a complex problem. Nevertheless, we show that this problem does have a solution and a remarkably simple one at that: the straightforward allocation of total costs proportional to player's demand rates satisfies all required properties. We see this as the main contribution of our paper.

Although our games are formulated in terms of *expected* costs, *realized* costs in any period of time may differ greatly from these expected costs. For the setting with optimized base stock levels, we therefore also study fair allocations of *realized* costs, and discuss its implications for truthful information disclosure. These issues have been previously considered in the context of collaborating newsvendors with no inventory carryover. In particular, fair divisions of realized costs are studied by Dror et al. (2008) and Chen and Zhang (2009, Remark 3), while schemes that induce truthful revelation of private demand information are studied by Norde et al. (2011). We, in contrast, tackle these practically relevant issues in an infinite-horizon continuous-review inventory model. Our approach differs from existing approaches in the newsvendor context. Specifically, we identify a process that fairly allocates costs as they materialize and, subsequently, we establish a link between this process and our proportional core allocation.

The remainder of this paper is organized as follows. In Section 2, we discuss the related literature. We introduce the inventory model that we study in Section 3. Next, in Section 4, we give some preliminaries on cooperative game theory. In Section 5, we introduce the spare parts pooling games wherein the base stock level of any coalition is optimized for the demand rate of that coalition. The next three sections form the main part of our work. We analyze the associated inventory model in Section 6. The results are used in Section 7 to show that the proportional rule always results in a core allocation. Next, in Section 8, we show how this allocation rule can be implemented in practice. In Section 9 we make a side step: we show that the key result of core non-emptiness holds also if each player brings a fixed number of spare parts to any coalition. Finally, we conclude in Section 10.

## 2 Related literature

There are two streams of related literature: the literature on spare parts inventory management and the literature on cooperative game theory applied to inventory or queueing systems.

Due to the high economic impact of spare parts all around the world, the amount of literature on spare parts inventory management is enormous. The first relevant paper is that of Feeney and Sherbrooke (1966), who derive the steady-state distribution of the number of items in resupply in the spare parts model that we consider. Several subsequent papers have studied pooling of spare parts between several locations, under the assumption that the system is owned by a single entity who decides whether or not to pool. Examples include Kukreja et al. (2001) and Wong et al. (2005); they show that if there are multiple stock points at one echelon level, it is generally worthwhile to use lateral transshipments between these stock points in order to reduce costs or increase the service level. Our model is in line with the models used in these two examples, although we do not explicitly take transshipment costs into account in our present paper. Wong et al. (2006) give an extensive overview of the literature on pooling in spare parts inventory models, and we refer to the books by Sherbrooke (2004) and Muckstadt (2005) for extensive overviews of the spare parts literature in general. Opposed to this literature, we will consider the setting with independent parties and address the issue of fair cost allocation.

The literature on cooperative games in inventory systems has recently been reviewed by Fiestras-Janeiro et al. (2011) and Dror and Hartman (2011). Four lines of research can be distinguished. First, the literature on games in which players face deterministic demand, use economic order quantity policies, and cooperate by using joint replenishments; see, e.g., Anily and Haviv (2007). Second, the vast literature on single-period newsvendor games, in which players face stochastic demand and may cooperate by coordinating orders and pooling inventory; see, e.g., Hartman et al. (2000), Slikker et al. (2001), Müller et al. (2002), Dror et al. (2008), Özen et al. (2008), and Chen and Zhang (2009). In contrast to these two lines of research, we consider an infinite-horizon model with stochastic demand. The third line of research is on inventory centralization games in a continuous-review setting with stochastic demand and penalty costs per backorder occurrence independent of duration. Hartman and Dror (1996) and references therein study such games via approximate evaluation. We, in contrast, perform exact evaluation of a setting in which backorder costs are paid for each unit of time a part is lacking.

The fourth, relatively scarce line of inventory game literature is motivated by spare parts applications. Wong et al. (2007) are the first to study a multi-location, continuous-review, infinite-horizon setting with several players who cooperate by pooling their parts. Wong et al. propose various cost allocation policies and numerically illustrate them, but their work lacks structural results. Karsten et al. (2009, 2011a) derive structural results for cooperative games in which resources, such as spare parts, can be pooled. As mentioned in Section 1, the model of Karsten et al. (2009, 2011a) differs from ours in one key aspect: we assume full backordering if a demand cannot be fulfilled immediately, whereas they assume that there is an emergency option, which results in lost sales for the inventory system under study. It is well-known in the inventory literature that results for a model with lost sales need not carry over to a model with backordering, or vice versa, and that the two models require different analysis. Indeed, to show that the core of their games is non-empty, Karsten et al. use properties of (new extensions of) the Erlang loss formula (i.e., the blocking probability of an $M/G/s/s$ queueing system), which has no direct relation with the model considered in the present paper. Karsten et al. (2009) assume that each player brings a fixed number of resources to each coalition, whereas Karsten et al. (2011a) assume that the number of resources in each coalition is optimized. In the present paper, we mainly focus on a setting with optimization, in line with the assumption made in Karsten et al. (2011a). Besides establishing core non-emptiness for games with backorders rather than lost sales, we also study implementation issues (in Sections 7.2 and 8) that were not considered by Karsten et al. (2011a). We briefly treat the setting with fixed base stock levels as well to show that core non-emptiness as derived by Karsten et al. (2009) for a model with lost sales carries over to our model with backordering.

Finally, we briefly mention the literature that applies cooperative game theory to analyze resource pooling in queueing facilities. A recent overview is provided by Karsten et al. (2011b), who themselves study a model in which several $M/M/s$ queues join forces. Özen et al. (2011) study the core of similar queueing games. The stream of literature on queueing games is relevant because there is a correspondence between the pipeline stock in our spare parts inventory model and the number of busy servers in an $M/G/\infty$ queueing model. Nevertheless, we have an inventory buffer confounding our analysis and the $M/G/\infty$ queue behaves fundamentally different from the $M/M/1$ and $M/M/s$ queues that have been considered in existing queueing games.

## 3 The $(S-1, S)$ inventory model with backlogging

We consider a single location that stocks one item. Initially, there are $S \in \mathbb{N}_0$ ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) parts on stock. The demand process is a Poisson process with stationary rate $\lambda > 0$. A demand is immediately fulfilled from stock if a part is available. Otherwise, it is backordered and fulfilled first come first serve. In either case, an order for a new part is instigated immediately. This means that the stock point operates under a continuous-review base stock policy with base stock level $S$ and one-for-one replenishments.

The stock point orders parts at an external, uncapacitated supplier. The time that elapses between demand occurrence and receival of the new part is called the lead time. Lead times are assumed to be independent and identically distributed (i.i.d.) according to some general distribution function, and we assume without loss of generality (by rescaling time) that its mean is 1 time period.

In the remainder, we will analyze the resulting inventory system in steady-state. First, we consider the number of parts on order, the so-called pipeline stock, denoted by $X(\lambda)$. By Palm (1938), $X(\lambda)$ is Poisson distributed with mean $\lambda$, i.e., for all $x \in \mathbb{N}_0$ it holds that

$$\mathbb{P}[X(\lambda) = x] = \frac{\lambda^x}{x!} e^{-\lambda}. \tag{1}$$

We are mainly interested in the number of backorders, $B(S, \lambda)$, and the stock on hand, $I(S, \lambda)$, both as functions of the base stock level $S$ and the demand rate $\lambda$. Backorders exist if the pipeline stock is larger than the base stock level, so $B(S, \lambda) = \max\{X(\lambda) - S, 0\}$. Similarly, $I(S, \lambda) = \max\{S - X(\lambda), 0\}$. Thus, using Equation (1), we can obtain the distributions and expectations of the number of backorders and the total stock on hand. For instance, the distribution of the number of backorders, $B(S, \lambda)$, is given by

$$\mathbb{P}[B(S, \lambda) = x] = \begin{cases} \sum_{y=0}^{S} \mathbb{P}[X(\lambda) = y] & \text{if } x = 0; \\ \mathbb{P}[X(\lambda) = x + S] & \text{if } x \in \mathbb{N}. \end{cases} \tag{2}$$

Accordingly, the expected number of backorders is

$$\begin{aligned} \mathbb{E}B(S, \lambda) &= \sum_{x=S+1}^{\infty} (x - S)\mathbb{P}[X(\lambda) = x] \\ &= \sum_{x=0}^{\infty} x \cdot \mathbb{P}[X(\lambda) = x] - \sum_{x=0}^{\infty} S \cdot \mathbb{P}[X(\lambda) = x] + \sum_{x=0}^{S} (S - x)\mathbb{P}[X(\lambda) = x] \\ &= \lambda - S + \sum_{x=0}^{S} (S - x)\mathbb{P}[X(\lambda) = x]. \end{aligned} \tag{3}$$

Similarly, the expected on-hand stock is

$$\mathbb{E}I(S,\lambda) = \sum_{x=0}^{S} (S-x)\mathbb{P}[X(\lambda) = x]. \tag{4}$$

We consider holding costs $h > 0$ per unit time per spare part in the on-hand stock. These costs encompass warehousing, insurance, and interest costs on the capital tied up by the inventory. Furthermore, we consider penalty costs $b > 0$ per unit time per backordered demand. (We disregard the part procurement price or holding costs for pipeline stock because these cost factors would represent constant terms, unaffected by decisions on base stock level or collaboration.) The long-term average costs per unit time are given by

$$K(S,\lambda) = h \cdot \mathbb{E}I(S,\lambda) + b \cdot \mathbb{E}B(S,\lambda). \tag{5}$$

Although the above-described model is general and might also apply for, e.g., inventories of luxury cars, its formulation and underlying assumptions are driven by inventory systems of expensive, low-demand spare parts meant for technologically advanced capital goods. Indeed, the critical assumptions (the stationary Poisson demand process, the continuous-review one-for-one replenishment strategy, and the independence of successive lead times) are justifiable for this spare parts setting (see, e.g., Wong et al., 2006; Karsten et al., 2009).

The assumption regarding the demand process in particular is standard in the spare parts literature. It is justified because a stock point typically serves multiple high-tech machines, whose merged stream of component failure processes corresponds very well to a Poisson process. Although no failures occur in a machine that is down due to a backordered part, the work for a broken machine is often largely taken over by the functional machines (albeit at penalty costs $b$ per unit time) and, moreover, the expected number of broken machines is typically small relative to the total number of machines. Together, this implies that the total component failure rate remains close to constant.

In a typical spare parts setting, demands are triggered by failures of either consumable or repairable machine components. Although we formulated our model for consumable parts, it is also applicable for repairable parts if — instead of placing orders for new parts — any failed component is immediately sent to an uncapacitated repair facility that returns the component to the stock point as a ready-for-use spare part after an i.i.d. repair lead time with mean 1. (Again, direct repair costs and holding costs for parts in repair would represent constant terms and can be disregarded without loss of generality.)

In this paper, we are mainly interested in the problem of fair allocation of shared costs in a spare parts inventory system operated by multiple players. We will tackle this problem for the above-described model by applying concepts from cooperative game theory.

# 4 Cooperative game theory

In this section, we treat concepts from cooperative game theory that are relevant to our work. A cooperative cost game with transferable utility, which we will simply refer to as a *game*, is a pair $(N, c)$. Here, $N$ is the non-empty finite set of *players*. A subset $M \subseteq N$ is called a coalition, and $N$ is also referred to as the *grand coalition*. We let $2^N_{-} = \{M \subseteq N \mid M \neq \emptyset\}$ denote the power set containing all non-empty coalitions. For any two sets $M$ and $L$, we write $M \subset L$ if $M$ is a *proper subset* of $L$, i.e., if $M \subseteq L$ and $M \neq L$. In ensuing analysis, it is sometimes convenient to consider only proper, non-empty subsets of the grand coalition. The set of such coalitions is given by $2^N_{--} = \{M \subset N \mid M \neq \emptyset\}$. We call any coalition in $2^N_{--}$ a *non-empty subcoalition*. Given the set of players $N$, a game $(N, c)$ is defined by the *characteristic cost function* $c$, which assigns to every coalition $M \subseteq N$ its costs $c(M)$. In our spare parts pooling context, the value $c(M)$ will be interpreted as the total long-term average costs per unit time of the joint inventory system if only the players in $M$ are involved in it. By convention, $c(\emptyset) = 0$.

Two interesting properties that a game might satisfy are subadditivity and concavity. A game is called *subadditive* if it is always beneficial to combine coalitions, i.e., if for any two coalitions $M, L \subseteq N$ with $M \cap L = \emptyset$ it holds that $c(M) + c(L) \geq c(M \cup L)$. In a subadditive game, cooperation by the grand coalition is socially optimal. A game is called *concave* if any player's marginal cost contribution does not increase for larger coalitions, i.e., if for each $i \in N$ and for all $M, L \subseteq N \setminus \{i\}$ with $M \subseteq L$ it holds that $c(M \cup \{i\}) - c(M) \geq c(L \cup \{i\}) - c(L)$.

In cooperative game theory, players are assumed to be able to draw up binding agreements, and side payments are allowed. A central problem is allocating $c(N)$ to the individual players in a stable way. Any vector $x = (x_i)_{i \in N} \in \mathbb{R}^N$ is called an *allocation* for game $(N, c)$ if it satisfies *efficiency*, i.e., if $\sum_{i \in N} x_i = c(N)$. The value $x_i$ is then interpreted as the costs assigned to player $i$. Two well-known allocation rules are the Shapley value (defined in Shapley, 1953) and the nucleolus (defined in Schmeidler, 1969). An allocation $x$ for a game $(N, c)$ is called *stable* if $\sum_{i \in M} x_i \leq c(M)$ for all $M \in 2^N_{--}$. Under a stable allocation, each group of players has to pay no more collectively than what they would face by acting independently as a group. The set of all stable allocations is called the *core*, introduced by Gillies (1959). A game may have an empty core, even if it is subadditive. If a game is concave, its core is non-empty and the Shapley value lies in the core (Shapley, 1971).

We next strengthen the notion of a stable cost allocation. In our experience, practitioners are usually interested in allocations under which each coalition becomes *strictly* better off as a result of cooperation. Indeed, an allocation $x$ under which some group of players is indifferent between cooperating or not (i.e., if $\sum_{i\in M} x_i = c(M)$ for some non-empty subcoalition $M$) may be hard to defend in practice because players may decide not to collaborate if they do not strictly benefit from it, out of spite. This issue is rarely addressed in the literature on cooperative game theory, with the notable exception of Zhao (2001), who introduces and characterizes the *relative interior of the core*. We will refer to this concept more succinctly as the *strict core*; it is defined as

$$\mathscr{C}(N,c) = \{x \in \mathbb{R}^N \mid \sum_{i\in N} x_i = c(N) \text{ and } \sum_{i\in M} x_i < c(M) \text{ for all } M \in 2^N_{--}\}.$$

Given a game $(N,c)$, we call any element of $\mathscr{C}(N,c)$ a *strictly stable* allocation. Such allocations remain stable for small perturbations of the characteristic cost function. The nucleolus always accomplishes strictly stable allocations for games with a non-empty strict core (Zhao, 2001, Lemma 1). The class of games with a non-empty strict core is characterized in the appendix via the notion of balancedness, in line with the characterization result of Bondareva (1963) and Shapley (1967) for games with a non-empty "traditional" core. This characterization differs from the characterizations put forward by Zhao (2001) and is more suitable for analysis of spare parts games with fixed base stock levels.

