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Spherical distribution vectors

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Abstract. In this paper we consider a locally compact second countable unimodular group G and a closed unimodular subgroup H. Let ρ be a finite dimensional unitary representation of H with closed image. For the unitary representation of G obtained by inducing ρ from H to G a decomposition in Hilbert subspaces of a certain space of distributions is given. It is shown that the representations relevant for this decomposition are determined by so-called (ρ, H) spherical distributions, which leads to a description of the decomposition on the level of these distributions.

Keywords: Hilbert subspace, Distribution vector, Spherical distribution, Plancherel formula

20G15, 20G20, 22E15, 22E46:

1. Introduction

Let G be a locally compact group and H be a closed subgroup. Now one is interested in unitary representations of G related to the homogeneous space $X := H/G$. A rich class of examples among these spaces are the symmetric spaces over locally compact fields. If both groups are unimodular, then inducing unitary representations ρ from H to G gives you an ample variety of examples of unitary representations of G. First results of a general nature concerned the H-invariant case, i.e. where ρ is the trivial one-dimensional representation, for compact H. A special subclass form the so-called Gelfand pairs, i.e. the pairs (G, H) for which the convolution algebra of continuous bi-H-invariant functions is commutative, see (Faraut, 1979) for an overview. A key role in this theory plays the so-called spherical function related to the representation. Also, still for the case H compact, attention was paid in the literature to nontrivial representations ρ, see for the case of the Riemannian symmetric spaces e.g. (van Dijk and Pasquale, 1999),(Camporesi, 1997) and his contribution to this special issue.

In the noncompact H-invariant situation, the proper generalization for real symmetric varieties turned out to be that of an H-invariant

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spherical distribution, see e.g. (van Dijk and Poel, 1986). They play a central role in the work of various contributors to the Plancherel formula for real symmetric spaces, see e.g.(Brylinski and Delorme, 1992; Carmona and Delorme, 1994; Delorme, 1998; Ōshima and Matsuki, 1984; van den Ban, 1988; van den Ban and Schlichtkrull, 1997). In this more general situation, this leads to the notion of a generalized Gelfand pair, see (Thomas, 1984), as those pairs for which $L^2(H/G)$ decomposes multiplicity free. For a survey of various examples of such pairs, we refer to (van Dijk, 1994).

Recently, see (van Dijk and Sharshov, 2000) and (Sharshov, 2000), concrete real examples with a noncompact group H and a nontrivial one-dimensional ρ were considered. Geometrically these representations correspond to the natural action of G on the square-integrable sections of nontrivial line bundles over X. The geometric picture of the generalization to be considered here, is that of a class of finite dimensional vector bundles over X.

In the present paper we present for fairly general spaces X and a wide class of representations ρ a decomposition of this induced representation in Hilbert subspaces of a certain space of distributions. The representations relevant for this decomposition are determined by an extension of the notion of spherical distribution, which leads to a description of the decomposition on the level of these distributions.

The precise content of the various subsections is as follows: the first subsection presents the type of homogeneous spaces H/G we will work with and describes the class of geometric representations π_ρ that will be decomposed with the theory of Hilbert subspaces. To be able to apply this theory one needs a suitable description of these representations and this is given in the next subsection.

Next the necessary ingredients from the theory of Hilbert subspaces are treated, and a description of the relevant class of Hilbert subspaces is given. The subsection ends with the description of the representation space as a direct integral of certain extremal Hilbert subspaces.

For each unitary representation one can introduce the space of C^∞-vectors and its antilinear dual, the space of distribution vectors. The characterization of which of these distribution vectors correspond uniquely to the relevant Hilbert subspaces is the main result of the next subsection. This set of distribution vectors, the so-called (ρ, H)-spherical distribution vectors, also has an interpretation as intertwining operators and this connection can be found in the sixth subsection.

One concludes with showing that these distribution vectors are in a unique correspondence with certain distributions on the group G, the so-called (ρ, H)-spherical distributions. The combination of this description with the direct integral decomposition found before, gives then a decomposition of the (ρ, H)-spherical distribution of our representation, which can be seen as an abstract Plancherel formula.

2. The representations

Let G be a unimodular second countable locally compact group and consider a closed unimodular subgroup H of G. On G resp. H we have Haar measures dg resp. dh. It is well-known then that the homogeneous space $X := H/G$ possesses a positive right G-invariant measure dx such that for all f in the space $C_c(G)$ of continuous functions on G with compact support

$$\int_G f(g)dg = \int_X \left\{ \int_H f(hx)dh \right\} dx. \tag{1}$$

An important class of examples of this situation are the following:

Example 2.1. Consider an affine algebraic group $\mathcal{G}$ defined over a locally compact field k. Then $G := \mathcal{G}_k$, the group of k-rational points of $\mathcal{G}$, is a locally compact group, which is unimodular e.g. if $\mathcal{G}$ is reductive. For this class of algebraic groups let $\sigma : G \to G$ be an involution of G and let $\mathcal{H} = \mathcal{G}_\sigma$ be the group of fixed points under σ. According to (Helminck and Wang, 1993), the group $\mathcal{H}$ is defined over k if and only if σ is defined over k. So, for the involutions defined over k we can consider the group $\mathcal{H}_k$. As the fixed point set of an automorphism of finite order of a reductive algebraic group is reductive, see (Steinberg, 1968), the choice $H = \mathcal{H}_k$ gives you an unimodular subgroup. Analogous to the real situation, we call the variety $X_k = \mathcal{H}_k/\mathcal{G}_k$ a symmetric k-variety.

Natural geometric objects related to the homogeneous spaces X are complex vector bundle $\mathcal{V}$ over them. To each such a bundle is associated the representation of G on the space of global sections of this vector bundle. If the structure group of the bundle can be reduced to the unitary group, then the representation of G on the square-integrable global sections of this bundle is a natural unitary representation. Concretely this means that we have a unitary representation ρ of H on the finite dimensional space of fibers V_ρ. In this correspondence the trivial line bundle gives rise to the one dimensional trivial representation of H. Global sections correspond then bijectively with functions $f : G \to V_\rho$ such that

$$f(hg) = \rho(h)(f(g)) \tag{2}$$

Consider now the space $L^2(\rho, X, dx)$ of classes of measurable $f : G \to V_\rho$ satisfying the condition (2) and

$$\int_X < f(x), f(x) >_\rho dx \quad < \infty, \tag{3}$$

where $< \,.\, , \,.\, >_\rho$ denotes the inner product on V_ρ. On this space $L^2(\rho, X, dx)$ we have the inner product

$$< f, g >= \int_X < f(x), f(x) >_\rho dx. \tag{4}$$

The group G acts on this space by right translations and this representation we denote by $\mathcal{R}_\rho$. Thanks to the right G-invariance of dx, this representation is unitary. It is a natural generalization of the right G-action on $L^2(X, dx)$, which corresponds to the case of the trivial line bundle.

Since the unitary representation ρ is completely reducible, it suffices for the decomposition of the space $L^2(\rho, X, dx)$ to consider irreducible ρ and this assumption we make from now on.

Let H_0 be the kernel of ρ and let dh_0 be a Haar measure on H_0. Since a normal subgroup of an unimodular group is also unimodular, this property holds also for H_0. Hence also the homogeneous manifold $X_0 := H_0/G$ possesses a positive right G-invariant measure that is denoted by dx_0 and is related to dg and dh_0 by a formula like (1). We denote the inner product of φ_1 and φ_2 in $L^2(X_0, dx_0)$ by $< \varphi_1, \varphi_2 >_0$ and the the action of G by right translations on $L^2(X_0, dx_0)$ by $\mathcal{R}$.

To be able to exploit the representation theory of the group H/H_0 we assume throughout the rest of this paper the following :

Property 2.2. The group H/H_0 is compact.

