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Shape preserving C^2 interpolatory subdivision schemes

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Shape Preserving C^2 Interpolatory Subdivision Schemes

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Stationary interpolatory subdivision schemes which preserve shape properties such as convexity or monotonicity are constructed. The schemes are rational in the data and generate limit functions that are at least C^2. The emphasis is on a class of six-point convexity preserving subdivision schemes that generate C^2 limit functions. In addition, a class of six-point monotonicity preserving schemes that also leads to C^2 limit functions is introduced. As the algebra is far too complicated for an analytical proof of smoothness, validation has been performed by a simple numerical methodology.

Keywords: rational stationary subdivision, shape preservation, smoothness, convexity, monotonicity, positivity

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1. Introduction

Several stationary nonlinear subdivision schemes for the purpose of shape preserving interpolation have been proposed in literature. Many of these schemes use only four points, however, the limit function generated by these schemes is at most C^1 in general. Examples of such schemes are a rational C^1 convexity preserving interpolatory subdivision scheme, see [13,14,10], and a monotonicity

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preserving subdivision scheme, see [11].
In this paper shape preserving interpolatory subdivision schemes are constructed that generate limit functions that are at least C^2. These schemes are less local than the subdivision schemes from literature: now six points are used. The main focus is on a class of six-point *convexity* preserving subdivision schemes that generate C^2 limit functions. In addition, a class of six-point monotonicity preserving schemes is introduced that also leads to C^2 limit functions. The smoothness properties of the subdivision schemes are analysed numerically, as the algebra for an analytical proof of smoothness is far too complicated.

Some shape preserving rational spline interpolation methods have been introduced in [8] (monotonicity preservation), and [3], [2] (convexity preservation). Shape preserving subdivision algorithms have been examined in literature, e.g., convexity preserving subdivision is examined in [15] and [6]. However, the proposed methods generate results that are only C^1 in general.
The goal of this paper is to examine C^2 interpolatory subdivision schemes. The new subdivision point is defined by making use of a two-point Hermite interpolant. The derivatives in this Hermite interpolating function however, are estimated by a four-point scheme on suitable derivative data: divided differences in the function values.
When we take a cubic two-point Hermite interpolant, and the derivatives are estimated using the well-known linear four-point scheme [5], a linear six-point scheme will be obtained. This scheme is known to be C^2-smooth from literature and its approximation order is four.
The same approach is performed for shape preserving subdivision schemes. First, the case of convexity preservation is discussed. The Hermite interpolant is taken as the rational Hermite interpolant, based on [2], which preserves convexity. The derivatives are estimated by using the class of four-point monotonicity preserving interpolatory subdivision schemes [11]. to the (monotone) divided differences in the function values. This leads to rational interpolatory subdivision schemes that preserve convexity. As the expressions arising in the smoothness analysis of the limit function become too complicated, these properties are analysed using a numerical approach. A simple methodology is presented for this purpose, and the method is compared with known results. Based on this numerical approach, the smoothness of the convex scheme turns out to be C^2. Numerical experiments show that the approximation order is four.

The construction of C^2 convexity preserving subdivision schemes is repeated in a similar way for monotonicity preserving subdivision. Again, a rational Hermite spline interpolant, see [8], defines the new subdivision point such that monotonicity is preserved. A suitable four-point interpolatory positivity preserving subdivision scheme is constructed. This scheme, which turns out to be C^1, is applied to the (positive) divided differences in the functions values. The six-point monotonicity preserving subdivision scheme that results from this process appears to be C^2, and its approximation order is four.

2. Problem definition

Consider a univariate initial data set $\{(x_i^{(0)}, f_i^{(0)})\}$ in $I\!R^2$, where the $\{x_i^{(0)}\}$ are equidistantly distributed points, i.e., $x_i^{(k)} = 2^{-k}ih$. The differences $h_i^{(k)}$ are defined by

$$h_i^{(k)} = dx_i^{(k)} = x_{i+1}^{(k)} - x_i^{(k)}, \tag{2.1}$$

and they satisfy $h^{(k)} := h_i^{(k)} = 2^{-k}h$.

The approach in this paper is to construct higher order subdivision schemes which generate limit functions that are at least C^2. We consider subdivision schemes that define the new points $f_{2i+1}^{(k+1)}$ depending on two old data values $f_i^{(k)}$ and $f_{i+1}^{(k)}$, and two derivative estimates $\widetilde{g}_i^{(k)}$ and $\widetilde{g}_{i+1}^{(k)}$. Each derivative estimate $\widetilde{g}_j^{(k)}$ is assumed to be determined by at most five data points: $f_{j-2}^{(k)}$, $f_{j-1}^{(k)}$, $f_j^{(k)}$, $f_{j+1}^{(k)}$ and $f_{j+2}^{(k)}$.

For linear subdivision schemes the reproduction of linear functions is a necessary condition for C^1. Since the subdivision schemes in this paper are required to be at least C^1, we restrict ourselves to subdivision schemes that are exact for linear functions: the schemes are considered to satisfy the following assumption:

Definition 2.1. (Assumption) A *nonlinear* interpolatory subdivision scheme can only be C^ℓ, if the scheme for the ℓ-th differences $\Delta^\ell f_i^{(k)}$ exists and reproduces constants, i.e., the scheme reproduces polynomials of degree ℓ.

