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Disproof of an admissibility conjecture of Weiss

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Abstract

Two conjectures on admissible control operators by George Weiss are disproved in this paper. One conjecture says that an operator B defined on an infinite-dimensional Hilbert space U is an admissible control operator if for every element $u \in U$ the vector Bu defines an admissible control operator. The other conjecture says that B is an admissible control operator if a certain resolvent condition is satisfied. The examples given in this paper show that even for analytic semigroups the conjectures do not hold. In the last section we show that this example leads to a semigroup example showing that the first estimate in the Hille-Yosida Theorem is not sufficient to conclude boundedness of the semigroup.

Keywords: Infinite-dimensional system, admissible control operator, conditional basis, C_0-semigroup.

Mathematics Subject Classification: 93C25, 93A05, 47D60

1 Introduction

For abstract differential equations the Cauchy problem is an important problem, i.e., given the abstract differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \qquad t \geq 0. \tag{1}$$

does for any input $u(t)$ and any initial condition x_0 there exists a unique solution $x(t)$ satisfying (weakly) the equation (1)? If A generates a C_0-semigroup on the Hilbert space H, and B is a bounded operator from the Hilbert space U to H, i.e., $B \in \mathcal{L}(U, H)$, then for any input $u \in L^2(0, \infty; U)$ and $x_0 \in H$, there exists a unique solution of (1). This solution is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-\rho)Bu(\rho)d\rho, \tag{2}$$

where $T(t)$ is the C_0-semigroup generated by A. An interesting question is whether this result still holds if B is not an element of $\mathcal{L}(U, H)$. As is clear from the linearity of the system, we only have to study the second part of the solution (2). Weiss [12] showed that if the solution of (1) must take values in H for any $u \in L^2(0, \infty; U)$, then $B \in \mathcal{L}(U, D(A^*)')$, where $D(A^*)$ is the domain of the adjoint of A, and $'$ denotes the dual space. Note that this implies that B is bounded if A is bounded on H.

One would like to characterize those B's for which (1) has a unique solution. These B's are called admissible.

Definition 1.1 *$B \in \mathcal{L}(U, D(A^*)')$ is called an* admissible control operator *for $T(t)$, if for some (and hence any) $t > 0$ there exists a constant $M > 0$ such that*

$$\left\| \int_0^t T(t-\rho)Bu(\rho)\, d\rho \right\|_H \leq M\|u\|_{L^2(0,\infty;U)}, \qquad u \in L^2(0, \infty; U).$$

For $u \equiv 0$ the Hille-Yosida Theorem gives necessary and sufficient conditions such that (1) has a strongly continuous solution. For non-zero inputs one would like to obtain simple conditions as well. Weiss [13] conjectured the following.

Conjecture 1.2 *Let $B \in \mathcal{L}(U, D(A^*)')$. Then the following statements are equivalent.*

1. *B is an admissible control operator for $T(t)$.*
2. *For every $u \in U$, Bu is an admissible control operator for $T(t)$.*

Clearly, Part 1 implies Part 2. Hansen and Weiss [6] showed that Part 2 implies Part 1 if $T(t)$ is a normal and analytic C_0-semigroup and in Weiss [13] it is shown that this implication also holds for left-invertible C_0-semigroups.

In Weiss [13] the following stronger conjecture appeared.

Conjecture 1.3 *Let $B \in \mathcal{L}(U, D(A^*)')$. Then the following statements are equivalent.*

1. *B is an admissible control operator for $T(t)$.*

2. *There exist constants $K, \rho > 0$ such that*

$$\|(sI - A)^{-1}B\| \le \frac{K}{\sqrt{\mathrm{Re}(s)}}, \qquad s \in \mathbb{C}, \mathrm{Re}(s) > \rho.$$

Weiss [13] showed that Part 1 implies Part 2, and that the converse implication holds for left-invertible semigroups, and for normal semigroups with a finite-dimensional input space (see also Weiss [14]). Moreover, Partington and Weiss [9] proved that Part 2 implies Part 1 if U is finite-dimensional and $T(t)$ is the right shift semigroup, and Jacob and Partington [7] used the techniques of [9] to show the implication for contraction semigroups and finite-dimensional input spaces.