The last concept that we wish to introduce is a (strict) population monotonic allocation scheme (cf. Sprumont, 1990). An allocation scheme for a game $(N,c)$ is a vector $y = (y_{i,M})_{i\in M, M\in 2^N_-}$ with $\sum_{i\in M} y_{i,M} = c(M)$ for all $M \in 2^N_-$, which specifies how to allocate the costs of every coalition to its members. This scheme is called a *population monotonic allocation scheme* (PMAS) if the amount that any player has to pay does not increase when the coalition to which he belongs grows. That is, $y_{i,M} \geq y_{i,L}$ for all $M, L \in 2^N_-$ with $M \subset L$ and $i \in M$. If this inequality is strict for the members of all such nested pairs of coalitions, then we call this scheme a *strictly population monotonic allocation scheme* (SPMAS). It is apparent from this definition that if a game $(N,c)$ admits a (strict) PMAS, say $y$, then $(y_{i,N})_{i\in N}$ is a (strictly) stable allocation, which implies that $(N,c)$ has a non-empty (strict) core.

# 5 Spare parts pooling games

Consider several players that may pool inventories of a common item. Each player witnesses a stationary Poisson demand process, and the demand processes of the players are assumed to be independent. The players have the same mean replenishment lead time, possibly because they use the same supplier or repair facility, and without loss of generality we rescale it to 1 time unit. To capture all relevant parameters of such a setting, we define a *spare parts situation with backordering* as a tuple $\varphi = (N,\, (\lambda_i)_{i\in N},\, h,\, b)$, where $N$ is the non-empty finite set of players, $\lambda_i > 0$ is the demand rate of player $i \in N$, $h > 0$ is the holding cost rate, and $b > 0$ is the backorder cost rate. We denote $\lambda_M = \sum_{i\in M} \lambda_i$ for any $M \in 2^N_-$.

Now, consider an arbitrary coalition $M \in 2^N_-$. We assume that the players in this coalition will collaborate by setting up a single stock point from which the combined demand streams of the coalition members are fulfilled first-come-first-serve. Since the superposition of independent Poisson processes is also a Poisson process, this single stock point will face a Poisson arrival process with merged rate $\lambda_M$. We assume throughout that players are interested in reducing their long-term average holding and backorder costs, and that other, smaller effects of setting up the pool are insignificant in comparison. For instance, if the tram operators of Amsterdam, Rotterdam, and The Hague decide to pool their spare parts in one central stock point, the lead time from this stock point to the trams will increase slightly, leading to somewhat higher transportation costs (which are usually small relative to holding and backorder costs anyway), but on the other hand operating a single central warehouse will be less costly than operating three separate warehouses.

The base stock level of the joint stock point is a decision variable. For any particular choice of the base stock level $S \in \mathbb{N}_0$, the behavior of the stock point would correspond to the model described in Section 3, and thus the expected relevant costs per time unit faced by coalition $M$ would be equal to $K(S, \lambda_M)$, with $K$ as defined in Equation (5). Naturally, there is a trade-off between holding and backorder costs. As we show later in Lemma 6.3, an optimal base stock level exists.

Assuming that any coalition picks a cost-minimizing base stock level, the game $(N, c^\varphi)$ corresponding to our spare parts situation with backordering $\varphi$ is defined by

$$c^\varphi(M) = \min_{S\in\mathbb{N}_0} K(S, \lambda_M) \tag{6}$$

for all $M \in 2^N_-$. We call this game the associated *spare parts pooling game.*

As mentioned, we are interested in methods to fairly distribute the collective expected costs of the grand coalition over the cooperating players in any spare parts situation with backordering $\varphi = (N,\, (\lambda_i)_{i\in N},\, h,\, b)$. A simple rule would be to divide these costs proportional to the demand rate of each player. Formally, we define this rule $\mathscr{P}$ by $\mathscr{P}_i(\varphi) = c^\varphi(N) \cdot \lambda_i/\lambda_N$ for each $i \in N$ in situation $\varphi$. Extending this idea to every coalition, we define the proportional allocation scheme rule $\mathcal{P}$ by

$$\mathcal{P}_{i,M}(\varphi) = c^\varphi(M) \cdot \lambda_i/\lambda_M \tag{7}$$

for each $M \in 2^N_-$ and $i \in M$ in situation $\varphi$. The following example illustrates the resulting allocations numerically and simultaneously shows that spare parts pooling games need not be concave, as the Shapley value is not necessarily in the core.

**Example 5.1.** Consider the spare parts situation with backordering $\varphi = (N, (\lambda_i)_{i \in N}, h, b)$ with player set $N = \{1, 2, 3\}$, demand rates $\lambda_1 = 0.1$, $\lambda_2 = 0.8005$, $\lambda_3 = \ln 2 (\approx 0.6931)$, and cost parameters $h = b = 1$. To illustrate the determination of an optimal base stock level and associated costs, consider the singleton coalition $\{3\}$. By Equation (1), $\mathbb{P}[X(\lambda_3) = 0] = 0.5$. Combining this with Equations (3), (4), and (5), we obtain for the case were player 3 would decide to stock zero parts that $\mathbb{E}B(0, \lambda_3) = \ln 2$, $\mathbb{E}I(0, \lambda_3) = 0$, and $K(0, \lambda_3) = \ln 2$. If player 3 would decide to use a base stock level of one instead, then $\mathbb{E}B(1, \lambda_3) = \ln 2 - 1 + 0.5$, $\mathbb{E}I(1, \lambda_3) = 0.5$, and $K(1, \lambda_3) = \ln 2$. As $K(0, \lambda_3) = K(1, \lambda_3)$ and this cost function is strictly convex in the base stock level (which we shall prove later in Lemma 6.3), minimal costs are achieved with a base stock level of either 0 or 1, and $c^\varphi(\{3\}) = \ln 2$. In the remainder of this example, we will round values to four decimals for notational convenience.

If player 1 would join to form coalition $\{1, 3\}$, then it can be verified that a base stock level of one for their combined stock point is optimal; thus, $c^\varphi(\{1, 3\}) = K(1, \lambda_1 + \lambda_3) \approx 0.6980$. Under the proportional allocation scheme rule $\mathcal{P}$, player 3 would have to pay $\mathcal{P}_{3,\{1,3\}}(\varphi) \approx 0.6100$ in coalition $\{1, 3\}$, which is lower than $\mathcal{P}_{3,\{3\}}(\varphi) \approx 0.6931$ (see Table 1). This strict population monotonicity can be verified for the members of all other nested pairs of coalitions as well, implying that $\mathcal{P}(\varphi)$ is strictly population monotonic. Accordingly, the spare parts pooling game $(N, c^\varphi)$ has a non-empty strict core containing $\mathscr{P}(\varphi)$.

| Coalition $M$ | Optimal base stock levels | $c^\varphi(M)$ | $\mathcal{P}_{1,M}(\varphi)$ | $\mathcal{P}_{2,M}(\varphi)$ | $\mathcal{P}_{3,M}(\varphi)$ |
|---|---|---|---|---|---|
| $\{1\}$ | 0 | 0.1000 | 0.1000 | * | * |
| $\{2\}$ | 1 | 0.6987 | * | 0.6987 | * |
| $\{3\}$ | 0 and 1 | 0.6931 | * | * | 0.6931 |
| $\{1, 2\}$ | 1 | 0.7132 | 0.0792 | 0.6340 | * |
| $\{1, 3\}$ | 1 | 0.6980 | 0.0880 | * | 0.6100 |
| $\{2, 3\}$ | 1 | 0.9428 | * | 0.5053 | 0.4375 |
| $N$ | 1 | 1.0000 | 0.0628 | 0.5023 | 0.4349 |

Table 1: *The spare parts pooling game and proportional allocation scheme of Example 5.1.*

However, the game's Shapley value $\Phi(N, c^\varphi)$, which assigns $\Phi_1(N, c^\varphi) \approx 0.0556$, $\Phi_2(N, c^\varphi) \approx 0.4774$, and $\Phi_3(N, c^\varphi) \approx 0.4670$, is not in the core of this game because $\Phi_2(N, c^\varphi) +$

$\Phi_3(N, c^{\varphi}) > c^{\varphi}(\{2,3\})$. Accordingly, this game is not concave; indeed, $c^{\varphi}(\{1,3\}) - c^{\psi}(\{3\}) < c^{\phi}(\{1,2,3\}) - c^{\phi}(\{2,3\})$. In other words, player 1's marginal cost contribution may increase if he joins a larger coalition. $\diamondsuit$

In this example, cost allocation could be carried out in a stable and population monotonic way via the proportional rules. To show that this is not a coincidence, we will exploit various new analytical properties of our inventory model's cost function, which are derived in the next section.

# 6 Analysis of the underlying inventory model

In this section, we first provide a characterization of the optimal base stock levels and subsequently derive partial derivatives of the cost function $K$ with respect to the demand rate. These intermediate results ultimately enable us to analyze how the cost performance under optimal base stock levels behaves as the demand rate varies on $\mathbb{R}_{++} = (0, \infty)$. The holding and backorder cost rates, $h$ and $b$, will remain fixed in the ensuing analysis.

We show in Lemma 6.3 that the optimal base stock levels are intricately related to the steady-state probability of having no backorders, $\mathbb{P}[B(S, \lambda) = 0]$. We therefore start by stating several properties of $\mathbb{P}[B(S, \lambda) = 0]$ in Lemmas 6.1 and 6.2; their proofs are straightforward and are given in the appendix.

**Lemma 6.1.** *Let the demand rate $\lambda > 0$ be fixed.*

*(i)* $\mathbb{P}[B(S, \lambda) = 0]$ *is strictly increasing as a function of $S$ (for $S$ on $\mathbb{N}_0$).*

*(ii)* $\lim_{S\to\infty} \mathbb{P}[B(S, \lambda) = 0] = 1$.

**Lemma 6.2.** *Let the base stock level $S \in \mathbb{N}_0$ be fixed.*

*(i)* $\mathbb{P}[B(S, \lambda) = 0]$ *is continuous and differentiable as a function of $\lambda$ (for $\lambda$ on $\mathbb{R}_{++}$).*

*(ii)* $\mathbb{P}[B(S, \lambda) = 0]$ *is strictly decreasing as a function of $\lambda$ (for $\lambda$ on $\mathbb{R}_{++}$).*

*(iii)* $\lim_{\lambda\downarrow 0} \mathbb{P}[B(S, \lambda) = 0] = 1$ *and* $\lim_{\lambda\to\infty} \mathbb{P}[B(S, \lambda) = 0] = 0$.

The following lemma states that the cost function in our model is convex in the base stock level and provides a standard characterization of the cost-minimizing base stock level(s) in terms of a newsvendor fractile. This characterization is illustrated in Figure 1. Such results are relatively well-known for the inventory model under consideration (see, e.g., Zipkin, 2000, p. 215) but we provide a proof in the appendix for completeness and to address the uniqueness and multiplicity of optimal base stock levels more formally, which will facilitate our analysis.

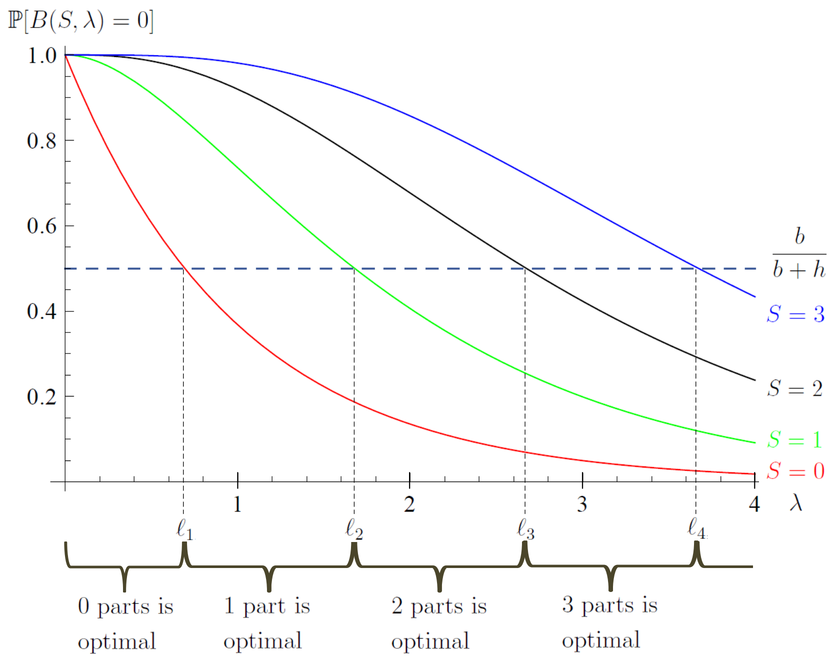


Figure 1: *The probability of having no backorders,* $\mathbb{P}[B(S,\lambda)=0]$*, as a function of the demand rate* $\lambda$ *for various base stock levels* $S$*.* ($h=b=1$.)

**Lemma 6.3.** *Let the demand rate* $\lambda > 0$ *be fixed.*

(*i*) $K(S,\lambda)$ *is strictly convex as a function of* $S$ *(for* $S$ *on* $\mathbb{N}_0$*).*

(*ii*) *There is at least one* $S \in \mathbb{N}_0$ *for which* $\mathbb{P}[B(S,\lambda)=0] \geq b/(b+h)$*; let* $S^*$ *denote the smallest such* $S$*. Then,* $S^*$ *is the unique optimal base stock level unless* $\mathbb{P}[B(S^*,\lambda)=0] = b/(b+h)$*; in that case, both* $S^*$ *and* $S^*+1$ *(and no other) are optimal.*

We now introduce some additional notation: for any $n \in \mathbb{N}$ (note that $0 \notin \mathbb{N}$), we define $\ell_n$ to be the unique positive real number that satisfies

$$\mathbb{P}[B(n-1,\ell_n)=0] = b/(b+h),$$

which is well-defined due to Lemma 6.2. Notice that, by Part (*ii*) of Lemma 6.3, for any $n \in \mathbb{N}$ it holds that $\ell_n$ is the demand rate for which both base stock levels $n-1$ and $n$ are optimal. The following lemma, proven in the appendix, formally establishes several additional properties, which are illustrated in Figure 1.