A simple calculation yields that we indeed examine the following class of six-point interpolatory subdivision schemes:

$$\begin{cases} f_{2i}^{(k+1)} = f_i^{(k)}, \\ f_{2i+1}^{(k+1)} = \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right) + h^{(k)}\mathcal{F}_1(\Delta f_i^{(k)}, \widetilde{g}_i^{(k)}, \widetilde{g}_{i+1}^{(k)}), \\ \widetilde{g}_i^{(k)} \quad = \mathcal{F}_2(\Delta f_{i-2}^{(k)}, \Delta f_{i-1}^{(k)}, \Delta f_i^{(k)}, \Delta f_{i+1}^{(k)}), \end{cases} \tag{2.2}$$

where the divided differences $\Delta f_i^{(k)}$ are defined by

$$\Delta f_i^{(k)} = \frac{f_{i+1}^{(k)} - f_i^{(k)}}{x_{i+1}^{(k)} - x_i^{(k)}} = \frac{df_i^{(k)}}{h_i^{(k)}}. \tag{2.3}$$

A scheme in this class is attractive, because it is a six-point scheme, whereas the derivatives are estimated by only a four-point scheme. Another reason to restrict to this class of schemes is that it turns out that it contains schemes that generate C^2 limit functions that preserve shape properties like convexity or monotonicity. The choice for the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ depends on the requirements, e.g., linearity of the scheme or the requirement of shape preserving properties. Suitable choices for these functions are discussed in the next sections.

Motivation. As the schemes presented in this paper are required to have a relatively simple generalisation to nonuniform data, the motivation uses a comparison with *nonuniform* subdivision schemes.

In [12], a class of convexity-preserving interpolatory subdivision schemes for nonuniform data has been constructed. One scheme,

$$\begin{cases} x_{2i}^{(k+1)} = x_i^{(k)}, \\ x_{2i+1}^{(k+1)} = \frac{1}{2}\left(x_i^{(k)} + x_{i+1}^{(k)}\right) + \frac{1}{2}h_i^{(k)}\mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)}), \\ f_{2i}^{(k+1)} = f_i^{(k)}, \\ f_{2i+1}^{(k+1)} = \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right) + \frac{1}{2}\mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})\left(f_{i+1}^{(k)} - f_i^{(k)}\right) \\ \qquad\qquad -h_i^{(k)}\mathcal{F}(s_i^{(k)}, s_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)}), \end{cases} \tag{2.4}$$

combined with the function

$$\mathcal{F}(x, y, r, R) = \frac{1}{2}\frac{1}{\frac{1+r}{(1+\mathcal{G}(r,R))x} + \frac{1+R}{(1-\mathcal{G}(r,R))y}}, \tag{2.5}$$

turns out to have approximation order three. This scheme is compared with the nonuniform scheme that directly comes from the rational Hermite interpolant (see [2])

$$u_i^{(k)}(x) = \frac{(x_{i+1}^{(k)} - x)f_i^{(k)} + (x - x_i^{(k)})f_{i+1}^{(k)}}{x_{i+1}^{(k)} - x_i^{(k)}} - \frac{1}{\frac{1}{(x-x_i^{(k)})(\Delta f_i^{(k)} - g_i^{(k)})} + \frac{1}{(x_{i+1}^{(k)}-x)(g_{i+1}^{(k)} - \Delta f_i^{(k)})}}, \tag{2.6}$$

by a simple evaluation at $x_{2i+1}^{(k+1)}$:

$$f_{2i+1}^{(k+1)} = \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right) + \frac{1}{2}\mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})\left(f_{i+1}^{(k)} - f_i^{(k)}\right) - \frac{1}{2}h_i^{(k)}\frac{1}{\frac{1}{(1+\mathcal{G}(r_i^{(k)},R_{i+1}^{(k)}))(\Delta f_i^{(k)} - \widetilde{g}_i^{(k)})} + \frac{1}{(1-\mathcal{G}(r_i^{(k)},R_{i+1}^{(k)}))(\widetilde{g}_{i+1}^{(k)} - \Delta f_i^{(k)})}}, \tag{2.7}$$

The question that arises is the following: how should the derivatives in this scheme be chosen such that it reduces to the scheme for nonequidistant data? A simple calculation shows that the scheme is obtained by estimating the derivatives using

$$g_i = \frac{h_i}{h_{i-1} + h_i}\Delta f_{i-1} + \frac{h_{i-1}}{h_{i-1} + h_i}\Delta f_i, \quad h_i = x_{i+1} - x_i, \tag{2.8}$$

which is the nonuniform variant of estimating the derivatives by a two-point scheme on divided differences $\Delta f_i^{(k)}$. For equidistant data, this yields:

$$\widetilde{g}_i^{(k)} = \frac{1}{2}\left(\Delta f_{i-1}^{(k)} + \Delta f_i^{(k)}\right). \tag{2.9}$$

A useful interpretation of (2.9) is that the derivative is estimated using a two-point scheme that preserves monotonicity. This scheme operates on successive *divided differences* $\Delta f_j^{(k)}$, and indeed, the divided differences from a monotone sequence for any convex data set. In order to preserve convexity, the derivatives have to be estimated in a monotonicity preserving way: $\Delta f_{i-1}^{(k)} \leq \widetilde{g}_i^{(k)} \leq \Delta f_i^{(k)}$. Any monotone two-point scheme for the derivative estimates can be written as $\widetilde{g}_i^{(k)} = \mathcal{M}(\Delta f_{i-1}^{(k)}, \Delta f_i^{(k)})$.