We give an example which shows that the Conjecture 1.2 does not hold in general, not even if we restrict the conjecture to analytic C_0-semigroups. As a consequence we get that the Conjecture 1.3 is wrong as well, since Weiss [13] showed that Conjecture 1.3 implies Conjecture 1.2. Hence it seems that in general there are no simple necessary and sufficient conditions for admissibility. We remark that in Grabowski and Callier [5] a (more involved) necessary and sufficient condition is given. Furthermore, in Zwart [15] the following sufficient condition is presented. If there exist $M > 0$, $\rho \in \mathbb{R}$, and $\alpha > \frac{1}{2}$ such that for all s with real part bigger than ρ we have that $\mathrm{Re}(s)^{\alpha}\|(sI - A)^{-1}B\| \le M$, then B is admissible.

If one studies the limit behavior of solutions of (1), a stronger concept than admissibility is needed, called *infinite-time admissibility.*

Definition 1.4 *$B \in \mathcal{L}(U, D(A^*)')$ is called an* infinite-time admissible control operator *for $T(t)$, if there exists a constant $M > 0$ such that*

$$\left\| \int_0^\infty T(\rho)Bu(\rho)\, d\rho \right\|_H \le M\|u\|_{L^2(0,\infty;U)}, \qquad u \in L^2(0, \infty; U).$$

Of course, every infinite-time admissible control operator for $T(t)$ is an admissible control operator for $T(t)$, and if $T(t)$ is exponentially stable, then admissibility and infinite-time admissibility are equivalent notions. We give an example showing that, even for compact operators A and B, that is, $T(t)$ is an uniformly continuous semigroup with a compact generator and $B \in \mathcal{L}(H)$ is compact, the infinite-time admissibility of Bu for every $u \in H$, does in general not imply the infinite-time admissibility of B. Note that the continuous semigroup $T(t)$ in this example is bounded and strongly stable. Moreover, this example shows that in general the condition

$$\|(sI - A)^{-1}B\| \le \frac{K}{\sqrt{\mathrm{Re}(s)}}, \qquad s \in \mathbb{C}, \mathrm{Re}(s) > 0,$$

for some $K > 0$, does not imply the infinite-time admissibility of B.

In the last section we show that our counterexamples leads to a semigroup example showing that the first estimate in the Hille-Yosida Theorem is not sufficient to conclude boundedness of the semigroup.

2 Construction of the counterexamples

Let H be a separable Hilbert space with orthonormal basis $\{\phi_n\}_{n\in\mathbb{N}}$. We define the conditional basis $\{e_n\}_{n\in\mathbb{N}}$ of H as follows:

$$\begin{aligned} e_{2n} &:= \phi_{2n}, \\ e_{2n-1} &:= \phi_{2n-1} + \sum_{k=n}^{\infty} \alpha_{k-n+1}\phi_{2k}, \end{aligned}$$

where $\alpha_n = \frac{1}{(n+1)\log(n+1)}$. In Singer [10, page 429] it is shown that $\{e_n\}_n$ is a conditional basis, but not a Riesz basis. Since $\{e_n\}_n$ is a basis we have that for every x there exists a unique sequence of coefficients $\{x_n\}_n$, $x_n \in \mathbb{C}$, such that

$$x = \sum_{n=1}^{\infty} x_n e_n = \lim_{N\to\infty} \sum_{n=1}^{N} x_n e_n.$$

Note that the sequence $\{\alpha_n\}_n$ satisfies

$$\sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} n\alpha_n^2 < \infty.$$

The following lemma is useful and can be found in Singer [10, pp. 429–430].

Lemma 2.1 *There exists a constant $\kappa > 0$ such that for all $n \in \mathbb{N}$ and all $\{\beta_k\}_{k=1}^{2n}$ we have*

$$\begin{aligned} \sum_{j=1}^{n} |\beta_{2j-1}|^2 + \sum_{j=1}^{n} \left| \beta_{2j} + \sum_{k=1}^{j} \beta_{2k-1}\alpha_{j-k+1} \right|^2 + \sum_{j=n+1}^{\infty} \left| \sum_{k=1}^{n} \beta_{2k-1}\alpha_{j-k+1} \right|^2 \\ = \left\| \sum_{j=1}^{2n} \beta_j e_j \right\|^2 \\ \leq \kappa \sum_{j=1}^{n} |\beta_{2j-1}|^2 + \sum_{j=1}^{n} \left| \beta_{2j} + \sum_{k=1}^{j} \beta_{2k-1}\alpha_{j-k+1} \right|^2. \end{aligned}$$

Note, that Lemma 2.1 immediately implies the following result. For an element

$$v = \sum_{n=1}^{\infty} v_n e_n \in H$$

we have

$$\sum_{j=1}^{\infty} |v_{2j-1}|^2 \leq \|v\|^2. \tag{3}$$

For diagonal operators on a basis of H there is the following nice result, which can be found in Benamara and Nikolski [1, Lemma 3.2.5].

