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Quasi-Stationary Distributions For A Class Of Discrete-Time Markov Chains

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Abstract

This paper is concerned with the circumstances under which a discrete-time absorbing Markov chain has a quasi-stationary distribution. We showed in a previous paper that a pure birth-death process with an absorbing bottom state has a quasi-stationary distribution – actually an infinite family of quasi-stationary distributions – if and only if absorption is certain and the chain is geometrically transient. If we widen the setting by allowing absorption in one step (*killing*) from any state, the two conditions are still necessary, but no longer sufficient. We show that the birth-death-type of behaviour prevails as long as the number of states in which killing can occur is finite. But if there are infinitely many such states, and if the chain is geometrically transient and absorption certain, then there may be 0, 1, or infinitely many quasi-stationary distributions. Examples of each type of behaviour are presented. We also survey and supplement the theory of quasi-stationary distributions for discrete-time Markov chains in general.

Keywords and phrases: absorption probability, birth-death process with killing, decay parameter, quasi-stationarity, rate of convergence.

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1 Introduction

A quasi-stationary distribution of an absorbing discrete-time Markov chain is any initial distribution on the non-absorbing states with the property that the state probabilities at time n, conditional on the chain being in one of the non-absorbing states, do not vary with n. Clearly, eventual absorption should be certain for a quasi-stationary distribution to exist, but also the geometric convergence to zero as $n \to \infty$ of the n-step transition probabilities of the chain is necessary. When the Markov chain is a birth-death process with an absorbing bottom state these two conditions happen to be necessary as well as sufficient. In fact, in this case there exists a one-parameter family of quasi-stationary distributions (see [4]).

In settings which are more general than that of birth-death processes additional assumptions are required to ensure the existence of a quasi-stationary distribution. Some interesting results in this vein have been reported in the literature (see the next section), but since the additional conditions that are brought to light are sufficient, but not necessary, the question as to which features of a Markov chain are essentially responsible for the existence of a quasi-stationary distribution has not been answered satisfactorily yet.

Our approach to finding an answer to this question is to build on what we know already rather than to focus on Markov chains in general and search for suitable restrictions. Concretely, we will study birth-death processes with killing, which are birth-death processes with an absorbing state and the additional feature that a transition to the absorbing state (*killing*) may occur from any, rather than just one state. This class of Markov chains seems only marginally larger than that of pure birth-death processes with an absorbing bottom state, but allows for considerably more varied behaviour. In particular, neat results such as those for pure birth-death processes are no longer valid in the generalized setting.

In the next section we will introduce some relevant concepts and state some results concerning quasi-stationarity and related issues in the general setting of discrete-time Markov chains. Most of these results (or their continuous-time analogues) can be found in the literature, and are collected here for convenience. But also some new results are included. Pertinent definitions and properties of discrete-time birth-death processes with killing are given in Section 3. In Section 4 we address the issue of absorption times for such processes, giving in particular a simple criterion for eventual absorption to be certain. In Section 5 we show that the neat result on quasi-stationary distributions for birth-death processes can be generalized to birth-death processes with killing *provided* the number of states from which a transition to the absorbing state may occur is finite. We also demonstrate that several types of behaviour may occur if there are infinitely many such states. Section 6 contains a worked-out example.

2 Preliminaries

Let $\mathcal{X} \equiv \{X(n),\ n = 0, 1, \ldots\}$ denote a homogeneous discrete-time Markov chain on $S \equiv \{0, 1, \ldots\}$ with matrix $P \equiv (P_{ij})$ of 1-step transition probabilities. We will assume that S constitutes an irreducible class, and that P is substochastic, that is,

$$\kappa_i \equiv 1 - \sum_{j \in S} P_{ij} \geq 0, \quad i \in S.$$

The quantities κ_i, henceforth called *killing probabilities*, may be regarded as the probabilities of absorption into a fictitious state ∂, say. A transition to the absorbing state is sometimes referred to in the literature as a *total catastrophe*. If $\kappa_i = 0$ for all states $i \in S$ then the matrix P is stochastic and $\mathcal{X}$ is an honest Markov chain on S. However, we will assume in what follows that $\kappa_i > 0$ for at least one state $i \in S$, so that ∂ is accessible from S, and hence S constitutes a transient class.

