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Chordality and 2-factors in tough graphs

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Abstract

A graph G is *chordal* if it contains no chordless cycle of length at least four and is *k-chordal* if a longest chordless cycle in G has length at most k. In this note it is proved that all $\frac{3}{2}$-tough 5-chordal graphs have a 2-factor. This result is best possible in two ways. Examples due to Chvátal show that for all $\varepsilon > 0$ there exists a $(\frac{3}{2} - \varepsilon)$-tough chordal graph with no 2-factor. Furthermore, examples due to Bauer and Schmeichel show that the result is false for 6-chordal graphs.

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1. Introduction

We begin with a few definitions and some notation. Other definitions will be given later, as needed. A good reference for any undefined terms is [7]. We consider only undirected graphs with no loops or multiple edges. Let G be a graph. Then G is *hamiltonian* if it has a *Hamilton* cycle, i.e., a cycle containing all of its vertices. It is *traceable* if it has a path containing all of its vertices. Let $\omega(G)$ denote the number of components of G. Then G is *t-tough* if $|S| \geqslant t\omega(G-S)$ for every subset S of the vertex set V of G with $\omega(G-S) > 1$. The *toughness* of G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough (taking $\tau(K_n) = (n-1)/2$ for all $n \geqslant 1$). A *k-factor*

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is a k-regular spanning subgraph. Of course, a Hamilton cycle is a 2-factor. We say G is *chordal* if it contains no chordless cycle of length at least four and is *k-chordal* if a longest chordless cycle in G has length at most k.

Our work was motivated by a desire to understand the relationship between the toughness of a graph and its cycle structure. For a survey of recent work in this area, see [3–5]. Toughness was introduced by Chvátal in [9]. An obvious connection between toughness and hamiltonicity is that being 1-tough is a necessary condition for a graph to be hamiltonian. Chvátal conjectured that there exists a finite constant t_0 such that every t_0-tough graph is hamiltonian. This conjecture is still open. Until recently it was believed that the smallest value of t_0 for which this might be true was $t_0 = 2$. We now know this is false.

Theorem 1.1 (Bauer et al. [1]). *For every $\varepsilon > 0$, there exists a $(\frac{9}{4} - \varepsilon)$-tough non-traceable graph.*

Chvátal also conjectured that every k-tough graph on n vertices with $n \geqslant k+1$ and kn even has a k-factor. This was established in [10].

Theorem 1.2 (Enomoto et al. [10]). *Let G be a k-tough graph on n vertices with $n \geqslant k+1$ and kn even. Then G has a k-factor.*

It was also shown in [10] that Theorem 1.2 is best possible.

Theorem 1.3 (Enomoto et al. [10]). *Let $k \geqslant 1$. For any $\varepsilon > 0$, there exists a $(k - \varepsilon)$-tough graph G on n vertices with $n \geqslant k+1$ and kn even which has no k-factor.*

The above results imply that while 2-tough graphs have 2-factors, there exists an infinite sequence of graphs without 2-factors having toughness approaching 2. In [11] it was shown that a similar statement holds for split graphs. A graph G is called a *split graph* if its vertices can be partitioned into an independent set and a clique.

Theorem 1.4 (Kratsch et al. [11]). *Every $\frac{3}{2}$-tough split graph is Hamiltonian.*

In [9, p. 223], Chvátal found a sequence $\{G_n\}_{n=1}^{\infty}$ of non-2-factorable graphs with $\tau(G_n) \to \frac{3}{2}$. These graphs were in fact split graphs.

Theorem 1.5. *There is a sequence $\{G_n\}_{n=1}^{\infty}$ of non-2-factorable split graphs with $\tau(G_n) \to \frac{3}{2}$.*

In this note we prove that all $\frac{3}{2}$-tough chordal graphs have a 2-factor. In fact we prove a bit more.

Theorem 1.6. *Let G be a $\frac{3}{2}$-tough 5-chordal graph. Then G has a 2-factor.*

Since all split graphs are chordal, the graphs Chvátal constructed in [9] are also chordal. Thus Theorem 1.6 is best possible with respect to toughness. Furthermore, the graphs $G_{l,m}$ in [2, p. 251] are 6-chordal graphs without a 2-factor. By choosing l and m large the toughness of these graphs can be made to approach 2 from below. Note that Theorem 1.6 is in some sense the definitive result of the form “If G is a t-tough k-chordal graph, then G has a 2-factor”: it follows from the examples in [9] that this is false for $t < \frac{3}{2}$ and any k, by Theorem 1.2 it is true for $t \geqslant 2$ and any k, and from the examples in [2] it follows that for $\frac{3}{2} \leqslant t < 2$ the best one can hope for is a result with $k = 5$.