The straightforward generalisation is to apply a *four-point* monotonicity preserving subdivision scheme to successive divided differences, instead of the simple two-point scheme.

Taking the two-point scheme (which is only C^0) for the divided differences, finally yields a convex limit function that is C^1, see chapter 3. Therefore it is reasonable that convex C^2 limit functions are obtained if a C^1 monotonicity preserving scheme for the divided difference is used to determine the derivative estimates. The derivative estimates in the class

$$\widetilde{g}_i^{(k)} = \mathcal{M}(\Delta f_{i-2}^{(k)}, \Delta f_{i-1}^{(k)}, \Delta f_i^{(k)}, \Delta f_{i+1}^{(k)}), \tag{2.10}$$

are discussed and analysed in section 5.

In the next section, we discuss linear six-point subdivision schemes in the class (2.2). The smoothness properties of these linear schemes are known from literature and can be analysed using standard techniques, e.g., based on Laurent polynomials: no further smoothness analysis is required for these schemes.

3. Linear six-point interpolatory subdivision

In this section six-point interpolatory subdivision schemes in the class (2.2) are examined which are linear in the data.

First, we state some known results for linear six-point schemes. A general class of linear six-point interpolatory subdivision schemes is given in [5]:

$$\begin{aligned} f_{2i+1}^{(k+1)} &= \left(\frac{1}{2} + w + 2\theta\right)\left(f_i^{(k)} + f_{i+1}^{(k)}\right) - (w+3\theta)(f_{i-1}^{(k)} + f_{i+2}^{(k)}) + \theta(f_{i-2}^{(k)} + f_{i+3}^{(k)}), \\ &= \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right) - h^{(k)}(w+2\theta)(\Delta f_{i+1}^{(k)} - \Delta f_{i-1}^{(k)}) + h^{(k)}\theta(\Delta f_{i+2}^{(k)} - \Delta f_{i-2}^{(k)}) \\ &= \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right) - h^{(k)}(w+\theta)(s_i^{(k)} + s_{i+1}^{(k)}) + h^{(k)}\theta(s_{i+2}^{(k)} + s_{i-1}^{(k)}), \end{aligned} \tag{3.1}$$

where the second differences $s_i^{(k)}$ are defined as

$$s_i^{(k)} = \Delta f_i^{(k)} - \Delta f_{i-1}^{(k)}. \tag{3.2}$$

Subdivision scheme (3.1) reproduces quadratic (even cubic) polynomials if $w = 1/16$. As a linear subdivision scheme is required to reproduce quadratic polynomials in order to be able to generate C^2 functions, we restrict to this value $w = 1/16$ and the scheme then is at least fourth order accurate. A sufficient range for C^2-convergence of (3.1) is $0 < \theta < 0.02$, see [5].

A special case is obtained by the six-point subdivision scheme (3.1) with $w = 1/16$ and $\theta = 3/256$. The scheme then reproduces quintic polynomials and has approximation order six. Since $\theta < 0.02$, see [5], this scheme is C^2.

For the purpose of this chapter, the subdivision value $f_{2i+1}^{(k+1)}$ is determined by the two-point cubic Hermite-interpolant:

$$f_{2i+1}^{(k+1)} = \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right) - \frac{1}{8}h^{(k)}(\widetilde{g}_{i+1}^{(k)} - \widetilde{g}_i^{(k)}).$$

The derivative estimates $\widetilde{g}_i^{(k)}$ are determined by applying the linear four-point scheme [5],

$$\begin{cases} f_{2i}^{(k+1)} = f_i^{(k)}, \\ f_{2i+1}^{(k+1)} = -wf_{i-1}^{(k)} + \left(\frac{1}{2} + w\right) f_i^{(k)} + \left(\frac{1}{2} + w\right) f_{i+1}^{(k)} - wf_{i+2}^{(k)}, \end{cases} \tag{3.3}$$

for some w_1, to four successive divided differences, i.e., as in (2.10). Then this results in the following six-point interpolatory subdivision scheme:

$$f_{2i+1}^{(k+1)} = \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right) - h^{(k)}\frac{1}{16}(1+2w_1)(s_i^{(k)} + s_{i+1}^{(k)}) + 2h^{(k)}\frac{1}{16}w_1(s_{i-1}^{(k)} + s_{i+2}^{(k)}),$$

The smoothness properties of this scheme are further discussed. The scheme automatically reproduces quadratic polynomials, which is necessary for C^2.
If we take $w_1 = 1/16$ this generates in the six-point scheme with $w = 1/16$ and $\theta = 1/128$, which is C^2 as $\theta < 0.02$ (and fourth order accurate), see [5]. However, the derivatives are only estimated second order accurate then. If the derivatives are estimated fourth order accurate, a simple calculation shows that the tension parameter has to be taken as $w_1 = 1/12$, and this generates the six-point C^2 scheme with $w = 1/16$ and $\theta = 1/96$, see [5]. Both schemes have approximation order four, where as the scheme determined by the quintic fit (with $\theta = 3/256$ is six-th order accurate.