**Lemma 6.4.** *Let* $n \in \mathbb{N}$*. Then, the following statements hold.*

(*i*) $\ell_n < \ell_{n+1}$.

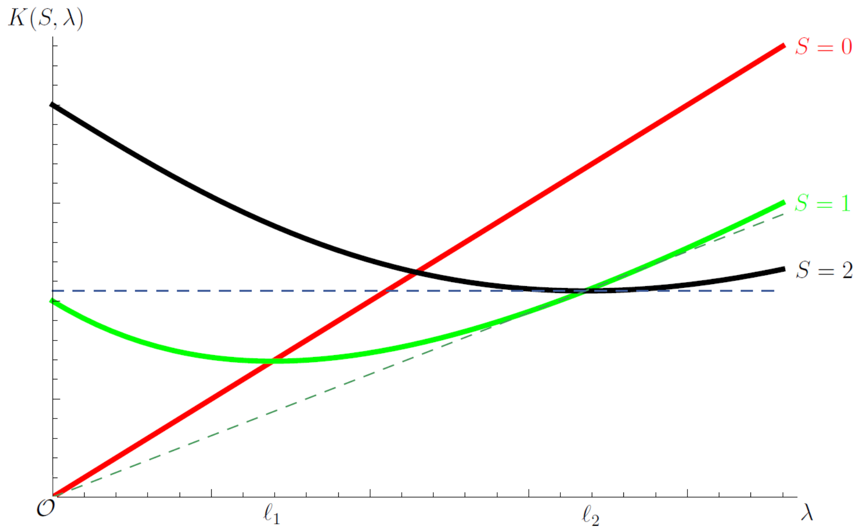


Figure 2: *The costs, $K(S, \lambda)$, as a function of the demand rate $\lambda$ for various base stock levels $S$. Also shown (dashed) are the tangent lines to $K(1, \lambda)$ and $K(2, \lambda)$ at $\lambda = \ell_2$.*

(*ii*) *For any $\lambda$ in the interval $(\ell_n, \ell_{n+1})$, the unique optimal base stock level for the inventory system with demand rate $\lambda$ is $n$.*

(*iii*) $\lim_{n\to\infty} \ell_n = \infty$.

The following lemma considers, for a fixed base stock level greater than zero, the expected steady-state costs per unit time as a function of the demand rate: the lemma states a simple expression for its derivative and shows that this cost function is strictly convex. This convexity is illustrated in Figure 2.

**Lemma 6.5.** *Let the base stock level $S \in \mathbb{N}$ be fixed.*

(*i*) $\dfrac{\partial}{\partial\lambda} K(S, \lambda) = b - (b + h) \cdot \mathbb{P}[B(S - 1, \lambda) = 0]$.

(*ii*) *$K(S, \lambda)$ is twice differentiable and strictly convex as a function of $\lambda$ (for $\lambda$ on $\mathbb{R}_{++}$).*

*Proof. Part* (*i*). First, observe that

$$K(S, \lambda) = b\lambda - bS + (b + h) \cdot \left[ Se^{-\lambda} + \sum_{x=1}^{S-1} (S - x)e^{-\lambda} \frac{\lambda^x}{x!} \right].$$

Taking the partial derivative with respect to the demand rate,

$$\begin{aligned}
\frac{\partial}{\partial\lambda}K(S,\lambda) &= b + (b+h)\cdot e^{-\lambda}\left[-S + \sum_{x=1}^{S-1}(S-x)\frac{\lambda^{x-1}}{(x-1)!} - \sum_{x=1}^{S-1}(S-x)\frac{\lambda^x}{x!}\right] \\
&= b + (b+h)\cdot e^{-\lambda}\left[-S + \sum_{x=0}^{S-2}(S-x-1)\frac{\lambda^x}{x!} - \sum_{x=0}^{S-2}(S-x)\frac{\lambda^x}{x!} - \frac{\lambda^{S-1}}{(S-1)!} + S\right] \\
&= b + (b+h)\cdot e^{-\lambda}\left[-\sum_{x=0}^{S-2}\frac{\lambda^x}{x!} - \frac{\lambda^{S-1}}{(S-1)!}\right] \\
&= b - (b+h)\cdot\sum_{x=0}^{S-1}\mathbb{P}[X(\lambda)=x] \\
&= b - (b+h)\cdot\mathbb{P}[B(S-1,\lambda)=0]. \qquad (8)
\end{aligned}$$

*Part (ii).* This follows upon combining Equation (8) with the facts that $\mathbb{P}[B(S-1,\lambda)=0]$ is differentiable and strictly decreasing in $\lambda$ (cf. Parts $(i)$ and $(ii)$ of Lemma 6.2). □

The following lemma provides insightful expressions for the partial derivatives of the cost function with respect to the demand rate, evaluated at any $\ell_n$ (the demand rate at which both base stock levels $n$ and $n-1$ are optimal). In particular, when this cost function is considered as a function of $\lambda$, the tangent line to $K(n,\lambda)$ at $\lambda=\ell_n$ is flat and the tangent line to $K(n-1,\lambda)$ at $\lambda=\ell_n$ goes through the origin, as illustrated in Figure 2.

**Lemma 6.6.** *Let $n\in\mathbb{N}$ be an arbitrary positive integer.*

$(i)$ $\frac{\partial}{\partial\lambda}K(n,\lambda)\big|_{\lambda=\ell_n} = 0.$

$(ii)$ $\frac{\partial}{\partial\lambda}K(n-1,\lambda)\big|_{\lambda=\ell_n} = \frac{K(n-1,\ell_n)}{\ell_n}.$

*Proof.* Part $(i)$. By Lemma 6.5, Part $(i)$, and by definition of $\ell_n$,

$$\frac{\partial}{\partial\lambda}K(n,\lambda)\big|_{\lambda=\ell_n} = b-(b+h)\cdot\mathbb{P}[B(n-1,\ell_n)=0] = b-(b+h)\cdot\frac{b}{b+h} = 0. \qquad (9)$$

Part $(ii)$. We distinguish between two cases. First, if $n=1$, then clearly $K(n-1,\lambda)=b\lambda$, and the result follows trivially. Second, if $n>1$, then we define $\Delta K(\lambda)=K(n,\lambda)-K(n-$

$1, \lambda)$ for all $\lambda > 0$. Differentiating this function and evaluating it at $\ell_n$, we obtain

$$\begin{aligned}
\frac{d}{d\lambda}\Delta K(\lambda)\big|_{\lambda=\ell_n} &= \frac{\partial}{\partial\lambda}K(n,\lambda)\big|_{\lambda=\ell_n} - \frac{\partial}{\partial\lambda}K(n-1,\lambda)\big|_{\lambda=\ell_n} \\
&= \Big(b - (b+h)\cdot\mathbb{P}[B(n-1,\ell_n)=0]\Big) - \Big(b - (b+h)\cdot\mathbb{P}[B(n-2,\ell_n)=0]\Big) \\
&= -(b+h)\cdot\left(\sum_{x=0}^{n-1}\mathbb{P}[X(\ell_n)=x] - \sum_{x=0}^{n-2}\mathbb{P}[X(\ell_n)=x]\right) \\
&= -(b+h)\mathbb{P}[X(\ell_n)=n-1], \qquad (10)
\end{aligned}$$

where the second equality follows from Part $(i)$ of Lemma 6.5. Combining Equations (9) and (10), we obtain

$$\begin{aligned}
\frac{\partial}{\partial\lambda}K(n-1,\lambda)\big|_{\lambda=\ell_n} &= \frac{\partial}{\partial\lambda}K(n,\lambda)\big|_{\lambda=\ell_n} - \frac{d}{d\lambda}\Delta K(\lambda)\big|_{\lambda=\ell_n} \\
&= (b+h)\mathbb{P}[X(\ell_n)=n-1]. \qquad (11)
\end{aligned}$$

We will now rewrite Expression (11), multiplied by $\ell_n$. For this, it is convenient to denote $S^* = n-1$ and $Z = (b+h)\left(\sum_{x=0}^{S^*}(S^*-x)\mathbb{P}[X(\ell_n)=x]\right)$. Then, we obtain

$$\begin{aligned}
\ell_n(b+h)\mathbb{P}[X(\ell_n)=S^*] &= (b+h)\left(\ell_n\cdot\mathbb{P}[X(\ell_n)=S^*] - \sum_{x=0}^{S^*}(S^*-x)\mathbb{P}[X(\ell_n)=x]\right) + Z \\
&= (b+h)e^{-\ell_n}\left(\ell_n\cdot\frac{(\ell_n)^{S^*}}{S^*!} - S^* - \sum_{x=1}^{S^*}(S^*-x)\frac{(\ell_n)^x}{x!}\right) + Z \\
&= (b+h)e^{-\ell_n}\left(\frac{(\ell_n)^{S^*+1}}{S^*!} + \sum_{x=1}^{S^*}\frac{(\ell_n)^x\cdot x}{x!} - S^* - \sum_{x=1}^{S^*}S^*\frac{(\ell_n)^x}{x!}\right) + Z \\
&= (b+h)e^{-\ell_n}\left(\sum_{x=1}^{S^*+1}\frac{(\ell_n)^x}{(x-1)!} - \sum_{x=0}^{S^*}\frac{(\ell_n)^x}{x!}\cdot S^*\right) + Z \\
&= (b+h)e^{-\ell_n}\left(\sum_{x=0}^{S^*}\frac{(\ell_n)^x}{x!}\cdot\ell_n - \sum_{x=0}^{S^*}\frac{(\ell_n)^x}{x!}\cdot S^*\right) + Z \\
&= (b+h)\sum_{x=0}^{S^*}\mathbb{P}[X(\ell_n)=x](\ell_n - S^*) + Z \\
&= (b+h)\mathbb{P}[B(S^*,\ell_n)=0](\ell_n - S^*) + Z \\
&= b(\ell_n - S^*) + Z = K(S^*,\ell_n). \qquad (12)
\end{aligned}$$

The penultimate equality holds because $\mathbb{P}[B(S^*,\ell_n)=0] = b/(b+h)$ by Part $(ii)$ of Lemma

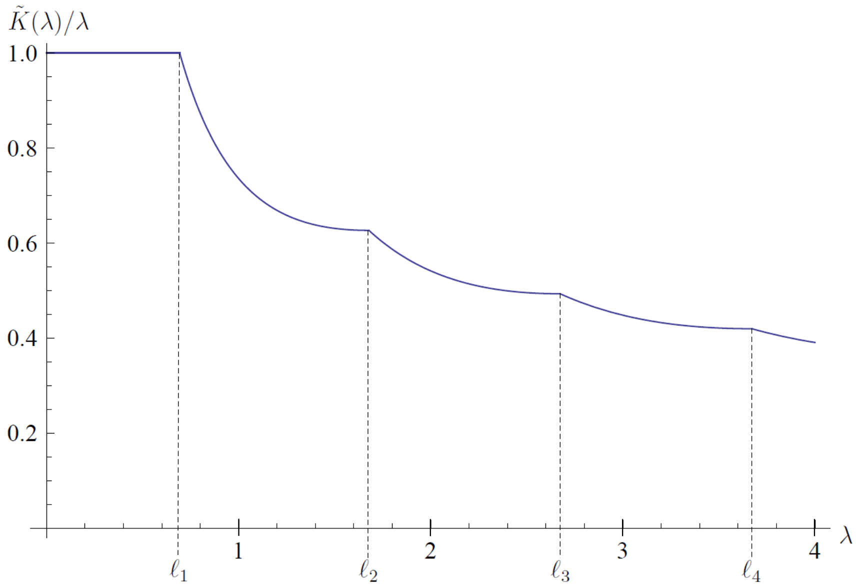


Figure 3: *The optimal per-demand costs, $\tilde{K}(\lambda)/\lambda$. ($h = b = 1$.)*

6.3. Equation (12) implies that

$$(b + h)\mathbb{P}[X(\ell_n) = n - 1] = \frac{K(n-1, \ell_n)}{\ell_n}. \tag{13}$$

Combining Equations (11) and (13) completes the proof. □

We finally consider how the cost of an inventory system with optimal base stock levels behaves as the demand rate varies. To this end, we define the optimal cost function $\tilde{K} : \mathbb{R}_{++} \to \mathbb{R}$ by

$$\tilde{K}(\lambda) = \min_{S \in \mathbb{N}_0} K(S, \lambda). \tag{14}$$

This function is well-defined due to Part (*ii*) of Lemma 6.3.

Now, we say that a function $f : \mathbb{R}_{++} \to \mathbb{R}$ is *elastic* if $f(x_1)/x_1 \geq f(x_2)/x_2$ for all $x_1, x_2 \in \mathbb{R}_{++}$ with $x_1 \leq x_2$. Intuitively, if $f(x)$ expresses the cost of, say, serving demand level $x$, then elasticity of $f$ says that the per-demand cost is non-increasing in the total demand served, i.e., $f$ exhibits economies of scale. The following theorem states that the optimal cost function in our spare parts inventory model, $\tilde{K}$, is elastic, as illustrated in Figure 3.

**Theorem 6.7.** *The function $\tilde{K}$ is elastic. In particular, $\tilde{K}(\lambda)/\lambda$ is constant for $\lambda$ on $(0, \ell_1]$ and strictly decreasing for $\lambda$ on $[\ell_1, \infty)$.*

*Proof.* First, for any $\lambda \in (0, \ell_1]$, a base stock level of zero is optimal, by the combination of Part (*ii*) of Lemma 6.2, Part (*ii*) of Lemma 6.3, and the definition of $\ell_1$. Hence, for $\lambda$ on $(0, \ell_1]$, $\tilde{K}(\lambda) = K(0, \lambda) = b\lambda$, and thus $\tilde{K}(\lambda)/\lambda$ is constant.