This is clearly equivalent to the image of ρ being closed and this condition was surely satisfied in the examples treated in (van Dijk and Sharshov, 2000) and (Sharshov, 2000)

If the group G is a type I group, then it is known that each unitary representation $(\mathcal{L}, \mathcal{H}_\mathcal{L})$ of G on a separable Hilbert space has an abstract direct integral decomposition

$$\mathcal{L} \simeq \int_{\hat{G}}^{\oplus} \mathcal{L}^\pi d\mu_\mathcal{L}(\pi) \simeq \int_{\hat{G}_k} m_\pi \pi d\mu_\mathcal{L}(\pi). \tag{5}$$

Here $\hat{G}_k$ is the unitary dual of $G, d\mu_\mathcal{L}$ is a Borel measure on $\hat{G}, (\pi, \mathcal{H}_\pi)$ is a representative of a class in $\hat{G}$, $\mathcal{L}^\pi$ is a multiple of π and m_π is the multiplicity of π in $\mathcal{L}^\pi$, see (Dixmier, 1994). In particular this holds for $(\mathcal{R}_\rho, L^2(\rho, X, dx))$. The ultimate aim of harmonic analysis on X is to make this decomposition for $(\mathcal{R}_\rho, L^2(\rho, X, dx))$ as explicit as it can be.

A first step in the direction of a decomposition like (5) is to have an idea of the nature of the elements of the relevant spaces $\mathcal{H}_\pi$, like are they functions e.g. or distributions? For the rather explicit representations $(\mathcal{R}_\rho, L^2(\rho, X, dx))$, one can throw some more light on these questions with the theory of Hilbert subspaces of K. Maurin and L. Schwartz. Before carrying this out, it is convenient to have a different form of the representations $(\mathcal{R}_\rho, L^2(\rho, X, dx))$.

3. A different realization of $\mathcal{R}_\rho$

The group H acts by left translations on the space X_0 and by transposition on the functions on X_0. Since dg is also left G-invariant, one sees from relation (1) that

$$\mathcal{L}(h)(f)(x) := f(h^{-1}x), \tag{6}$$

defines a unitary representation of H on $L^2(X_0, dx_0)$. It clearly factorizes over H/H_0. Let $d\tilde{h}$ denote the normalized Haar measure on H/H_0. Then we have an algebra morphism from the convolution algebra of continuous functions on H/H_0 to the bounded linear operators on $L^2(X_0, dx_0)$. It is defined by

$$\mathcal{L}(\varphi)(f)(x_0) = \int_{H/H_0} \varphi(\tilde{h})\mathcal{L}(\tilde{h})(f)(x_0)d\tilde{h} =: \varphi * f(x_0), \tag{7}$$

with φ continuous on H/H_0.

For $u, v \in V_\rho$, let $e_{v,u}$ be the matrix coefficient of the representation ρ of H/H_0 given by

$$e_{v,u}(h) = d_\rho < \rho(h)(u), v >_\rho, \tag{8}$$

where $d_\rho = dim(V_\rho)$. They satisfy the orthogonality relations, see (Borel, 1972)

$$\int_{H/H_0} e_{v_1,u_1}(\tilde{h})\overline{e_{v_2,u_2}(\tilde{h})}d\tilde{h} = d_\rho \overline{< v_1, v_2 >_\rho} < u_1, u_2 >_\rho . \tag{9}$$

These relations imply that one has with respect to convolution on H/H_0

$$e_{v_1,u_1} * e_{v_2,u_2} =< u_1, v_2 >_\rho e_{v_1,u_2}. \tag{10}$$

Let $\{f_i \mid 1 \leq i \leq d_\rho\}$ be an orthonormal basis of the space V_ρ. For simplicity, we denote for each i and j the function e_{f_i,f_j} by e_{ij}. From the definition we see for all j that

$$\begin{aligned} e_{j1}(ht) &= < \rho(t)(f_1), \sum_{k=1}^{d_\rho} \rho_{kj}(h^{-1}) f_k >_\rho \\ &= \sum_{k=1}^{d_\rho} \rho_{jk}(h) < \rho(t)(f_1), f_k >_\rho, \end{aligned} \tag{11}$$

where the $\{\rho_{jk}(h)\}$ are the matrix coefficients of $\rho(h)$ w.r.t. the orthonormal basis $\{f_i\}$. Consider the vector function $e = \sum_j e_{j1} f_j : H \to V_{d_\rho}$. Then for all h and t in H the relations (11) can be written as

$$e(ht) = \rho(h)(e(t)), \tag{12}$$

precisely the same transformation behaviour under left translations from H as the functions in $L^2(\rho, X, dx)$. Thus one is led to consider inside $L^2(X_0, dx_0)$ the following closed subspace

$$L^2(e_{11}, X_0, dx_0) = \{\varphi \mid e_{11} * \varphi := \mathcal{L}(e_{11})(\varphi) = \varphi\}. \tag{13}$$

From (12) follows directly that for each $\varphi \in L^2(e_{11}, X_0, dx_0)$ the function

$$A(\varphi) := \frac{1}{\sqrt{d_\rho}} \sum_{j=1}^{d_\rho} (e_{j1} * \varphi)\, f_j \tag{14}$$

satisfies (2). Reversely, if $\varphi : H \to V_\rho$ is written as $\varphi = \sum_i \phi_i f_i$, then the transformation properties (2) together with (9) imply that for all i $e_{ii} * \varphi_i = \varphi_i$ and $e_{i1} * \varphi_1 = \varphi_i$. Note that for all $f \in L^2(e_{11}, X_0, dx_0)$ and for all i, $1 \le i \le d_\rho$, there holds

$$\begin{aligned} < \mathcal{L}(e_{i1})(f), \mathcal{L}(e_{i1})(f) >_0 = \\ &= \int_{H/H_0} \int_{H/H_0} e_{i1}(\tilde{t}) \overline{e_{i1}(\tilde{s})} < \mathcal{L}(\tilde{t})(f), \mathcal{L}(\tilde{s})(f) >_0 d\tilde{t} d\tilde{s} \\ &= \int_{H/H_0} e_{i1}(\tilde{t}) \{ \int_{H/H_0} < f, \mathcal{L}(\tilde{u})(f) >_0 \overline{e_{i1}(\tilde{t}\tilde{u})} d\tilde{u} \} d\tilde{t} \\ &= \int_{H/H_0} \overline{e_{1i} * e_{i1}(\tilde{u})} < f, \mathcal{L}(\tilde{u})(f) >_0 d\tilde{u} \\ &= < f, f >_0 \end{aligned}$$

We may assume that the measures dx and dx_0 are chosen such that $d\tilde{h} dx = dx_0$. The map $A : L^2(e_{11}, X_0, dx_0) \to L^2(\rho, X, dx)$ is a unitary bijection, for

$$\begin{aligned} < A(\varphi), A(\varphi) > &= \int_X \frac{1}{d_\rho} \{ \sum_{i=1}^{d_\rho} |e_{i1} * \varphi(x)|^2 \} dx \\ &= \frac{1}{d_\rho} \int_{H/H_0} \int_X \{ \sum_{i=1}^{d_\rho} |e_{i1} * \varphi(x)|^2 \} dx \\ &= \frac{1}{d_\rho} \sum_{i=1}^{d_\rho} \int_{X_0} |e_{i1} * \varphi(x_0)|^2 dx_0 \\ &= < \varphi, \varphi >_0 . \end{aligned} \tag{15}$$

Clearly A also commutes with the right G-action on both spaces. Therefore we will work from now on with $(\mathcal{R}, L^2(e_{11}, X_0, dx_0))$ instead of $(\mathcal{R}_\rho, L^2(\rho, X, dx)$.

