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Memorandum No. 1491

On factors of 4-connected claw-free graphs

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July 1999

ISSN 0169-2690

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On factors of 4-connected claw-free graphs

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July 15, 1999

Abstract

We consider the existence of several different kinds of factors in 4-connected claw-free graphs. This is motivated by the following two conjectures which are in fact equivalent by a recent result of the third author. Conjecture 1 (Thomassen): Every 4-connected line graph is hamiltonian, i.e. has a connected 2-factor. Conjecture 2 (Matthews and Sumner): Every 4-connected claw-free graph is hamiltonian. We first show that Conjecture 2 is true within the class of hourglass-free graphs, i.e. graphs that do not contain an induced subgraph isomorphic to two triangles meeting in exactly one vertex. Next we show that a weaker form of Conjecture 2 is true, in which the conclusion is replaced by the conclusion that there exists a connected spanning subgraph in which each vertex has degree two or four. Finally we show that Conjecture 1 and 2 are equivalent to seemingly weaker conjectures in which the conclusion is replaced by the conclusion that there exists a spanning subgraph consisting of a bounded number of paths.

Keywords: claw-free graph, line graph, Hamilton cycle, Hamilton path, factor

AMS Subject Classifications (1991): 05C45, 05C38

1 Introduction

We use [1] for terminology and notation not defined here.

Most of the results in this paper are motivated by the following two conjectures due to Thomassen [13] and Matthews and Sumner [10], respectively.

*Research supported by grant GA ČR No. 201/97/0407

Conjecture 1
Every 4-connected line graph is hamiltonian.

Conjecture 2
Every 4-connected claw-free graph is hamiltonian.

A recent result on closures due to the third author [11] (Theorem 3 below) implies that Conjecture 1 and Conjecture 2 are equivalent.

We first introduce some terminology and notation. All *multigraphs* considered in the sequel are finite, undirected, and loopless. We use the term *graph* for a multigraph $G = (V, E)$ in order to indicate that G is *simple*, i.e. there is at most one edge joining two vertices. As usual, $V(G)$ or V denotes the *vertex set* and $E(G)$ or E the *edge set* of a multigraph G. Let $A, B \subseteq V$ and $a, b \in V$. With $[A, B]_G$ we denote the set of edges between vertices of A and B in G, and we let $[a, b]_G := [\{a\}, \{b\}]_G$. If $[a, b]_G = \{e\}$ for some $e \in E$, then we also use ab or $[a, b]_G$ for e.

The *submultigraph* $G[A]$ induced by the set $A \subseteq V(G)$ is defined by $G[A] := (A, [A, A]_G)$, and the *degree* of some vertex $a \in V$ is denoted by $d_G(a) := |[\{a\}, V \setminus \{a\}]_G|$. Let $N_G(A) := \{c \in V \setminus A \mid [A, \{c\}]_G \neq \emptyset\}$, and let $N_G(a) := N_G(\{a\})$. Clearly, $d_G(a) = |N_G(a)|$ provided that G is a graph. The submultigraph $G[N_G(a)]$ is called the *neighborhood* of a in G. By $d_G(a, b)$ we denote the *distance* of a, b in G, i.e. the length of a shortest path between a and b in G. If a, b are not in the same component of G, we simply set $d_G(a, b) := \infty$.

A *claw* in the multigraph G is a set of four distinct vertices a, b, c, y such that a, b, c are *independent* in G, i.e. pairwise nonadjacent in G, and $a, b, c \in N_G(y)$. G is called *claw-free* if there exists no claw in G. Clearly, a multigraph is *claw-free* if it contains no induced subgraph isomorphic to $K_{1,3}$, but the converse is guaranteed only in graphs.

A spanning submultigraph H of G is called a *factor* of G, and a *2-factor* (of G) if all vertices of H have degree 2 in H. Hence a *Hamilton cycle* is a connected 2-factor. A *circuit* C of G is a closed trail (possibly consisting of a single vertex), and it is said to be *(edge) dominating* if every edge of G is incident with some vertex of C. If, moreover, $V(G) = V(C)$ holds then C is a *spanning circuit*.

The *local completion of a graph G at a vertex v* is the operation of joining all pairs of nonadjacent vertices in $N_G(v)$, i.e. replacing the neighborhood of v by the complete graph on $N_G(v)$.

In [11] the following has been proved.