We write $\mathbb{P}_i(.)$ for the probability measure of the process when $X(0) = i$ and $\mathbb{E}_i(.)$ for the expectation with respect to this measure. For any distribution $\mu \equiv (\mu_i,\ i \in S)$, we let $\mathbb{P}_\mu(.) \equiv \sum_i \mu_i \mathbb{P}_i(.)$. The n-step transition probabilities of the process $\mathcal{X}$ are denoted by $P_{ij}(n) \equiv \mathbb{P}_i(X(n) = j)$. Hence $P_{ij}(1) = P_{ij}$, and the matrix $P(n) \equiv (P_{ij}(n),\ i, j \in S)$ of n-step transition probabilities satisfies $P(n) = P^n,\ n \geq 0$. By $T \equiv \inf\{t \geq 0 : X(t) = \partial\}$ we denote the *absorption* *time*, the (possibly defective) random variable representing the time at which absorption in state ∂ occurs.

A proper probability distribution $\mu \equiv (\mu_j,\ j \in S)$ over S is called *x-invariant* for P (on S) if

$$\sum_{i \in S} \mu_i P_{ij} = x\mu_j, \quad j \in S. \tag{1}$$

Obviously, if μ is x-invariant for P we must have $x < 1$, and, by the irreducibility of S, $x > 0$ and $\mu_j > 0$ for all $j \in S$. On the other hand, if $0 < x < 1$ an x-invariant distribution for P need not exist, and if it *does* exist it need not be unique. Kesten [12, Theorem 1] has established conditions on P which ensure that there is at most one x-invariant distribution.

The distribution $\mu \equiv (\mu_j,\ j \in S)$ is said to be a *quasi-stationary distribution* for $\mathcal{X}$ if, for all $n = 0, 1, \ldots,$

$$\mathbb{P}_\mu(X(n) = j \,|\, T > n) = \mu_j, \quad j \in S. \tag{2}$$

Evidently, a quasi-stationary distribution can exist only if $\mathbb{P}_i(T > n) \to 0$ as $n \to \infty$ for all $i \in S$, that is, absorption is certain. Defining

$$\kappa_\mu \equiv \mathbb{P}_\mu(T = 1) = \sum_{i \in S} \mu_i \kappa_i, \tag{3}$$

we can now formulate the following theorem.

Theorem 1 Let $\mu \equiv (\mu_j,\ j \in S)$ be a proper probability distribution over S, then the following statements are equivalent:

(*i*) μ is a quasi-stationary distribution for $\mathcal{X}$;

(*ii*) μ is x-invariant for P for some x, $0 < x < 1$;

(*iii*) μ is $(1 - \kappa_\mu)$-invariant for P;

(*iv*) for all $j \in S$ and $n = 0, 1, \ldots,$ and some x, $0 < x < 1$, one has

$$\mathbb{P}_\mu(X(n) = j) = x^n \mu_j; \tag{4}$$

(*v*) for all $j \in S$ and $n = 0, 1, \ldots$ one has

$$\mathbb{P}_\mu(X(n) = j) = (1 - \kappa_\mu)^n \mu_j. \tag{5}$$

This theorem has (essentially) been known for a long time in the setting of finite Markov chains, when, actually, there is precisely one quasi-stationary distribution (see [2], where the quasi-stationary distribution is called a *stationary conditional distribution*). The equivalence of (i), (iii) and (v) was established in [4] in the setting of discrete-time birth-death processes with infinite state space. Other sources, for example [16], mention partial results for the more general setting at hand. The full theorem may be established by an appeal to related results in continuous time (see in particular [17, Proposition 3.1 and Theorem 3.1]), but for completeness' sake we provide a direct proof.

Proof of Theorem 1 To show that (i) implies (v), let $\mu \equiv (\mu_j,\ j \in S)$ be a quasi-stationary distribution for $\mathcal{X}$. For all $j \in S$ we then have $\mathbb{P}_\mu(X(0) = j) = \mu_j$ and

$$\mathbb{P}_\mu(X(n) = j) = \mathbb{P}_\mu(T > n)\mu_j, \quad n = 1, 2, \dots.$$

Consequently, for all $n = 0, 1, \dots,$

$$\begin{aligned}
\mathbb{P}_\mu(X(n+1) = j) &= \mathbb{P}_\mu(T > n+1)\mu_j \\
&= \left(\mathbb{P}_\mu(T > n) - \sum_{i \in S} \mathbb{P}_\mu(X(n) = i)\kappa_i\right)\mu_j \\
&= \mathbb{P}_\mu(T > n)\left(1 - \sum_{i \in S} \mu_i \kappa_i\right)\mu_j \\
&= (1 - \kappa_\mu)\mathbb{P}_\mu(X(n) = j),
\end{aligned}$$

from which (v) follows by induction.