4. Hilbert subspaces of $L^2(\rho, X, dx)$

Recall that Bruhat, see (Bruhat, 1961), has introduced for each locally compact group G_1 and each homogeneous space F/G_1, where F is a closed subgroup of G_1, the spaces of test functions $\mathcal{D}(G_1)$ and $\mathcal{D}(F/G_1)$ with an appropriate topology. It unifies the cases that G_1 is a Lie group, where it equals the space of C^∞-functions with compact support, and that of totally disconnected spaces, in which case it consists of the locally constant functions with compact support. Therefore the notations $C_c^\infty(G_1)$ respectively $C_c^\infty(F/G_1)$ are also common in this last setting. The elements of their continuous antilinear duals are called distributions on G_1 resp. F/G_1 and these spaces are denoted by $\mathcal{D}^1(G_1)$ and $\mathcal{D}^1(F/G_1)$. Basic examples of distributions that we will use, are the point distributions $\varepsilon_g,\ g \in G$, given by

$$\varepsilon_g(\varphi) = \overline{\varphi(g)} \text{ for } \varphi \in \mathcal{D}(G)\ .$$

Further there is for each compact subgroup $\mathbb{K}$ of G the distribution $e_{\mathbb{K}}$ given by

$$e_{\mathbb{K}}(f) = \int_{\mathbb{K}} \overline{f(k)} dk,$$

where dk is the normalized Haar measure on $\mathbb{K}$. The group G acts on $\mathcal{D}(X_0)$ by right translation and it leaves the subspace

$$\mathcal{D}(e_{11}, X_0) = \{\phi \in \mathcal{D}(X_0) \mid e_{11} * \phi = \phi\}. \tag{16}$$

invariant. By transposing this representation $\mathcal{R}_\infty$ of G on $\mathcal{D}(X_0)$ one arrives at the representation $\mathcal{R}_{-\infty}$ of G on $\mathcal{D}^1(X_0)$, i.e. for $T \in \mathcal{D}^1(X_0)$

$$\mathcal{R}_{-\infty}(g)(T)(\varphi) = T(\mathcal{R}_\infty(g^{-1})\varphi).$$

Likewise one can dualize the left H-action on $\mathcal{D}(X_0)$ to a representation $\mathcal{L}_{-\infty}$ of H on $\mathcal{D}^1(X_0)$ and one verifies directly that the antilinear dual of the subspace $\mathcal{D}(e_{11}, X_0)$ can be identified with

$$\mathcal{D}^1(e_{11}, X_0) = \{T \in \mathcal{D}^1(X_0), \int_{H/H_0} e_{11}(\tilde{h})\mathcal{L}_{-\infty}(\tilde{h})(T)d\tilde{h} = T\}. \tag{17}$$

Hence, if we take an $f \in L^2(e_{11}, X_0, dx_0)$ and consider the distribution $T = f(x)dx$ on X_0, then it belongs to $\mathcal{D}^1(e_{11}, X_0)$ and

$$\mathcal{R}_{-\infty}(g)(f(x)dx) = f(xg)dx.$$

In other words the embedding $j : f(x) \mapsto f(x)dx$ of $L^2(e_{11}, X_0, dx_0)$ into $\mathcal{D}^1(e_{11}, X_0)$ is a G-morphism. Hence $L^2(e_{11}, X_0, dx_0)$ is a G-invariant Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$. Let $\mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$ be the collection of G-invariant Hilbert subspaces of $\mathcal{D}^1(e_{11}, X_0)$. It is well-known, see (Schwartz, 1964a), that the Hilbert subspaces of $\mathcal{D}^1(X_0)$ are completely determined by their reproducing kernel $jj^* : \mathcal{D}(e_{11}, X_0) \to \mathcal{D}^1(e_{11}, X_0)$, where j^* is the adjoint map of the embedding $j : \mathcal{H} \hookrightarrow \mathcal{D}^1(X_0)$. From the Schwartz kernel theorem (Schwartz, 1964b) one sees that each such a Hilbert subspace $\mathcal{H}$ of $\mathcal{D}^1(X_0)$ corresponds bijectively to a distribution $K \in \mathcal{D}^1(X_0 \times X_0)$ defined by

$$K(\varphi \otimes \overline{\psi}) = (j^*\varphi, j^*\psi)_{\mathcal{H}}. \tag{18}$$

Here $(\cdot, \cdot)_{\mathcal{H}}$ is the inner product on $\mathcal{H}$. From this relation one sees directly that K is a distribution of positive type

$$K(\varphi \otimes \overline{\varphi}) = (j^*(\varphi) \mid j^*(\varphi))_{\mathcal{H}} \geq 0.$$

The G-invariance of the corresponding Hilbert subspace translates into

$$K(\mathcal{R}(g)\varphi \otimes \mathcal{R}(g)\psi) = K(\varphi \otimes \psi).$$

Finally, one has to require of the distribution K that it renders you a G-invariant Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$, not just of $\mathcal{D}^1(X_0)$. Therefore it has to satisfy still the following relation

$$K(e_{11} * \varphi \otimes \overline{e_{11} * \psi}) = K(\varphi \otimes \overline{\psi}). \tag{19}$$

W.r.t. addition these distributions form a closed convex cone Γ_G. Let $\mathrm{ext}(\Gamma_G)$ be the set of extremal rays of Γ_G. Those are the distributions K that satisfy

$$0 \leq K^1 \leq K, K^1 \in \Gamma_G \Rightarrow K^1 = \alpha K. \tag{20}$$

The relevance of $\mathrm{ext}(\Gamma_{G_k})$ follows from

Theorem 4.1. *Let $(\pi, \mathcal{H}_\pi)$ be a G-invariant Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$ and let $K_\pi \in \Gamma_{G_k}$ be the corresponding distribution. Then there holds*

$$(\pi, \mathcal{H}_\pi) \text{ is irreducible } \Leftrightarrow K_\pi \text{ is extremal}$$

A proof of this theorem can be found in (Klamer, 1979). Since the group G is second countable, we know that $\mathcal{D}^1(e_{11}, X_0)$ is the dual of a nuclear barrelled space and hence, according to (Thomas, 1984), there exists a Hausdorff topological space S and an admissible parametrization of $\mathrm{ext}(\Gamma_{G_k}), s \to K_s$, such that, if $\mathcal{H}_s$ is the Hilbert space corresponding to K_s, then there holds

Theorem 4.2. *For every $\mathcal{H} \in \mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$ there exists a Radon measure m on S such that*

$$\mathcal{H} = \int_S^{\oplus} \mathcal{H}_s dm(s). \tag{21}$$

In particular for the Hilbert subspace $L^2(e_{11}, X_0, dx_0)$ this theorem gives you a decomposition of $L^2(e_{11}, X_0, dx_0)$ in minimal unitary G-models as in (5). In view of Theorem 4.2 it is important to have an idea of which representations can be realized as a Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$. In the real case, these are the representations that possess a non-zero cyclic H-invariant distribution vector. We will introduce a similar notion in the present setting and we will determine also a useful characterization.

5. C^∞-vectors and distribution vectors

Let $(\pi, \mathcal{H}_\pi)$ be a continuous representation of G on the Hilbert space $\mathcal{H}_\pi$. If G is a Lie group, then the space of C^∞-vectors of $(\pi, \mathcal{H}_\pi)$ is given by

$$\mathcal{H}_\pi^\infty = \{v \in \mathcal{H}_\pi \mid g \to \pi(g)v \text{ is } C^\infty \}.$$