Theorem 3
Let G be a claw-free graph, v a vertex of G whose neighborhood is connected, and G' the graph obtained from G by local completion at v. Then

(i) G' is claw-free, and

(ii) for every cycle C' of G' there exists a cycle C of G such that $V(C') \subseteq V(C)$.

For a claw-free graph G, we define the *closure* $\mathrm{cl}(G)$ of G as the graph obtained from G by iteratively performing local completions at vertices with connected neighborhoods until no more edges can be added. As shown in [11], $\mathrm{cl}(G)$ is uniquely determined by G, and $\mathrm{cl}(G)$ is the line graph of a triangle-free graph. Moreover, in [11] it is shown that Theorem 3 has the following consequences. Let $c(G)$ denote the *circumference* of G, i.e. the length of a longest cycle of G.

Theorem 4
Let G be a claw-free graph. Then

(i) $c(cl(G)) = c(G)$.

(ii) If $cl(G)$ is complete and $|V(G)| \geq 3$, then G is hamiltonian.

(iii) Every nonhamiltonian claw-free graph is a factor of a nonhamiltonian line graph.

Theorem 4(iii) together with a result of Zhan [15] and, independently, Jackson [5] implies that every 7-connected claw-free graph is hamiltonian. Moreover it yields the mentioned equivalence of Conjecture 1 and Conjecture 2.

In the sequel we prove several results concerning the existence of certain factors in 4-connected claw-free graphs or multigraphs.

In the next section we give a short proof of Conjecture 2 within the subclass of *hourglass-free* graphs, i.e. graphs that do not contain an induced subgraph isomorphic to the *hourglass*, a graph consisting of two triangles meeting in exactly one vertex. This result also follows from a recent result due to the second author [7].

In Section 3 we prove the validity of a weaker form of Conjecture 2 in which we replace the conclusion by the conclusion that there exists a connected factor in which each vertex has degree 2 or 4.

Finally, in Section 4 we show that Conjecture 1 and 2 are equivalent to seemingly weaker conjectures in which we replace the conclusion by the conclusion that there exists a factor consisting of a bounded number of paths.

2 Hourglass-free graphs

Our aim in this section is to prove that all 4-connected claw-free hourglass-free graphs are hamiltonian. For this purpose we need the fact that all 4-connected *inflations* are hamiltonian.

We start this section by introducing some additional terminology. A multigraph G is called *essentially k-edge connected* if it is connected and if every edge cut E' of G such that $G-E'$ has at least two components containing an edge, has at least k edges. It is well-known and easy to check that a line graph $L(G)$ of a multigraph G is k-connected if and only if G is essentially k-edge connected. The *inflation* $I(G)$ of a graph G is the graph obtained from G by replacing all vertices $v_1, v_2, \ldots, v_n$ of G by disjoint complete graphs on $d(v_i)$ vertices $v_{i,1}, v_{i,2}, \ldots v_{i,d(v_i)}$, and all edges v_iv_j by disjoint edges $v_{i,p}v_{j,q}$ $(i, j \in \{1, \ldots, n\}; p \in \{1, \ldots, d(v_i)\}; q \in \{1, \ldots, d(v_j)\})$.

Alternatively, as shown in [10, Lemma 2], $I(G)$ is the line graph of the *subdivision graph* $S(G)$, i.e. the graph obtained from G by subdividing each edge of G once. We use the term *inflation* for a graph that is isomorphic to the inflation of some graph. It is obvious that inflations are claw-free and hourglass-free.

The following result has been observed by several graph theorists, but we have not found it in literature (therefore, we include its proof).

Lemma 5
Every 4-connected inflation is hamiltonian.

Proof Let G be a 4-connected inflation. Then $G = L(S(H))$ for some essentially 4-edge connected subdivision $S(H)$ of a 4-edge connected graph H. As shown in [13], using the result of Kundu [8] that H has two edge-disjoint spanning trees, it is easy to show that H contains a spanning circuit, hence $S(H)$ contains a dominating circuit. By a result of Harary and Nash-Williams [3] this implies $G = L(S(H))$ is hamiltonian. ■

The connectivity bound in Lemma 5 cannot be decreased, since there are nonhamiltonian 3-connected inflations, e.g. the inflation of the Petersen graph. These graphs also show that the connectivity bound in the next result is best possible.

Theorem 6
Every 4-connected claw-free hourglass-free graph is hamiltonian.