Lemma 2.2 *Let $\{\varphi_n\}_n$ be a basis of H. If Q is defined as*

$$Q\varphi_n = q_n \varphi_n$$

with $\{q_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$, and the total variation of the sequence $\{q_n\}$ is finite, i.e.,

$$Var(q_n) := \sum_{n=1}^{\infty} |q_{n+1} - q_n| < \infty,$$

then Q can be extended to a linear bounded operator on H, and

$$\|Q\| \leq K(Var(q_n) + \limsup |q_n|),$$

where K is a constant independent of the sequence $\{q_n\}$.

In order to calculate the total variation, the following observation is useful. If f is a continuous function which is non-decreasing or non-increasing on the interval (a, b), and if the sequence $\{q_n\}_n \subset (a, b)$ is non-decreasing or non-increasing, then

$$Var(f(q_n)) \leq |f(a) - f(b)|.$$

A sequence $\{\gamma_n\}_n \subset \mathbb{C}_+$, here $\mathbb{C}_+$ denotes the open right half plane, is an *interpolating sequence* if for every bounded sequence $\{\gamma_n\}_n \subset \mathbb{C}$ there exists a holomorphic and bounded function f on $\mathbb{C}_+$ such that

$$f(\gamma_n) = a_n, \qquad n \in \mathbb{N}.$$

More information on interpolating sequences can be found in Garnett [4]. Concerning interpolating sequences in the right half plane we need the following result (see Garnett [4, page 316]).

Lemma 2.3 *Let $\{\gamma_n\}_n$ be an interpolating sequence in the right half plane $\mathbb{C}_+$. Then there exists a subspace $U_0 \subset L^2(0, \infty)$ such that*

1. $\{\sqrt{\mathrm{Re}(\gamma_n)}\hat{u}(\gamma_n)\}_n \in \ell^2$ *for every* $u \in L^2(0,\infty)$*, and for some* $M_2 > 0$

$$\|\{\sqrt{\mathrm{Re}(\gamma_n)}\hat{u}(\gamma_n)\}_n\|_{\ell^2} \leq M_2\|u\|_{L^2(0,\infty)}, \qquad u \in L^2(0,\infty).$$

2. For some constant $M_1 > 0$ *we have*

$$M_1\|u\|_{L^2(0,\infty)} \leq \|\{\sqrt{\mathrm{Re}(\gamma_n)}\hat{u}(\gamma_n)\}_n\|_{\ell^2}, \qquad u \in U_0.$$

3. For every $\{a_n\}_n \in \ell^2$ *there exists an* $u \in U_0$ *such that*

$$\sqrt{\mathrm{Re}(\gamma_n)}\hat{u}(\gamma_n) = a_n.$$

We are now in a position to present the example showing that infinite-time admissibility of Bu for $T(t)$ for every $u \in U$ in general not imply the infinite-time admissibility of B for $T(t)$. Note, that in the presented example the operators A and B are compact elements of $\mathcal{L}(H)$ and that $T(t)$ is bounded and strongly stable.

Example 2.4 Let $\{\mu_n\}_n \subset (-1,0)$ be a monotonically increasing sequence with $\lim_{n\to\infty}\mu_n = 0$ such that $\{-\mu_n\}_n$ is an interpolating sequence of the right half plane $\mathbb{C}_+$ and $\sum_{n=1}^{\infty}\sqrt{-\mu_n} < \infty$. We could for example choose $\mu_n := -2^{-n}$, see Garnett [4, page 288]. We now define A by

$$Ae_n = \mu_n e_n, \qquad n \in \mathbb{N},$$

where $\{e_n\}$ is the conditional basis as defined at the beginning of this section.

Since the sequence $\{\mu_n\}_n$ is monotonically increasing it is easy to see that $\{\mu_n\}_n$ is of bounded variation. Now by Lemma 2.2, we get that A has a linear bounded extension to H, that is $A \in \mathcal{L}(H)$. Let $T(t)$ be the C_0-semigroup generated by A, that is

$$T(t)e_n = e^{\mu_n t}e_n, \qquad t \geq 0, n \in \mathbb{N}.$$

We define the operator B by

$$Be_n = \begin{cases} \sqrt{-\mu_n}e_n & , \ n = 2k-1, k \in \mathbb{N}, \\ 0 & , \ \text{otherwise.} \end{cases}$$

It is easy to see that the sequence $(\sqrt{-\mu_1}, 0, \sqrt{-\mu_3}, 0, \sqrt{-\mu_5}, 0, \ldots)$ is of bounded variation. Lemma 2.2 shows that B has a linear bounded extension to H, that is $B \in \mathcal{L}(H)$. We now get

1. $T(t)$ is bounded and strongly stable.