It is a dense subspace of $\mathcal{H}_\pi$ on which both the Lie group and the Lie algebra act and it can be given a topology, see (Cartier, 1976), for which it is a Fréchet space. We will introduce the analogue of this space in our setting.. If G_1 is a locally compact group, then $F(G_1)$ denotes the collection of normal compact subgroups $\mathbb{K}$ of G_1 such that $G/\mathbb{K}$ is a Lie group. We say a sequence $\{\mathbb{K}_n\}$ in $F(G_1)$ converges to the identity, if $\mathbb{K}_{n+1} \subset \mathbb{K}_n$ for all n and $\cap_{n\geq 1}\mathbb{K}_n = e$, the identity element in G. Let G_e be the connected component of the identity element. Since G is assumed to be second countable, there exists a sequence $\{\mathbb{K}_n\}$ in $F(G)$ converging to the identity as soon as G/G_e is compact, see (Montgomery and Zippin, 1955). In that case the group G is isomorphic to the projective limit of the Lie groups $G/\mathbb{K}_n$. Such a locally compact group is called a Yamabe group. If G itself is not a Yamabe group, then it contains at least an open subgroup G_Y that is one. Let $\{\mathbb{K}_n\}$ be a sequence in $F(G_Y)$ that converges to the identity. Then we first introduce the space

$$\mathcal{H}_\pi(\mathbb{K}_n) = \{v \in \mathcal{H}_\pi \mid \pi(\mathbb{K}_n)v = v \text{ and } g \to \pi(g)v \in C^\infty(G/\mathbb{K}_n, \mathcal{H}_\pi)\},$$

where a function on $gG_Y/\mathbb{K}_n,\ \ g \in G$, is called C^∞ if it is the translate of a C^∞-function on $G_Y/\mathbb{K}_n$. We give it the Fréchet space structure alluded to above. Next we consider the inductive limit of those spaces

$$\mathcal{H}_\pi^\infty = \varinjlim_{\mathbb{K}_n} \mathcal{H}_\pi(\mathbb{K}_n). \tag{22}$$

and call it the space of C^∞*-vectors* of $(\pi, \mathcal{H}_\pi)$ as it is a generalization of the notion from the Lie group case. It is independent of the choice of the Yamabe subgroup G_Y and the sequence $\{\mathbb{K}_n\}$ in $F(G_Y)$ converging to the identity. Note that, if G is a totally disconnected group like the groups $\mathcal{G}_k$, with k nonarchimedean, from example 2.1, then

$$\mathcal{H}_\pi^\infty = \{v \in \mathcal{H}_\pi \mid g \to \pi(g)v \text{ is locally constant } \}.$$

On $\mathcal{H}_\pi^\infty$ we put the inductive limit topology and the embedding of $\mathcal{H}_\pi^\infty$ into $\mathcal{H}_\pi$ is continuous. As a subspace of $\mathcal{H}_\pi$ the space $\mathcal{H}_\pi^\infty$ is dense. For, the $\mathbb{K}_1$-finite vectors are dense in $\mathcal{H}_\pi$ and since $\mathbb{K}_1$ is the projective limit of the $\{\mathbb{K}_1/\mathbb{K}_n\}$, each such a vector is fixed under some $\mathbb{K}_n$ for n sufficiently large and hence can be approximated arbitrarily close by an element of $\mathcal{H}_\pi(\mathbb{K}_n)$ due to the result in the Lie group case. The topological antilinear dual of $\mathcal{H}_\pi^\infty$ is called the space of *distribution vectors* of $(\pi, \mathcal{H}_\pi)$ and is denoted by $\mathcal{H}_\pi^{-\infty}$. Since $\mathcal{H}_\pi^\infty$ is dense in $\mathcal{H}_\pi$, we get a continuous embedding $\mathcal{H}_\pi \hookrightarrow \mathcal{H}_\pi^{-\infty}$.

The space $\mathcal{H}_\pi^\infty$ is G-invariant and the restriction of π to $\mathcal{H}_\pi^\infty$ is also denoted by π_∞. If one forgets in the totally disconnected case the topology on the space $\mathcal{H}_\pi^\infty$, then the representation $(\pi_\infty, \mathcal{H}_\pi^\infty)$ belongs to the category of *algebraic* representations of G such as it was introduced in (Bernstein and Zelevinsky, 1976).

By transposition we have a representation $\pi_{-\infty}$ of G on $\mathcal{H}_\pi^{-\infty}$, i.e.

$$\langle \pi_{-\infty}(g)T, v\rangle = \langle T, \pi_\infty(g^{-1})v\rangle \tag{23}$$

As above let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G on the Hilbert space $\mathcal{H}_\pi$.

Lemma 5.1. *For each $\varphi \in \mathcal{D}(G)$ and each $v \in \mathcal{H}_\pi$ the vector $\pi(\varphi)v$ belongs to $\mathcal{H}_\pi^\infty$. Moreover the space $G(\mathcal{H}_\pi)$ spanned by all these vectors is dense in $\mathcal{H}_\pi^\infty$. It is called the Gårding space of $(\pi, \mathcal{H}_\pi)$.*

Proof. Let G_Y be an open Yamabe subgroup of G and let $\{\mathbb{K}_n\}$ be a sequence in $F(G_Y)$ that converges to the identity. Then φ is by definition a finite sum of translates of functions in $C_c^\infty(G_Y/\mathbb{K}_m)$, for a sufficiently small m. Hence it is sufficient to consider functions $\varphi \in \mathcal{D}(G_Y)$. In particular, one has that $\varphi * e_{\mathbb{K}_m} = \varphi$, so that we may assume that $\pi(e_{\mathbb{K}_m})(v) = v$. This reduces the problem to the unitary representation of the Lie group $G_Y/\mathbb{K}_m$ in the Hilbert space $\pi(e_{\mathbb{K}_m})(\mathcal{H}_\pi)$ and there the result is well-known, see (Cartier, 1976). Similarly, one reduces the density of the Gårding space to the Lie group situation. □

The foregoing fact enables you to define for each $\varphi \in \mathcal{D}(G)$ and each $T \in \mathcal{H}_\pi^{-\infty}$ the distribution vector $\pi_{-\infty}(\varphi)(T) \in \mathcal{H}_\pi^{-\infty}$ by

$$\begin{aligned} \langle \pi_{-\infty}(\varphi)(T), v\rangle &= \int_{G_k} \varphi(g)\langle \pi_{-\infty}(g)T, v\rangle dg \\ &= \langle T, \pi_\infty(\overset{\vee}{\varphi}_0)(v)\rangle. \end{aligned} \tag{24}$$

for all $v \in \mathcal{H}_\pi^\infty$. Here $\check{\varphi}$ is defined by $\check{\varphi}(g) = \varphi(g^{-1})$. As in the Lie group case there holds

Lemma 5.2. *The distribution vector $\pi_{-\infty}(\varphi)(T)$ for $\varphi \in \mathcal{D}(G)$ and $T \in \mathcal{H}_\pi^{-\infty}$ belongs to $\mathcal{H}_\pi^\infty$.*

Proof. Note first of all that $\pi_{-\infty}(\varphi)(T)$ is defined on all of $\mathcal{H}_\pi$. As in lemma 5.1 it suffices to prove the lemma for all $\varphi \in \mathcal{D}(G_Y)$, where G_Y is a Yamabe subgroup of G. From the definition of $\mathcal{D}(G_Y)$ follows that there is a $\mathbb{K}$ in $F(G_Y)$ such that $e_\mathbb{K} * \varphi * e_\mathbb{K} = \varphi$. This implies first of all that $\pi_{-\infty}(\varphi)\pi_{-\infty}(e_\mathbb{K})T = \pi_{-\infty}(\varphi)T$ so that we may just as well replace T by $\pi_{-\infty}(e_\mathbb{K})T$, which belongs to $\pi_{-\infty}(e_\mathbb{K})(\mathcal{H}_\pi)^{-\infty}$. Further we have $\pi_{-\infty}(e_\mathbb{K})\pi_{-\infty}(\varphi)T = \pi_{-\infty}(\varphi)T$ and hence that for all $v \in \mathcal{H}_\pi$

$$\langle \pi_{-\infty}(\varphi)T, \pi_\infty(e_\mathbb{K})(v)\rangle = \langle \pi_{-\infty}(\varphi)T, v\rangle.$$

In other words $\pi_{-\infty}(\varphi)(T)$ is a continuous linear form on the closed subspace $\pi(e_\mathbb{K})(\mathcal{H}_\pi)$. Therefore there exists a $w \in \pi(e_\mathbb{K})(\mathcal{H}_\pi)$ such that for all $v \in \pi(e_\mathbb{K})(\mathcal{H}_\pi)$

$$\langle \pi_{-\infty}(\varphi)(T), v\rangle = (w, v)_\pi.$$

Thus we have reduced the question to that for a function φ in the space $C_c^\infty(G_Y/\mathbb{K})$ and the distribution vector $\pi_{-\infty}(e_\mathbb{K})T$ of the unitary representation of $G_Y/\mathbb{K}$ on $\pi(e_\mathbb{K})(\mathcal{H}_\pi)$ and there the result is well-known. □