Proof Let G be a 4-connected claw-free hourglass-free graph. Then by a result in [2] $\mathrm{cl}(G)$ is also claw-free and hourglass-free. Hence by Theorem 4 we can assume that $G = \mathrm{cl}(G)$. This implies that the neighborhood of each vertex of G induces either a complete graph or a disjoint union of two complete graphs. Since G is hourglass-free, in the latter case one of the complete graphs is a K_1. Hence G contains two types of edges, namely edges that are contained in a complete subgraph on more than 2 vertices, and edges that are contained in a K_2 only. Moreover, all maximal complete subgraphs on more than two vertices contain two types of vertices, namely vertices with a complete neighborhood (contained in the subgraph) which are called *simplicial* vertices, and vertices with precisely one neighbor outside the subgraph. It is not difficult to check that the graph G' obtained from G by deleting all simplicial vertices is a 4-connected inflation. Hence G' is hamiltonian by Lemma 5. Clearly, a Hamilton cycle in G' contains at least one edge of each maximal complete subgraph on more than 2 vertices, and all the maximal complete subgraphs of G containing simplicial vertices correspond to such subgraphs. Hence a Hamilton cycle of G' can easily be extended to a Hamilton cycle in G. ■

3 Connected factors with degree restrictions

By Theorem 3.1 in [6], every connected claw-free graph has a 2-*walk*, i.e. a (closed) walk which passes every vertex at most twice. Clearly, the edges of a 2-walk induce a connected factor of maximum degree at most 4.

The aim of this section is to prove that every 4-connected claw-free graph contains a connected factor with vertices of degree 2 or 4. We start with a series of lemmas on *congruent* factors of multigraphs, i.e. factors of a multigraph G which have the same parity of degrees at every vertex. Lemma 7 will allow us to apply the closure introduced in Section 1 later on. (Note that $\mathrm{cl}(G)$ can be constructed from G by iteratively adding the missing edge in a subgraph $K_4 - e$.)

Lemma 7
Let F be a connected factor of a multigraph G and let e be an edge contained in some complete subgraph K_4 of G. Then $G - e$ has a connected factor F' such that $d_{F'}(x) \equiv d_F(x) \bmod 2$ for all $x \in V(G)$.

Proof For two multigraphs G_1, G_2 we define $G_1 \cup G_2 := (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$, $G_1 \cap G_2 := (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$, and $G_1 \Delta G_2 := (G_1 \cup G_2) - E(G_1 \cap G_2)$. ($G_1 \Delta G_2$ is the *symmetric difference* of G_1 and G_2.)

Let w, x, y, z be the vertices of the subgraph $H \cong K_4$ which contains e, say $e \in [w, x]$. The conclusion of the lemma is obviously true if $e \notin E(F)$. So we may assume $e \in E(F)$. We define the following four w, x-subpaths of H: $Q := w, y, x$, $R := w, z, x$, $S := w, y, z, x$, and $T := w, z, y, x$. It is easy to see that if F' is the symmetric difference of $F - e$ and any of these paths, then $d_{F'}(u) \equiv d_F(u) \bmod 2$ holds for all $u \in V(H)$. Hence it suffices to prove that the symmetric difference F' of one of these paths and $F - e$ contains a connected spanning subgraph of H. We denote $(F - e) \cap H$ by H'.

If $d_{H'}(y) = 3$, then $F' := (F - e)\Delta R$ will serve, if $d_{H'}(y) = 0$ and $d_{H'}(z) \neq 0$ then $F' := (F - e)\Delta Q$ will do, and if $d_{H'}(y) = d_{H'}(z) = 0$ then $D' := (F - e)\Delta T$ will. So we may assume that y and, by symmetry, z have degree 1 or 2 in H'.

Without loss of generality, we may assume that $d_{H'}(w) \geq d_{H'}(x)$. We distinguish three cases.

Case 1. $d_{H'}(w) = 2$ and $d_{H'}(x) \geq 1$. Without loss of generality, x is adjacent to y in H'. Since $d_{H'}(y) \neq 3$, there is no edge between y and z in H. It follows that $F' := (F - e)\Delta S$ is an appropriate factor.

Case 2. $d_{H'}(w) = 2$ and $d_{H'}(x) = 0$. If y is adjacent to z in H', then $F' := (F - e)\Delta Q$ will do; otherwise $F' := (F - e)\Delta S$ will.

