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ABSTRACT

Fisher's information measure for the item difficulty parameter in the Rasch model and its marginal and conditional formulations are investigated. It is shown that expected item information in the unconditional model equals information in the marginal model, provided the assumption of sampling examinees from an ability distribution is made. For the logistic ability distribution considered in this paper, item information in the two models can be expressed in a closed form. Also, it is shown that for a random examinee expected item information in the conditional model is always less than that in the other two models, albeit the difference quickly decreases with an increase in test length. If the distribution of the item difficulties in the test deviates more and more from the ability distribution, item information in all three models takes smaller and smaller values. Results from a simulation study of tests with 5 and 20 items demonstrate these features numerically. Six tables present the results of the simulation study, and one graph illustrates item information in the marginal model. (Author/SLD)

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Item Information in the Rasch Model

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Abstract

Fisher's information measure for the item difficulty parameter in the Rasch model as well as its marginal and conditional formulations is investigated. It is shown that expected item information in the unconditional model equals information in the marginal model, provided the assumption of sampling examinees from an ability distribution is made. For the logistic ability distribution considered in this paper, item information in the two models can be expressed in a closed form. Also, it is shown that for a random examinee expected item information in the conditional model is always less than in the other two models, albeit that the difference quickly decreases with an increase in test length. If the distribution of the item difficulties in the test deviates more and more from the ability distribution, item information in all three models takes smaller and smaller values. Results from a simulation study demonstrate these features numerically.

Keywords: Item Response Theory, Rasch Model, Fisher's Information

Item Information in the Rasch Model

Consider a test consisting of I dichotomously scored items and let N examinees take the test. Let the random variables (rv) X_{ji} ($j=1,\ldots,N$; $i=1,\ldots,I$) take the value 1 if examinee j answers item i correctly and 0 otherwise. As usual, local independence and the assumption of examinees answering the items independently of one other is made throughout. For the form of the item response function, i.e., the probability that an item is answered correctly as a function of the (unidimensional) latent ability, the one-parameter logistic is chosen. In other words, we assume that the Rasch (1980) model holds. To be more specific, let the item difficulty parameters be denoted by $\sigma_1, \ldots, \sigma_I$ and the examinee ability parameter by $\theta_1, \ldots, \theta_N$, then

(1) $$P(X_{ji} = 1;\theta_j,\sigma_i) = \exp(\theta_j-\sigma_i)/[1 + \exp(\theta_j-\sigma_i)],$$

where $-\infty < \theta_j, \sigma_i < +\infty$.

In this paper, the main interest lies in obtaining information about the item difficulties $\sigma_1, \ldots, \sigma_I$, whereas the abilities $\theta_1, \ldots, \theta_N$ will be considered as nuisance parameters. It will be shown that, depending on the way the nuisance parameters are treated, different forms of Fisher's information measure for the item parameter can be defined. The objective is to compare these information measures. The results of the comparison are relevant to the test constructor who wants to estimate item parameters under the most informative model. However, a direct comparison of the different information measures is, for obvious reasons, not possible. Therefore, the comparison will be made either for the experiment of replicated item administrations to a fixed examinee or for the experiment of an examinee randomly sampled from a certain ability distribution. The reason for the choice of the experiments is elucidated below. For these experiments, an information analysis can be worked out either analytically or numerically in a fairly easy way.

In the next section, the different information measures in the Rasch model will be presented. Then, the mathematical relations between the different information measures is discussed. As it turns out, all relevant relations are consequences of a simple theorem which, loosely speaking, states that average conditional information is always less than marginal information. The following section gives (closed-form) results for the special case of a logistic distribution for the ability parameter. Simulated data is used to illustrate the results numerically. The last section gives some guidelines for selecting an appropriate model.

Different Information Measures in the Rasch Model

Fisher's information in the sample for a (scalar) parameter of interest ξ is defined as

$$(2) \qquad I(\xi) = E_{\xi}\left[\frac{\partial}{\partial\xi} \ln L(\xi;X)\right]^2,$$

where $L(\xi;X)$ is the likelihood function. Under certain regularity conditions (e.g., Bickel & Doksum, 1977), which are usually met, the well-known information inequality for a statistic $T = f(X)$ can be derived:

$$(3) \qquad Var_{\xi}(T) \geq \frac{[\frac{\partial}{\partial\xi} E_{\xi}(T)]^2}{I(\xi)} .$$

So, Fisher's information can be thought of as providing a lower bound to the variance of a statistic; the less the amount of information in the sample, the larger the lower bound, and hence, likely, the variance itself. For more detailed information, see a standard textbook on mathematical statistics.