Unlike the case with split graphs, however, it is not true that all $\frac{3}{2}$-tough chordal graphs are hamiltonian.

Theorem 1.7 (Bauer et al. [1]). *For every $\varepsilon > 0$ there exists a $(\frac{7}{4} - \varepsilon)$-tough chordal nontraceable graph.*

Recently, Chen et al. [8] have shown that every 18-tough chordal graph is hamiltonian. We now conjecture the following.

Conjecture. *Every 2-tough chordal graph is hamiltonian and for every $\varepsilon > 0$ there exists a $(2 - \varepsilon)$-tough chordal nonhamiltonian graph.*

Returning to 2-factors, it is natural to ask how large the minimum vertex degree of a t-tough $(1 \leqslant t < 2)$ graph can be, if the graph contains no 2-factor. This problem was answered in [2] for $1 \leqslant t \leqslant \frac{3}{2}$ and for infinitely many t satisfying $\frac{3}{2} \leqslant t < 2$. A key lemma (Lemma 8) in [2] is the basis for the proof of our main result. Of course, any paper dealing with sufficient conditions for a graph to have a regular factor relies heavily on a well-known theorem of Belck [6] and Tutte [12]. This result is given in Section 2. The proof of our main result appears in Section 3.

2. Preliminary results

Let G be a graph. If A and B are subsets of V or subgraphs of G, and $v \in V$, we use $e(v,B)$ to denote the number of edges joining v to a vertex of B, and $e(A,B)$ to denote $\sum_{v\in A} e(v,B)$. We use $\langle A \rangle$ to denote the subgraph of G induced by A. A vertex $v \in V$ will be called *complete* if v is adjacent to every other vertex in V, and is called *simplicial* if the subgraph induced by the neighborhood of v is complete.

Our proof of Theorem 1.6 relies heavily on a theorem that characterizes those graphs not containing a 2-factor. This theorem is a special case of the theorems of Belck [6] and Tutte [12]. For disjoint subsets A, B of $V(G)$ let odd (A,B) denote the number of components H of $G - (A \cup B)$ with $e(H,B)$ odd, and let

$$\Theta(A,B) = 2|A| + \sum_{y\in B} d_{G-A}(y) - 2|B| - \mathrm{odd}(A,B).$$

Theorem 2.1 (Belck [6] and Tutte [12]). *Let G be any graph. Then*

(i) *for any disjoint sets $A,B \subseteq V(G), \Theta(A,B)$ is even*;

(ii) *the graph G does not contain a 2-factor if and only if $\Theta(A,B) \leqslant -2$ for some disjoint pair of sets $A,B \subseteq V(G)$.*

We call a pair (A,B) of disjoint subsets of $V(G)$ with $\Theta(A,B) \leqslant -2$ a *Tutte pair* for G. Note that in any Tutte pair (A,B) for G we have $B \neq \emptyset$, since by definition $\sum_{y\in B} d_{G-A}(y) \geqslant \text{odd}\,(A,B)$ and so $\Theta(A,B) \leqslant -2$ implies $|B| > |A| \geqslant 0$. We define a Tutte pair (A,B) to be *minimal* if $\Theta(A,B') \geqslant 0$ for any proper subset $B' \subseteq B$. Clearly any graph without a 2-factor contains a minimal Tutte pair.

The next lemma follows easily from a result in [10]. The proof also appears in [2].

Lemma 2.2. *Let G be a graph having no 2-factor. If (A,B) is a minimal Tutte pair for G, then B is an independent set.*

To facilitate the proof in the next section we define a Tutte pair (A,B) to be a *strong Tutte pair* if B is an independent set.

3. Proof of Theorem 1.6

We begin with the following lemma, which is also implicit in [2].