Next, let $n \in \mathbb{N}$. By Part (*ii*) of Lemma 6.4, for any $\lambda \in [\ell_n, \ell_{n+1})$, it holds that $\tilde{K}(\lambda) = K(n, \lambda)$. We now fix $\tilde{\lambda} \in [\ell_n, \ell_{n+1})$ arbitrarily. Note that, by Part (*i*) of Lemma 6.4, $\tilde{\lambda} < \ell_{n+1}$. Using this, we find that

$$\begin{aligned} K(n, \tilde{\lambda}) &> K(n, \ell_{n+1}) + (\tilde{\lambda} - \ell_{n+1}) \cdot \frac{\partial}{\partial \lambda} K(n, \lambda)\big|_{\lambda = \ell_{n+1}} \\ &= K(n, \ell_{n+1}) + (\tilde{\lambda} - \ell_{n+1}) \cdot \frac{K(n, \ell_{n+1})}{\ell_{n+1}} \\ &= K(n, \ell_{n+1}) \cdot \frac{\tilde{\lambda}}{\ell_{n+1}}. \end{aligned} \tag{15}$$

The inequality holds because any strictly convex, twice differentiable function — properties satisfied by $K(n, \lambda)$ as a function of $\lambda$ for $\lambda$ on $\mathbb{R}_{++}$, cf. Part (*ii*) of Lemma 6.5 — lies strictly above any of its tangent lines (except, of course, at the point where the tangent line touches the function's curve, but that does not pose a problem since $\tilde{\lambda} < \ell_{n+1}$). The first equality holds by Part (*ii*) of Lemma 6.6.

Using this, we obtain

$$\begin{aligned} \frac{\partial}{\partial \lambda} \left( \frac{K(n, \lambda)}{\lambda} \right)\bigg|_{\lambda = \tilde{\lambda}} &= \left( \tilde{\lambda} \cdot \frac{\partial}{\partial \lambda} K(n, \lambda)\big|_{\lambda = \tilde{\lambda}} - K(n, \tilde{\lambda}) \right) / \tilde{\lambda}^2 \\ &< \left( \tilde{\lambda} \cdot \frac{\partial}{\partial \lambda} K(n, \lambda)\big|_{\lambda = \ell_{n+1}} - K(n, \tilde{\lambda}) \right) / \tilde{\lambda}^2 \\ &= \left( \tilde{\lambda} \cdot \frac{K(n, \ell_{n+1})}{\ell_{n+1}} - K(n, \tilde{\lambda}) \right) / \tilde{\lambda}^2 \\ &< \left( \tilde{\lambda} \cdot \frac{K(n, \tilde{\lambda})}{\tilde{\lambda}} - K(n, \tilde{\lambda}) \right) / \tilde{\lambda}^2 = 0. \end{aligned}$$

The first inequality holds by Part (*ii*) of Lemma 6.5. The subsequent equality holds by Part (*ii*) of Lemma 6.6. The second inequality holds by Inequality (15). We conclude that $\tilde{K}(\lambda)/\lambda$ is strictly decreasing for $\lambda$ on $[\ell_n, \ell_{n+1})$.

As, by Part (*ii*) of Lemma 6.3, both $n$ and $n+1$ are optimal base stock levels for demand rate $\ell_{n+1}$, it holds that $\tilde{K}(\ell_{n+1}) = K(n, \ell_{n+1}) = K(n+1, \ell_{n+1})$. Furthermore, it follows from Part (*ii*) of Lemma 6.5 that both $K(n, \lambda)$ and $K(n+1, \lambda)$ as functions of $\lambda$ are continuous at $\lambda = \ell_{n+1}$. We conclude that $\tilde{K}$ is continuous at $\ell_{n+1}$.

To summarize, we have established that $\tilde{K}(\lambda)/\lambda$ is non-increasing for $\lambda$ on $(0, \ell_1]$ and that, for arbitrary positive integer $n$, this function is strictly decreasing on $[\ell_n, \ell_{n+1})$ and continuous at $\ell_{n+1}$. Now, since by Part $(iii)$ of Lemma 6.4 it holds that $\bigcup_{n\in\mathbb{N}}[\ell_n, \ell_{n+1}) = [\ell_1, \infty)$, this implies that $\tilde{K}(\lambda)/\lambda$ is strictly decreasing for $\lambda$ on $[\ell_1, \infty)$. Elasticity of $\tilde{K}$ follows, and the proof is complete. □

# 7 Fair allocations of expected costs

We first give our main results for spare parts pooling games in Section 7.1. Subsequently, we study who reaps the benefits of cooperation in Section 7.2. Finally, we discuss connections with so-called single-attribute games and newsvendor games in Section 7.3.

## 7.1 Stability and population monotonicity

Using Theorem 6.7, we can can now show that spare parts pooling games have a non-empty core and that the proportional allocation scheme rule $\mathcal{P}$ (see Equation 7) accomplishes a PMAS. As stated in the following theorem, the population monotonicity is strict if demand rates are sufficiently high for any coalition with two or more players to have an optimal base stock level greater than zero. To formally state this and later results, we let

$$S^*(\lambda) = \min\{S \in \mathbb{N}_0 : K(S, \lambda) = \tilde{K}(\lambda)\}$$

denote the (smallest) optimal base stock level for any demand rate $\lambda > 0$.

**Theorem 7.1.** *Let $\varphi = (N, (\lambda_i)_{i\in N}, h, b)$ be a spare parts situation with backordering.*

- $(i)$ *The associated spare parts pooling game $(N, c^\varphi)$ has a non-empty core containing $\mathscr{P}(\varphi)$, and $\mathcal{P}(\varphi)$ is a PMAS.*
- $(ii)$ *If $S^*(\lambda_L) > 0$ for each $L \in 2^N_-$ with $|L| \geq 2$, then $\mathcal{P}(\varphi)$ is an SPMAS and $(N, c^\varphi)$ has a non-empty strict core containing $\mathscr{P}(\varphi)$.*

*Proof.* Part $(i)$. We use a straightforward implication of $\tilde{K}$'s elasticity, thereby specializing a known implication (see, e.g., Hamlen et al., 1977, p. 621, or Özen et al., 2011, Theorem 1) for general cost sharing problems to our spare parts pooling games. Let $M, L \in 2^N_-$ with $M \subset L$, and let $i \in M$. Then $\mathcal{P}(\varphi)$ is a PMAS because, by Theorem 6.7,

$$\mathcal{P}_{i,L}(\varphi) = c^\varphi(L)\frac{\lambda_i}{\lambda_L} = \tilde{K}(\lambda_L)\frac{\lambda_i}{\lambda_L} \leq \tilde{K}(\lambda_M)\frac{\lambda_i}{\lambda_M} = c^\varphi(M)\frac{\lambda_i}{\lambda_M} = \mathcal{P}_{i,M}(\varphi). \tag{16}$$

Core inclusion of $\mathscr{P}(\varphi)$ follows from the closing sentence of Section 4.

Part $(ii)$. For arbitrary $M, L \in 2^N_-$ with $M \subset L$, assume that $S^*(\lambda_L) > 0$. This implies that $\lambda_L$, the collective demand rate of coalition $L$, is strictly larger than $\ell_1$, the demand rate for which both base stock levels 0 and 1 are optimal. Therefore, the inequality in (16) is strict by Theorem 6.7. Accordingly, $\mathcal{P}(\varphi)$ is an SPMAS, and $\mathscr{P}(\varphi)$ is strictly stable. $\square$

Theorem 7.1 states an important result, because a proportional allocation rule is easy to understand and computationally attractive. Moreover, it satisfies the appealing property of immunity to manipulations of the players via artificial splitting and merging. This means that, if collective costs are divided proportionally according to the rule $\mathscr{P}$, no group of players will have an incentive to artificially represent themselves together as a single player, or vice versa. Indeed, their total cost allocation will remain the same because splitting or merging does not affect their sum of demand rates. So in a collaboration between, e.g., company A with a single business unit and company B with two business units, the costs assigned to company A by rule $\mathscr{P}$ are unaffected by whether the business units comprising company B claim they should be treated as one player together or two players separately. Finally, in line with a result in Karsten et al. (2011a) for games in Erlang loss queues, $\mathscr{P}$ can be axiomatically characterized as the *unique* continuous efficient rule satisfying this non-manipulability property, which strengthens its fairness as a allocation rule.

## 7.2 Who reaps the benefits?

One might wonder who actually reaps most of the benefits of the collaboration. With benefits we mean the difference between the costs incurred by a player when acting alone and the cost assigned to this player under rule $\mathscr{P}$. Thus, the benefits for a player with demand rate $\lambda > 0$ when participating in a spare parts pool with aggregate demand rate $\Lambda \geq \lambda$ are given by

$$\mathcal{B}(\lambda, \Lambda) = \tilde{K}(\lambda) - \lambda \cdot \tilde{K}(\Lambda)/\Lambda.$$

Now, will the smallest player (i.e., the player with the lowest demand rate) always benefit most, or will the largest player always take the lion's share? The following example shows that it could actually be neither of them. Thus, the proportional rule appealingly does not categorically favor smaller or larger players.

**Example 7.1.** Reconsider the spare parts situation with backordering $\varphi$ as described in Example 5.1. For this situation, the players' benefits (rounded to four decimals) are $\mathcal{B}_1 = \mathcal{B}(\lambda_1, \lambda_N) = 0.0372$, $\mathcal{B}_2 = \mathcal{B}(\lambda_2, \lambda_N) = 0.1964$, $\mathcal{B}_3 = \mathcal{B}(\lambda_3, \lambda_N) = 0.2582$. Since $\lambda_3 < \lambda_2$ and $\mathcal{B}_3 > \mathcal{B}_2$, a small player might reap more benefits than a large player. Yet, since $\lambda_3 > \lambda_1$ and $\mathcal{B}_3 > \mathcal{B}_1$, a large player might reap more benefits than a small player as well. This is represented graphically in Figure 4. $\lozenge$

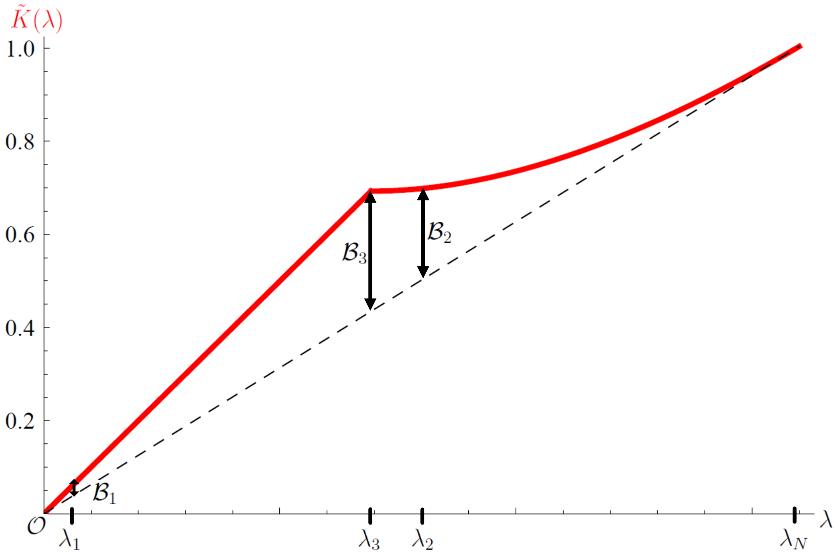


Figure 4: *The optimal costs $\tilde{K}(\lambda)$, a dotted line through the origin with slope $\tilde{K}(\lambda_N)/\lambda_N$, and the benefits for the players in Example 7.1.*

This example suggests that the largest benefits are typically reaped by a player with demand rate equal to $\ell_n$ for some $n \in \mathbb{N}$, i.e., a demand rate for which two base stock levels are optimal. In the following theorem, we show that this holds in general, provided that it is not optimal for the grand coalition to stock zero parts (in which case there would clearly be no pooling benefits at all). The proof is deferred to the appendix.

**Theorem 7.2.** *Consider a spare parts pool with total demand rate $\Lambda > \ell_1$. Then, there exists an $n \in \{1, \dots, S^*(\Lambda)\}$ such that $\mathcal{B}(\ell_n, \Lambda) > \mathcal{B}(\lambda, \Lambda)$ for all $\lambda \in (0, \Lambda]$ with $\lambda \neq \ell_n$ for some $n \in \mathbb{N}$.*

From this theorem, we immediately obtain the following corollary, which concerns situations where the grand coalition optimally stocks a single part. This is quite common for low-demand, expensive spare parts.

**Corollary 7.3.** *Let $\varphi = (N, (\lambda_i)_{i \in N}, h, b)$ be a spare parts situation such that $S^*(\lambda_N) = 1$ and $\lambda_i = \ell_1$ for some $i \in N$. Then, $\mathcal{B}(\lambda_i, \lambda_N) > \mathcal{B}(\lambda_j, \lambda_N)$ for all $j \in N$ with $\lambda_j \neq \lambda_i$.*

## 7.3 An alternative proof approach

The fact that spare parts pooling games have a non-empty core with a proportional PMAS (Part (*i*) of Theorem 7.1) can be proven in a different way, via a connection with so-called single-attribute games and newsvendor games, together with the contraposition of a recent result in Özen et al. (2011). This alternative proof approach, which is detailed in the appendix, employs a result that applies more generally: a certain class of newsvendor games admits a PMAS in which costs are assigned proportional to player's mean demands. However, it does not provide insights into the structure of the problem as our analysis in Section 6 did. Moreover, our structural analysis in Section 6 allowed us to identify a *strictly* stable allocation (Part (*ii*) of Theorem 7.1) and the player who benefits most (Section 7.2); these additional results do not follow from the alternative proof approach.

# 8 An allocation process for cost realizations

The previous section showed that the proportional allocation rule satisfies the first three requirements posed in the introduction (Section 1). In this section, we treat the remaining requirement. First, we show that the proportional allocation rule can be easily implemented in practice via a simple cost division per realization. Although our games have been formulated in expected terms to investigate a priori attractiveness of pooling, fair assignments of *realized* costs in any finite time period will be required to sustain cooperation in practice. To propose a method for assigning realized costs, we make the natural assumption of a First-In-First-Out (FIFO) stock discipline: whenever more than one part is available in the on-hand stock when a demand is placed, the demand is fulfilled by the oldest part in the on-hand stock. We now propose the following method to allocate costs as they materialize in an inventory system with any base stock level $S \in \mathbb{N}_0$ operated by the grand coalition in any spare parts situation with backordering $(N, (\lambda_i)_{i \in N}, h, b)$.

**Process 8.1.** Realized costs for the grand coalition are assigned as follows:

- Each player, upon placing a demand when the on-hand stock is positive, pays all holding costs incurred for the part taken (according to FIFO). That is, if the taken part was delivered at the stock point at a time $\tau$ and the player's demand occurs at time $t$, then this player pays $h(t - \tau)$ upon placing his demand.

- Each player, upon placing a demand that is backordered, pays all backorder costs incurred for this backorder. That is, if the demand occurs at time $t$ and the associated backorder is later fulfilled via delivery of a new part at time $\tau$, then this player will have to pay $b(\tau - t)$ over the duration of his backorder.

This process assigns the holding costs incurred for some part in an intuitively fair way to the player who directly benefits from the part. No holding costs are assigned to a player who does not benefit from on-hand inventory (as a result of not facing any demands over a certain time period or due to unfortunate stock-outs at his demand epochs). Additionally, since players fully incur all costs for their own backorders, the process eliminates the need for transfer payments of backorder costs, thereby avoiding disputes about their exact magnitude — this is an important property for the capital goods context wherein backorder costs typically comprise the downtime costs of a player's machine due to unavailability of a spare part. Moreover, as stated in the following lemma, the process can indeed be used to *implement* the proportional allocation.