Hence each $T \in \mathcal{H}_\pi^{-\infty}$ defines a linear map $A_T : \mathcal{D}(G) \to \mathcal{H}_\pi$ by $A_T(\varphi) = \pi_{-\infty}(\check{\varphi})(T)$. By reduction to the Lie group case one shows that it is continuous. With respect to left translations on $\mathcal{D}(G)$ the map A_T behaves as follows

$$A_T(\varepsilon_g * \varphi) = \pi_{-\infty}(\check{\varphi})\pi_{-\infty}(g^{-1})T, \tag{25}$$

for all $g \in G$. The map A_T also intertwines the action of G by right translation on $\mathcal{D}(G)$ and by the representation π on $\mathcal{H}_\pi$, i.e. for all $g \in G$ and all $\varphi \in \mathcal{D}(G)$

$$A_T(\varphi * \varepsilon_{g^{-1}}) = \pi(g)(A_T(\varphi)). \tag{26}$$

All continuous maps from $\mathcal{D}(G)$ to $\mathcal{H}_\pi$ with the property (26) have this form, for there holds

Theorem 5.3. *Let $A : \mathcal{D}(G) \to \mathcal{H}_\pi$ be a continuous map that satisfies for all $g \in G$ and all $\varphi \in \mathcal{D}(G)$, $A(\varphi * \varepsilon_{g^{-1}}) = \pi(g)(A_T(\varphi))$. Then there is a unique distribution vector $T \in \mathcal{H}_\pi^{-\infty}$ such that $A = A_T$.*

Proof. Let G_Y again be an open Yamabe subgroup of G and let $\{\mathbb{K}_n\}$ be a sequence in $F(G_Y)$ that converges to the identity. Since A satisfies property (26) and any $\varphi \in \mathcal{D}(G)$ is a finite sum of translates of functions in $C_c^\infty(G_Y/\mathbb{K}_m)$, for a sufficiently small m, it is sufficient to show that $A : \mathcal{D}(G_Y) \to \mathcal{H}_\pi$ has the form described in the theorem. In other words we consider from now on $G = G_Y$. By definition $\mathcal{D}(G_Y)$ is the inductive limit of the $C_c^\infty(G_Y/\mathbb{K}_m)$ and thanks to property (26) we have for each $\varphi \in C_c^\infty(G_Y/\mathbb{K}_m)$

$$A(\varphi) = \pi(e_{\mathbb{K}_m})A(\varphi). \tag{27}$$

Thus the restriction of A to the space $C_c^\infty(G_Y/\mathbb{K}_m)$ maps into the Hilbert space $\pi(e_{\mathbb{K}_m})(\mathcal{H}_\pi)$, on which we have a natural unitary representation of the Lie group $G_Y/\mathbb{K}_m$. In the Lie group case the result is well-known and can be found e.g. in (Cartier, 1976). Hence there is for each m a $T_m \in \pi(e_{\mathbb{K}_m})(\mathcal{H}_\pi)^{-\infty}$ such that for all $\varphi \in C_c^\infty(G_Y/\mathbb{K}_m)$ $A(\varphi) = \pi_{-\infty}(\check{\varphi}T_m$ and T_m is unique due to the fact that the Gårding space is dense in the C^∞-vectors. As $\mathcal{H}_\pi^\infty$ is the inductive limit of the $\pi(e_{\mathbb{K}_m})(\mathcal{H}_\pi)^\infty$, the distribution vectors $\mathcal{H}_\pi^{-\infty}$ are the projective limit of the $\pi(e_{\mathbb{K}_m})(\mathcal{H}_\pi)^{-\infty}$ and thus the T_m determine a unique T in $\mathcal{H}_\pi^{-\infty}$, such that $A(\varphi) = \pi_{-\infty}(\check{\varphi})T$ for all $\varphi \in \mathcal{D}(G_Y)$. This concludes the proof of the theorem. □

Now that we have the action of G on $\mathcal{H}_\pi^{-\infty}$ we define

$$(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11}) = \left\{ T \in \mathcal{H}_\pi^{-\infty} \;\middle|\; \begin{array}{l} \pi_{-\infty}(h)T = T \text{ for all } h \in H_0, \\ \pi_{-\infty}(e_{11})T = T \end{array} \right\}.$$

Note that, if H is compact, then $\pi(e_{11})$ is a well-defined orthogonal projection of the space $\mathcal{H}_\pi$ and the conditions on a $T \in (\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ simply mean that it factorizes over $\pi(\check{e_{11}})$. Hence

Lemma 5.4. *For compact H, we have $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11}) = \pi(\check{e_{11}})(\mathcal{H}_\pi)$.*

Clearly in the noncompact case the operator $\pi(\check{e_{11}})$ can not be given a sense and that is why one has to proceed more carefully. Before coming to the characterization of $\mathrm{Hilb}_G(\mathcal{D}^1(e_{11}, X_0))$ in terms of distribution vectors we introduce still

Definition 5.5. A distribution vector T in $\mathcal{H}_\pi^{-\infty}$ is called *cyclic* if the space

$$\{\pi_{-\infty}(\varphi)(T) \mid \varphi \in \mathcal{D}(G)\}$$

is lying dense in $\mathcal{H}_\pi$.

With the help of this notion, one can see from the space $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ if a unitary representation is a Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$, for there holds

Theorem 5.6. *Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G. Then the set of non-zero cyclic elements of $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ is in bijective correspondence with the continuous G-equivariant embeddings $j : \mathcal{H}_\pi \hookrightarrow \mathcal{D}^1(e_{11}, X_0)$.*

Proof. Let T be a non-zero cyclic element in $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ and let $A_T : \mathcal{D}(G) \to \mathcal{H}_\pi$ be given by $A_T(\varphi) = \pi_{-\infty}(\check{\varphi})(T)$. Recall from (Bruhat, 1961) that the map $P_{H_0} : \mathcal{D}(G) \to \mathcal{D}(X_0)$ defined by

$$P_{H_0}(\varphi_0)(g) = \int_{H_0} \varphi_0(h_0 g) dh_0. \tag{28}$$

is a continuous surjection. Since T is H_0-invariant, it follows from property (25) that the map A_T factorizes over P_{H_0},i.e. $A_T(\varphi) = \tilde{A}_T(P_{H_0}(\varphi))$. Since P_{H_0} is a G-morphism for the right action of G on both spaces, the map $\tilde{A}_T : \mathcal{D}(X_0) \to \mathcal{H}_\pi$ is also G-equivariant. By applying once more relation (25) and the property $\pi_{-\infty}(\check{e_{11}})T = T$, one gets

$$\begin{aligned} A_T(\varphi) &= \int_{H/H_0} e_{11}(\tilde{h}) \pi_{-\infty}(\check{\varphi}) \pi_{-\infty}(\tilde{h}^{-1}) T d\tilde{h} \\ &= \int_{H/H_0} e_{11}(\tilde{h}) A_T(\varepsilon_{\tilde{h}} * \varphi) d\tilde{h} \\ &= \tilde{A}_T(e_{11} * P_{H_0}(\varphi)). \end{aligned} \tag{29}$$