Lemma 3.1. *Let v be a simplicial vertex in a non-complete graph G. Then $\tau(G-v) \geqslant \tau(G)$.*

Proof. First denote $G-v$ by G_v. Note that if G_v is complete, then

$$\tau(G_v) = \frac{|V(G_v)|-1}{2} = \frac{|V(G)|-2}{2} \geqslant \tau(G).$$

Suppose $\tau(G_v) < \tau(G)$. Then there exists $X \subseteq V(G_v)$ such that $\omega(G_v - X) \geqslant 2$ and $|X|/\omega(G_v - X) < \tau(G)$. However $\omega(G-X) \geqslant \omega(G_v - X) \geqslant 2$, since the neighbors of v in G induce a complete subgraph. But this gives $|X|/\omega(G-X) \leqslant |X|/\omega(G_v - X) < \tau(G)$, a contradiction. □

Proof of Theorem 1.6. Let G be a $\frac{3}{2}$-tough 5-chordal graph having no 2-factor and (A,B) be a strong Tutte pair for G, existing by Lemma 2.2. Thus $\Theta(A,B) \leqslant -2$. Let $C = V(G)-(A\cup B)$. Since B is an independent set of vertices, $\sum_{y\in B} d_{G-A}(y) = e(B,C)$. Hence by Theorem 2.1,

$$2|A| + e(B,C) \leqslant 2|B| + \text{odd}(A,B) - 2. \tag{1}$$

Among all possible choices, we choose G and the strong Tutte pair (A,B) as follows:

(i) $|V(G)|$ is minimal;

(ii) $|E(G)|$ is maximal, subject to (i);

(iii) $|B|$ is minimal, subject to (i) and (ii);
(iv) $|A|$ is maximal, subject to (i)–(iii).

We now show that G has properties (a)–(g) below.

(a) For any $x \in B$ and any component H of $\langle C \rangle, e(x,H) \leqslant 1$.

Proof of (*a*): Let $x \in B$ with $d_{G-A}(x) = k$, and let $C_1, C_2, \dots, C_j$ denote the components of $\langle C \rangle$ to which x is adjacent. If $j \leqslant k-1$, delete x from B and add x to C (thus redefining B and C). Since odd (A,B) has decreased by at most $j \leqslant k-1$, it is easy to check that $\Theta(A,B)$ has increased by at most 1. Thus, we still have $\Theta(A,B) \leqslant -2$ (by Theorem 2.1(i)) and we contradict (iii).

(b) The vertices of A are complete.

Proof of (*b*): If not, form a new graph G' by adding the edges required to make the vertices of A complete. Clearly G' is still $\frac{3}{2}$-tough and (A,B) is still a strong Tutte pair for G'. Obviously, no chordless cycle of G' can contain a vertex of A. Since G is 5-chordal, it follows that G' is also 5-chordal. Thus we contradict (ii).

(c) For any $y \in C, e(y,B) \leqslant 1$.

Proof of (*c*): Suppose that $e(y,B) \geqslant 2$ for some $y \in C$. Delete y from C and add y to A (thus redefining A and C). It is easy to check that (A,B) remains a strong Tutte pair. Thus we contradict (iv).

(d) Each component of $\langle C \rangle$ is a complete graph.

Proof of (*d*): If not, form a new graph G' by adding the edges required to make each component $C_1, C_2, \dots, C_s$ of $\langle C \rangle$ a complete graph. Clearly, G' is still 3/2-tough and (A,B) is still a strong Tutte pair for G'. Assuming G' is not 5-chordal, let C^* be a shortest chordless cycle in G' of length at least 6. Clearly C^* can not contain a vertex of A, nor can it have more than two vertices from any component of $\langle C \rangle$. Since B is independent, C^* is of the form

$$C^* : b_1 T_1' b_2 T_2' \cdots b_k T_k' b_1,$$

where, for $1 \leqslant i \leqslant k$, each T_i' represents an edge $t_i^1 t_i^2$ of a component C_i in G'.

Form the cycle C^{**} in G by taking C^* and substituting T_i for $T_i' (1 \leqslant i \leqslant k)$, where T_i is a shortest $t_i^1 - t_i^2$ path in C_i in G. The graph G is 5-chordal, so C^{**} has a chord. Since any chord of C^{**} must join a vertex of B and a vertex of C and C^* is a chordless cycle in G', we may assume, without loss of generality, that there exists a chord $b_1 u$ of C^{**} such that

- u is an internal vertex of some T_i, say of T_m, and
- the cycle $b_1 T_1 b_2 T_2 \cdots b_m U b_1$, where U is the $t_m^1 - u$ subpath of T_m, is chordless.