Case 3. $d_{H'}(w) = 1$. Without loss of generality, w is adjacent to y in H. If x is not adjacent to z, then $F' := (F - e)\Delta R$ will do; in the other case, $d_{H'}(x) = 1$ as well, and $F' := (F - e)\Delta T$ contains a connected spanning subgraph of H', since it contains all edges of H' except possibly an edge beetween y, z. ∎

Lemma 8 guarantees the existence of a connected low degree factor in a claw-free multigraph which is congruent to a given one.

Lemma 8
Let F be a connected factor of a claw-free multigraph G. Then there exists a connected factor F' of G without vertices of degree exceeding 4 such that $d_{F'}(x) \equiv d_F(x) \bmod 2$ for all $x \in V(G)$.

Proof Throughout the proof, we call a connected factor F' with $d_{F'}(x) \equiv d_F(x) \bmod 2$ for all $x \in V(G)$ a *good* factor. Among all good factors we choose one, say F', with a minimum number of edges. We claim that F' contains no vertex of degree exceeding 4.

Suppose to the contrary that $x \in V(G)$ had degree at least 5 in F'. We distinguish two cases.

Case 1. $F' - x$ is connected. First note that there is no pair of distinct edges $e, f \in E(F')$ between x and some $y \in V(G)$, for otherwise $F' - e - f$ would be a good factor, contradicting the choice of F. So $|N_{F'}(x)| \geq 5$. Let $e \in [y, z]_G$ be an edge in $G[N_{F'}(x)]$. Then $e \in E(F')$, too, for otherwise $(F' - [x, y] - [x, z]) + e$ would be a good factor, a contradiction. Furthermore, e is a bridge of $F' - x$, for otherwise $F' - [x, y] - [x, z] - e$ is a good factor, which is absurd again. So every edge in $G[N_{F'}(x)]$ is a bridge of $F' - x$, and in particular, $G[N_{F'}(x)]$ contains no cycle. But then $N_{F'}(x)$ must contain three independent vertices (since $|N_{F'}(x)| \geq 5$), which form a claw together with x, a contradiction.

Case 2. $F' - x$ is not connected. First note that there is no triple $e, f, h \in E(F')$ between x and some $y \in V(G)$, for otherwise $F' - e - f$ would be a good factor. Let C, D be distinct components of $F' - x$, and let $Y := N_{F'}(x) \cap V(C)$ and $Z := N_{F'}(x) \cap V(D)$. There is no edge in G between a vertex of Y and one of Z, for otherwise there were edges $e \in [x, y]_{F'}$, $f \in [x, z]_{F'}$, $h \in [y, z]_G \setminus E(F')$ for some $y \in Y$, $z \in Z$, and $(F' - e - f) + h$ would be a good factor, a contradiction. In particular, C and D are the *only* components of $F' - x$. Since G is claw-free, Y and Z are complete in G. Without loss of generality, we may assume that there are at least three edges between x and vertices of Y (otherwise we swap the roles of Y and Z). Then Y must be complete in F' as well, for otherwise there would be edges $e \in [x, y]_{F'}$, $f \in [x, z]_{F'}$, $h \in [y, z]_G \setminus E(F')$, and so $(F' - e - f) + h$ would be a good factor, a contradiction. It follows that there cannot be a pair e, f of distinct edges between x and $y \in Y$, for otherwise $F' - e - f$ would be a good factor, a contradiction. So $|Y| \geq 3$. But then $F' - [x, y] - [x, z] - e$ is a good factor for arbitrary $e \in [y, z]_{F'} \neq \emptyset$, $y, z \in Y$, our final contradiction. ■

Lemma 9 deals with the existence of a connected even factor in 4-connected line graphs of multigraphs.

Lemma 9
Every 4-connected line graph of a multigraph contains a connected factor which has degree two or four at each vertex.

Proof Let G be a multigraph such that $L(G)$ is 4-connected. Suppose that x is a vertex of degree 3 in G. If a neighbor y of x has degree less than 3, then $G - \{x, y\}$ must be edgeless, since $L(G)$ is 4-connected. In this case, the assertion of the theorem can be checked easily by exhaustion. So *doubling* an edge e incident with x, i.e. adding a further, new edge e^+ with the same endvertices as e, will not produce a vertex of degree less than four at one of its ends. So there exists a set $E' \subseteq E(G)$ such that doubling each edge of E' (once) produces a graph G' without vertices of degree 3, with $E(G') = E(G) \cup \{e^+ \mid e \in E'\}$, and with $V(G') = V(G)$. Furthermore, no edge $e \in E'$ has endvertices of degree one or two in G.