The goal of this chapter is to construct six-point subdivision schemes which are shape-preserving. In contrast with linear six-point schemes discussed in this section, we cannot use smoothness properties from literature. On the other hand, the algebraic expressions that arise from an analytical proof of C^1 and especially C^2-smoothness become complicated. To deal with this problem, a numerical approach is required for proving, or at least validating, the smoothness properties. Such a numerical method is presented in the next section.

4. A numerical approach for smoothness analysis

In the previous chapters, some four-point interpolatory subdivision schemes have been presented that preserve the shape in the data. To prove the smoothness and approximation properties of these schemes however, the complexity of the algebraic expressions turns out to be much involved. Especially, the rational six-point schemes constructed in this chapter give rise to unmanageable expressions, which means that a numerical method is unavoidable to determine the smoothness properties of the subdivision schemes. Such a numerical approach is briefly discussed in this section. The method is validated with known results for linear subdivision schemes as well as the rational subdivision schemes from the previous chapters.

Numerical determination of the Hölder regularity. The numerical approach for the analysis of smoothness properties of subdivision schemes which is set up in this section deals with the notion of Hölder regularity H_r:

Definition 4.1 (Hölder regularity). An ℓ times continuously differentiable function $f : \Omega \subset I\!R \rightarrow I\!R$ is said to have Hölder regularity $H_r = \ell + \alpha$, if

$$\exists C < \infty \ \text{ such that } \ \left| \frac{\partial^\ell f(x_1)}{\partial x^\ell} - \frac{\partial^\ell f(x_2)}{\partial x^\ell} \right| \leq C \left| x_1 - x_2 \right|^\alpha, \quad \forall x_1, x_2 \in \Omega.$$

Next, the definition of Hölder regularity is applied to subdivision schemes, which requires a suitable definition for the discrete case.
For any data set on the k-th iteration, $\{x_i^{(k)}, f_i^{(k)}\}_i$, consider any two successive data points, say $x_1 = 2^{-k} i h$ and $x_2 = 2^{-k}(i+1)h$. According to definition 4.1, a subdivision scheme is said to have Hölder regularity $H_r = \alpha$, if:

$$\exists C < \infty \ \text{ such that } \ \max_i \left| f_{i+1}^{(k)} - f_i^{(k)} \right| \leq C (2^{-k} h)^\alpha, \ \forall k.$$

The Hölder regularity $\ell + \alpha$ of a subdivision scheme is more complicated. First, the subdivision scheme is required to be at least C^ℓ, and the ℓ-th derivatives in definition 4.1 are replaced by divided differences according to $f^\ell(x_i^{(k)}) \approx \ell! \Delta^\ell f_i^{(k)} + \mathcal{O}(h)$; this $\mathcal{O}(h)$ turns out to be irrelevant.
The Hölder regularity $\ell + \alpha$ is defined using ℓ-th divided differences:

Definition 4.2 (Hölder regularity of subdivision). A C^ℓ subdivision scheme is said to have Hölder regularity $\ell + \alpha_\ell$, if

$$\exists C < \infty \ \text{ such that } \ \lim_{k \rightarrow \infty} \ell! \left| \Delta^\ell f_{i+1}^{(k)} - \Delta^\ell f_i^{(k)} \right| \leq C (2^{-k} h)^{\alpha_\ell}.$$

When $\alpha_\ell = 1$ in definition 4.2, this means that the subdivision scheme is almost $C^{\ell+1}$.

This definition suggest to define an algorithm to determine the Hölder regularity of a subdivision scheme. Therefore, define

$$\rho_\ell^{(k)} = \ell! \cdot \max_i |\Delta^\ell f_{i+1}^{(k)} - \Delta^\ell f_i^{(k)}|,$$

and we assume that the maximum values are attained, i.e., $\rho_\ell^{(k)} \approx C(2^{-k}h)^\alpha$. The contraction factor λ_ℓ satisfies

$$\lambda_\ell = \frac{\rho_\ell^{(k+1)}}{\rho_\ell^{(k)}} \approx \frac{C(2^{-(k+1)}h)^{\alpha_\ell}}{C(2^{-k}h)^{\alpha_\ell}} = 2^{-\alpha_\ell},$$

and hence

$$\alpha_\ell := -\log_2\left(\frac{\rho_\ell^{(k+1)}}{\rho_\ell^{(k)}}\right). \tag{4.1}$$

provides a good estimate for the Hölder regularity $\ell + \alpha_\ell$.
Note that the calculation of the Hölder regularity $H_r = \ell + \alpha_\ell$ only makes sense if $\alpha_j \approx 1$, $j = 0, \ldots, \ell - 1$. When we briefly write e.g., $H_r \approx 1.442$, we mean that $\alpha_0 = 1$ and $\alpha_1 = 0.442$.