Hence $\tilde{A}_T$ also factorizes over the map $\psi \mapsto e_{11} * \psi$ that projects $\mathcal{D}(X_0)$ onto $\mathcal{D}(e_{11}, X_0)$ and also here the G-equivariance is preserved. So we have a continuous G-equivariant map $\tilde{A}_T : \mathcal{D}(e_{11}, X_0) \to \mathcal{H}_\pi$ and by taking its adjoint we get a continuous G-equivariant injection $\tilde{A}_T^* : \mathcal{H}_\pi \to \mathcal{D}^1(e_{11}, X_0)$. For, by definition, we have

$$\langle \tilde{A}_T^*(v), e_{11} * P_{H_0}(\varphi) \rangle = \langle v, A_T(\varphi) \rangle = \langle v, \pi_{-\infty}(\check{\varphi})(T) \rangle, \tag{30}$$

and, since T is cyclic, we see that $\tilde{A}_T^*(v) = 0$ implies $v = 0$. Moreover, the map $\tilde{A}_T^*(v)$ is G-equivariant, as

$$\begin{aligned} \langle \pi_{-\infty}(g) \tilde{A}_T^*(v), e_{11} * P_{H_0}(\varphi) \rangle &= \langle v, A_T(\varphi * \varepsilon_g) \rangle \\ &= \langle v, \pi_\infty(g^{-1}) \pi_{-\infty}(\check{\varphi}) T \rangle \\ &= \langle \pi(g)(v), A_T(\varphi) \rangle = \langle \tilde{A}_T^* \pi(g)(v), \varphi \rangle. \end{aligned} \tag{31}$$

For the reverse statement we start with a continuous G-equivariant embedding $j : \mathcal{H}_\pi \hookrightarrow \mathcal{D}^1(e_{11}, X_o)$. Then its adjoint $j^* : \mathcal{D}(e_{11}, X_0) \to$

$\mathcal{H}_\pi$ is, as we saw above, also G-equivariant. The same can be said of the continuous map $A : \mathcal{D}(G) \to \mathcal{H}_\pi$, given by $A(\varphi) := j^*(e_{11} * P_{H_0}(\varphi))$. According to theorem 5.3, there exists a unique $T \in \mathcal{H}_\pi^{-\infty}$ such that $A(\varphi) = \pi_{-\infty}(\check{\varphi})(T)$. First we show that T is H_0-invariant. Here we use that for all $h \in H_0$ and all $\varphi \in \mathcal{D}(G)$, there holds $P_{H_0}(\varepsilon_h * \varphi) = P_{H_0}(\varphi)$, for that gives

$$\pi_{-\infty}(\check{\varphi})T = A(\varphi) = A(\varepsilon_h * \varphi) = \pi_{-\infty}(\check{\varphi})\pi_{-\infty}(h^{-1})T. \tag{32}$$

From the uniqueness result in theorem 5.3 follows then $\pi_{-\infty}(h^{-1})T = T$. The second invariance property of T uses the same uniqueness result and can be seen directly from

$$\begin{aligned}
\int_{H/H_0} e_{11}(\tilde{h}^{-1})\pi_{-\infty}(\check{\varphi})\pi_{-\infty}(\tilde{h})T d\tilde{h} = \\
&= \int_{H/H_0} e_{11}(\tilde{h}) A(\varepsilon_{\tilde{h}} * \varphi) d\tilde{h} \qquad (33)\\
&= \int_{H/H_0} e_{11}(\tilde{h}) j^*(e_{11} * \varepsilon_{\tilde{h}} * P_{H_0}(\varphi)) d\tilde{h} \\
&= \pi_{-\infty}(\check{\varphi})T.
\end{aligned}$$

To see that T is cyclic, one supposes that there is a $v \in \mathcal{H}_\pi$ such that for all $\varphi \in \mathcal{D}(G)$

$$(\pi_{-\infty}(\varphi)T, v)_\pi = 0.$$

By definition we have

$$(v, j^*(e_{11} * P_{H_0}(\varphi)))_\pi = \langle j(v), e_{11} * P_{H_0}(\varphi)\rangle = 0.$$

Since the map $\varphi \mapsto e_{11} * P_{H_0}(\varphi)$ is a surjection from $\mathcal{D}(G)$ to $\mathcal{D}(e_{11}, X_0)$, we get $j(v) = 0$ and hence $v = 0$, for j was an injection. This proves the last remaining property of T and concludes the proof of the theorem. □

We will call the nonzero cyclic elements of $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ the (ρ, H)-*spherical distribution vectors* of $(\pi, \mathcal{H}_\pi)$.

6. $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ as intertwining operators

Next we translate the space $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ into a useful form. For if $T \in (\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ and $v \in \mathcal{H}_\pi^\infty$, then we can define a function $I_T(v) : G \to \mathbb{C}$ by

$$\overline{I_T(v)}(g) = \langle \pi_{-\infty}(g)^{-1}(T), v\rangle = \langle T, \pi_\infty(g)(v)\rangle.$$

Since $v \in \mathcal{H}_\pi^\infty$ the map $g \mapsto \pi_\infty(g)(v)$ is C^∞ and hence each $I_T(v)$ is a C^∞-function on G. Moreover, from the H_0-invariance of T, we see that

$$\overline{I_T(v)}(hg) = \langle \pi_{-\infty}(g^{-1})\pi_{-\infty}(h^{-1})T, v\rangle = \overline{I_T(v)}(g). \tag{34}$$

In other words, $I_T(v)$ belongs to $C^\infty(H_0\backslash G)$. Now T also satisfies $\check{e_{11}} * T = T$ and this property results into

$$e_{11} * I_T(v)(g) = \int_{H/H_0} \overline{\check{e_{11}}(\tilde{h})\langle \pi_{-\infty}(g^{-1}\tilde{h})T, v\rangle} d\tilde{h} = I_T(v)(g). \tag{35}$$

Thus the function $I_T(v)$ belongs to the space

$$C^\infty(e_{11}, X_0) = \left\{ f \in C^\infty(G) \;\middle|\; \begin{array}{l} f(hg) = f(g), \text{ for all } h \in H_0, \\ \text{and } \; e_{11} * f = f \end{array} \right\}, \tag{36}$$

where the group G acts upon by right translations. The map $I_T : \mathcal{H}_\pi^\infty \to C^\infty(e_{11}, X_0)$ also commutes with the action of G on both spaces.

$$\begin{aligned} \overline{I_T(\pi_\infty(g)v)}(x) &= \langle \pi_{-\infty}(x^{-1})(T), \pi_\infty(g)v\rangle \\ &= \langle T, \pi_\infty(xg)(v)\rangle \\ &= \overline{I_T(v)}(xg) \\ &= \overline{\mathcal{R}(g)(I_T(v))}(x). \end{aligned} \tag{37}$$

The map $T \to I_T$ from $(\mathcal{H}_\pi^\infty)^{H_0}(e_{11})$ to $\mathrm{Hom}_G(\mathcal{H}_\pi^\infty, C^\infty(e_{11}, X_0))$ is clearly linear and even a bijection. For, if $I_T(v) = 0$ for all $v \in \mathcal{H}_\pi^\infty$, then we have that $\langle T, v\rangle = 0$. Hence $T = 0$.

If A belongs to $\mathrm{Hom}_G(\mathcal{H}_\pi^\infty, C^\infty(\check{e_{11}}, X_0))$, then $\alpha : v \to \overline{A(v)(e)}$ belongs to $\mathcal{H}_\pi^{-\infty}$. Since $A(v)$ belongs to $C^\infty(e_{11}, X_0)$, the linear form α is first of all H_0-invariant:

$$\begin{aligned} \langle \pi_{-\infty}(h)(\alpha), v\rangle &= \langle \alpha, \pi_\infty(h^{-1})(v)\rangle \\ &= \overline{A(\pi_\infty(h^{-1})(v)(e)} \\ &= \overline{A(v)(h^{-1})} = \overline{A(v)(e)} = \langle \alpha, v\rangle \end{aligned} \tag{38}$$

and secondly behaves under left convolution with $\check{e_{11}}$ as the elements of $(\mathcal{H}_\pi^\infty)^{H_0}(e_{11})$

$$\begin{aligned} \langle \check{e_{11}} * \alpha, v\rangle &= \int_{H/H_0} \overline{e_{11}(\tilde{t})\langle \pi(\tilde{t})(\alpha), v\rangle} d\tilde{t} \\ &= \int_{H/H_0} \overline{e_{11}(\tilde{t}) A(\pi(\tilde{t}^{-1})(v))(e)} d\tilde{t} \\ &= \int_{H/H_0} \overline{e_{11}(\tilde{t}) A(v)(\tilde{t}^{-1})} d\tilde{t} = \langle \alpha, v\rangle \end{aligned} \tag{39}$$

From the G-equivariance of A follows then that $A = I_\alpha$. Hence we have shown:

Proposition 6.1. *Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G on a Hilbert space $\mathcal{H}_\pi$. Then $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ is isomorphic to the space of intertwining operators* $\mathrm{Hom}_G(\mathcal{H}_\pi^\infty, C^\infty(e_{11}, X_0))$.