As already mentioned, item information in the Rasch model depends on the way the ability parameters are treated; i.e., as fixed (possibly unknown) parameters or as realizations of a (possibly unknown) rv. These possible views on the person parameter as well as their consequences for the definition of the item information measure will now be considered in more detail.

First, if the ability parameters $\theta_1,\ldots,\theta_N$ are considered as fixed but unknown parameters, information on a single item difficulty parameter σ is given by

$$(4) \qquad I(\sigma;\theta_1,\ldots,\theta_N) = \Sigma_{j=1}^{N} P_{j\sigma}(1-P_{j\sigma}),$$

where $P_{j\sigma}$ is the probability that person j answers an item with difficulty σ correctly (e.g., Hambleton & Swaminathan, 1985; Lord, 1980). Note that in this case, for fixed σ, information is an unknown parameter that can only be estimated by replacing $\theta_1,\ldots,\theta_N$ by reasonable estimates $\hat{\theta}_1,\ldots,\hat{\theta}_N$. Note that these estimates need not be consistent for $I\to\infty$, though (Fischer, 1974; Haberman, 1977).

It is a well-known fact that in the Rasch model the total score $S_j = X_{j+}$ ($j=1,\ldots,N$) is a sufficient statistic for the ability θ_j, and that the conditional likelihood given $S_1=s_1, \ldots, S_N=s_N$ equals

$$(5)\qquad L_c(\sigma_1,\ldots,\sigma_I;\ q_1,\ldots,q_I|s_1,\ldots,s_N) = \exp(-\Sigma_{i=1}^{I}\ q_i\sigma_i)/\ \Pi_{r=0}^{I}\ \gamma_r^{\ n_r}(\exp(-\sigma_1),\ \ldots,\ \exp(-\sigma_I))$$

where $\gamma_r(\exp(-\sigma_1),\ \ldots,\ \exp(-\sigma_I))$ is the elementary symmetric function of order r (Andersen, 1980, chap. 6; Fisher, 1974, chap. 13), n_r the number of examinees in the sample with total score $S_j = r$ and $q_i = x_{+i}$, the number of correct answers on item i.

For this conditional likelihood, Fisher's information on item difficulty σ_i is defined as

$$(6)\qquad I(\sigma_i|s_1,\ldots,s_N) = \Sigma_{r=0}^{I}\ n_r P_{ri}(1-P_{ri}),$$

where $P_{ri} = P(X_{ji} = 1 | X_{j+} = r)$ (Fischer, 1974). Note that the conditional information in (6) is an observable realization of the rv $I(\sigma_i | S1, \ldots, SN)$. This means that (6) can be used for constructing confidence intervals for the item difficulty parameter, but not for determining the sample size.

If the persons taking the test are randomly sampled from a certain population, it is reasonable to consider the abilities as realizations $\theta_1, \ldots, \theta_N$ of the rv's $\Theta_1, \ldots, \Theta_N$. These rv's will then be considered as independently and identically distributed with density $g(\theta)$. The density may or may not be known, and is usually called the ability density. Later the case of a logistic form for $g(\theta)$ will be considered in depth; here its form is left unspecified as yet. This new look at the abilities leads to considering the unconditional information in (4) as an observable realization $I(\sigma; \theta_1, \ldots, \theta_N)$ of the rv $I(\sigma; \Theta_1, \ldots, \Theta_N)$, where σ is the vector of item difficulties $(\sigma_1, \ldots, \sigma_I)$. Furthermore, it leads to a different information measure. The marginal likelihood function is

$$(7) \qquad L_m(\sigma) = \prod_{j=1}^{N} \int_{-\infty}^{\infty} \frac{\exp(x_{j+}\theta - \sum_{i=1}^{I} x_{ji}\sigma_i)}{\prod_{i=1}^{I} (1+\exp(\theta - \sigma_i))} g(\theta) d\theta$$

The information in this marginal distribution on the difficulty parameter σ is

$$(8) \qquad I(\sigma;g) = \int_{-\infty}^{\infty} \exp(\theta-\sigma)[1+\exp(\theta-\sigma)]^{-2} g(\theta)d\theta.$$

For fixed σ, marginal information will be considered either as a known quantity (if g is known) or as an unknown parameter (if g is unknown). In the latter case, the information measure can be estimated consistently using a consistent estimate of the ability density (Engelen, 1987). In either case, the information measures can be used to construct approximate confidence intervals for σ. Only if g is (approximately) known in advance, it can be used to determine the sample size (van der Linden, 1988).