**Lemma 8.1.** *Under Process 8.1, the share of long-term average costs (of an inventory system operated by the grand coalition with base stock level $S \in \mathbb{N}_0$) borne by player $i \in N$ is $\lambda_i/\lambda_N$. In particular, if the grand coalition optimally stocks $S^*(\lambda_N)$ parts, the long-term average costs assigned to each player under Process 8.1 coincide with the assignment of expected costs under the proportional allocation rule $\mathscr{P}$.*

*Proof.* Follows directly from the well-known property that Poisson arrivals see time averages (Wolff, 1982). □

A final appealing property of this process is that it removes any incentive for players to lie about their demand rates a priori. Although thus far we have adhered to the assumption of full and open information (a standard assumption in cooperative game theory), in reality a player's demand rate may be private information. During initial negotiations, when all cooperating players have to state their demand rate for the purpose of joint base stock level optimization, a player might lie if the collaboration would be implemented via an inappropriate cost realization assignment method. For example, under a method that charges each player a yearly fee based on his stated demand rates independent of that player's realized demand volume in that year, a player might have a reason to understate his actual demand rate a priori. However, under Process 8.1, truth telling is a Nash equilibrium (formally defined in Nash, 1951) in the associated noncooperative game in which each player in $N$ has to state any demand rate in $\mathbb{R}_{++}$. The payoff to each player in this game for any strategy profile $(\hat{\lambda}_i)_{i\in N}$, containing each player's stated demand rates, is equal to the long-term average costs assigned under Process 8.1 in the inventory system with base stock level $S^*(\sum_{i\in N} \hat{\lambda}_i)$.

**Theorem 8.2.** *The strategy profile $(\lambda_i)_{i\in N}$, in which each player $i \in N$ states his true demand rate $\lambda_i$, is a Nash equilibrium in this noncooperative game.*

*Proof.* Consider a player $i \in N$ and suppose that all other players $j \in N \setminus \{i\}$ announce their true demand rate $\lambda_j$. By lying, i.e., stating any demand rate $\mathcal{L} \in \mathbb{R}_{++}$ other than

$\lambda_i$, player $i$ can only effect a possibly suboptimal base stock level since $K(S^*(\lambda_N), \lambda_N) \leq K(S^*(\sum_{j \in N \setminus \{i\}} \lambda_j + \mathcal{L}), \lambda_N)$ by definition of $S^*$. Yet, the fraction of long-term average costs assigned to player $i$ under Process 8.1 is equal to $\lambda_i/\lambda_N$ by Lemma 8.1, i.e., is independent of the demand rate that he states. Thus, player $i$ minimizes his costs by stating $\lambda_i$. This completes the proof. □

In similar fashion, we can derive that Process 8.1 gives each player an incentive to immediately disclose any change in his expected demand rate, which is relevant for collaborations in a dynamic world where a player's number of installed machines may change over time or if forecasts improve.

# 9 Games with exogenously given base stock levels

In this section, we investigate the validity of one of our main results — nonemptiness of the strict core — if we drop the assumption that base stock levels can be adjusted. This assumption allowed each coalition to pick a cost-minimizing base stock level and may appear natural, but adjustment of the stock levels is not always possible in practice. For instance, players might already — before considering collaboration — possess a stock of repairable spare parts, which cannot be sold or produced anymore due to their specificity. In that case, players might set up a pooling group in which their entire stock of repairable parts is taken over in full, but the base stock level cannot be adjusted. Stock levels might also be unadjustable if contracts with a service provider or government quota specify a fixed base stock level. Furthermore, users of capital goods contemplating spare parts pooling might not want to enter a deep, long-term integration with jointly optimized stocks right from the start; rather, they might prefer to build trust by starting with a short-term trial in which their current inventories are pooled. In this section, we focus on such collaborations with unadjustable stocking levels.

To analyze such settings, we adjust the model described in Section 5 by additionally endowing each player in our game with an exogenously given, unadjustable number of (repairable) spare parts a priori. Formally, we define a spare parts situation with fixed base stock levels as a tuple $\gamma = (N, (\lambda_i)_{i \in N}, (S_i)_{i \in N}, h, b)$, where $N$, $\lambda$, $h$, and $b$ are as in Section 5 and $S_i \in \mathbb{N}_0$ is the amount of (repairable) spare parts that player $i \in N$ brings to any coalition. For each coalition $M \in 2^N_-$, we denote $\lambda_M = \sum_{i \in M} \lambda_i$ and $S_M = \sum_{i \in M} S_i$. Accordingly, the game $(N, c^\gamma)$ associated with this situation is defined by

$$c^\gamma(M) = K(S_M, \lambda_M) \tag{17}$$

for all $M \in 2^N_-$, with $K$ as defined in Equation (5).

With $\Gamma$ we denote the set of all such situations with at least two players, i.e., $|N| > 1$, and with at least one spare part amongst the players, i.e., $S_N > 0$. In the remainder, we will focus on such situations in order to avoid the trivial cases where no pooling benefits can be obtained. We call any game $(N, c^\gamma)$ induced by some $\gamma \in \Gamma$ a game with fixed base stock levels. Our main result for such games is that they always admit a strictly stable allocation.

**Theorem 9.1.** *Games with fixed base stock levels have a non-empty strict core.*

The formal proof of this theorem is lengthy and therefore relegated to the appendix. Here, we briefly describe the line of the proof. The first step is to consider the subclass of situations wherein players have identical demand rates. For this subclass, we analyze balanced combinations of cumulative pipeline stock probabilities. Via applications of the binomial theorem and the exponential function's Taylor expansion, we are able to show that (weighted) backlogs and on-hand stocks are lower in the grand coalition than in a balanced combination of pooling arrangements between smaller coalitions. This ultimately implies existence of a strict core allocation for this subclass. Next, we consider the subclass of situations wherein players have non-identical but rational-valued demand rates. For any such situation, we show how to split the players into sub-players with identical demand rates in such a way that a strict core element from the corresponding game with identical demand rates can be transformed into a strict core element for the original setting. This last result is finally generalized to real-valued demand rates via a continuity argument.

Theorem 9.1 immediately implies that pooling the inventory and demand streams of several spare parts inventory systems (with common lead time and cost structure) is beneficial. That is, such pooling leads to a cost reduction, consistent with intuition. Note that this does not follow from any of the preceding sections' results due to our current focus on exogenously given base stock levels. In the next theorem, we additionally show that such pooling leads to a reduction in both expected backorders and on-hand inventory. The intuition behind this is that pooling allows one player's backorder to cancel against another player's on-hand part.

**Theorem 9.2.** *Let $\lambda_1, \lambda_2 > 0$ and let $S_1, S_2 \in \mathbb{N}_0$ with $S_1 + S_2 > 0$. Then the following strict subadditivity properties hold:*

*(i)* $K(S_1, \lambda_1) + K(S_2, \lambda_2) > K(S_1 + S_2, \lambda_1 + \lambda_2)$, *assuming common positive holding costs $h$ and backorder costs $b$.*

*(ii)* $\mathbb{E}B(S_1, \lambda_1) + \mathbb{E}B(S_2, \lambda_2) > \mathbb{E}B(S_1 + S_2, \lambda_1 + \lambda_2)$.

*(iii)* $\mathbb{E}I(S_1, \lambda_1) + \mathbb{E}I(S_2, \lambda_2) > \mathbb{E}I(S_1 + S_2, \lambda_1 + \lambda_2)$.

# 10 Conclusion

We have studied the cost allocation problem in a spare parts inventory model with backordering. We have derived new structural properties of the resulting cost function, in particular concerning its behavior for varying demand rates, which may be relevant beyond the context of our games. Using these properties, we were able to show that pooling is not only beneficial from the whole system's point of view but also for each separate group of players. In other words, the associated cooperative games have non-empty cores. This result holds both if each coalition uses an optimal base stock level and if each player brings a fixed number of (repairable) spare parts to each coalition. For the former setting, we have further shown that cost allocation according to each player's demand rate leads to a core allocation, and that this cost allocation has appealing properties for implementation in practice. Indeed, it satisfies the four requirements posed in the introduction (Section 1).

For further research, it may be interesting to extend the model to two echelon levels: a one warehouse, multiple retailers setting. The spare parts at the central warehouse may be owned by a coalition of retailers, or by a third party. If this third party is the original equipment manufacturer, then it may also be interesting to allow this party to exert additional design effort to improve component reliability. This may be beneficial from the whole system's point of view, but it also raises the question of what share of the benefits the manufacturer is entitled to. Cooperative game theory may provide the tools to determine (existence of) fair allocations of collective costs.

# A Appendix

## A.1 Technical results for Section 4.

To characterize the class of games with a non-empty strict core, we first need to define balanced maps and balanced collections for any player set $N$ with two or more players. A map $\kappa : 2^N_{--} \to [0,1]$ is called *balanced for* $N$ if $\sum_{M \in 2^N_{--} : i \in M} \kappa(M) = 1$ for all $i \in N$, which may be interpreted as describing an artificial arrangement where each player $i \in N$ divides 1 unit (of, say, time/effort) over the non-empty subcoalitions that include $i$, with $\kappa(M)$ the fraction dedicated to coalition $M$. For each balanced map $\kappa$ for $N$, we define $\mathbb{B}(\kappa) = \{M \in 2^N_{--} \mid \kappa(M) > 0\}$. Any collection $\mathbb{B} \subseteq 2^N_{--}$ of coalitions is called *balanced for* $N$ if there is a balanced map $\kappa$ for $N$ such that $\mathbb{B} = \mathbb{B}(\kappa)$. As an example, for $N = \{1, 2, 3\}$, $\mathbb{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ is a balanced collection because the map $\kappa : 2^N_{--} \to [0, 1]$, described by $\kappa(M) = \frac{1}{2}$ if $|M| = 2$ and $\kappa(M) = 0$ otherwise, is a balanced map satisfying

$\mathbb{B}(\kappa) = \mathbb{B}$. A balanced collection $\mathbb{B}$ for $N$ is called *minimally balanced* if there does not exist another balanced collection for $N$ that is a proper subset of $\mathbb{B}$. Correspondingly, a balanced map $\kappa$ is called *minimally balanced for* $N$ if $\mathbb{B}(\kappa)$ is a minimally balanced collection for $N$. The set of minimally balanced maps for $N$ is denoted by $\mathcal{W}^N$. We can now state a characterization result in line with Bondareva (1963) and Shapley (1967).

**Proposition A.1.** *Let $(N, c)$ be a game with two or more players. $(N, c)$ has a non-empty strict core if and only if $c(N) < \sum_{M \in \mathbb{B}(\kappa)} \kappa(M)c(M)$ for each $\kappa \in \mathcal{W}^N$.*

*Proof.* ($\Leftarrow$). Suppose that $c(N) < \sum_{M \in \mathbb{B}(\kappa)} \kappa(M)c(M)$ for each $\kappa \in \mathcal{W}^N$. Then, let $\kappa^* \in \operatorname{argmin}_{\kappa \in \mathcal{W}^N} \sum_{M \in \mathbb{B}(\kappa)} \kappa(M)c(M)$. Such a $\kappa^*$ exists because the number of minimally balanced collections for $N$ is no larger than the number of subsets of $2^N_{--}$ and because each minimally balanced collection is associated with a unique minimally balanced map (Shapley, 1967). Let $\epsilon^* = \sum_{M \in \mathbb{B}(\kappa^*)} \kappa^*(M)c(M) - c(N)$; note that $\epsilon^* > 0$. We will aim to identify a strictly stable allocation for game $(N, c)$. To this end, we define the auxiliary game $(N, c^*)$ by $c^*(M) = c(M)$ for all proper subsets $M \subset N$, and $c^*(N) = c(N) + \epsilon^*$. Then, for any $\kappa \in \mathcal{W}^N$, we obtain

$$c^*(N) = c(N) + \epsilon^* = \sum_{M \in \mathbb{B}(\kappa)} \kappa^*(M)c(M) - \epsilon^* + \epsilon^* \leq \sum_{M \in \mathbb{B}(\kappa)} \kappa(M)c(M),$$

where the inequality holds by choice of $\kappa^*$ as a minimizer. Hence, by Bondareva (1963) and Shapley (1967), the game $(N, c^*)$ has a non-empty core as well, so the nucleolus $\nu(N, c^*)$ is in the core of $(N, c^*)$, cf. Schmeidler (1969).

Now, we define, for each $i \in N$, $x_i = \nu_i(N, c^*) - \epsilon^*/|N|$. Note that $x = (x_i)_{i \in N}$ is an efficient allocation for game $(N, c)$ since

$$\sum_{i \in N} x_i = \sum_{i \in N} \nu_i(N, c^*) - \epsilon^* = c^*(N) - \epsilon^* = c(N),$$

where the second equality holds because $\nu(N, c^*)$ is an efficient allocation for the game $(N, c^*)$. Moreover, $x$ is a strictly stable allocation for game $(N, c)$ since, for any $M \in 2^N_{--}$, it holds that

$$\sum_{i \in M} x_i < \sum_{i \in M} \nu_i(N, c^*) \leq c^*(M) = c(M),$$

where the first inequality holds because $\epsilon^* > 0$ and the second inequality holds because $\nu(N, c^*)$ is a stable allocation for the game $(N, c^*)$. We conclude that $x$ is a strictly stable allocation for game $(N, c)$. Thus, the game $(N, c)$ has a non-empty strict core.