By (a) we have $1 < m < k$. But then $b_1 T_1' b_2 T_2' \cdots b_m t_m^1 u b_1$ is a chordless cycle in G' of length at least 6 which is shorter than C^*, contradicting the choice of C^*. Thus G' is 5-chordal and we contradict (ii).

(e) For any $y \in C$, $e(y,B) = 1$ (and thus $e(B,C) = |C|$).

Proof of (*e*): Suppose now that C contains a vertex y with $e(y,B)=0$. It follows from (b) and (d) that v is simplicial. Hence by Lemma 3.1, $\tau(G-y)\geqslant\tau(G)$. Furthermore, (A,B) is still a strong Tutte pair for the 5-chordal graph $G-y$. Hence, by (i), the graph $G-y$ contradicts the choice of G.

(f) $|B|\geqslant 2$.

Proof of (*f*): We saw earlier that $|B|>|A|\geqslant 0$, and so $|B|\geqslant 1$. Suppose $B=\{x\}$. Since (A,B) is a Tutte pair with $|B|=1$ and $|A|=0$, we have $e(B,C)\leqslant \mathrm{odd}(A,B)$ by (1). If $e(B,C)\geqslant 2$, then $\omega(G-B)\geqslant \mathrm{odd}(A,B)\geqslant e(B,C)\geqslant 2>|B|$, and G is not 1-tough. If $e(B,C)=1$, then G is not 1-tough either. Hence $|B|\geqslant 2$.

(g) odd $(A,B)=\omega(\langle C\rangle)$.

Proof of (*g*): Suppose there exists a component C_i in $\langle C\rangle$ with $e(C_i,B)=|C_i|$, an even integer. Let y be any vertex in C_i. Add y to A, thus redefining A and C. It is easy to see that (A,B) is still a strong Tutte pair for G. Thus we contradict (iv).

Hence G and its minimal Tutte pair (A,B) have properties (a)–(g). Set $s=\omega(\langle C\rangle)=$ odd (A,B).

Consider the components $C_1,C_2,\dots,C_s$ of $\langle C\rangle$ and let $y_j\in V(C_j)$. Define $X=A\cup C-\{y_1,\dots,y_s\}$. Since B is independent and $e(y_i,B)=1$ for $1\leqslant i\leqslant s$, we have $\omega(G-X)=|B|\geqslant 2$. For convenience let $a=|A|$, $b=|B|$ and $c=|C|$. Using properties (e), (g) and inequality (1), we have

$$\frac{3}{2}\leqslant\frac{|X|}{\omega(G-X)}=\frac{a+c-s}{b}=\frac{a+e(B,C)-\mathrm{odd}(A,B)}{b}\leqslant\frac{2b-a-2}{b}.$$

Hence

$$b\geqslant 2a+4. \tag{2}$$

To complete the proof we establish the following.

Claim. $b\geqslant c-s+1$.

Once the claim is established, it follows that

$$\frac{3}{2}\leqslant\frac{|X|}{\omega(G-X)}=\frac{a+c-s}{b}\leqslant\frac{a+b-1}{b}.$$

Thus

$$b\leqslant 2a-2. \tag{3}$$

The fact that (2) and (3) are contradictory completes the argument.

Proof of Claim. Form a bipartite graph F from G by deleting A and contracting each component of $\langle C\rangle$ into a single vertex. By (a), F has no multiple edges. The key observation is that since G is 5-chordal, F is a forest. Otherwise, let C_F be a shortest cycle in F. Then C_F is of the form

$$C_F: b_1T_1b_2T_2\cdots b_pT_pb_1,$$

where each $T_i, 1 \leqslant i \leqslant p$, represents the contracted component C_i. By (d) and (e), it follows that the 2 edges incident with each T_i in C_F correspond to edges $b_i t_i^1, b_{i+1} t_i^2$, where $t_i^1 t_i^2$ is an edge in C_i. It follows that G has a chordless cycle of length at least 6, a contradiction.

Hence

$$\sum_{v \in C} d_F(v) = c = |E(F)| \leqslant |V(F)| - 1 = b + s - 1.$$

Thus $b + s - 1 \geqslant c$ and the claim is established. □

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