   Proof: By Lemma 2.2 we have for $t \geq 0$

$$\|T(t)\| \leq 2K,$$

and thus the C_0-semigroup $T(t)$ is bounded.

Next we show that $T(t)$ is strongly stable. Let $x = \sum_{n=1}^{\infty} f_n e_n \in H$ and $\varepsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that $x_N := \sum_{n=1}^{N} f_n e_n$ satisfies

$$\|x - x_N\|_H < \varepsilon.$$

Thus for sufficiently large $t > 0$ we have

$$\begin{aligned} \|T(t)x\| &\leq \|T(t)x - T(t)x_N\| + \|T(t)x_N\| \\ &\leq \|T(t)\| \, \|x - x_N\| + \left\| \sum_{n=1}^{N} e^{\mu_n t} f_n e_n \right\| \\ &\leq 2K\varepsilon + \sum_{n=1}^{N} e^{\mu_n t} |f_n| \, \|e_n\| \\ &\leq 2K\varepsilon + \varepsilon, \end{aligned}$$

and so $T(t)$ is strongly stable. ■

2. The operator A is compact.

   Proof: Define A_n, $n \in \mathbb{N}$, by

$$A_n e_k = \begin{cases} \mu_k e_k, & k \leq n \\ 0 \quad , & k > n \end{cases}.$$

   Using Lemma 2.2 we see that $A_n \in \mathcal{L}(H)$. Moreover, the operator A_n has rank n. Again using Lemma 2.2 we get the estimate

$$\|A - A_n\| \leq 2K|\mu_{n+1}| \to 0 \text{ as } n \to \infty,$$

   which shows that A is compact. ■

3. Similarly to Part 2 it can be shown that B is compact.

4. B is not an infinite-time admissible control operator for $T(t)$.

   Proof: Assume that B is an infinite-time admissible control operator for $T(t)$. Then there exists a constant $L > 0$ such that

$$\left\| \int_0^{\infty} T(\tau) B u(\tau) \, d\tau \right\|_H \leq L \|u\|_{L^2(0,\infty;H)}, \qquad u \in L^2(0,\infty;H). \tag{4}$$

   We now choose scalar functions $u_{2k-1} \in U_0$, $k \in \mathbb{N}$, such that

$$\sqrt{-\mu_j}\, \hat{u}_{2k-1}(-\mu_j) = \delta_{j,2k-1}, \qquad k, j \in \mathbb{N}.$$

Using Lemma 2.3, Part 2 and 3, this is possible with

$$\|u_{2k-1}\|_{L^2(0,\infty)} \leq \frac{1}{M_1}, \qquad k \in \mathbb{N}. \tag{5}$$

Moreover, we define $\tilde{u}_n \in L^2(0,\infty;H)$ by

$$\tilde{u}_n(t) := \sum_{k=1}^{n} u_{2k-1}(t)e_{2k-1}, \qquad t \in [0,\infty).$$

Next we calculate the norm of $\tilde{u}_n$.