($\Rightarrow$). Suppose that $(N, c)$ has a non-empty strict core. Consider any $x \in \mathscr{C}(N, c)$ and $\kappa \in \mathcal{W}^N$. Then, by definition of a balanced map,

$$c(N) = \sum_{i \in N} x_i = \sum_{i \in N} \sum_{M \in \mathbb{B}(\kappa): i \in M} \kappa(M) x_i = \sum_{M \in \mathbb{B}(\kappa)} \kappa(M) \sum_{i \in M} x_i < \sum_{M \in \mathbb{B}(\kappa)} \kappa(M) c(M),$$

which holds as $x$ is a strictly stable allocation for game $(N, c)$. This completes the proof. □

## A.2 Proofs for Section 6.

*Proof of Lemma 6.1.* Part $(i)$ follows immediately from Equation (2) as $\mathbb{P}[X(\lambda) = y] > 0$ for all $y \in \mathbb{N}_0$. Part $(ii)$ holds because $\lim_{S \to \infty} \mathbb{P}[B(S, \lambda) = 0] = \sum_{y=0}^{\infty} \mathbb{P}[X(\lambda) = y] = 1$ as the infinite sum covers the entire support of the Poisson distribution. □

*Proof of Lemma 6.2.* Part $(i)$ follows immediately from the fact that, for each $y \in \mathbb{N}_0$, $\mathbb{P}[X(\lambda) = y]$ is continuous and differentiable in $\lambda$. For Part $(ii)$, we distinguish between two cases. First, if $S = 0$, it holds that

$$\frac{\partial}{\partial \lambda} \mathbb{P}[B(S, \lambda) = 0] = -e^{-\lambda} < 0.$$

Second, if $S \in \mathbb{N}$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial \lambda} \mathbb{P}[B(S, \lambda) = 0] &= \frac{\partial}{\partial \lambda} \sum_{y=0}^{S} \frac{\lambda^y}{y!} e^{-\lambda} \\
&= -e^{-\lambda} + \sum_{y=1}^{S} \frac{\lambda^{y-1}}{(y-1)!} e^{-\lambda} - \sum_{y=1}^{S} \frac{\lambda^y}{y!} e^{-\lambda} \\
&= -e^{-\lambda} + \sum_{y=0}^{S-1} \frac{\lambda^y}{y!} e^{-\lambda} - \sum_{y=1}^{S} \frac{\lambda^y}{y!} e^{-\lambda} \\
&= \sum_{y=0}^{S-1} \left[ \left( \frac{\lambda^y}{y!} - \frac{\lambda^y}{y!} \right) e^{-\lambda} \right] - \frac{\lambda^S}{S!} e^{-\lambda} < 0.
\end{aligned}$$

We conclude that $\mathbb{P}[B(S, \lambda) = 0]$ is strictly decreasing in $\lambda$. Part $(iii)$ follows from

$$\lim_{\lambda \downarrow 0} \mathbb{P}[B(S, \lambda) = 0] = \sum_{y=0}^{S} \lim_{\lambda \downarrow 0} \mathbb{P}[X(\lambda) = y] = \lim_{\lambda \downarrow 0} \mathbb{P}[X(\lambda) = 0] + \sum_{y=1}^{S} 0 = 1,$$

since $\lambda^y e^{-\lambda} \to 0$ as $\lambda \downarrow 0$ for any $y \in \mathbb{N}$, and from

$$\lim_{\lambda \to \infty} \mathbb{P}[B(S, \lambda) = 0] = \sum_{y=0}^{S} \lim_{\lambda \to \infty} \mathbb{P}[X(\lambda) = y] = \sum_{y=0}^{S} 0 = 0,$$

since $\lambda^y e^{-\lambda} \to 0$ as $\lambda \to \infty$ for any $y \in \mathbb{N}_0$. This completes the proof. □

*Proof of Lemma 6.3. Part* (*i*). We start by defining $\Delta K(S) = K(S+1,\lambda) - K(S,\lambda)$ for all $S \in \mathbb{N}_0$. Consider Equations (3) and (4), and notice that an equivalent way of writing them is by letting their summations run to $S-1$ instead of $S$. Using this in combination with Equations (2) and (5), we obtain for all $S \in \mathbb{N}_0$ that

$$\begin{aligned}
\Delta K(S,\lambda) &= h \cdot [\mathbb{E}I(S+1,\lambda) - \mathbb{E}I(S,\lambda)] + b \cdot [\mathbb{E}B(S+1,\lambda) - \mathbb{E}B(S,\lambda)] \\
&= h \left[ \sum_{x=0}^{S} (S+1-x)\mathbb{P}[X(\lambda)=x] - \sum_{x=0}^{S}(S-x)\mathbb{P}[X(\lambda)=x] \right] \\
&\quad + b \left[ -(S+1) + \sum_{x=0}^{S}(S+1-x)\mathbb{P}[X(\lambda)=x] + S - \sum_{x=0}^{S}(S-x)\mathbb{P}[X(\lambda)=x] \right] \\
&= h \sum_{x=0}^{S} \mathbb{P}[X(\lambda)=x] + b \sum_{x=0}^{S} \mathbb{P}[X(\lambda)=x] - b \\
&= (b+h)\mathbb{P}[B(S,\lambda)=0] - b. \qquad (18)
\end{aligned}$$

From Equation (18), from Part (*i*) of Lemma 6.1, and from the fact that $b > 0$ and $h > 0$, we conclude that $\Delta K(S)$ is strictly increasing in $S$. Therefore, $K(S,\lambda)$ is strictly convex in $S$.

*Part* (*ii*). By Lemma 6.1, Part (*ii*), there always exists an $S \in \mathbb{N}_0$ for which $\mathbb{P}[B(S,\lambda) = 0] \geq b/(b+h)$. By the convexity result established in Part (*i*) above, it immediately follows from Equation (18) that the cost $K(S,\lambda)$ as a function of $S$ achieves a minimum at the smallest $S \in \mathbb{N}_0$ for which $(b+h)\mathbb{P}[B(S,\lambda)=0]-b \geq 0$, which corresponds to the definition of $S^*$ in the lemma.

We now distinguish two cases. First, if $\mathbb{P}[B(S^*,\lambda)=0] > b/(b+h)$, then $\Delta K(S^*,\lambda) > 0$, which implies that $S^*$ is the unique optimal base stock level. Second, if $\mathbb{P}[B(S^*,\lambda)=0] = b/(b+h)$, then $\Delta K(S^*,\lambda) = 0$, and thus $K(S^*,\lambda) = K(S^*+1,\lambda)$, which implies that both $S^*$ and $S^*+1$ are optimal base stock levels, and these are the only two optimal base stock levels due to the strict convexity established in Part (*i*). □

*Proof of Lemma 6.4. Part* (*i*). By Lemma 6.1, Part (*i*), and by definition of $\ell_n$, it holds that

$$\mathbb{P}[B(n,\ell_n)=0] > \mathbb{P}[B(n-1,\ell_n)=0] = b/(b+h). \qquad (19)$$

Because $\mathbb{P}[B(n,\lambda)=0]$ is strictly decreasing as a function of $\lambda$ (by Part (*ii*) of Lemma 6.2) while, by definition, $\mathbb{P}[B(n,\ell_{n+1})=0] = b/(b+h)$, Equation (19) implies that $\ell_n < \ell_{n+1}$.

*Part* (*ii*). Let $\lambda \in (\ell_n, \ell_{n+1})$. Thus, $\lambda < \ell_{n+1}$; this, in combination with Part (*ii*) of Lemma 6.2, implies that

$$\mathbb{P}[B(n,\lambda)=0] > \mathbb{P}[B(n,\ell_{n+1})=0] = b/(b+h).$$

Yet, for each $S \in \mathbb{N}_0$ with $S < n$, we observe, again by Part (*ii*) of Lemma 6.2, that

$$\mathbb{P}[B(S, \lambda) = 0] < \mathbb{P}[B(S, \ell_{S+1}) = 0] = b/(b + h),$$

where the inequality holds because $\lambda > \ell_n \geq \ell_{S+1}$. (To see that $\ell_n \geq \ell_{S+1}$, note that either $S + 1 = n$, in which case $\ell_n = \ell_{S+1}$, or $S + 1 < n$, in which case $\ell_n > \ell_{S+1}$ by Part (*i*) of this lemma.) Hence, by Part (*ii*) of Lemma 6.3, the unique optimal base stock level is $n$.

*Part* (*iii*). By the monotonicity result established in Part (*i*) above, there are two possibilities: $\lim_{n\to\infty} \ell_n = \infty$ or $\lim_{n\to\infty} \ell_n = A$ for some $A \in \mathbb{R}$. Suppose that the latter is true, aiming for a contradiction, and consider any $\lambda > A$. Then, by Lemma 6.1, there must exist an $S \in \mathbb{N}_0$ such that $\mathbb{P}[B(S, \lambda) = 0] > b/(b+h)$. Given this $S$, there must exist a $\Lambda > \lambda$ such that $\mathbb{P}[B(S, \Lambda) = 0] = b/(b+h)$ by virtue of Lemma 6.2. Then, by definition, $\ell_{S+1} = \Lambda$. But since $\ell_{S+1} = \Lambda > \lambda > A$, this yields a contradiction with the assumption that $\lim_{n\to\infty} \ell_n = A$. We conclude that $\lim_{n\to\infty} \ell_n = \infty$. □

## A.3 Proofs for Section 7.2.

In the process of proving Theorem 7.2, we will use Lemma A.2.

**Lemma A.2.** *Let a pooling group's total demand rate $\Lambda > \ell_1$ be fixed.*

(*i*) *For all $\lambda \in (0, \ell_1)$, $\mathcal{B}(\lambda, \Lambda) < \mathcal{B}(\ell_1, \Lambda)$.*

(*ii*) *If $S^*(\Lambda) \geq 2$, then for any $n \in \{1, \dots, S^*(\Lambda) - 1\}$ and $\lambda \in (\ell_n, \ell_{n+1})$, at least one of the following two inequalities hold: $\mathcal{B}(\lambda, \Lambda) < \mathcal{B}(\ell_{n+1}, \Lambda)$ and $\mathcal{B}(\lambda, \Lambda) < \mathcal{B}(\ell_n, \Lambda)$.*

(*iii*) *For all $\lambda \in (\ell_{S^*(\Lambda)}, \Lambda]$, $\mathcal{B}(\lambda, \Lambda) < \mathcal{B}(\ell_{S^*(\Lambda)}, \Lambda)$.*

*Proof.* Part (*i*). For any $\lambda \in (0, \ell_1)$, it holds that $S^*(\lambda) = S^*(\ell_1) = 0$, which implies that $\tilde{K}(\lambda)/\lambda = \tilde{K}(\ell_1)/\ell_1 = b$. Using this, we find for any $\lambda \in (0, \ell_1)$ that

$$\mathcal{B}(\lambda, \Lambda) = \lambda \left( \frac{\tilde{K}(\lambda)}{\lambda} - \frac{\tilde{K}(\Lambda)}{\Lambda} \right) = \lambda \left( \frac{\tilde{K}(\ell_1)}{\ell_1} - \frac{\tilde{K}(\Lambda)}{\Lambda} \right) < \ell_1 \left( \frac{\tilde{K}(\ell_1)}{\ell_1} - \frac{\tilde{K}(\Lambda)}{\Lambda} \right) = \mathcal{B}(\ell_1, \Lambda)$$

where the inequality holds because $\tilde{K}(\ell_1)/\ell_1 > \tilde{K}(\Lambda)/\Lambda$ by Theorem 6.7.

Part (*ii*). Assume that $S^*(\Lambda) \geq 2$, and let $n \in \{1, \dots, S^*(\Lambda) - 1\}$. We distinguish between two cases, depending on whether the slope of the line connecting the points $(\ell_n, \tilde{K}(\ell_n))$ and $(\ell_{n+1}, \tilde{K}(\ell_{n+1}))$ is steeper than the slope of the line connecting the points $(0, 0)$ and $(\Lambda, \tilde{K}(\Lambda))$. We denote the difference between these slopes by $\Delta$, i.e.,

$$\Delta = \frac{\tilde{K}(\ell_{n+1}) - \tilde{K}(\ell_n)}{\ell_{n+1} - \ell_n} - \frac{\tilde{K}(\Lambda)}{\Lambda}.$$

Notice that, by definition, $\tilde{K}(\ell_{n+1}) = K(n, \ell_{n+1})$ and $\tilde{K}(\ell_n) = K(n, \ell_n)$.

First case: $\Delta > 0$. Then, for any $\lambda \in (\ell_n, \ell_{n+1})$, we obtain

$$\begin{aligned}
\mathcal{B}(\lambda, \Lambda) &= \tilde{K}(\lambda) - \lambda \frac{\tilde{K}(\Lambda)}{\Lambda} \\
&= K\left(n, \frac{\ell_{n+1} - \lambda}{\ell_{n+1} - \ell_n} \ell_n + \frac{\lambda - \ell_n}{\ell_{n+1} - \ell_n} \ell_{n+1}\right) - \lambda \frac{\tilde{K}(\Lambda)}{\Lambda} \\
&< \frac{\ell_{n+1} - \lambda}{\ell_{n+1} - \ell_n} K(n, \ell_n) + \frac{\lambda - \ell_n}{\ell_{n+1} - \ell_n} K(n, \ell_{n+1}) - \lambda \frac{\tilde{K}(\Lambda)}{\Lambda} \\
&< \frac{\ell_{n+1} - \lambda}{\ell_{n+1} - \ell_n} K(n, \ell_n) + \frac{\lambda - \ell_n}{\ell_{n+1} - \ell_n} K(n, \ell_{n+1}) - \lambda \frac{\tilde{K}(\Lambda)}{\Lambda} + (\ell_{n+1} - \lambda)\Delta \\
&= \tilde{K}(\ell_{n+1}) - \ell_{n+1} \frac{\tilde{K}(\Lambda)}{\Lambda} = \mathcal{B}(\ell_{n+1}, \Lambda),
\end{aligned}$$

where the first inequality holds by Part $(ii)$ of Lemma 6.5 and the second inequality holds because $\Delta > 0$ by assumption.

Second case: $\Delta \le 0$. For any $\lambda \in (\ell_n, \ell_{n+1})$ we find, analogous to the first case,

$$\begin{aligned}
\mathcal{B}(\lambda, \Lambda) &< \frac{\ell_{n+1} - \lambda}{\ell_{n+1} - \ell_n} K(n, \ell_n) + \frac{\lambda - \ell_n}{\ell_{n+1} - \ell_n} K(n, \ell_{n+1}) - \lambda \frac{\tilde{K}(\Lambda)}{\Lambda} \\
&\le \frac{\ell_{n+1} - \lambda}{\ell_{n+1} - \ell_n} K(n, \ell_n) + \frac{\lambda - \ell_n}{\ell_{n+1} - \ell_n} K(n, \ell_{n+1}) - \lambda \frac{\tilde{K}(\Lambda)}{\Lambda} + (\ell_n - \lambda)\Delta \\
&= \tilde{K}(\ell_n) - \ell_n \frac{\tilde{K}(\Lambda)}{\Lambda} = \mathcal{B}(\ell_n, \Lambda).
\end{aligned}$$

Combining the two cases completes the proof of Part $(ii)$.