By [7], there exists a dominating circuit of G which contains all vertices of degree at least 4 in G', and here we can achieve that if it contains exactly one of e and e^+, then it contains e. Among all dominating circuits with these properties we choose one, say F, with as few edges as possible. It follows that if F contains both edges e and e^+ for some $e \in E'$, then $F - e - e^+$ is disconnected. The edges of F induce a dominating circuit T, which we orient according to one way of traversing the circuit, starting at an arbitrary vertex. Since $F-e-e^+$ is disconnected whenever e and e^+ are in F for some $e \in E'$, e and e^+ are oriented oppositely (if they are both in F).

Now we produce a sequence T' of edges of G by inserting some of the edges not in $E(F)$ (not necessarily once) at some position into the sequence of edges corresponding to T, according to the following rules:
1) If e and f with $f = e^+$ or $e = f^+$ are consecutive on T, then we insert two edges of $E(G) \setminus E(F)$ incident with the outvertex of e (i.e. the invertex of f) at the position in between e and f (such edges exist).
2) If e and f, and f^+ and e^+ are both consecutive on T, then we insert an edge incident with the outvertex of f^+ at the position in between f^+ and e^+ (such an edge exists).

The sequence T' need not be a circuit. Note that every inserted edge occurs at most twice in T' and all others occur once in T'; those which have been inserted twice never occur consecutively in T'. Neither e and e^+ nor e^+ and e are consecutive in T', and if e and f are consecutive in T', then f^+ and e^+ are not.

Now we construct T'' from T' by inserting sequentially the remaining edges: If there is an edge e in $E(G)$ not inserted so far into T, then we insert it at a position between f and g, whenever e, f and g have a common endvertex. If this is not possible, then e has a common endvertex with the first and the last edge of T'', and we add e at the end of T''. All edges inserted in the latter way into T' occur only once.

Finally, we construct T''' from T'' by replacing each doubled edge e^+, $e \in E'$, by the original edge e.

T''' is a sequence of edges of G with the following properties:
1) Any two consecutive edges have a common vertex, and the first and the last one have a common vertex.
2) Two consecutive edges of T''' are distinct.
3) If $e, f \in E'$ are consecutive in T''', then f and e are not.
4) Every edge of G occurs in T' at least once, at most $3 \cdot |E'|$ edges occur twice, and no edge of G occurs more than twice.

Therefore, the edges of T''' form a connected factor of $L(G)$ with vertices of degree 2 or 4, and with at most $3|E'|$ vertices of degree 4. ■

In general, one cannot expect an upper bound for $|E'|$ better than the number $v_3(G)$ of vertices of degree 3 in G, which leads, according to the proof of Lemma 9, to an upper bound of $3 \cdot v_3(G)$ for the number of vertices of degree 4 in the factor. Unfortunately, this bound may equal $|V(L(G))|$, for example if G is an essentially 4-edge-connected bipartite graph where one color class consists of vertices of degree 3.

If one provides more structure on G, then one can improve this bound. For example, if in

G the vertices of degree 3 are independent, then one gets $|E'| \le v_3(G)$ by similar arguments as above. This implies, for example, that a 4-connected line graph with minimum degree 5 contains a connected factor with more than 2/3 of its vertices having degree 2 and all others having degree 4.

Now we are able to establish the main result of this section.

Theorem 10
Every 4-connected claw-free graph contains a connected factor which has degree two or four at each vertex.

Proof Let G be a 4-connected claw-free graph. Then $\mathrm{cl}(G)$ is a 4-connected line graph. By Lemma 9, $\mathrm{cl}(G)$ contains a connected factor which has degree two or four at each vertex. By Lemma 7, G contains a connected factor which has even degree at each vertex. Finally, by Lemma 8, the assertion follows. ■

By the results of [7] it is also possible to prove the stronger result that between every pair of distinct vertices in a 4-connected line graph there exists a spanning trail which passes every vertex at most twice.