$$\begin{aligned}
\|\tilde{u}_n\|^2_{L^2(0,\infty;H)} &= \int_0^\infty \left\| \sum_{k=1}^{n} u_{2k-1}(t)e_{2k-1} \right\|^2 dt \\
&\leq \int_0^\infty \left\{ \kappa \sum_{k=1}^{n} |u_{2k-1}(t)|^2 + \sum_{j=1}^{n} \left| \sum_{k=1}^{j} u_{2k-1}(t)\alpha_{j-k+1} \right|^2 \right\} dt \\
&\qquad \text{(using Lemma 2.1)} \\
&\leq \kappa \sum_{k=1}^{n} \frac{1}{M_1^2} + \sum_{j=1}^{n} \int_0^\infty \left| \sum_{k=1}^{j} u_{2k-1}(t)\alpha_{j-k+1} \right|^2 dt \text{ (using (5))} \\
&\leq \kappa \frac{1}{M_1^2} n + \frac{1}{M_1^2} \sum_{j=1}^{n} \sum_{l=1}^{\infty} \left| \sum_{k=1}^{j} \sqrt{-\mu_l}\,\hat{u}_{2k-1}(-\mu_l)\alpha_{j-k+1} \right|^2 \\
&\qquad \text{(using Part 2 of Lemma 2.3)} \\
&= \kappa \frac{1}{M_1^2} n + \frac{1}{M_1^2} \sum_{j=1}^{n} \sum_{l=1}^{\infty} \left| \sum_{k=1}^{j} \delta_{l,2k-1}\alpha_{j-k+1} \right|^2 \\
&= \kappa \frac{1}{M_1^2} n + \frac{1}{M_1^2} \sum_{j=1}^{n} \sum_{l=1}^{\infty} |\delta_{l,1}\alpha_j + \delta_{l,3}\alpha_{j-1} + \ldots + \delta_{l,2j-1}\alpha_1|^2 \\
&= \kappa \frac{1}{M_1^2} n + \frac{1}{M_1^2} \sum_{j=1}^{n} \sum_{l=1}^{j} |\alpha_l|^2 \\
&\leq \kappa \frac{1}{M_1^2} n + \frac{n}{M_1^2} \sum_{l=1}^{\infty} |\alpha_l|^2 = \tilde{\kappa} n,
\end{aligned}$$

where $\tilde{\kappa} > 0$ is independent of $n \in \mathbb{N}$. Thus (4) implies

$$\left\| \int_0^\infty T(\tau)B\tilde{u}_n(\tau)\,d\tau \right\|_H^2 \leq L\tilde{\kappa} n. \tag{6}$$

However,

$$\begin{aligned}
\int_0^\infty T(\tau)B\tilde{u}_n(\tau)\,d\tau &= \sum_{k=1}^n \int_0^\infty T(\tau)Be_{2k-1}u_{2k-1}(\tau)\,d\tau \\
&= \sum_{k=1}^n \int_0^\infty \sqrt{-\mu_{2k-1}}e^{\mu_{2k-1}\tau}u_{2k-1}(\tau)\,d\tau e_{2k-1} \\
&= \sum_{k=1}^n \sqrt{-\mu_{2k-1}}\hat{u}_{2k-1}(-\mu_{2k-1})e_{2k-1} \\
&= \sum_{k=1}^n e_{2k-1},
\end{aligned}$$

and so

$$\begin{aligned}
\left\|\int_0^\infty T(\tau)B\tilde{u}_n(\tau)\,d\tau\right\|_H^2 &= \left\|\sum_{k=1}^n e_{2k-1}\right\|_H^2 \\
&\geq n + \sum_{j=1}^n \left|\sum_{k=1}^j \alpha_{j-k+1}\right|^2 \qquad \text{(using Lemma 2.1)} \\
&= n + \sum_{j=1}^n \left|\sum_{k=1}^j \alpha_k\right|^2 \geq \sum_{j=1}^n \left|\int_2^{j+2} \frac{1}{k\log k}\,dk\right|^2 \\
&\geq \sum_{j=1}^n |\log(\log j + 2)|^2,
\end{aligned}$$

which contradicts (6). Thus B is not an infinite-time admissible control operator for $T(t)$. ∎

5. For every $v \in H$ we have that Bv is an infinite-time admissible control operator for $T(t)$.

   Proof: Let $v \in H$. Then there exist scalars $\{f_n\}_n$ such that

$$v = \sum_{n=1}^\infty f_n e_n.$$

   Let $v_N := \sum_{n=1}^{2N-1} f_n e_n$. For $u \in L^2(0,\infty)$ we get

$$\begin{aligned}
&\left\|\int_0^\infty T(\tau)Bv_N u(\tau)\,d\tau\right\|_H \\
&= \left\|\int_0^\infty \sum_{n=1}^N e^{\mu_{2n-1}\tau} f_{2n-1}\sqrt{-\mu_{2n-1}}e_{2n-1}u(\tau)\,d\tau\right\|_H
\end{aligned}$$