Numerical validation. The simple numerical approach proposed in this section is applied to subdivision schemes for which the smoothness properties are known. In the recent preprint [17] some Lagrange and Hermite interpolatory subdivision schemes are examined, and bounds on the Hölder regularity have been obtained numerically. For some linear subdivision schemes, the results in [17] are compared with the numerical method proposed in this section.
First we examine linear schemes and calculate the numerical results that are obtained for H_r, and compare these results with the literature. The numerical validation is continued for nonlinear subdivision schemes for which no methodology for smoothness analysis is known to us in literature:

- The linear four-point scheme [5]. For the value $w = 1/16$, the numerical value for the Hölder regularity is $H_r \approx 2.000$, since $\alpha_0 \approx 1$, $\alpha_1 \approx 1$ and $\alpha_2 \approx 0$. Indeed, note that it is known from literature, e.g., [4], that this scheme is almost C^2. For example, if $w = 1/32$, we obtained that $H_r \approx 1.228$: indeed a C^1 scheme.
- The second example concerns the linear six-point scheme, see (3.1), with $w = 1/16$. We give the numerical results for three values of θ: $\theta = 3/256$ yields $H_r \approx 2.8301$, $\theta = 1/128$ yields $H_r \approx 2.3919$ and $H_r \approx 2.6309$ for $\theta = 1/96$.

For the case $\theta = 3/256$, the smoothness results can be compared with lower and upper bounds in [17]: $2.8094 \leq H_r \leq 2.8301$. The upper bound turns out to be sharp.

- The first example for numerical validation of nonlinear subdivision schemes considers an equidistant convex data set for which the second differences satisfy $d^2 f_{2i}^{(0)} = \sqrt{\beta}$ and $d^2 f_{2i+1}^{(0)} = 1/\sqrt{\beta}$. We show the relation of β with the Hölder regularity of the convexity preserving four-point scheme in [14], i.e., the scheme

$$f_{2i+1}^{(k+1)} = \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right) - \frac{1}{4}\frac{1}{\frac{1}{f_{i+1}^{(k)} - 2f_i^{(k)} + f_{i-1}^{(k)}} + \frac{1}{f_{i+2}^{(k)} - 2f_{i+1}^{(k)} + f_i^{(k)}}}, \tag{4.2}$$

which is known to be C^1.
Using the numerical methodology for several values of β we determine α_1 for which the scheme has regularity $H_r = 1 + \alpha_1$. The contractivity factor that arises in the proof of C^1-smoothness in [14], is $q^{(0)}/(1 + q^{(0)}) = \beta/(1 + \beta)$, where β equals the ratios of second differences $d^2 f_j^{(0)}$.
From the numerical results then is obtained that $\alpha_0 = 1$ and

$$\alpha_1 = -\log_2\left(\frac{q^{(0)}}{1 + q^{(0)}}\right) = -\log_2\left(\frac{\beta}{1+\beta}\right),$$

which indeed shows a strong relation with the single step proof in [14]: $\lambda_1 = 2^{-\alpha_1}$.

- Another example for a nonlinear subdivision scheme deals with monotonicity preserving subdivision schemes, see [11], i.e., schemes of the form

$$f_{2i+1}^{(k+1)} = \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right) + \frac{1}{2}(f_{i+1}^{(k)} - f_i^{(k)})\mathcal{G}\left(\frac{f_i^{(k)} - f_{i-1}^{(k)}}{f_{i+1}^{(k)} - f_i^{(k)}}, \frac{f_{i+2}^{(k)} - f_{i+1}^{(k)}}{f_{i+1}^{(k)} - f_i^{(k)}}\right), \tag{4.3}$$

Numerical experiments on several monotone data show that the class of four order accurate schemes is C^1: for all schemes in the class

$$\mathcal{G}(x,y) = \frac{x - y}{\ell_1 + (1 + \ell_2)(x+y) + \ell_3 xy}, \ \ell_1 + 2\ell_2 + \ell_3 = 6, \ \ell_1, \ell_2, \ell_3 \geq 0, \tag{4.4}$$

it is obtained that $H_r = 1.466722\ldots$ independent of the initial data. The value for the Hölder regularity corresponds with a contraction factor of $\lambda \approx 0.724$.

- The final example concern a so-called *trinary* subdivision scheme based on an algorithm that in every step inserts two points instead of one in every interval. A general class of trinary interpolatory subdivision schemes is given by:

$$\begin{cases} f_{3i}^{(k+1)} = f_i^{(k)}, \\ f_{3i+1}^{(k+1)} = \frac{2}{3} f_i^{(k)} + \frac{1}{3} f_{i+1}^{(k)} - \mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}), \\ f_{3i+2}^{(k+1)} = \frac{1}{3} f_i^{(k)} + \frac{2}{3} f_{i+1}^{(k)} - \mathcal{F}(d_{i+1}^{(k)}, d_i^{(k)}). \end{cases} \tag{4.5}$$

  Linear four-point interpolatory subdivision schemes can be constructed which generate C^2 limit functions: the function $\mathcal{F}(x, y) = (1/9 - c)x + cy$ is proved in [16] to generate a C^2 scheme for the range $1/27 < c < 2/45$.
  If $\mathcal{F}$ satisfies $0 \le \mathcal{F}(x, y) \le 1/6 \min\{x, 2y\}$ and $\mathcal{F}(x, y) \le 2\mathcal{F}(y, x)$, $\forall x, y \ge 0$, subdivision scheme (4.5) preserves convexity. A suitable ansatz therefore is

$$\mathcal{F}(x, y) = \frac{1}{\frac{\gamma_1}{x} + \frac{\gamma_2}{y}}$$

  and for the parameter values $\gamma_1 = 6$ and $\gamma_2 = 3$ this scheme turns out to be convexity preserving independent of the initial data. The numerical approach sketched above indicates that this trinary subdivision scheme is also C^2.