4 Factors consisting of a bounded number of paths

In this section we prove that Conjecture 1 and 2 are equivalent to seemingly weaker conjectures in which the conclusion that G is hamiltonian is replaced by the conclusion that G contains a factor consisting of a number of paths bounded by a constant, or, more generally, by a function which is sublinear in the number of vertices of the graph. In particular we show that every k-connected claw-free graph is hamiltonian if and only if every k-connected claw-free graph is traceable, i.e. contains a Hamilton path. For convenience we use the term *r-path-factor* for a factor consisting of at most r paths. A *path-factor* is an r-path factor for some r, and its *endvertices* are the vertices of degree less than 2 of its components.

We start with an auxiliary result. Here a *k-clique* of a graph G is a subset of k vertices of G inducing a complete subgraph in G.

Lemma 11
Let $k \ge 2$ be an integer. If there exists a k-connected nonhamiltonian claw-free graph on n vertices, then there exists a k-connected nonhamiltonian claw-free graph on at most $2n-2$ vertices containing a k-clique.

Proof Let G be a k-connected nonhamiltonian claw-free graph on n vertices, and assume that $G = \mathrm{cl}(G)$. Hence G is the line graph of some triangle-free graph H. We may assume $k \ge 4$, since for If $k \le 3$ the claw-freeness clearly implies that there is a k-clique in G. If all vertices of H have degree at least 4, then it is easy to see that H is 4-edge connected; by the result of [14] G is hamiltonian. If there is a vertex in H with precisely one neighbor u, then the edges incident with u induce a clique in G with at least k vertices. Hence we may assume there is a vertex x of degree 2 or 3 in H. Therefore G contains a vertex whose neighborhood consists of disjoint cliques R and Q with $|R| \geq |Q| \in \{1, 2\}$. If some vertex of G is contained in a k-clique, then we are done. Hence we may assume that $|R| = k - 2$ and $|Q| = 2$. Now consider two copies G_1 and G_2 of G with the same fixed vertex x called x_i in G_i $(i = 1, 2)$ and the same partition of $N(x)$ into two cliques Q_i, R_i in G_i with $|Q_i| = 2$ and $|R_i| = k - 2$ for $i = 1, 2$, respectively. Define the graph G' on $2n - 2$ vertices obtained from G_1 and G_2 by deleting x_1 and x_2, and joining all vertices of Q_1 to all vertices of Q_2, and joining all vertices of R_1 to all vertices of R_2. Denote by E' the set of edges joining vertices of $G_1 - x_1$ and $G_2 - x_2$. Then one easily checks that G' is claw-free and k-connected, and that G' contains a k-clique. We complete the proof by showing that G' is nonhamiltonian.

Suppose to the contrary that G' has a Hamilton cycle C. Then $F_i := C \cap (G_i - x_i)$ is a path-factor of $G_i - x_i$ with all endvertices in $Q_i \cup R_i$. Either F_1 contains no path between the vertices of Q_1, or F_2 contains no path between the endvertices, for otherwise these two paths, together with two edges of E', would form a proper subcycle of C, which is absurd. Without loss of generality, F_1 contains no path between the endvertices of Q_1.

Case 1. Q_1 contains no endvertex of F_1. Then $F_1 \cup \{x_1\}$ is a path-factor of G_1 with all endvertices in the clique $R_1 \cup \{x_1\}$.

Case 2. Q_1 contains endvertices of exactly one component P of F_1. Then Q_1 contains precisely one endvertex of P, and hence $(F_1 - P) \cup (P + x_1)$ is a path-factor of G_1 with all endvertices in the clique $R_1 \cup \{x_1\}$.

Case 3. Q_1 contains endvertices of two distinct components $P \neq P'$ of F_1. Then $(F_1 - P - P') \cup (P + x_1 + P')$ is a path-factor of G_1 with all endvertices in the clique R_1.

Since a graph on at least 3 vertices is hamiltonian if and only if it has a path-factor with all endvertices being contained in the same clique, it follows in either case that G_1 is hamiltonian, a contradiction. ■

We use the above lemma to prove the following result.

Theorem 12
Let $k \geq 2$ and $r \geq 1$ be two integers. Then the following statements are equivalent.

(1) There is a k-connected claw-free nonhamiltonian graph.

(2) There is a k-connected claw-free graph without an r-path-factor.

Moreover, if there is an example for (1) on n vertices, then there is an example for (2) with at most $(2r + 1)(2n - 2)$ vertices.