This correspondence is used in the case of the real symmetric varieties, see e.g. (Brylinski and Delorme, 1992) and (van den Ban, 1988), to construct H-invariant distribution vectors for concrete series of representations, but this can be done for more general classes of groups.

7. Distribution vectors as distributions on G

Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G and let $j : \mathcal{H}_\pi \to \mathcal{D}^1(e_{11}, X_0)$ be a continuous G-equivariant embedding. We denote the to j corresponding non-zero cyclic element of $(\mathcal{H}_\pi^{-\infty})^{H_0}(e_{11})$ by T. As in the real case we can associate with T a special distribution σ_T on G. For $\varphi \in \mathcal{D}(G)$ we know from Lemma 5.2 that $\pi_{-\infty}(\varphi)(T) \in \mathcal{H}_\pi^\infty$ and then we define $\sigma_T \in \mathcal{D}^1(G)$ by

$$\langle \sigma_T, \varphi \rangle = \langle T, \pi_{-\infty}(\varphi)(T) \rangle.$$

Remark 7.1. If H is compact, then we know from lemma 5.4 that T corresponds to a cyclic vector $v \in \pi(\check{e_{11}})(\mathcal{H}_\pi)$ and in that case the distribution σ_T equals

$$\langle \sigma_T, \varphi \rangle = \int_G \overline{\varphi(g)} (v, \pi(g)(v))_\pi dg.$$

In other words, we have $\sigma_T = (v, \pi(g)(v))_\pi dg$. Following the terminology of the invariant context, this last function is called *the spherical function* of the representation.

From the fact that T is H_0-invariant follows the bi-H_0-invariance of σ_T

$$\begin{aligned} \langle \sigma_T, \varepsilon_{h_1} * \varphi * \varepsilon_{h_2} \rangle &= \langle T, \pi_{-\infty}(\varepsilon_{h_1} * \varphi * \varepsilon_{h_2})(T) \rangle \\ &= \langle T, \pi_\infty(h_1) \pi_{-\infty}(\varphi) \pi_{-\infty}(h_2)(T) \rangle \\ &= \langle \pi_{-\infty}(h_1^{-1})(T), \pi_{-\infty}(\varphi)(T) \rangle = \langle \sigma_T, \varphi \rangle. \end{aligned} \tag{40}$$

As T satisfies $\check{e_{11}} * T = T$ we get that σ_T transforms as follows

$$\begin{aligned} \int_{H/H_0} \check{e_{11}}(\tilde{t}) \varepsilon_{\tilde{t}} * \sigma_T(\varphi) d\tilde{t} &= \int_{H/H_0} \check{e_{11}}(\tilde{t}) \langle T, \pi_{-\infty}(\varepsilon_{\tilde{t}^{-1}} * \varphi) T \rangle \\ &= \int_{H/H_0} \check{e_{11}}(\tilde{t}) \langle T, \pi_\infty(\tilde{t}^{-1}) \pi_{-\infty}(\varphi)(T) \rangle \\ &= \langle T, \pi_{-\infty}(\varphi)(T) \rangle. \end{aligned} \tag{41}$$

The distribution σ_T is not only invariant under convolution with $\check{e_{11}}$ from the left but also from the right, as one sees from

$$\begin{aligned}\int_{H/H_0} \check{e_{11}}(\tilde{t})\sigma_T * \varepsilon_{\tilde{t}}(\varphi)d\tilde{t} &= \int_{H/H_0} \check{e_{11}}(\tilde{t})\langle T, \pi_{-\infty}(\varphi * \varepsilon_{\tilde{t}^{-1}})T\rangle \\ &= \int_{H/H_0} \langle T, \check{e_{11}}(\tilde{t}^{-1})\pi_{-\infty}(\varphi)\pi_{-\infty}(\tilde{t}^{-1})(T)\rangle \\ &= \langle T, \pi_{-\infty}(\varphi)(T)\rangle.\end{aligned}$$

Note that besides these transformation properties of the distribution σ_T there also holds for all φ and ψ in $\mathcal{D}(G_k)$ that

$$\begin{aligned}\langle \sigma_T, \varphi * \psi\rangle &= \int_{G_k} \langle \sigma_T, \varphi(g)\varepsilon_g * \psi\rangle dg \\ &= \int_{G_k} \overline{\varphi(g)}\langle T, \pi_\infty(g)\pi_{-\infty}(\varphi)(T)\rangle dg \\ &= \int_G \overline{\varphi(g^{-1})}\langle \pi_{-\infty}(g)(T), \pi_{-\infty}(\varphi)T\rangle dg \\ &= (\pi_{-\infty}(\widetilde{\varphi})(T), \pi_{-\infty}(\psi)T)_\pi,\end{aligned} \tag{42}$$

where $(\cdot \mid \cdot)_\pi$ is the inner product on $\mathcal{H}_\pi$ and $\widetilde{\varphi} \in \mathcal{D}(G)$ is given by $\widetilde{\varphi}(g) = \overline{\varphi(g^{-1})}$. Hence the distribution σ_T is positive definite, i.e.

$$\langle \sigma_T, \widetilde{\varphi} * \varphi\rangle \geq 0.$$

As T is cyclic, σ_T is uniquely determined by T.

Reversely, let σ be a nonzero positive definite bi-H_0-invariant distribution on G that satisfies $\check{e_{11}} * \sigma = \sigma * \check{e_{11}} = \sigma$. It will be shown that σ determines a G-invariant Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$. Recall that for σ_T as above we have

$$\begin{aligned}\langle \sigma_T, \overline{\varphi} * \check{\psi}\rangle &= (\pi_{-\infty}(\check{\varphi})(T), \pi_{-\infty}(\check{\psi})T)_\pi, \\ &= (j^*(e_{11} * P_{H_0}(\varphi)), j^*(e_{11} * P_{H_0}(\psi)))_\pi.\end{aligned} \tag{43}$$

Therefore we consider for φ and $\psi \in \mathcal{D}(G)$ first the sesquilinear form $\sigma(\overline{\varphi} * \check{\psi})$ on $\mathcal{D}(G)$ and show that it factorizes over $\mathcal{D}^1(e_{11}, X_0)$.

By definition we have

$$
\begin{aligned}
\sigma(\overline{\varphi} * \check{\psi}) &= \int_G \varphi(g)\sigma(\varepsilon_g * \check{\psi})dg \\
&= \int_{X_0} \{\int_{X_0} \varphi(h_0 x_0) dh_0\} \sigma(\varepsilon_{x_0} * \check{\psi}) dx_0 \\
&= \int_{X_0} P_{H_0}(\varphi)(x_0) \{\int_{H/H_0} \check{e_{11}}(\tilde{t}) \varepsilon_{\tilde{t}} * \sigma(\varepsilon_{x_0} * \check{\psi}) d\tilde{t}\} dx_0 \qquad (44) \\
&= \int_{X_0} \{\int_{H/H_0} \check{e_{11}}(\tilde{t}) P_{H_0}(\varphi)(\tilde{t}x_0) d\tilde{t}\} \sigma(\varepsilon_{x_0} * \check{\psi}) dx_0 \\
&= \int_{X_0} e_{11} * P_{H_0}(\varphi)(x_0) \sigma(\varepsilon_{x_0} * \check{\psi}) dx_0.
\end{aligned}
$$