5. Six-point convexity preserving subdivision

Six-point convexity preserving interpolatory subdivision schemes are constructed in this section, following the ideas presented in section 2.

Consider the class of subdivision schemes (2.2). We define function values $f_{2i+1}^{(k+1)}$ as the two-point rational Hermite-interpolant, see (2.6), evaluated at the midpoint $x_{2i+1}^{(k+1)}$ (see also (2.7)). Hence,

$$f_{2i+1}^{(k)} = \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right) - \frac{1}{2} h^{(k)} \frac{1}{\frac{1}{\Delta f_i^{(k)} - \widetilde{g}_i^{(k)}} + \frac{1}{\widetilde{g}_{i+1}^{(k)} - \Delta f_i^{(k)}}}, \tag{5.1}$$

where $\widetilde{g}_j^{(k)}$ are *estimates* of derivatives.
In section 2, it is shown that if the derivative estimates are determined by the two-point scheme on successive divided differences, this generates the convexity preserving four-point scheme (4.2).
In addition, we suggested in section 2 to apply a four-point monotonicity preserving subdivision scheme. Such monotonicity preserving subdivision schemes have

been proposed in [11], i.e., (4.3) with (4.4) as a class of C^1 rational four-point schemes, briefly written as:

$$\widetilde{g}_i^{(k)} = \mathcal{M}(\Delta f_{i-2}^{(k)}, \Delta f_{i-1}^{(k)}, \Delta f_i^{(k)}, \Delta f_{i+1}^{(k)}).$$

As each derivative estimate then depends on five data points, the resulting subdivision scheme becomes a six-point scheme. Note that this six-point scheme is a Lagrange scheme and not a Hermite-interpolatory scheme, as the derivatives change: e.g., $\widetilde{g}_{2i}^{(k+1)} \neq \widetilde{g}_i^{(k)}$ in general.

According to this construction and using the definitions (3.2) and (5.3), we finally arrive at the derivative estimate:

$$\widetilde{g}_i^{(k)} = \frac{1}{2}\left(\Delta f_{i-1}^{(k)} + \Delta f_i^{(k)}\right) + \frac{1}{2} s_i^{(k)} \mathcal{G}(q_{i-1}^{(k)}, Q_i^{(k)}), \tag{5.2}$$

where the ratios $q_j^{(k)}$ are defined by

$$q_i^{(k)} = \frac{s_i^{(k)}}{s_{i+1}^{(k)}} \quad \text{and} \quad Q_i^{(k)} = \frac{1}{q_i^{(k)}}, \tag{5.3}$$

and $\mathcal{G}$ is determined by e.g., (4.4). For these explicit derivative estimates, the following theorem can be formulated:

Theorem 5.1. The stationary six-point interpolatory subdivision scheme

$$f_{2i+1}^{(k+1)} = \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right) - \frac{1}{4} h^{(k)} \frac{1}{\frac{1}{s_i^{(k)}\left(1-\mathcal{G}(q_{i-1}^{(k)}, Q_i^{(k)})\right)} + \frac{1}{s_{i+1}^{(k)}\left(1+\mathcal{G}(q_i^{(k)}, Q_{i+1}^{(k)})\right)}} \tag{5.4}$$

preserves convexity. Furthermore, the scheme reproduces quadratic functions, it generates C^2 limit functions, and it has approximation order four.

Proof. Convexity preservation is easily checked from the construction and this directly yields that the scheme converges and generates continuous limit functions. Reproduction of quadratic polynomials is guaranteed, as then the ratios $q_i^{(k)}$ are equal to 1 and hence $\mathcal{G} = 0$ in that case.

Further properties are not examined analytically, as the algebraic expressions involved are too complicated. Using the numerical method from section 4, it is shown that the scheme is C^2: we obtained that the Hölder regularity is $H_r = 2.392\ldots$ for all data sets.

A simple Taylor expansion on initial data and the stability of this scheme yields that the scheme is fourth order accurate. □

It is briefly summarised why subdivision scheme (5.4) is believed to generate C^2 limit functions:

- The numerical approach from section 4 gives the Hölder regularity $H_r = 2.392\ldots$.
- The linear six-point subdivision scheme (3.1) with $w = 1/16$ and $\theta = 1/128$ is constructed in a similar but linear way. This scheme is known to be C^2, see [5].
- The four-point interpolatory monotonicity preserving subdivision scheme is applied to the divided differences, and this scheme is C^1. Therefore, it is reasonable that the resulting scheme for the function values is C^2.