$$
\begin{aligned}
&= \left\| \sum_{n=1}^{N} f_{2n-1}\sqrt{-\mu_{2n-1}} \int_0^\infty e^{\mu_{2n-1}\tau} u(\tau)\, d\tau e_{2n-1} \right\|_H \\
&= \left\| \sum_{n=1}^{N} f_{2n-1}\sqrt{-\mu_{2n-1}}\hat{u}(-\mu_{2n-1}) e_{2n-1} \right\|_H \\
&\leq \sum_{n=1}^{N} |f_{2n-1}\sqrt{-\mu_{2n-1}}\hat{u}(-\mu_{2n-1})| \\
&\leq \left(\sum_{n=1}^{N} |f_{2n-1}|^2 \right)^{1/2} \left(\sum_{n=1}^{N} |\sqrt{-\mu_{2n-1}}\hat{u}(-\mu_{2n-1})|^2 \right)^{1/2} \\
&\leq M_2 \|v_N\| \, \|u\|_{L^2(0,\infty)} \qquad \text{(using Lemma 2.1 and Lemma 2.3).}
\end{aligned}
$$

Thus Bv_N is an infinite-time admissible control operator for $T(t)$. Similarly, we can show that for $N > M$ we have

$$
\left\| \int_0^\infty T(\tau) B v_N u(\tau)\, d\tau - \int_0^\infty T(\tau) B v_M u(\tau)\, d\tau \right\|_H
\leq \left\| \sum_{n=2M+1}^{2N-1} f_n e_n \right\|_H \|u\|_{L^2(0,\infty)},
$$

and so using the fact that the space of admissible control operators with input space $\mathbb{C}$ is complete, see Weiss [12, Remark 4.6], we get that Bv is an admissible control operator for $T(t)$. ■

6. There exists a constant $K > 0$ such that

$$
\|(sI - A)^{-1}B\| \leq \frac{K}{\sqrt{\mathrm{Re}(s)}}, \qquad s \in \mathbb{C}, \mathrm{Re}(s) > 0.
$$

Proof: In the previous part we proved that for every $v \in H$ Bv is an infinite-time admissible control operator for $T(t)$. Thus Weiss [14] showed that for every $v \in H$, $\|v\| = 1$, there exists a constant $M_v > 0$ such that

$$
\|(sI - A)^{-1}Bv\| \leq M_v/\sqrt{\mathrm{Re}(s)} \qquad \text{for all } s \in \mathbb{C}_+.
$$

Using the uniform boundedness theorem this implies the existence of a constant $M > 0$ such that

$$
\|(sI - A)^{-1}B\| \leq M/\sqrt{\mathrm{Re}(s)} \qquad \text{for all } s \in \mathbb{C}_+.
$$

■

Next we give an example showing that Conjecture 1.2 and 1.3 do not hold in general. Note that the C_0-semigroup $T(t)$ used in this example is analytic.

Example 2.5 Let $\{\mu_n\}_n \subset (-\infty, -1)$ be a monotonically decreasing sequence with $\lim_{n\to\infty} \mu_n = -\infty$ such that $\{-\mu_n\}_n$ is an interpolating sequence of $\mathbb{C}_+$. We could for example choose $\mu_n := -2^n$, see Garnett [4, page 288].

For $t \geq 0$ we define $T(t)$ by

$$T(t)e_n := e^{\mu_n t} e_n, \qquad n \in \mathbb{N}.$$

Since the sequence $\{\mu_n\}_n$ is monotonically decreasing and since $\lim_{n\to\infty} \mu_n = -\infty$, we get by Lemma 2.2 that $T(t)$ has a linear bounded extension to H. Thus $T(t) \in \mathcal{L}(H)$, and

$$\|T(t)\| \leq Ke^{-t}, \qquad t \geq 0. \tag{7}$$

Clearly, $T(0) = I$ and $T(t)T(s) = T(t+s)$ for $t, s \geq 0$. We will show that $T(t)$ is strongly continuous. For $x \in H$, there exists a sequence $\{f_n\}_n$ of scalars such that

$$x = \sum_{n=1}^{\infty} f_n e_n.$$

Choose $\varepsilon > 0$ and choose N such that $\|x - x_N\|_H < \varepsilon$, where $x_N := \sum_{n=1}^{N} f_n e_n$ Next choose $t_0 > 0$ such that $\sum_{n=1}^{N} |e^{\mu_n t_0} - 1|\,|f_n|\,\|e_n\| \leq \varepsilon$. Then we have for $t \in (0, t_0)$ that

$$\begin{aligned}
\|T(t)x - x\| &\leq \|T(t)x - T(t)x_N\| + \|T(t)x_N - x_N\| + \|x_N - x\| \\
&\leq Ke^{-t_0}\varepsilon + \sum_{n=1}^{N} |e^{\mu_n t_0} - 1|\,|f_n|\,\|e_n\| + \varepsilon \\
&\leq [Ke^{-t} + 2]\varepsilon.
\end{aligned}$$