Mathematical Relations between Different Information Measures

In the preceding section, three different information measures for the item parameters were presented. Naturally, the question arises if any relation exists between these measures. The answer is positive: The link between these measures is provided by the 'ability' parameter, since a different way of treating this parameter leads to a different information measure. Two different cases will be considered: (1) $\theta_1,\ldots,\theta_N$ are unknown but fixed parameters; and (2) $\theta_1,\ldots,\theta_N$ are realizations of $\Theta_1,\ldots,\Theta_N$ with (known or unknown) density $g(\theta)$.

If $\theta_1,\ldots,\theta_N$ are unknown but fixed parameters, the pertinent comparison is between the unconditional information measure $I(\sigma;\theta_1,\ldots,\theta_N)$ in (4) and the conditional measure $I(\sigma|s1,\ldots,sN)$ in (6). However, the latter depends on the observed realization of $S_1,\ldots,S_N$; if the same examinees were to respond to the same items again, these sufficient statistics would likely to take different values. As a result, $I(\sigma|s1,\ldots,sN)$ would also take a different value, whereas the unconditional information measure $I(\sigma;\theta_1,\ldots,\theta_N)$ would remain constant. The obvious approach in this case is to compare $I(\sigma;\theta_1,\ldots,\theta_N)$ with

$$E_{\theta_1,\ldots,\theta_N}[I(\sigma|S1,\ldots,SN)]$$

as a typical value of the former.

If $\theta_1,\ldots,\theta_N$ are realizations of $\Theta_1,\ldots,\Theta_N$ with common density g, comparisons between all three information measures in (4), (6) and (8) seem to be obvious. However, the marginal information measure $I(\sigma;g)$ is a constant for each possible value of σ, whereas the unconditional and conditional measures $I(\sigma;\theta_1,\ldots,\theta_N)$ and $I(\sigma|s1,\ldots,sN)$ are likely to take different values for each realization of $\Theta_1,\ldots,\Theta_N$. Along the same lines, it now seems obvious to compare $I(\sigma;g)$ with the expected unconditional and conditional information measures

$$E_g[I(\sigma;\Theta_1,\ldots,\Theta_N)]$$

and

$$E_g[I(\sigma|S_1,\ldots,S_N)],$$

respectively.

Fundamental to the relations between these different (expected) information measures is the following general theorem:

Theorem. Let the random vector X have a density (with respect to some σ-finite measure) $p_\sigma(x)$, for $\sigma \in \Sigma$. Let $S=s(X)$ have density $f_\sigma(s)$ and let $g_\sigma(x|s)$ be the conditional density of X given $S=s$. Finally, let $I(\sigma)$ and $I(\sigma|s)$ be Fisher's information about σ in the densities $p_\sigma(x)$ and $g_\sigma(x|s)$, respectively. Then, $I(\sigma) \geq E[I(\sigma|S)]$, with equality if and only if $\frac{\partial}{\partial\sigma} \ln p_\sigma(X)$ and S are uncorrelated.

Proof. We have $\ln p_\sigma(x) = \ln f_\sigma(s) + \ln g_\sigma(x|s)$. Thus,

$$I(\sigma) = \mathrm{Var}[\frac{\partial}{\partial\sigma} \ln p_\sigma(X)]$$

$$= \mathrm{Var}[E(\frac{\partial}{\partial\sigma} \ln p_\sigma(X|S)] + E[\mathrm{Var}(\frac{\partial}{\partial\sigma} \ln p_\sigma(X|S)]$$

$$\geq E[\mathrm{Var}(\frac{\partial}{\partial\sigma} \ln p_\sigma(X|S)]$$

$$= E[\mathrm{Var}(\frac{\partial}{\partial\sigma} \ln g_\sigma(X|S))]$$

$$= E[I(\sigma|S)]$$

Equality holds iff $\mathrm{Var}\{E(\frac{\partial}{\partial\sigma} \ln p_\sigma(X)|S)] = 0$, that is iff $\frac{\partial}{\partial\sigma} \ln p_\sigma(X)$ and S are uncorrelated. ∎

The theorem immediately leads to the following inequalities between unconditional and conditional information for the case of $\theta_1,\ldots,\theta_N$ fixed (Proposition 1) and between marginal and expected conditional information for the case of $\theta_1,\ldots,\theta_N$ as a realization of $\Theta_1,\ldots,\Theta_N$ (Proposition 2):

Proposition 1. If $\theta_1,\ldots,\theta_N$ are (unknown but) fixed parameters, the expected value of the conditional information in (6) cannot exceed the (unknown) unconditional information in (4), that is

(9) $$E_{\theta_1,\ldots,\theta_N} [I(\sigma|S_1,\ldots,S_N)] \leq I(\sigma;\theta_1,\ldots,\theta_N).$$

Proposition 2. If the abilities $\theta_1,\ldots,\theta_N$ are realizations of independently and identically distributed rv's $\Theta_1,\ldots,\Theta_N$ with density $g(\theta)$, the expectation of the conditional information in (6) cannot exceed the (known or unknown) marginal information in (8), that is

(10) $E_g[I(\sigma|S_1,\ldots,S_N)] \leq I(\sigma;g)$.