Part $(iii)$. Let $\lambda \in (\ell_{S^*(\Lambda)}, \Lambda]$. We redefine $\Delta$ as

$$\Delta = \frac{\tilde{K}(\Lambda) - \tilde{K}(\ell_{S^*(\Lambda)})}{\Lambda - \ell_{S^*(\Lambda)}} - \frac{\tilde{K}(\Lambda)}{\Lambda}$$

Since $\Lambda > \ell_{S^*(\Lambda)} \ge \ell_1$, Theorem 6.7 implies that $\tilde{K}(\Lambda)/\Lambda < \tilde{K}(\ell_{S^*(\Lambda)})/\ell_{S^*(\Lambda)}$. Using this, we obtain

$$\begin{aligned}
\Delta &= \frac{1}{\Lambda - \ell_{S^*(\Lambda)}} \left(\tilde{K}(\Lambda) - \tilde{K}(\ell_{S^*(\Lambda)})\right) - \frac{1}{\Lambda - \ell_{S^*(\Lambda)}} \left(\tilde{K}(\Lambda) - \frac{\ell_{S^*(\Lambda)}}{\Lambda} \tilde{K}(\Lambda)\right) \\
&= \frac{1}{\Lambda - \ell_{S^*(\Lambda)}} \left(\ell_{S^*(\Lambda)} \frac{\tilde{K}(\Lambda)}{\Lambda} - \tilde{K}(\ell_{S^*(\Lambda)})\right) \\
&< \frac{1}{\Lambda - \ell_{S^*(\Lambda)}} \left(\ell_{S^*(\Lambda)} \frac{\tilde{K}(\ell_{S^*(\Lambda)})}{\ell_{S^*(\Lambda)}} - \tilde{K}(\ell_{S^*(\Lambda)})\right) = 0.
\end{aligned}$$

Using that $\Delta < 0$, we find, in line with the second case in Part $(ii)$,

$$\begin{aligned}\mathcal{B}(\lambda,\Lambda) &\le \frac{\Lambda-\lambda}{\Lambda-\ell_{S^*(\Lambda)}}K(S^*(\Lambda),\ell_{S^*(\Lambda)}) + \frac{\lambda-\ell_{S^*(\Lambda)}}{\Lambda-\ell_{S^*(\Lambda)}}K(S^*(\Lambda),\Lambda) - \lambda\frac{\tilde{K}(\Lambda)}{\Lambda} \\ &< \frac{\Lambda-\lambda}{\Lambda-\ell_{S^*(\Lambda)}}K(S^*(\Lambda),\ell_{S^*(\Lambda)}) + \frac{\lambda-\ell_{S^*(\Lambda)}}{\Lambda-\ell_{S^*(\Lambda)}}K(S^*(\Lambda),\Lambda) - \lambda\frac{\tilde{K}(\Lambda)}{\Lambda} + (\ell_{S^*(\Lambda)}-\lambda)\Delta \\ &= \tilde{K}(\ell_{S^*(\Lambda)}) - \ell_{S^*(\Lambda)}\frac{\tilde{K}(\Lambda)}{\Lambda} = \mathcal{B}(\ell_{S^*(\Lambda)},\Lambda).\end{aligned}$$

We conclude that $\mathcal{B}(\lambda,\Lambda) < \mathcal{B}(\ell_{S^*(\Lambda)},\Lambda)$. □

*Proof of Theorem 7.2.* Let $n^* \in \operatorname{argmax}_{n\in\{1,\dots,S^*(\Lambda)\}} \mathcal{B}(\ell_n,\Lambda)$. Let $\lambda \in (0,\Lambda]$ with $\lambda \neq \ell_n$ for some $n \in \mathbb{N}$. If $\lambda < \ell_1$, then

$$\mathcal{B}(\lambda,\Lambda) < \mathcal{B}(\ell_1,\Lambda) \le \mathcal{B}(\ell_{n^*},\Lambda),$$

where the first inequality is due to Lemma A.2, Part $(i)$, and the second inequality holds by choice of $n^*$ as a maximizer. Similarly, if $\lambda > \ell_1$, then $\mathcal{B}(\lambda,\Lambda) < \mathcal{B}(\ell_{n^*},\Lambda)$ by Part $(ii)$ or $(iii)$ of Lemma A.2, and by choice of $n^*$ as a maximizer. □

## A.4 Technical results for Section 7.3.

To describe the alternative proof approach, we first need a few definitions. Given any function $f : \mathbb{R}_{++} \to \mathbb{R}$, we define the *family of single-attribute games embedded in* $f$ (cf. Özen et al., 2011) to be the set of all games $(N,c)$ for which there exists a vector $(a_i)_{i\in N} \in \mathbb{R}^N_{++}$ such that, for all $M \in 2^N_-$, $c(M) = f(a_M)$, where $a_M = \sum_{i\in M} a_i$. Here, $a_i$ is interpreted as the attribute of player $i \in N$, and $f$ expresses the costs to serve any level of attributes. Further, a game $(N,c)$ is called a *newsvendor game* (cf. Hartman et al., 2000) if there exists oversupply cost $c_o \ge 0$, undersupply cost $c_u \ge 0$, and stochastic demand $D_i$ for each player $i \in N$ such that

$$c(M) = \min_{s\ge 0}\left(c_o \cdot \mathbb{E}\max\Big\{s - \sum_{i\in M} D_i, 0\Big\} + c_u \cdot \mathbb{E}\max\Big\{\sum_{i\in M} D_i - s, 0\Big\}\right)$$

for all $M \in 2^N_-$. We can now show the following two propositions.

**Proposition A.3.** *Consider some function $f : \mathbb{R}_{++} \to \mathbb{R}$, and suppose that all games in the family of single-attribute games embedded in $f$ are newsvendor games. Then, for any player set $N$ and attribute vector $a \in \mathbb{R}^N_{++}$, the game $(N,c)$ defined by $c(M) = f(a_M)$ for all $M \in 2^N_-$ has a non-empty core, and the allocation scheme assigning $c(M)\cdot a_i/a_M$ to any $M \in 2^N_-$ and $i \in M$ is a PMAS for this game.*

*Proof.* The second sentence of Theorem 1 of Özen et al. (2011) states that if $f$ is not elastic, then there exists a game in the family of single-attribute games embedded in $f$ that has an empty core. (Although Özen et al. (2011) focus on value games rather than cost games, their results straightforwardly carry over to cost games by simply reversing all inequalities involved.) By contraposition, this means that if each game in the family of single-attribute games embedded in $f$ has a non-empty core, then $f$ is elastic.

But in our case, all games in this family are newsvendor games by assumption, and newsvendor games are known to have a non-empty core in general — a result that was derived independently by Müller et al. (2002) and Slikker et al. (2001). We conclude that $f$ is elastic. This elasticity implies, by Theorem 1 of Özen et al. (2011), that any game in this family has a non-empty core admitting the desired proportional PMAS. □

**Proposition A.4.** *Let $\varphi = (N, (\lambda_i)_{i\in N}, h, b)$ be a spare parts situation with backordering. Then, the associated spare parts pooling game is a newsvendor game.*

*Proof.* We set oversupply cost $c_o = h$, undersupply cost $c_u = b$, and stochastic demand $D_i$ for any player $i \in N$ distributed according to a Poisson distribution with mean $\lambda_i$. Then, for any $M \in 2^N_-$, we obtain

$$\begin{aligned} c^\varphi(M) &= \min_{S\in\mathbb{N}_0} \big(h \cdot \mathbb{E}I(S, \lambda_M) + b \cdot \mathbb{E}B(S, \lambda_M)\big) \\ &= \min_{S\in\mathbb{N}_0} \big(h \cdot \mathbb{E}\max\{S - X(\lambda_M), 0\} + b \cdot \mathbb{E}\max\{X(\lambda_M) - S, 0\}\big) \\ &= \min_{S\in\mathbb{N}_0} \left(c_o \cdot \mathbb{E}\max\left\{S - \sum_{i\in M} D_i, 0\right\} + c_u \cdot \mathbb{E}\max\left\{\sum_{i\in M} D_i - S, 0\right\}\right) \\ &= \min_{s\geq 0} \left(c_o \cdot \mathbb{E}\max\left\{s - \sum_{i\in M} D_i, 0\right\} + c_u \cdot \mathbb{E}\max\left\{\sum_{i\in M} D_i - s, 0\right\}\right) \end{aligned}$$

The first equality holds by Equations (5) and (6). The third equality holds because both $X(\lambda_M)$ and $\sum_{i\in M} D_i$ are Poisson distributed with mean $\lambda_M$. The final equality holds because, even if we allow optimization over the real numbers, the optimal order size will always be integer-valued due to the discreteness of the demand distribution. We conclude that the spare parts pooling game $(N, c^\varphi)$ is a newsvendor game. □

Consider the class of spare parts situations with backordering for given holding and backorder costs $h$ and $b$. Since all associated spare parts pooling games comprise the family of single-attribute games embedded in the optimal cost function $\tilde{K}$, with attributes corresponding to the players' demand rates, the combination of the preceding two propositions yields an alternative proof for Part ($i$) of Theorem 7.1.

The following two examples illustrate applications of Proposition A.3 beyond the spare parts context. These examples concern newsvendor games for which the demand information of any player can be fully represented as a single real number (i.e., a single attribute) and the demand distribution of each coalition belong to the same class. Example A.1 deals with normally distributed demands, while Example A.2 deals with Poisson distributed demands.

**Example A.1.** For given oversupply cost $c_o \geq 0$, undersupply cost $c_u \geq 0$, and variance-to-mean ratio $C$, consider the function $f : \mathbb{R}_{++} \to \mathbb{R}$ defined by

$$f(\mu) = \min_{s\geq 0} \left( c_o \cdot \mathbb{E} \max \left\{ s - D(\mu), 0 \right\} + c_u \cdot \mathbb{E} \max \left\{ D(\mu) - s, 0 \right\} \right),$$

where $D(\mu) \sim \mathcal{N}(\mu, \sigma^2)$ with $\sigma^2 = \mu C$. Then, the family of single-attribute games embedded in $f$ is comprised of all newsvendor games with oversupply cost $c_o$, undersupply cost $c_u$, and stochastic demand $D_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ with $\sigma_i^2 = \mu_i C$ and $\mu_i > 0$ for all players $i$. By Proposition A.3, for any such newsvendor game $(N, c)$, the allocation scheme assigning $c(M) \cdot \mu_i / \sum_{j\in M} \mu_j$ to any $M \in 2^N_-$ and $i \in M$ is a PMAS. ♢

**Example A.2.** For given oversupply cost $c_o \geq 0$, undersupply cost $c_u \geq 0$, and variance-to-mean ratio $C$, consider the function $f : \mathbb{R}_{++} \to \mathbb{R}$ defined by

$$f(\lambda) = \min_{s\geq 0} \left( c_o \cdot \mathbb{E} \max \left\{ s - D(\lambda), 0 \right\} + c_u \cdot \mathbb{E} \max \left\{ D(\lambda) - s, 0 \right\} \right),$$

where $D(\lambda) \sim Pois(\lambda)$. Then, the family of single-attribute games embedded in $f$ is comprised of all newsvendor games with oversupply cost $c_o$, undersupply cost $c_u$, and stochastic demand $D_i \sim Pois(\lambda_i)$ for all players $i$. By Proposition A.3, for any such newsvendor game $(N, c)$, the allocation scheme assigning $c(M) \cdot \lambda_i / \sum_{j\in M} \lambda_j$ to any $M \in 2^N_-$ and $i \in M$ is a PMAS. ♢

## A.5 Proofs for Section 9.

In the process of proving Theorem 9.1 (games with fixed base stock levels admit a strictly stable allocation), we first need several intermediate results, as described in Section 9. We start with a technical lemma that considers minimally balanced combinations of cumulative pipeline stock probabilities for situations with symmetric demand rates.

**Lemma A.5.** *Let $(N, (\lambda_i)_{i\in N}, (S_i)_{i\in N}, h, b) \in \Gamma$ with $\lambda_i = \lambda_j = \lambda$ for all $i, j \in N$. Let $\kappa \in \mathcal{W}^N$ be a minimally balanced map. Then,*

$$\sum_{M\in\mathbb{B}(\kappa)} \kappa(M) \sum_{x=0}^{S_M} (S_M - x)\mathbb{P}[X(\lambda_M) = x] > \sum_{x=0}^{S_N} (S_N - x)\mathbb{P}[X(\lambda_N) = x]. \tag{20}$$

*Proof.* For notational ease, we will systematically write $n = |N|$ and $m = |M|$ for any $M \in 2^N_{--}$. We will also employ the following notational convention: for any function $f$ on the integers, $\sum_{v=x}^{y} f(v) = 0$ if the sum is empty, i.e., if $x > y$. Moreover, we denote the left side of Inequality (20), multiplied by $e^{n\lambda}$, as $Z$, i.e., $Z = \sum_{M\in\mathbb{B}(\kappa)} \kappa(M) \sum_{x=0}^{S_M} (S_M - x)\mathbb{P}[X(m\lambda) = x]e^{n\lambda}$. Rewriting $Z$, plugging in the Taylor series of the exponential function, and using the binomial theorem twice, we obtain