As a consequence of our assumption that S constitutes an irreducible and transient class, we have $0 < \kappa_\mu < 1$ when (v) holds true, so that $(v) \Rightarrow (iv)$. The implication $(iv) \Rightarrow (ii)$ is trivial, while (iii) easily follows from (ii) by summing (1) over $j \in S$.

Finally, we will show $(iii) \Rightarrow (i)$. So suppose μ is $(1 - \kappa_\mu)$-invariant for P. Evidently, (2) is valid for $n = 0$. Assuming that (2) holds true for $n = k$, it follows that

$$\mathbb{P}_\mu(X(k+1) = j) = \sum_{i \in S} \mathbb{P}_\mu(X(k) = i) P_{ij}$$

$$
\begin{aligned}
&= \mathbb{P}_\mu(T > k) \sum_{i \in S} \mu_i P_{ij} \\
&= (1 - \kappa_\mu) \mathbb{P}_\mu(T > k) \mu_j.
\end{aligned}
$$

Summing over all $j \in S$ subsequently implies that $(1-\kappa_\mu)\mathbb{P}_\mu(T > k) = \mathbb{P}_\mu(T > k+1)$ and, hence, that (2) holds true for $n = k+1$. The validity of (i) follows by induction. □

A major challenge is to find conditions on P for a quasi-stationary distribution to exist. In the remainder of this section we describe the present status of this problem and fill in some gaps.

It is well known (see [15]) that there exists a real number ρ, $0 < \rho \le 1$, such that

$$\lim_{n \to \infty} (P_{ij}(n))^{1/n} = \rho, \quad i, j \in S. \tag{6}$$

The number ρ is called the *decay parameter* of the Markov chain $\mathcal{X}$ in S, and the chain is said to be *geometrically transient* if $\rho < 1$. Moreover (see [19, Theorem 4.1]), x must be in the interval $\rho \le x < 1$ when μ is x-invariant for P on S. Hence, Theorem 1 implies that we must have

$$\rho \le 1 - \kappa_\mu < 1, \tag{7}$$

when μ is a quasi-stationary distribution for $\mathcal{X}$, so that geometric transience is necessary for the existence of a quasi-stationary distribution.

Of interest to us is also the speed of convergence of the n-step absorption probabilities $\mathbb{P}_i(X(n) = \partial) = \mathbb{P}_i(T \le n)$ to their limits $\tau_i \equiv \mathbb{P}_i(T < \infty) = \lim_{n\to\infty} \mathbb{P}_i(T \le n)$, $i \in S$, the *(eventual) absorption probabilities.* We note that $\tau_i > 0$ because of our assumptions that ∂ is accessible and S irreducible. For future reference we also note at this point that by conditioning on the first event in $\mathcal{X}$ we obtain the relations

$$1 - \tau_i = \sum_{j \in S} P_{ij}(1 - \tau_j), \quad i \in S. \tag{8}$$

We denote the rate of convergence to zero of the quantities $\tau_i - \mathbb{P}_i(X(n) = \partial) = \mathbb{P}_i(n < T < \infty)$ by ρ_∂, that is,

$$\rho_\partial^{-1} = \inf \left\{ s > 1 : \sum_{n=0}^{\infty} s^n \mathbb{P}_i(n < T < \infty) = \infty \right\}, \quad i \in S, \tag{9}$$

and note that ρ_∂ is independent of i by an irreducibility argument. A simple argument reveals that ρ_∂ may also be expressed as

$$\rho_\partial^{-1} \equiv \inf\left\{s > 1 : \ \mathbb{E}_i\left(s^T\right) = \infty\right\}, \quad i \in S, \tag{10}$$

so that ρ_∂ is also the rate of convergence to zero of the probabilities $\mathbb{P}_i(T = n)$.

It can be shown (see Theorem 3 below) that $\rho = \rho_\partial$ when

$$\#\{i \in S : \ \kappa_i > 0\} < \infty, \tag{11}$$

but equality does not prevail in general (see Section 6 for a counterexample). We can, however, establish the following.