The following proposition is a direct consequence of the fact that $E(E(Y|X)) = E(Y)$, or, equivalently, that a change of the order of integration is permitted. It shows that the marginal information measure in (10) may be replaced by the expected unconditional measure:

Proposition 3. If the abilities $\theta_1,\ldots,\theta_N$ are realizations of independently and identically distributed rv's $\Theta_1,\ldots,\Theta_N$ with density $g(\theta)$, the expectation of the unconditional information in (4) equals the (known or unknown) marginal information in (8), that is

(11) $E_g[I(\sigma;\Theta_1,\ldots,\Theta_N)] = I(\sigma;g)$.

Note again that different 'averaging' takes place in (9) and (10) - (11). The latter case will now be explored further for a common logistic density for the ability parameters.

A Comparison of Different Information Measures for a Logistic Ability Distribution

This section consists of three parts. In the first part, item information is computed for the marginal model with a logistic ability distribution. The second part deals with the computation of expected information for the conditional model with the same ability distribution. A numerical comparison between the two is given in the final part.

Item Information in the Marginal Model

Item information in the marginal model, given by the integral in (8), is difficult to evaluate and has to be approximated numerically for most choices of $g(\theta)$. However, as will be shown here, the choice of a logistic distribution function permits computation of the integral in closed form. The logistic distribution does not differ much from the normal distribution adopted by authors as Andersen and Madsen (1977), Sanathanan and Blumenthal (1978), and Thissen (1982).

The logistic density is given by

$$(12) \qquad g(\theta) = \beta^{-1}\exp[-(\theta-\alpha)/\beta][1 + \exp[-(\theta-\alpha)/\beta]^{-2},$$

where α is the mean and $\beta^2\pi^2/3$ equals the variance of the distribution. The logistic density with parameters α and β will henceforth be denoted as $L(\theta;\alpha,\beta)$.

Since the Rasch model is not identifiable, a constraint has to be imposed on either the location of the ability distribution or the item parameters. Because it will be shown that this constraint does not lead to severe problems, this subject will be dropped for the moment; for convenience, the choice β=1 is made. Now, (8) can be computed as

$$I(\sigma) = \int_{-\infty}^{\infty} \frac{\exp(\theta-\sigma)\ \exp(\alpha-\theta)}{[1+\exp(\theta-\sigma)]^2[1+\exp(\alpha-\theta)]^2} d\theta.$$

Substituting $\exp(\theta-\sigma) = z$ and writing $\exp(\alpha-\sigma) = \tau$ gives for $I(\sigma)$

$$\tau \int_0^{\infty} (1 + z)^{-2}(z + \tau)^{-2} z dz.$$

For $\tau = 1$, i.e., $\alpha = \sigma$, the integrand is $(1 + z)^{-4}z$, and it follows that $I(\sigma) = 1/6$. For $\tau \neq 1$, the integrand is a simple rational function, and hence $I(\sigma)$ can be computed by using a fraction formula. For $\tau \neq 1$, this leads to

$$I(\sigma) = \tau[(\tau+1)(\tau-1)^{-3}\ln\tau - 2(\tau-1)^{-2}].$$

Combining all results leads to

$$(13) \qquad I(\sigma) = \begin{cases} \left[(\alpha-\sigma)\dfrac{1+\exp(\alpha-\sigma)}{(\exp(\alpha-\sigma)-1)^3} - \dfrac{2}{(\exp(\alpha-\sigma)-1)^2}\right] \exp(\alpha-\sigma), & \alpha \neq \sigma \\ 1/6, & \alpha = \sigma. \end{cases}$$

In order to check the continuity of this result in $\tau = 1$, the limit for $\tau \to 1$, or, equivalently, for $\sigma \to \alpha$, is taken, using the Taylor expansion of $\ln(\alpha-\sigma+1)$. Writing $y = \tau-1$, and substituting