Note that the derivative estimate (5.2) with (4.4) is only second order accurate if $\ell_1 + 2\ell_2 + \ell_3 = 6$. For the case if $\ell_1 + 2\ell_2 + \ell_3 = 4$, the estimate (5.2) with (4.4) is easily checked to be fourth order accurate. Then, subdivision scheme (5.4) still satisfies theorem 5.1. The numerical analysis shows that the regularity of the scheme satisfies $H_r = 2.63091\ldots$, which indicates that a smoother scheme has been obtained by the fourth order accurate derivative estimates.

6. Six-point monotonicity preserving subdivision

In this section we repeat the same constructive approach from the previous section for the purpose of deriving a *monotonicity* preserving subdivision scheme that generates C^2 limit functions. The method is based on a monotonicity preserving rational spline Hermite interpolant. For a suitable determination of the derivative estimates, a *positivity* preserving four-point interpolatory subdivision scheme is required. Therefore, positivity preserving subdivision is first discussed in section 6.1. Then, in section 6.2, the resulting subdivision schemes for positive data are used to construct C^2 monotonicity preserving subdivision schemes.

6.1. Positivity preserving interpolatory subdivision schemes

In this section positivity preserving interpolatory subdivision schemes are examined.

The general class of four-point interpolatory positivity preserving subdivision schemes is given by:

$$\begin{cases} f_{2i}^{(k+1)} = f_i^{(k)}, \\ f_{2i+1}^{(k+1)} = \mathcal{P}(f_{i-1}^{(k)}, f_i^{(k)}, f_{i+1}^{(k)}, f_{i+2}^{(k)}), \end{cases} \tag{6.1}$$

where the function $\mathcal{P}$ has to be further specified.
A simple class of schemes is given by two-point schemes that only depend on $f_i^{(k)}$ and $f_{i+1}^{(k)}$. The simplest two-point scheme is

$$f_{2i+1}^{(k+1)} = \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right), \tag{6.2}$$

which preserves convexity, monotonicity and positivity. However, the limit function is only continuous, as the scheme generates the piecewise linear interpolant to the given data.
Another positivity preserving subdivision scheme is the scheme based on the harmonic mean, see [1,7]:

$$f_{2i+1}^{(k+1)} = \frac{2 f_i^{(k)} f_{i+1}^{(k)}}{f_i^{(k)} + f_{i+1}^{(k)}}. \tag{6.3}$$

However, it can easily be proved that a two-point subdivision scheme cannot generate C^1 limit functions. As the purpose is C^1 positivity preserving subdivision schemes, we therefore proceed with the construction of four-point schemes.

In contrast with convexity preserving subdivision and monotonicity preserving subdivision, there are not many conditions and invariances that can be naturally imposed on the function $\mathcal{P}$ to restrict the general class of schemes (6.1).
As in (6.3), the function $\mathcal{P}$ is assumed to be bilinear in the numerator and linear in the denominator. In addition, $\mathcal{P}$ is assumed to satisfy the symmetry condition $\mathcal{P}(f_1, f_2, f_3, f_4) = \mathcal{P}(f_4, f_3, f_2, f_1)$. The functions $\mathcal{P}$ then automatically have the property of homogeneity, i.e., $\mathcal{P}(\lambda f_1, \lambda f_2, \lambda f_3, \lambda f_4) = \lambda \mathcal{P}(f_1, f_2, f_3, f_4)$.
These observations suggest to restrict the function $\mathcal{P}$ to class

$$\mathcal{P}(f_1, f_2, f_3, f_4) = \frac{a_1 f_2 f_3 + a_2(f_1 f_2 + f_3 f_4) + a_3(f_1 f_3 + f_2 f_4) + a_4 f_1 f_4}{a_5(f_2 + f_3) + a_6(f_1 + f_4)}, \tag{6.4}$$

and this class of subdivision schemes is further restricted by additional conditions on the coefficients a_j in (6.4).
For *linear* subdivision schemes, exactness for linear polynomials is a necessary condition for C^1. Therefore, we assume that $\mathcal{P}$ in (6.4) satisfies the condition

for reproduction of linear functions. This yields the following conditions on the coefficients: $a_1 + 2a_2 + 2a_3 + a_4 = 2a_5 + 2a_6$ and $a_1 - 6a_2 + 6a_3 + 9a_4 = 0$.
A necessary condition for approximation order three is obtained by taking initial data from a smooth function and requiring that the results after one subdivision are third order accurate, which yields: $12a_3 + 16a_4 + 3a_5 - 5a_6 = 0$. These three conditions reduce $\mathcal{P}$ in (6.4) to a class of schemes that satisfies the necessary condition on the initial data for approximation order four. Note that, no scheme in this class reproduces quadratic functions.
Necessary and sufficient for preservation of positivity of subdivision scheme (6.1) with (6.4) is that $a_j \geq 0$, $j = 1, \ldots 6$, which respectively yields

$$4a_4 + 9a_5 + a_6 \geq 0,\ 8a_4 + 3a_5 + 11a_6 \geq 0,\ 5a_6 - 3a_5 - 16a_4 \geq 0,\ a_4, a_5, a_6 \geq 0.$$

In order to further simplify the class of subdivision schemes, we restrict to the case $a_3 = 0$ and $a_4 = 0$, which then uniquely determines $\mathcal{P}$:

$$\mathcal{P}(f_1, f_2, f_3, f_4) = 2\frac{6f_2f_3 + f_2f_1 + f_3f_4}{5(f_2 + f_3) + 3(f_1 + f_4)}, \tag{6.5}$$

and the following result is obtained:

Theorem 6.1. The stationary four-point interpolatory subdivision scheme (6.1) with (6.5) reproduces linear polynomials and preserves positivity.
Furthermore, the scheme generates C^1 limit functions and has approximation order four.