Theorem 2 The rates of convergence ρ and ρ_∂ associated with the Markov chain $\mathcal{X}$ satisfy the inequalities

$$\rho \leq \rho_\partial \leq 1.$$

Proof We note that $1 \geq \mathbb{P}_i(T > n) \geq P_{ii}(n)$, so if $\tau_i \equiv \mathbb{P}_i(T < \infty) = 1$, that is, absorption is certain, the result immediately follows with (9). Now let $\bar{\mathcal{X}} \equiv [\mathcal{X} \,|\, T < \infty]$. It is readily seen that the process $\bar{\mathcal{X}}$ is a Markov chain with 1-step transition probabilities $\bar{P}_{ij} = P_{ij}\tau_j/\tau_i$, and hence n-step transition probabilities

$$\bar{P}_{ij}(n) = \frac{\tau_j}{\tau_i} P_{ij}(n), \quad i, j \in S, \tag{12}$$

while

$$\mathbb{P}_i(T \leq n \,|\, T < \infty) = \frac{1}{\tau_i}\mathbb{P}_i(T \leq n), \quad i \in S. \tag{13}$$

Evidently, eventual absorption for $\bar{\mathcal{X}}$ is certain, so that, by the result above and with evident notation, $\bar{\rho} \leq \bar{\rho}_\partial \leq 1$. But (12) and (13) imply that $\bar{\rho} = \rho$ and $\bar{\rho}_\partial = \rho_\partial$, respectively, so that the statement follows. □

As announced a sufficient condition for $\rho = \rho_\partial$ is given in the next theorem, which is the discrete-time counterpart of [10, Theorem 3.3.2 (iii)]. Certain absorption is an implicit assumption in [10], but the argument used in the proof of the previous theorem shows that it can be dispensed with.

Theorem 3 If $\mathcal{X}$ is such that absorption can occur *in one step* from at most a finite number of states, then $\rho = \rho_\partial$.

Theorem 2 enables us to improve upon (7) as follows, showing in particular that $\rho_\partial < 1$ is necessary for a quasi-stationary distribution to exist.

Theorem 4 If $\mu \equiv (\mu_j,\ j \in S)$ is a quasi-stationary distribution for $\mathcal{X}$ then

$$\rho \leq \rho_\partial \leq 1 - \kappa_\mu < 1.$$

Proof We must prove the second inequality only. With the help of Theorem 1, statement (v), we find that, for any $i \in S$ and $n = 0, 1, \ldots,$

$$(1-\kappa_\mu)^n = \mathbb{P}_\mu(T > n) \geq \mu_i \mathbb{P}_i(T > n),$$

if μ is a quasi-stationary distribution. The inequality immediately follows with (9), since absorption is certain when a quasi-stationary distribution exists. □

In analogy with Ferrari et al. [9, p. 515] (note that the definition used on p. 504 is formally different) we will call a quasi-stationary distribution μ *minimal* if $\kappa_\mu = 1 - \rho_\partial$. Theorem 4 now has the following obvious corollary.

Corollary If there exists a ρ-invariant quasi-stationary distribution then it is minimal, and $\rho = \rho_\partial$.

For discrete-time *birth-death processes* it is known that when absorption at ∂ is certain, geometric transience is necessary *and sufficient* for the existence of a quasi-stationary distribution. Moreover, for any number x in the interval $\rho \leq x < 1$, there is a unique quasi-stationary distribution μ such that $\kappa_\mu = 1 - x$ (see [4]). These results can actually be generalized to Markov chains that are *skip-free to the left*, that is, Markov chains in which the matrix $P \equiv (P_{ij})$ of 1-step transition probabilities satisfies $P_{ij} = 0$ if $j < i - 1$.

Theorem 5 Let $\mathcal{X}$ be a Markov chain that is skip-free to the left for which absorption at ∂ is certain. Then a quasi-stationary distribution exists if and only if $\rho < 1$. Moreover, for each x in the interval $\rho \leq x < 1$, there is a unique x-invariant quasi-stationary distribution.