Now we have for all $h \in H_0$ that $\sigma(\varepsilon_x * \check{\psi} * \varepsilon_h) = \sigma(\varepsilon_x * \check{\psi})$, since σ is right H_0-invariant. Hence there holds $\sigma(\varepsilon_x * (P_{H_0}\check{}(\psi))) = \sigma(\varepsilon_x * \check{\psi})$ and as $\sigma * \check{e_{11}} = \sigma$, we have moreover

$$
\begin{aligned}
\sigma(\varepsilon_x * P_{H_0}\check{}(\psi)) &= \int_{H/H_0} \check{e_{11}}(\tilde{t}) \sigma(\varepsilon_x * P_{H_0}\check{}(\psi) * \varepsilon_{\tilde{t}^{-1}}) d\tilde{t} \\
&= \sigma(\int_{H/H_0} e_{11}(\tilde{t}) \varepsilon_x * (\varepsilon_{\tilde{}} * \check{P}_{H_0}(\psi t)) d\tilde{t}) \qquad (45) \\
&= \sigma(\varepsilon_x * e_{11} * \check{P}_{H_0}(\psi)).
\end{aligned}
$$

Therefore there is a sesquilinear form B on $\mathcal{D}^1(e_{11}, X_0)$ such that for all φ and $\psi \in \mathcal{D}(G)$

$$B(e_{11} * P_{H_0}(\varphi), e_{11} * P_{H_0}(\psi)) = \sigma(\overline{\varphi} * \check{\psi}). \qquad (46)$$

Let I be the corresponding hermitian form on $\mathcal{D}^1(e_{11}, X_0)$, i.e.

$$I(f, g) := \frac{1}{2}(B(f, g) + \overline{B(g, f)}). \qquad (47)$$

Then the fact that σ is positive-definite gives for all $f = e_{11} * P_{H_0}(\psi) \in \mathcal{D}^1(e_{11}, X_0)$ that $I(f, f) = \sigma(\overline{\psi} * \check{\psi}) \geq 0$. Note that the form B and hence also I is G-invariant, i.e. for all f and $k \in \mathcal{D}^1(e_{11}, X_0)$ and each $g \in G$

$$B(f * \varepsilon_g, k * \varepsilon_g) = B(f, k). \qquad (48)$$

For, the projection $\varphi \mapsto e_{11} * P_{H_0}(\varphi)$ is a G-equivariant map from $\mathcal{D}(G)$ to $\mathcal{D}^1(e_{11}, X_0)$ and thus we get

$$
\begin{aligned}
B(e_{11} * P_{H_0}(\varphi) * \varepsilon_g, e_{11} * P_{H_0}(\psi) * \varepsilon_g) &= \sigma(\overline{\varphi * \varepsilon_g} * (\psi \check{*} \varepsilon_g)) \\
&= \sigma(\overline{\varphi} * \varepsilon_g * \varepsilon_{g^{-1}} * \check{\psi}) \qquad (49) \\
&= \sigma(\overline{\varphi} * \check{\psi}).
\end{aligned}
$$

Inside $\mathcal{D}^1(e_{11}, X_0)$, one has the subspace

$$\mathcal{D}_0 = \{f \in \mathcal{D}^1(e_{11}, X_0),\ I(f,f) = 0\},$$

which is G-invariant because of equation (48). On the quotient space $\mathcal{D}^1(e_{11}, X_0)/\mathcal{D}_0$ the form I induces an inner product. We denote the class of $f \in \mathcal{D}^1(e_{11}, X_0)$ in this quotient space by $[f]$. The action of G on this quotient space preserves the inner product and the map $f \mapsto j^*_\sigma(f) := [f]$ is G-equivariant. Let $\mathcal{H}_\sigma$ be the completion of the space $\mathcal{D}^1(e_{11}, X_0)/\mathcal{D}_0$ w.r.t. this inner product. Then the adjoint of j^*_σ gives you a Hilbert subspace of $\mathcal{D}^1(e_{11}, X_0)$. We call the class of positive definite bi-H_0-invariant distributions σ on G, satisfying moreover $\sigma * \check{e_{11}} = \check{e_{11}} * \sigma = \sigma$, that of (ρ, H)-*spherical distributions*. We summarize the foregoing result in a

Theorem 7.2. *The map $\sigma \mapsto \mathcal{H}_\sigma$ that associates with each (ρ, H)-spherical distribution the unitary G-module $\mathcal{H}_\sigma$, is a bijection between this class of distributions on G and the collection of G-invariant Hilbert subspaces of $\mathcal{D}^1(e_{11}, X_0)$.*

Example 7.3. Our main interest is in the Hilbert subspace $L^2(e_{11}, X_0, dx_0)$ of $\mathcal{D}^1(e_{11}, X_0)$. The positive definite bi-H_0-invariant distribution in this case is

$$\tau_0(\varphi) = e_{11} * P_{H_0}(\overline{\varphi})(e) = \{\int_{H/H_0} e_{11}(\tilde{t})\{\int_{H_0} \overline{\varphi}(\tilde{t}^{-1}h_0)dh_0\}d\tilde{t}\}, \quad (50)$$

where e is the point H_0 of X_0. From the defining formula of τ_0 it is clear that τ_0 is bi-H_0-invariant. Taking the convolution with $\check{e_{11}}$ leaves also τ_0 invariant, for

$$\begin{aligned}\check{e_{11}} * \tau_0(\varphi) &= \int_{H/H_0} \check{e_{11}}(\tilde{r})\tau_0(\varepsilon_{\tilde{r}^{-1}} * \varphi)d\tilde{r} \\ &= \{\int_{H/H_0} \check{e_{11}}(\tilde{r})\{\int_{H/H_0} e_{11}(\tilde{t})\{\int_{H_0} \overline{\varphi}(\tilde{r}\tilde{t}^{-1}h_0)dh_0\}d\tilde{t}d\tilde{r}\}\} \\ &= \{\int_{H/H_0} e_{11} * e_{11}(\tilde{s})P_{H_0}(\overline{\varphi})(\tilde{s})d\tilde{s}\} = \tau_0(\varphi)\end{aligned}$$

and a similar computation gives you $\tau_0 * \check{e_{11}} = \tau_0$. To see that τ_0 links to $L^2(e_{11}, X_0, dx_0)$, we compute

$$\begin{aligned}\tau_0(\overline{\varphi} * \check{\psi}) &= \int_{H/H_0} e_{11}(\tilde{t})\{\int_{H_0} \overline{\overline{\varphi} * \check{\psi}}(\tilde{t}^{-1}h_0)dh_0\}d\tilde{t} \\ &= \int_G \varphi(g)\{\int_{H/H_0} e_{11}(\tilde{t})\{\int_{H_0} \overline{\psi}(h_0^{-1}\tilde{t}g)dh_0\}d\tilde{t}\}dg \\ &= \int_{X_0} P_{H_0}(\varphi)(x_0)\overline{e_{11} * P_{H_0}(\psi)(x_0)}dx_0 \\ &= \int_{X_0} e_{11} * P_{H_0}(\varphi)(x_0)\overline{e_{11} * P_{H_0}(\psi)(x_0)}dx_0,\end{aligned} \tag{51}$$

where the last equality is a consequence of the fact that convolution from the left with e_{11} is an orthogonal projection on $\mathcal{D}(X_0)$ for the inner product defined by dx_0. In particular we see that τ_0 is positive definite and that it induces on $\mathcal{D}(e_{11}, X_0)$ the inner product just mentioned. Clearly also the induced G-action is that by right translations. The completion of $\mathcal{D}(e_{11}, X_0)$ gives you then $L^2(e_{11}, X_0, dx_0)$. If we combine the theorems (4.2) and (5.6), then we get a decomposition

$$\tau_0 = \int_S^{\oplus} \sigma_s dm(s), \tag{52}$$

where the σ_s are the (ρ, H)-spherical distributions corresponding to the irreducible G-modules $\mathcal{H}_s, s \in S$. This decomposition generalizes the one occurring in the Bochner-Godement theorem for the Gelfand pair (G, H), see (Faraut, 1979), which decomposes the spherical function corresponding to the right action on $L^2(H/G)$ into pure ones.

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