Proof. Positivity preservation and reproduction of polynomials of degree one follow from the construction.
The smoothness properties are examined numerically: the experiments based on the approach in section 4 show that the regularity of the scheme is scheme is $H_r = 2.000\ldots$, i.e., the scheme is almost C^2. Numerical experiments show that the approximation order is four. □

Next, this monotonicity preserving subdivision scheme is used for the construction of C^2 monotonicity preserving subdivision schemes.

6.2. *Construction of C^2 monotone schemes*

The rational two-point Hermite interpolant in [8] that preserves monotonicity is given by

$$u_i(x) = \frac{\Delta f_i f_{i+1} t^2 + (f_i g_{i+1} + f_{i+1} g_i)\, t(1-t) + \Delta f_i f_i (1-t)^2}{\Delta f_i t^2 + (g_{i+1} + g_i) t(1-t) + \Delta f_i (1-t)^2}, \tag{6.6}$$

This spline is evaluated at the parameter value $x_{2i+1}^{(k+1)}$, which generates the subdivision scheme:

$$f_{2i+1}^{(k+1)} = \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right) + \frac{1}{2} h^{(k)} \Delta f_i^{(k)} \frac{\widetilde{g}_i^{(k)} - \widetilde{g}_{i+1}^{(k)}}{\widetilde{g}_i^{(k)} + 2\Delta f_i^{(k)} + \widetilde{g}_{i+1}^{(k)}}. \tag{6.7}$$

Estimating $\widetilde{g}_j^{(k)}$ using two-point schemes yields monotonicity preserving subdivision schemes in the class (4.3) with (4.4). For example, determining $\widetilde{g}_j^{(k)}$ using (6.2) yields $\ell_1 = 6$ and $\ell_2 = \ell_3 = 0$. Application of (6.3) for the derivative estimates gives $\ell_1 = \ell_2 = 1$ and $\ell_3 = 3$. Both schemes are rational, stationary, four-point C^1 subdivision schemes that preserve monotonicity.
As in the previous section, for the construction of C^2 shape preserving subdivision schemes, *four-point* schemes are used to determine the derivative estimates, In order to obtain a six-point scheme that preserves monotonicity, these derivative estimates have to be calculated by a scheme that preserves positivity, i.e.,

$$\widetilde{g}_j^{(k)} = \mathcal{P}(\Delta f_{j-2}^{(k)}, \Delta f_{j-1}^{(k)}, \Delta f_j^{(k)}, \Delta f_{j+1}^{(k)})).$$

The positivity preserving subdivision scheme (6.1) with (6.5) is suited for this purpose.
The resulting six-point monotonicity preserving subdivision scheme becomes:

$$f_{2i+1}^{(k+1)} = \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right) + \frac{1}{2} h^{(k)} \Delta f_i^{(k)} \mathcal{G}(r_{i-1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)}, R_{i+2}^{(k)}), \tag{6.8}$$

where the function $\mathcal{G}$ is a complicated rational function with the following property:

$$\mathcal{G}(r_1, r_2, r_2, r_1) = 0. \tag{6.9}$$

The following theorem can be formulated:

Theorem 6.2. The stationary six-point interpolatory subdivision scheme (6.8) reproduces quadratic polynomials and preserves monotonicity.

Furthermore, the scheme generates C^2 limit functions and has approximation order four.

Proof. Preservation of strict monotonicity is easily checked from the construction and this yields that the scheme converges and generates continuous limit functions. Reproduction of linear polynomials is guaranteed, as then the ratios $r_i^{(k)}$ are equal to 1 and according to (6.9), $\mathcal{G} = 0$ in that case. Straightforward algebra shows that the scheme is also exact for quadratic polynomials.
The smoothness is examined numerically: using the numerical method described in section 4, it has been obtained that the scheme is C^2: the Hölder regularity satisfies $H_r = 2.392\ldots$.
In addition, the approximation order equals four, which straightforwardly follows from the stability of the scheme and approximation order four after one iteration, see [9]. □

As in section 5, the derivatives are only estimated second order accurate. A positivity preserving scheme which yields fourth order accurate derivative estimates is easily checked to be provided by the function

$$\mathcal{P}(f_1, f_2, f_3, f_4) = 3\frac{6f_2f_3 + f_2f_1 + f_3f_4}{7(f_2 + f_3) + 5(f_1 + f_4)}.$$

Then, subdivision scheme (6.8) still satisfies theorem 6.2. The numerical analysis shows that the regularity of the scheme satisfies $H_r = 2.63091\ldots$, which indicates that a smoother scheme has been obtained by the fourth order accurate derivative estimates.

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