Proof It is clear that we only have to show that the existence of a k-connected claw-free nonhamiltonian graph on n vertices implies the existence of a k-connected claw-free graph without an r-path-factor on at most $(2r + 1)(2n - 2)$ vertices.

Let G be a k-connected claw-free nonhamiltonian graph on n vertices. Then by Lemma 11 there is a k-connected claw-free nonhamiltonian graph H on at most $2n-2$ vertices containing a k-clique Q. We may assume that $H = \mathrm{cl}(H)$. Let G_r be the graph obtained from $2r + 1$ disjoint copies of H by joining all vertices corresponding to the k-clique Q in all copies, forming a $(2r + 1)k$-clique. Clearly, G_r is claw-free and k-connected and has at most $(2r + 1)(2n - 2)$

vertices. We complete the proof by showing that G_r admits no r-path-factor. Suppose to the contrary that P is an r-path-factor of G_r. Then P has at most $2r$ vertices of degree zero or one. Since G_r contains $2r+1$ disjoint copies of H, this implies that for at least one copy of H, $V(H) \setminus Q$ contains no endvertices of P. It is obvious that we can construct a Hamilton cycle in this copy of H, contradicting the assumption that H is nonhamiltonian. ■

Theorem 12 has a number of interesting consequences, the first of which is obvious and given without proof.

Corollary 13
Let $k \geq 2$ be an integer. Then the following statements are equivalent.

(1) Every k-connected claw-free graph is hamiltonian.

(2) Every k-connected claw-free graph is traceable.

In particular Corollary 13 shows that Conjecture 2 is equivalent to the conjecture that every 4-connected claw-free graph is traceable. We can weaken the conclusion a little further. The next consequences of Theorem 12 can be obtained by examining the order of the graph G_r in the proof of the theorem.

Corollary 14
Let $k \geq 2$ be an integer, and let $f(n)$ be a function of n with the property that $\lim_{n\to\infty} \frac{f(n)}{n} = 0$. Then the following statements are equivalent.

(1) Every k-connected claw-free graph is hamiltonian.

(2) Every k-connected claw-free graph on n vertices has an $f(n)$-path-factor.

(3) Every k-connected claw-free graph on n vertices has a 2-factor with at most $f(n)$ components.

(4) Every k-connected claw-free graph on n vertices has a spanning tree with at most $f(n)$ vertices of degree one.

(5) Every k-connected claw-free graph on n vertices has a path of length at least $n - f(n)$.

Proof We only prove that (2) implies (1). The other cases are similar and left to the reader.

Suppose (2) is true and suppose there exists a k-connected claw-free nonhamiltonian graph on m vertices. Then by Theorem 12 there is a k-connected claw-free graph G_r without an r-path-factor on $n_r \leq (2r+1)(2m-2)$ vertices. If we let r tend to infinity, then G_r is a graph on n_r vertices without an r-path-factor, while $\lim_{r\to\infty} \frac{r}{n_r} \geq \frac{1}{4m-4}$ for a fixed integer $m > 1$. This contradicts the assumption that (2) is true. ■

In particular Corollary 14 shows that Conjecture 2 is true if one could show that, e.g., every 4-connected claw-free graph on n vertices admits a factor consisting of a number of paths which is sublinear in n.

Recently, in [4] it has been shown that a claw-free graph G has an r-path-factor if and only if $\operatorname{cl}(G)$ has an r-path-factor. Similarly, in [12] it has been shown that a claw-free graph G has a 2-factor with at most r components if and only if $\operatorname{cl}(G)$ has such a 2-factor. These results immediately imply the equivalence of the following statements related to Conjecture 1.

Corollary 15
Let $k \geq 2$ be an integer, and let $f(n)$ be a function of n with the property that $\lim_{n\to\infty} \frac{f(n)}{n} = 0$. Then the following statements are equivalent.

(1) Every k-connected line graph is hamiltonian.

(2) Every k-connected line graph on n vertices has an $f(n)$-path-factor.

(3) Every k-connected line graph on n vertices has a 2-factor with at most $f(n)$ components.

In particular Corollary 15 shows that Conjecture 1 is true if one could show that, e.g., every 4-connected line graph on n vertices admits a 2-factor consisting of a number of components which is sublinear in n. The equivalences between (1) and (2) of Corollary 14 and of Corollary 15 appear also in a sequence of equivalences in [9].

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