$$\ln(y+1) = y - y^2/2 + y^3/3 + o(y^4),$$

gives

$$\lim_{\sigma\to\alpha} I(\sigma) =$$

$$\lim_{y\to 0} \{(y+1)y^{-3} [(y+2)(y-y^2/2 + y^3/3 + o(y^4)) - 2y]\} =$$

$$\lim_{y\to 0} (y+1)y^{-3}[y^3/6 + o(y^4)].$$

Hence,

$$\lim_{\sigma\to\alpha} I(\sigma) = 1/6.$$

Three different features of (13) can be noticed. First, and most important, $I(\sigma)$ is a function of $\alpha-\sigma$ only, implying that only the difference between the mean of the ability distribution and the difficulty of the item is of importance. Second, a translation of the ability/difficulty scale does not change the result. Third, marginal item information is symmetric about zero; substituting $-(\alpha-\sigma)$ for $(\alpha-\sigma)$ gives the same value. Note that the second feature is also a property of the Rasch model without the assumption of an ability distribution. Together these features show that marginal information has only to be computed for the case $\alpha = 0$ and for positive values of the difficulty parameter. In Figure 1, for $\alpha = 0$ the information function in (13) is plotted on the interval ⟨−3,3⟩. Observe the symmetry around the y-axis and the nice form of the plot. Larger or smaller

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Insert Figure 1 about here

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values of α only introduce a translation of the graph along the x-axis. Some numerical values of (11) for different values of α are given in Table 1.

Insert Table 1 about here

For comparison with the conditional model, it is noted that $I(\sigma)$ is a function of the difficulty of the item in question only, and does not depend on the difficulties of the other items in the test. Neither does it depend on the values of the response variables realized in the sample. Hence, item information in the marginal model, or in the unconditional model with sampling of examinees (cf. Proposition 3), can be obtained independently of the test by computing (11) as a function of σ for different values of α.

Item Information in the Conditional Model

Item information in the conditional model (6) is a function of n_r and P_{ri}. However,

(14) $P_{ri} = \epsilon_i \gamma_{r-1}^{(i)}(\epsilon)/\gamma_r(\epsilon)$,

with $\epsilon_i = \exp(-\sigma_i)$, γ_r being the rth-order elementary symmetric function of the ϵ_i's, and $\gamma_{r-1}{}^{(i)}$ the (r–1)th-order function deleting the parameter of item i (Fisher, 1974, chap. 14). Note that the sum in (14) actually ranges from r = 1 to r = I–1, because $P_{0i} = 0$ and $P_{Ii} = 1$. Note further that (6) depends on the observed vector $(n_0,\ldots,n_I)$ and, through (14), also on the difficulties of the other items in the test.

Suppose the examinees are sampled from an ability distribution with a logistic distribution function $L(\theta;\alpha,\beta)$. Now, since in each distinct sample of size N the item parameters are estimated conditionally on the sample distribution of the number-correct score, information on the item parameters should be evaluated across sampling. Hence, observing that (6) is linear in n_r, it follows for the expected value of the conditional information that

(15) $E_L[I(\sigma_i|S_1,\ldots,S_N)] = \sum_{r=1}^{I-1} P_{ri}(1-P_{ri})E_L(N_r)$.

For a fixed examinee with ability θ, the number-right score distribution is the generalized binomial with parameters $P(\theta_1),\ldots,P(\theta_I)$ given by (1), which has the moment generating function

(16) $\varphi(t) = \Pi_{i=0}^{I}[(Q(\theta_i) + P(\theta_i))e^t]$.

with $Q(\theta_i) = 1 - P(\theta_i)$ (Kendall & Stuart, 1977, sect. 5.10). The average total score distribution with respect to $L(\theta;\alpha,\beta)$ gives the values of $E_L(N_r)$ needed to compute (15).

A Numerical Comparison

Unlike the information function for the marginal model with logistic ability in (13), a representation of (15) in closed form seems not possible for logistic ability. Hence, in order to compare item information in the conditional model with information in the marginal and unconditional models, a numerical comparison was made. Since the distribution of the N_r's in (15) also depends on the number of items as well as the distribution of their parameter values in the test, comparisons were made for different cases: Tests with 5 and 20 items were simulated. The distribution of the item parameter values was chosen to be uniform, skewed to the right, or normal on the intervals $\langle -3,3\rangle$, $\langle -3,1\rangle$, $\langle -1,1\rangle$ and $\langle -5,-4\rangle$. The last interval was selected to simulate the case of an ill-matched, far-too-easy test. To realize the shape of the above item parameter distributions as closely as possible for a finite number of items, the actual item parameter values were chosen to be equal to the expected values of the order statistics on the interval of possible values for the distributions considered. For example, −2 is the expected value of the smallest value in a random sample of size 5 from a uniform distribution on $\langle -3,3\rangle$ (see Table 2 below).

For each of the tests, (15) was calculated by generating the abilities of 10,000 examinees from the L(θ;0,1) distribution. For each examinee the expected distribution of total scores was computed using the generalized binomial in (16). The average of these distributions over all examinees gave the expected total score distribution for the ability distribution concerned, i.e., the expected values of the N_r's in (15).