Thus $T(t)$ is a C_0-semigroup on H. From (7) we see that $T(t)$ is exponentially stable. Let A be the generator of $T(t)$. It is easy to see that A is given by

$$Ae_n = \mu_n e_n, \qquad n \in \mathbb{N}.$$

We define the operator B by

$$Be_n = \begin{cases} \sqrt{-\mu_n}\, e_n & , \ n = 2k-1, k \in \mathbb{N}, \\ 0 & , \ \text{otherwise.} \end{cases}$$

Noting that $D(A^*)'$ is the completion of H with respect to the norm

$$\|x\|_{D(A^*)'} = \|A^{-1}x\|_H,$$

it is easy to see that $B \in \mathcal{L}(H, D(A^*)')$. We now get

1. The C_0-semigroup $T(t)$ is analytic.

   Proof: Since the semigroup is uniformly bounded, it is sufficient, [8, Theorem 2.5.2], to show that

$$\|(sI - A)^{-1}\| \leq \frac{M}{|\operatorname{Im} s|}, \qquad s \in \mathbb{C}_+.$$

   Let $s = s_r + is_i \in \mathbb{C}_+$. Clearly,

$$(sI - A)^{-1} e_n = \frac{1}{s - \mu_n} e_n, \qquad n \in \mathbb{N}.$$

   In order to show the above estimate, we first prove that

$$\gamma_n := \frac{1}{s - \mu_n}, \qquad n \in \mathbb{N},$$

   is of bounded variation. We get

$$\gamma_n = \frac{1}{s - \mu_n} = \frac{\bar{s}}{|s - \mu_n|^2} - \frac{\mu_n}{|s - \mu_n|^2},$$

   and we define

$$h_1 : \mathbb{R}_- \to \mathbb{R}_+, \qquad h_1(x) := \frac{1}{|s - x|^2},$$
$$h_2 : \mathbb{R}_- \to \mathbb{R}_-, \qquad h_2(x) := \frac{x}{|s - x|^2}.$$

   Clearly, h_1 is monotonically increasing on $(-\infty, 0)$, and $h_1(-\infty) = 0$ and $h_1(0) = \frac{1}{|s|^2}$. Moreover, we have

$$h_2'(x) = \frac{|s - x|^2 + 2x(s_r - x)}{|s - x|^4}.$$

   Thus

$$h_2'(x) = 0 \Leftrightarrow |s - x|^2 + 2x(s_r - x) = 0 \Leftrightarrow -x^2 + |s|^2 = 0 \Leftrightarrow x = -|s|.$$

   Thus h_2 is monotonically decreasing on $(-\infty, -|s|)$ and monotonically increasing on $(-|s|, 0)$. Moreover, $h_2(-\infty) = h_2(0) = 0$, and thus $|h_2|$ has its maximum in $-|s|$. Note that

$$|h_2(-|s|)| = \frac{|s|}{|s + |s||^2} \leq \frac{1}{|s|}.$$

Using Lemma 2.1 we get the following estimate for $\|(sI-A)^{-1}\|$.

$$\begin{aligned}
&\|(sI-A)^{-1}\| \\
&\leq \; K(\mathrm{Var}\,(\{\gamma_n\}) + |\lim_{n\to\infty}\gamma_n|) \leq K\left(\sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|\right) \\
&\leq \; K\left(|s|\sum_{n=1}^{\infty}|h_1(\mu_{n+1})-h_1(\mu_n)| + \sum_{n=1}^{\infty}|h_2(\mu_{n+1})-h_2(\mu_n)|\right) \\
&\leq \; \frac{3K}{|s|} \leq \frac{3K}{|\operatorname{Im} s|}
\end{aligned}$$

where $K>0$ is independent of s. Thus the statement is proved. ■

2. B is not an admissible control operator for $T(t)$.

   Since $T(t)$ is exponentially stable it is enough to show that B is not an infinite-time admissible control operator for $T(t)$. The proof of this statement is the same as the proof of Part 4 of Example 2.4. ■

3. For every $v \in H$ we have that Bv is an admissible control operator for $T(t)$.

   Since $T(t)$ is exponentially stable it is enough to show that for every $v \in H$ we have that Bv is an infinite-time admissible control operator for $T(t)$. Again the proof is the same as the proof of Part 5 of Example 2.4. ■