Proof Kijima [14] has shown that, up to a multiplicative constant, there is a unique positive solution to the system (1) for $x \geq \rho$. Since state ∂ can be reached *in one step* from state zero only, the argument on p. 414 of [19] may be used to show that for each x in the interval $\rho \leq x < 1$, this solution must be summable, and hence, after normalization, constitutes a quasi-stationary distribution. □

A more general setting is that of Markov chains in which *asymptotic remoteness* prevails, that is,

$$\lim_{i\to\infty} \mathbb{P}_i(T \leq n) = 0 \quad \text{for all } n > 0. \tag{14}$$

By the discrete-time analogue of [9, Theorem 1.1] (see also Kesten [12, Theorem A]) $\rho_\partial < 1$ is necessary and sufficient for the existence of a quasi-stationary distribution when absorption at ∂ is certain. In fact, the discrete-time analogues of [9, Theorem 4.1 and Proposition 5.1(a)] tell us that the existence of a quasi-stationary distribution implies the existence of a *minimal* quasi-stationary distribution. We also know from the discrete-time counterpart of [9, Corollary 5.3] that asymptotic remoteness is yet another sufficient condition for $\rho = \rho_\partial$ if $\rho_\partial < 1$. In summary, when the Markov chain is such that asymptotic remoteness prevails, absorption at ∂ is certain, and $\rho_\partial < 1$, then $\rho = \rho_\partial$ and there exists a minimal (and hence ρ-invariant) quasi-stationary distribution.

Another approach towards obtaining sufficient conditions for the existence of a quasi-stationary distribution is to confine attention to *R-recurrent* Markov chains, which are Markov chains satisfying

$$\sum_{n=0}^{\infty} R^n P_{ij}(n) = \infty \quad \text{for some } i, j \in S \tag{15}$$

(and, hence, for all $i, j \in S$), where $R \equiv 1/\rho$. R-recurrence implies (see [20]) that there exists, up to normalization, a unique positive solution to the system (1) with $x = \rho$. However, besides $\rho_\partial < 1$ and certain absorption, additional restrictions on P are required to ensure summability, and hence the existence of a (unique) ρ-invariant quasi-stationary distribution. One such sufficient condition, given in [19, p. 414], is (11). Actually, in [19] the Markov chain is assumed to be *R-positive*, that is,

$$\lim_{n\to\infty} R^n P_{ii}(n) > 0 \quad \text{for some } i \in S \tag{16}$$

(and, hence, for all $i \in S$), but this can be relaxed to R-recurrence.

Since R-recurrence is usually difficult to verify, attempts have been made to replace it by a condition which is stated directly in terms of P. The most powerful result to date seems to be Kesten's result [12, Theorem 2]. Insofar as it concerns quasi-stationary distributions, this theorem states that a unique ρ-invariant quasi-stationary distribution exists if, besides (11) and certain absorption, certain restrictions on the sizes of downward jumps, and a type of uniform irreducibility condition are satisfied; we refer to [12] for details.

Evidently, any quasi-stationary distribution for $\mathcal{X}$ is also a *limiting conditional distribution*, which is a (proper) distribution $\mu \equiv (\mu_j,\ j \in S)$ such that for some initial distribution ν over S

$$\lim_{n\to\infty} \mathbb{P}_\nu(X(n) = j \,|\, X(n) \in S) = \mu_j, \quad j \in S. \tag{17}$$

But the reverse is also true (see [19, Theorem 4.1]), so our quest for conditions on P for a quasi-stationary distribution to exist may also be brought to bear on limiting conditional distributions.

Of particular interest is the discrete-time analogue of [9, Proposition 5.1(b)], which states that if $\rho_\partial < 1$ and, for some $i \in S$, the limits

$$\mu_j = \lim_{n\to\infty} \frac{P_{ij}(n)}{\sum_{k\in S} P_{ik}(n)}, \quad j \in C, \tag{18}$$

exist and constitute a distribution (and, hence, a quasi-stationary distribution), then $\mu \equiv (\mu_j,\ j \in S)$ must be a minimal quasi-stationary distribution. The existence of the limits in (18) has been proven in various settings, usually more restricted, however, than those required for the existence of a quasi-stationary distribution (see, for example, [19], [3], [18], [13], [5], [8] and [16]).

3 Birth-death processes with killing

In this and subsequent sections $\mathcal{X} \equiv \{X(n),\ n = 0, 1, \ldots\}$ will denote a *birth-death process with killing*, that is, the matrix $P \equiv (P_{ij})$ of 1-step transition probabilities has the tridiagonal structure

$$P = \begin{pmatrix} r_0 & p_0 & 0 & 0 & . & . & . \\ q_1 & r_1 & p_1 & 0 & . & . & . \\ 0 & q_2 & r_2 & p_2 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}. \tag{19}$$

We will assume throughout that $p_i > 0$, $q_{i+1} > 0$ and $r_i \geq 0$ for $i \geq 0$. We define $q_0 \equiv 0$ and let

$$\kappa_i = 1 - p_i - q_i - r_i \geq 0, \quad i \in S.$$

The probabilities p_i, q_i and r_i are the birth, death and self-transition probabilities, respectively, in state $i \in S$, while, as before, κ_i is the killing probability in state i, that is, the probability of absorption into a fictitious state ∂. As mentioned in Section 2, we will assume throughout that $\kappa_i > 0$ for at least one state $i \in S$, so that ∂ is accessible from S, and S constitutes a transient class.