Tables 2–5 give the results; they show the percentages

Insert Tables 2–5 about here

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by which the (expected) information in the marginal and unconditional model in (13) exceeds the conditional model for the given values of σ in the test. For example, for the 20-item test with a skewed distribution of the item parameter values in Table 2, the items with parameters values −1.85 and 0.52 yielded values for Fisher's (expected) information measure that were 9% and 6% larger for the marginal/unconditional model than for the unconditional model. From the tables it is clear that for all simulated tests conditional information was less than marginal or unconditional information. The differences were larger for the tests of 5 items; the 20-item tests typically resulted in differences in the 5–10% range. Also, conditional information tended to be relatively large for items with parameter values close to the mean of the ability distribution. This feature is manifest in each of the tables, but can also be observed when the results for tests with parameter values on ⟨−1,1⟩ are compared with those on ⟨−3,3⟩.

It is recalled that in the conditional model, the information measure in (15) is not based on response vectors with all items correct or wrong; such data are simply "conditioned out" by the model. In the marginal model, however, all data are used to estimate the item parameters. Since the probability of a response vector with all items correct or wrong depends on the number of items in the test and the distribution of their parameter values relative to the ability distribution, the pattern in Tables 2–5 could be explained by this phenomenon. Table 6 gives the percentages

Insert Table 6 about here

of response vectors with all items wrong or correct. Although the percentages resemble the pattern in Tables 2–5, the loss of information in the conditional model is not completely explained by the loss of data.

Conclusions

The general impression from the numerical results is that for practical test lengths and distributions of item parameter values that are on target, the loss of information in the conditional model relative to the marginal and unconditional models is less than ten percent. Roughly speaking, this means that for the conditional model the sample sizes have to exceed those for the marginal and conditonal models by the same percentage to guarantee an equal amount of information.

The results in this paper suggest the following guidelines for selecting one of the three Rasch models available:

(i) If the abilities are considered as unknown but fixed parameters, conditional information should be used. This is a known function of σ whose value depends on the observed sufficient statistics in the sample. The unconditional information measure is less useful; it depends on the unknown abilities, which can not be estimated consistently unless the test length also tends to infinity (Haberman, 1977). Since conditional information is known only after the data have been observed, it can be used to construct confidence intervals for the item parameters, but not to determine optimal sample sizes.

(ii) If the abilities are sampled from a distribution with unknown density function g, the choice of the proper information measure depends on the sample size. Engelen (1987) has shown that g can be estimated consistently along with the item parameters. Hence, for large samples, Proposition 2 suggests the use of the estimated marginal measure $I(\sigma;\hat{g})$; because of Proposition 3, this is equal to the expected unconditional measure. Again, the information measures can only be used to construct confidence intervals, not for determining sample sizes.

(iii) If the abilities are sampled from a known density function g, Proposition 2 motivates the choice of $I(\sigma;g)$ for all sample sizes. Now the information measure can also be used to determine sample sizes.

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Table 1

Item information in the marginal model with a logistic ability distribution at different locations.

σ	α			
	0.50	1.00	1.50	2.00
−2.88	0.059	0.044	0.032	0.022
−2.64	0.067	0.051	0.037	0.027
−2.40	0.076	0.058	0.043	0.032
−2.16	0.086	0.067	0.050	0.037
−1.92	0.096	0.076	0.058	0.043
−1.68	0.106	0.085	0.066	0.050
−1.44	0.116	0.095	0.075	0.057
−1.20	0.126	0.105	0.084	0.065
−0.96	0.135	0.115	0.094	0.074
−0.72	0.144	0.125	0.104	0.083
−0.48	0.152	0.135	0.114	0.093
−0.24	0.158	0.143	0.124	0.103
0.00	0.163	0.151	0.134	0.113
0.24	0.166	0.157	0.143	0.123
0.48	0.167	0.162	0.150	0.133
0.72	0.166	0.165	0.157	0.142
0.96	0.163	0.167	0.162	0.150
1.20	0.159	0.166	0.165	0.156
1.44	0.153	0.164	0.167	0.162
1.68	0.145	0.159	0.166	0.165
1.92	0.137	0.153	0.164	0.167
2.16	0.127	0.146	0.160	0.166
2.40	0.118	0.138	0.154	0.164
2.64	0.107	0.128	0.147	0.160
2.88	0.097	0.118	0.138	0.154

Table 2

Information in the marginal model relative to the conditional model for distributions of item parameter values on ⟨−3,3⟩