$$\begin{aligned}
Z &= \sum_{M\in\mathbb{B}(\kappa)} \kappa(M) \sum_{x=0}^{S_M} (S_M - x)\frac{(m\lambda)^x}{x!} \cdot e^{(n-m)\lambda} \\
&= \sum_{M\in\mathbb{B}(\kappa)} \kappa(M) \sum_{x=0}^{S_M} (S_M - x)\frac{(m\lambda)^x}{x!} \cdot \sum_{y=0}^{\infty} \frac{((n-m)\lambda)^y}{y!} \\
&> \sum_{M\in\mathbb{B}(\kappa)} \kappa(M) \sum_{x=0}^{S_M} (S_M - x)\frac{(m\lambda)^x}{x!} \cdot \sum_{y=0}^{S_N - x} \frac{((n-m)\lambda)^y}{y!} \\
&= \sum_{M\in\mathbb{B}(\kappa)} \kappa(M) \sum_{\alpha=0}^{S_N} \lambda^\alpha \sum_{v=0}^{\min\{\alpha,S_M\}} (S_M - v)\frac{m^v}{v!} \cdot \frac{(n-m)^{\alpha-v}}{(\alpha-v)!} \\
&= \sum_{M\in\mathbb{B}(\kappa)} \kappa(M) \sum_{\alpha=0}^{S_N} \frac{\lambda^\alpha}{\alpha!} \left[ \sum_{v=0}^{\min\{\alpha,S_M\}} (S_M - v) m^v (n-m)^{\alpha-v} \frac{\alpha!}{v!(\alpha-v)!} \right] \\
&= \sum_{M\in\mathbb{B}(\kappa)} \kappa(M) \sum_{\alpha=0}^{S_N} \frac{\lambda^\alpha}{\alpha!} \left[ \sum_{v=0}^{\min\{\alpha,S_M\}} S_M m^v (n-m)^{\alpha-v} \binom{\alpha}{v} \right. \\
&\qquad \left. - \sum_{v=1}^{\alpha} v m^v (n-m)^{\alpha-v} \frac{\alpha!}{v!(\alpha-v)!} + \sum_{v=S_M+1}^{\alpha} v m^v (n-m)^{\alpha-v} \binom{\alpha}{v} \right] \\
&= \sum_{M\in\mathbb{B}(\kappa)} \kappa(M) \sum_{\alpha=0}^{S_N} \frac{\lambda^\alpha}{\alpha!} \left[ \sum_{v=0}^{\min\{\alpha,S_M\}} S_M m^v (n-m)^{\alpha-v} \binom{\alpha}{v} \right. \\
&\qquad \left. - \sum_{v=0}^{\alpha-1} (v+1) m^{v+1} (n-m)^{\alpha-v-1} \frac{\alpha!}{(v+1)!(\alpha-v-1)!} + \sum_{v=S_M+1}^{\alpha} v m^v (n-m)^{\alpha-v} \binom{\alpha}{v} \right] \\
&= \sum_{M\in\mathbb{B}(\kappa)} \kappa(M) \sum_{\alpha=0}^{S_N} \frac{\lambda^\alpha}{\alpha!} \left[ \sum_{v=0}^{\min\{\alpha,S_M\}} S_M m^v (n-m)^{\alpha-v} \binom{\alpha}{v} \right. \\
&\qquad \left. - m\alpha \sum_{v=0}^{\alpha-1} m^v (n-m)^{\alpha-1-v} \frac{(\alpha-1)!}{v!(\alpha-1-v)!} + \sum_{v=S_M+1}^{\alpha} v m^v (n-m)^{\alpha-v} \binom{\alpha}{v} \right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{M\in\mathbb{B}(\kappa)} \kappa(M) \sum_{\alpha=0}^{S_N} \frac{\lambda^\alpha}{\alpha!} \Bigg[ \sum_{v=0}^{\min\{\alpha,S_M\}} S_M m^v (n-m)^{\alpha-v} \binom{\alpha}{v} \\
&\qquad -m\alpha(m+n-m)^{\alpha-1} + \sum_{v=S_M+1}^{\alpha} v m^v (n-m)^{\alpha-v}\binom{\alpha}{v} \Bigg] \\
&\geq \sum_{M\in\mathbb{B}(\kappa)} \kappa(M) \sum_{\alpha=0}^{S_N} \frac{\lambda^\alpha}{\alpha!} \Bigg[ \sum_{v=0}^{\min\{\alpha,S_M\}} S_M m^v (n-m)^{\alpha-v} \binom{\alpha}{v} \\
&\qquad -m\alpha n^{\alpha-1} + \sum_{v=S_M+1}^{\alpha} S_M m^v (n-m)^{\alpha-v}\binom{\alpha}{v} \Bigg] \\
&= \sum_{M\in\mathbb{B}(\kappa)} \kappa(M) \sum_{\alpha=0}^{S_N} \frac{\lambda^\alpha}{\alpha!} \left[ \sum_{v=0}^{\alpha} S_M m^v (n-m)^{\alpha-v} \binom{\alpha}{v} - m\alpha n^{\alpha-1} \right] \\
&= \sum_{M\in\mathbb{B}(\kappa)} \kappa(M) \sum_{\alpha=0}^{S_N} \frac{\lambda^\alpha}{\alpha!} \left[ S_M (m+n-m)^{\alpha} - m\alpha n^{\alpha-1} \right] \\
&= \sum_{\alpha=0}^{S_N} \frac{\lambda^\alpha}{\alpha!} \sum_{M\in\mathbb{B}(\kappa)} \kappa(M) \left( S_M n^\alpha - \frac{m}{n}\alpha n^\alpha \right) \\
&= \sum_{\alpha=0}^{S_N} \frac{(n\lambda)^\alpha}{\alpha!} \sum_{M\in\mathbb{B}(\kappa)} \kappa(M) \left( S_M - \frac{m}{n}\alpha \right) \\
&= \sum_{\alpha=0}^{S_N} \frac{(n\lambda)^\alpha}{\alpha!} \left( S_N - \frac{n}{n}\alpha \right) = \sum_{x=0}^{S_N} (S_N - x)\mathbb{P}[X(\lambda_N) = x] e^{n\lambda}.
\end{aligned}
$$

The first inequality is strict because (i) $S_N > 0$ by assumption on $\Gamma$, and thus there exists an $M \in \mathbb{B}(\kappa)$ for which $S_M > 0$; and (ii) every $M \in \mathbb{B}(\kappa)$ is a proper subset of $N$ since $\kappa$ is minimally balanced, and thus $(n - m) > 0$ for each $M \in \mathbb{B}(\kappa)$. The result is a product of two finite sums; note that in the allowed sum ranges, $x + y$ never exceeds $S_N$. In the subsequent equality, we group together terms with common powers of $\lambda$. Here, we use that any expanded term $\lambda^\alpha$ occurs for $x = 0$ and $y = \alpha$ (corresponding to $v = 0$), for $x = 1$ and $y = \alpha - 1$ (corresponding to $v = 1$), and so on up to either $x = \alpha$ and $y = 0$ in case $\alpha \leq S_M$ or up to $x = S_M$ and $y = \alpha - S_M$ in case $S_M < \alpha \leq S_N$. In steps thereafter, we rewrite our expression to enable suitable invocations of the binomial theorem. The second inequality is obtained upon replacing $v$ by $S_M$. The penultimate equality holds because $\sum_{M\in\mathbb{B}(\kappa)} \kappa(M) S_M = S_N$ and $\sum_{M\in\mathbb{B}(\kappa)} \kappa(M) m = n$. Finally, multiplying both sides of the above-derived inequality by $e^{-n\lambda} > 0$ completes the proof. $\square$

We next show that under the assumption of identical demand rates, while allowing for different base stock levels, games with fixed base stock levels possess a strictly stable allocation.

**Lemma A.6.** *Let $\gamma = (N, (\lambda_i)_{i\in N}, (S_i)_{i\in N}, h, b) \in \Gamma$ with $\lambda_i = \lambda_j = \lambda$ for all $i, j \in N$. The associated game $(N, c^\gamma)$ has a non-empty strict core.*

*Proof.* Let $\kappa \in \mathcal{W}^N$ be a minimally balanced map. Then,

$$\begin{aligned}
\sum_{M\in\mathbb{B}(\kappa)} \kappa(M)c^\gamma(M) &= \sum_{M\in\mathbb{B}(\kappa)} \kappa(M)\left(b\lambda_M - bS_M + (b+h)\sum_{x=0}^{S_M}(S_M - x)\mathbb{P}[X(\lambda_M) = x]\right) \\
&= b\lambda_N - bS_N + (b+h)\sum_{M\in\mathbb{B}(\kappa)} \kappa(M)\sum_{x=0}^{S_M}(S_M - x)\mathbb{P}[X(\lambda_M) = x] \\
&> b\lambda_N - bS_N + (b+h)\sum_{x=0}^{S_N}(S_N - x)\mathbb{P}[X(\lambda_N) = x] \\
&= c^\gamma(N),
\end{aligned}$$

where the inequality holds by Lemma A.5. By Proposition A.1, this strict inequality implies that $(N, c^\gamma)$ has a non-empty strict core. □

We now make use of the preceding lemma to show a generalization: if players have have non-identical but rational-valued demand rates, the associated game with fixed base stock levels also possesses a strictly stable allocation.

**Lemma A.7.** *Let $\gamma = (N, (\lambda_i)_{i\in N}, (S_i)_{i\in N}, h, b) \in \Gamma$ with $\lambda_i \in \mathbb{Q}$ for all $i \in N$. The associated game $(N, c^\gamma)$ has a non-empty strict core.*

*Proof.* For all $i \in N$, we know that $\lambda_i \in \mathbb{Q}$ and hence we can pick $a_i, b_i \in \mathbb{N}$ such that $\lambda_i = a_i/b_i$. Let $\Lambda = (\prod_{i\in N} b_i)^{-1}$, and let for all $i \in N$

$$K_i = \frac{a_i}{b_i} \cdot \frac{1}{\Lambda}.$$

Notice that $K_i \in \mathbb{N}$ for all $i \in N$. Now, we will construct a new situation with identical demand rates $\bar{\gamma} \in \Gamma$ by splitting each player $i \in N$ into $K_i$ (sub)players such that each (sub)player has a demand rate of $\Lambda$. We define $\bar{\gamma} = (\bar{N}, (\bar{\lambda}_j)_{j\in\bar{N}}, (\bar{S}_j)_{j\in\bar{N}}, \bar{h}, \bar{b})$ by

- $\bar{N} = \bigcup_{i\in N} N_i$ with $N_i = \{(i, 1), (i, 2), \ldots, (i, K_i)\}$ for all $i \in N$;
- $\bar{\lambda}_j = \Lambda$ for all $j \in N$;
- $\bar{S}_{(i,1)} = S_i$ for all $i \in N$, and $\bar{S}_{(i,k)} = 0$ for all $k \in \{2, ..., K_i\}$ and all $i \in N$;

- $\bar{h} = h$;
- $\bar{b} = b$.

For all $M \in 2^N_-$, define $L(M)$ to be the set of (sub)players in $\bar{N}$ created from players in $M$, i.e., $L(M) = \bigcup_{i \in M} N_i$. Now, for any non-empty subcoalition $M \in 2^N_{--}$, it holds that

$$c^{\bar{\gamma}}(L(M)) = K\left(\sum_{j \in L(M)} S_j, |L(M)|\Lambda\right) = K\left(\sum_{i \in M} S_i, \sum_{i \in M} \lambda_i\right) = c^{\gamma}(M)$$

and, similarly, $c^{\bar{\gamma}}(L(N)) = c^{\gamma}(N)$. Now, let $y \in \mathscr{C}(\bar{N}, c^{\bar{\gamma}})$, which exists by Lemma A.6. For all $i \in N$ define $x_i = \sum_{k=1}^{K_i} y_{(i,k)}$. Consider any $M \in 2^N_{--}$. By making use of the facts that $y \in \mathscr{C}(\bar{N}, c^{\bar{\gamma}})$, $c^{\bar{\gamma}}(L(N)) = c^{\gamma}(N)$, and $c^{\bar{\gamma}}(L(M)) = c^{\gamma}(M)$, as derived above, we obtain

$$\sum_{i \in N} x_i = \sum_{i \in N} \sum_{k=1}^{K_i} y_{(i,k)} = \sum_{j \in \bar{N}} y_j = c^{\bar{\gamma}}(\bar{N}) = c^{\bar{\gamma}}(L(N)) = c^{\gamma}(N) \text{ (efficiency); and}$$

$$\sum_{i \in M} x_i = \sum_{i \in M} \sum_{k=1}^{K_i} y_{(i,k)} = \sum_{j \in L(M)} y_j < c^{\bar{\gamma}}(L(M)) = c^{\gamma}(M) \text{ (strict stability).}$$

We conclude that $x \in \mathscr{C}(N, c^{\gamma})$. Therefore, $(N, c^{\gamma})$ has a non-empty strict core. □

We can now generalize the result of Lemma A.7 to real-valued demand rates in order to prove Theorem 9.1.

*Proof of Theorem 9.1.* Let $\gamma = (N, (\lambda_i)_{i \in N}, (S_i)_{i \in N}, h, b) \in \Gamma$. We employ a straightforward continuity argument. Pick, for all $i \in N$, a sequence $\{\Lambda_i^m\}_{m=1}^{\infty}$ of positive numbers such that $\Lambda_i^m \in \mathbb{Q}$ for all $m \in \mathbb{N}$ and $\lim_{m \to \infty} \Lambda_i^m = \lambda_i$. For all $m \in \mathbb{N}$ we define, by replacing demand rates $(\lambda_i)_{i \in N}$ in $\gamma$ by $(\Lambda_i^m)_{i \in N}$, a new situation $\gamma^m = (N, (\Lambda_i^m)_{i \in N}, (S_i)_{i \in N}, h, b)$.

Let, for all $m \in \mathbb{N}$, $\nu(N, c^{\gamma^m})$ be the nucleolus of game $(N, c^{\gamma^m})$. By Lemma A.7, $\nu(N, c^{\gamma^m}) \in \mathscr{C}(N, c^{\gamma^m})$. Thus, for all $m \in \mathbb{N}$ and all $M \in 2^N_{--}$, it holds that

$$\sum_{i \in M} \nu_i(N, c^{\gamma^m}) < c^{\gamma^m}(M). \tag{21}$$

It follows from Part $(ii)$ of Lemma 6.5 that, for given $S \in \mathbb{N}_0$, $K(S, \lambda)$ is continuous in $\lambda$; therefore, the characteristic cost function is continuous in the demand rates of all players. Moreover, the nucleolus is a continuous function of the characteristic cost function (Theorem 5, Schmeidler, 1969). By combining this continuity and Inequality (21), we obtain for the nucleolus $\nu(N, c^{\gamma})$ of game $(N, c^{\gamma})$ that

$$\sum_{i \in M} \nu_i(N, c^{\gamma}) = \sum_{i \in M} \lim_{m \to \infty} \nu_i(N, c^{\gamma^m}) < \sum_{i \in M} \lim_{m \to \infty} c^{\gamma^m}(M) = \sum_{i \in M} c^{\gamma}(M)$$

for any $M \in 2^N_{--}$. We conclude that $\nu(N, c^\gamma)$ is a strictly stable allocation for game $(N, c^\gamma)$. Hence, the game $(N, c^\gamma)$ has a non-empty strict core. □

*Proof of Theorem 9.2.* Consider the spare parts situation with fixed base stock levels $\gamma = (N, (\lambda_i)_{i\in N}, (S_i)_{i\in N}, h, b)$ with $N = \{1, 2\}$, wherein player 1 has demand rate $\lambda_1$ and $S_1$ parts and player 2 has demand rate $\lambda_2$ and $S_2$ parts. Then, by Theorem 9.1, it holds that

$$K(S_1, \lambda_1) + K(S_2, \lambda_2) = c^\gamma(\{1\}) + c^\gamma(\{2\}) > c^\gamma(N) = K(S_1, S_2, \lambda_1 + \lambda_2), \qquad (22)$$

which shows validity of Part $(i)$. Subtracting $b(\lambda_1 + \lambda_2 - S_1 - S_2)$ from both sides of (22) and dividing by $h + b > 0$, we obtain

$$\sum_{x=0}^{S_1}(S_1 - x)\mathbb{P}[X(\lambda_1) = x] + \sum_{x=0}^{S_2}(S_2 - x)\mathbb{P}[X(\lambda_2) = x] > \sum_{x=0}^{S_1+S_2}(S_1 + S_2 - x)\mathbb{P}[X(\lambda_1 + \lambda_2) = x].$$

Comparison with Equations (3) and (4) completes the proof of Parts $(ii)$ and $(iii)$, respectively. □


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