3 A semigroup example

A direct consequence of the Hille-Yosida Theorem is that a C_0-semigroup $T_e(t)$ is uniformly bounded if and only if there exists a constant M such that its generator A_e satisfies

$$\|(sI-A_e)^{-n}\| \leq \frac{M}{\mathrm{Re}(s)^n} \qquad \text{for all } n \in \mathbb{N} \text{ and } s \in \mathbb{C}_+.$$

An interesting question is whether the first inequality is sufficient to conclude the boundedness of the C_0-semigroup. Using the example of the previous section we will show that this is in general not true. Hence we construct a C_0-semigroup for which the infinitesimal generator A_e satisfies

$$\|(sI-A_e)^{-1}\| \leq \frac{M}{\mathrm{Re}(s)} \qquad \text{for all } s \in \mathbb{C}_+, \tag{8}$$

but the semigroup satisfies

$$\lim_{t\to\infty}\|T_e(t)\| = \infty. \tag{9}$$

Consider the operators A and B of Example 2.4, and let $T(t)$ denote the bounded semigroup generated by A. With these operators we define the semigroup $T_e(t)$ on $H \oplus L^2(0,\infty;H)$ as

$$T_e(t)\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} T(t)x + \int_0^t T(t-\tau)Bf(\tau)d\tau \\ f(t+\cdot) \end{pmatrix}.$$

Using that B is a bounded operator, and hence B is an admissible control operator for $T(t)$, Engel [3] showed that $T_e(t)$ is a C_0-semigroup on $H \oplus L^2(0,\infty;H)$. Since B is not infinite-time admissible we know that $T_e(t)$ cannot be a bounded semigroup, we will show that it satisfies (9). By taking the Laplace transform of the semigroup, we see that the resolvent is given by

$$(sI - A_e)^{-1}\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} (sI-A)^{-1}x + (sI-A)^{-1}B\hat{f}(s) \\ \widehat{f(t+\cdot)}(s) \end{pmatrix}$$

Since the left shift is a contraction semigroup we know that $L^2(0,\infty;H)$-norm of $\widehat{f(t+\cdot)}(s)$ is bounded by $\|f\|$ times $1/\mathrm{Re}(s)$. Furthermore, since $T(t)$ is a bounded semigroup, a similar estimate holds for $(sI-A)^{-1}$. Thus we have that for $s \in \mathbb{C}_+$

$$\left\|(sI - A_e)^{-1}\begin{pmatrix} x \\ f \end{pmatrix}\right\|^2 \leq \frac{M^2}{\mathrm{Re}(s)^2}\|x\|^2 + \|(sI-A)^{-1}B\|^2\|\hat{f}(s)\|^2 + \frac{1}{\mathrm{Re}(s)^2}\|f\|^2. \tag{10}$$

Since $f \in L^2(0,\infty;H)$ we have that

$$\|\hat{f}(s)\| \leq \|f\|/\sqrt{2\mathrm{Re}(s)} \qquad \text{for all } s \in \mathbb{C}_+. \tag{11}$$

In Example 2.4 we proved the existence of a constant $K > 0$ such that

$$\|(sI-A)^{-1}B\| \leq K/\sqrt{\mathrm{Re}(s)} \qquad \text{for all } s \in \mathbb{C}_+. \tag{12}$$

Combining (10)–(12) gives that A_e satisfies the estimate (8). Since the corresponding semigroup does not satisfy (9), we have shown that the first estimate in the Hille-Yosida Theorem is not sufficient to conclude the boundedness of the semigroup.

4 Underlying ideas

Although it is extremely hard to tell why the counterexample is constructed in the way as it is, we would like to give some indications. For definitions and background information on the system theoretic concepts we refer to Curtain and Zwart [2].

If a strongly stable system is infinite-time admissible and exactly controllable, then there exists a coercive solution of the Lyapunov equation

$$AL + LA^* = -BB^*.$$

In particular, this implies that A is similar to a contraction. Now it is well-know that not every infinitesimal generator on a Hilbert space is similar to a contraction. Last year A. Simard [11] showed that the operator e^A, with A given as in Example 2.4 is not similar to a contraction. Hence this could work as A operator. Since for diagonal generators $A = \mathrm{diag}(\mu_n)$ on a Riesz basis, the input operator $\mathrm{diag}(\sqrt{-\mu_n})$ is admissible, it seemed logically to try this on a conditional basis. During the process of writing we found out that it was better to use a variation of this candidate.

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