No. of Items	Type of Distribution					
	Uniform		Skewed		Normal	
	σ	%	σ	%	σ	%
5	−2.00	50	−2.34	40	−1.68	43
	−1.00	39	−1.65	38	−0.80	34
	0.00	37	−0.80	42	0.00	32
	1.00	38	0.32	60	0.80	33
	2.00	48	3.00	28	1.68	42
20	−2.89	11	−2.85	8	−2.71	10
	−2.27	10	−2.69	8	−2.43	10
	−1.85	9	−2.53	8	−2.14	9
	−1.52	8	−2.37	7	−1.86	9
	−1.23	8	−2.20	7	−1.57	8
	−0.97	8	−2.02	7	−1.29	8
	−0.74	7	−1.84	7	−1.00	8
	−0.52	7	−1.65	7	−0.71	7
	−0.31	7	−1.45	7	−0.43	7
	−0.10	7	−1.24	7	−0.14	7
	0.10	6	−1.03	7	0.14	7
	0.31	6	−0.80	7	0.43	7
	0.52	6	−0.55	7	0.71	7
	0.74	6	−0.29	7	1.00	7
	0.97	7	0.00	8	1.29	7
	1.23	7	0.32	8	1.57	7
	1.52	7	0.67	9	1.86	7
	1.85	7	1.10	9	2.14	7
	2.27	8	1.66	11	2.43	8
	2.89	9	3.00	18	2.71	8

Table 3

Information in the marginal model relative to the conditional model for distributions of item parameter values on ⟨−3,1⟩

No. of Items	Type of Distribution					
	Uniform		Skewed		Normal	
	σ	%	σ	%	σ	%
5	−2.33	31	−2.58	30	−1.97	28
	−1.67	28	−2.10	29	−1.46	26
	−1.00	29	−1.53	32	−1.00	27
	−0.34	34	−0.79	40	−0.54	30
	0.34	46	1.00	94	−0.03	38
20	−2.81	8	−2.90	7	−2.67	7
	−2.62	7	−2.80	7	−2.31	7
	−2.43	7	−2.69	7	−2.07	7
	−2.24	7	−2.58	7	−1.88	7
	−2.05	7	−2.46	7	−1.71	6
	−1.86	7	−2.35	7	−1.56	6
	−1.67	7	−2.23	7	−1.43	6
	−1.48	7	−2.10	7	−1.30	6
	−1.29	7	−1.97	7	−1.18	6
	−1.10	6	−1.83	7	−1.06	6
	−0.90	6	−1.68	7	−0.94	6
	−0.71	6	−1.53	7	−0.82	6
	−0.52	6	−1.37	7	−0.70	6
	−0.33	7	−1.19	7	−0.57	6
	−0.14	7	−1.00	7	−0.44	6
	0.05	7	−0.79	8	−0.29	6
	0.24	7	−0.55	8	−0.12	7
	0.43	7	−0.26	9	0.07	7
	0.62	8	0.11	10	0.31	7
	0.81	9	1.00	15	0.67	9

Table 4

Information in the marginal model relative to the conditional model for distributions of item parameter values on (–1,1)

No. of Items	Type of Distribution					
	Uniform		Skewed		Normal	
	σ	%	σ	%	σ	%
5	–0.67	28	–0.79	28	–0.69	29
	–0.33	26	–0.55	27	–0.39	26
	0.00	25	–0.26	27	0.00	25
	0.33	26	0.10	27	0.69	26
	0.67	28	1.00	34	0.39	28
20	–0.90	6	–0.95	6	–1.81	7
	–0.81	6	–0.90	6	–0.93	6
	–0.71	6	–0.84	6	–0.76	6
	–0.62	6	–0.79	6	–0.62	6
	–0.52	6	–0.73	6	–0.50	6
	–0.43	5	–0.67	5	–0.40	6
	–0.33	5	–0.61	5	–0.30	5
	–0.24	5	–0.55	5	–0.21	5
	–0.14	5	–0.48	5	–0.13	5
	–0.05	5	–0.41	5	–0.04	5
	0.05	5	–0.34	5	0.04	5
	0.14	5	–0.26	5	0.13	5
	0.24	5	–0.18	5	0.21	5
	0.33	5	–0.09	5	0.30	5
	0.43	5	0.00	5	0.40	5
	0.52	5	0.10	5	0.50	5
	0.62	5	0.22	5	0.62	5
	0.71	5	0.37	5	0.76	5
	0.81	5	0.55	6	0.93	6
	0.90	5	1.00	6	1.81	6

Table 5

Information in the marginal model relative to the conditional model for distributions of item parameter values on ⟨−5,−4⟩

No. of Items	Type of Distribution					
	Uniform		Skewed		Normal	
	σ	%	σ	%	σ	%
5	−4.83	23	−4.94	24	−4.74	25
	−4.67	26	−4.78	25	−4.61	27
	−4.50	29	−4.63	27	−4.50	29
	−4.33	33	−4.45	30	−4.39	32
	−4.17	38	−4.00	45	−4.26	35
20	−4.95	8	−4.98	8	−4.92	8
	−4.90	8	−4.95	8	−4.83	8
	−4.86	8	−4.92	8	−4.77	8
	−4.81	8	−4.89	8	−4.72	8
	−4.76	8	−4.87	8	−4.67	8
	−4.72	8	−4.84	9	−4.64	8
	−4.67	8	−4.80	9	−4.61	8
	−4.62	8	−4.78	9	−4.58	8
	−4.57	8	−4.74	9	−4.55	9
	−4.52	9	−4.71	9	−4.52	9
	−4.48	9	−4.67	9	−4.49	9
	−4.43	9	−4.63	9	−4.46	9
	−4.38	9	−4.59	9	−4.43	9
	−4.33	9	−4.55	9	−4.39	9
	−4.29	9	−4.50	9	−4.36	9
	−4.24	9	−4.45	9	−4.32	9
	−4.19	10	−4.39	10	−4.28	9
	−4.14	10	−4.32	10	−4.23	9
	−4.10	10	−4.22	10	−4.17	10
	−4.05	10	−4.00	11	−4.08	10

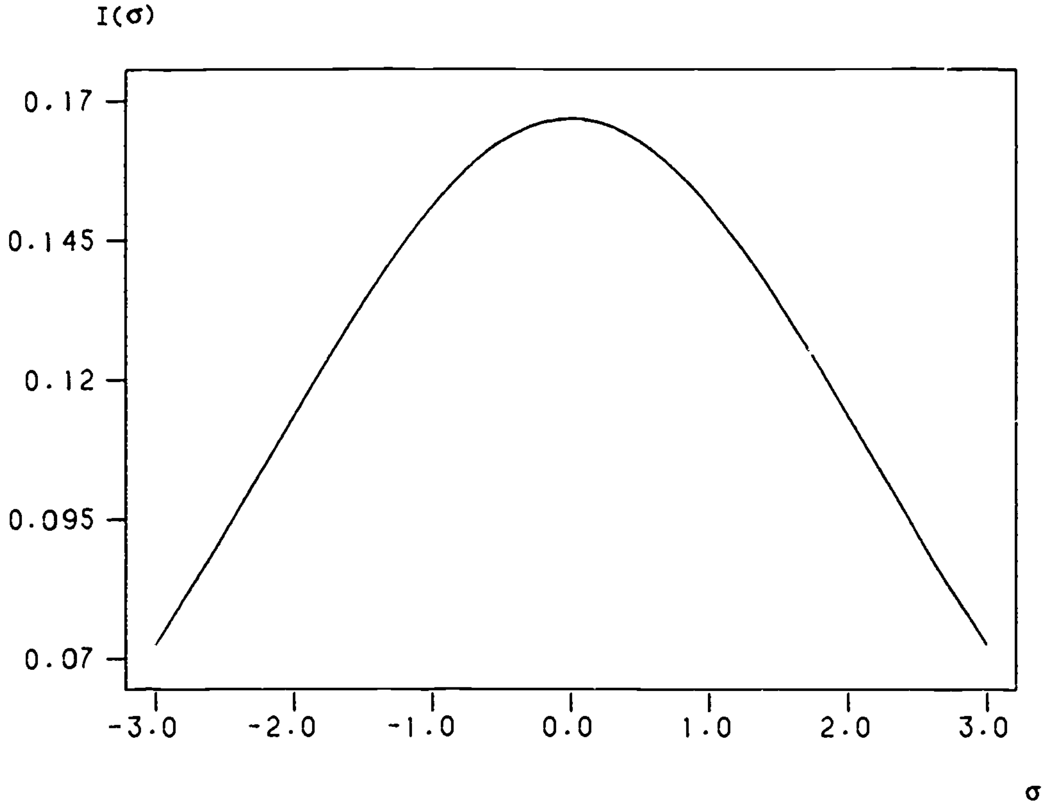
Table 6

Percentages of response vectors with all items correct or all items incorrect

Test Length	Range of Item Parameter Values	Type of Distribution Uniform	Skewed	Normal
5	⟨−3,3⟩	20	15	24
	⟨−3,1⟩	32	29	35
	⟨−1,1⟩	31	30	31
	⟨−5,−4⟩	87	87	87
20	⟨−3,3⟩	5	4	4
	⟨−3,1⟩	9	12	10
	⟨−1,1⟩	9	9	8
	⟨−5,−4⟩	69	71	69

Figure Caption

Figure 1. Item information in the marginal model with a logistic ability distribution (α=0)



I(σ)
0.17
0.145
0.12
0.095
0.07
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
σ

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