A prominent role in what follows will be played by the polynomials $\{Q_j\}$ which are uniquely determined by the 1-step transition probabilities of $\mathcal{X}$ via the recurrence relation

$$\begin{aligned} &xQ_j(x) = q_j Q_{j-1}(x) + r_j Q_j(x) + p_j Q_{j+1}(x), \quad j \geq 1, \\ &Q_0(x) = 1, \quad p_0 Q_1(x) = x - r_0. \end{aligned} \tag{20}$$

We let

$$\pi_0 \equiv 1 \;\text{ and }\; \pi_j \equiv \frac{p_0 p_1 \dots p_{j-1}}{q_1 q_2 \dots q_j}, \quad j \geq 1, \tag{21}$$

and observe that

$$\begin{aligned} &p_j \pi_j (Q_{j+1}(x) - Q_j(x)) = \\ &\qquad p_{j-1}\pi_{j-1}(Q_j(x) - Q_{j-1}(x)) + (\kappa_j - 1 + x)\pi_j Q_j(x), \quad j \geq 1, \\ &p_0 \pi_0 (Q_1(x) - Q_0(x)) = (\kappa_0 - 1 + x)\pi_0 Q_0(x), \end{aligned}$$

so that

$$p_j \pi_j (Q_{j+1}(x) - Q_j(x)) = \sum_{k=0}^{j} (\kappa_k - 1 + x)\pi_k Q_k(x), \quad j \geq 0. \tag{22}$$

It follows that

$$Q_{n+1}(x) = 1 + \sum_{j=0}^{n} \frac{1}{p_j \pi_j} \sum_{k=0}^{j} (\kappa_k - 1 + x)\pi_k Q_k(x), \quad n \geq 0, \tag{23}$$

and in particular

$$Q_{n+1}(1) = 1 + \sum_{j=0}^{n} \frac{1}{p_j \pi_j} \sum_{k=0}^{j} \kappa_k \pi_k Q_k(1), \quad n \geq 0, \tag{24}$$

a result which will be used in the next section.

Karlin and McGregor [11] have shown that the transition probabilities $P_{ij}(n)$ may be represented in the form

$$P_{ij}(n) = \pi_j \int_{-1}^{1} x^n Q_i(x) Q_j(x) \psi(dx), \quad n \geq 0, \ i, j \in S, \tag{25}$$

where ψ is the (unique) measure of total mass 1 and infinite support in the interval $[-1, 1]$ with respect to which the polynomials $\{Q_j\}$ are orthogonal. Of particular interest to us is the fact that the decay parameter of the process $\mathcal{X}$ equals the largest point in the support of the measure ψ, that is,

$$\rho = \sup \operatorname{supp}(\psi) \tag{26}$$

(see [4, Theorem 3.1]). As a consequence (see, for example, [4]) ρ may also be characterized in terms of the polynomials $\{Q_j\}$ by

$$x \geq \rho \iff Q_j(x) > 0 \ \text{ for all } \ j \geq 0. \tag{27}$$

Since $Q_j(x)$ is a polynomial with positive leading coefficient the preceding actually implies

$$y > x \geq \rho \iff Q_j(y) > Q_j(x) > 0 \ \text{ for all } \ j \geq 0, \tag{28}$$

which will prove useful in Section 5.

4 Absorption probability

Before turning our attention to quasi-stationary distributions we have to find out under which condition absorption in state ∂ is certain, because certain absorption is necessary for the existence of quasi-stationary distributions. In fact, in this section we will determine the eventual absorption probabilities τ_i of the birth-death process with killing $\mathcal{X}$.

Writing $\xi_i \equiv 1 - \tau_i$ and employing (8) we obtain the recurrence relations

$$\xi_i = p_i \xi_{i+1} + r_i \xi_i + q_i \xi_{i-1}, \quad i \geq 1,$$
$$\xi_0 = p_0 \xi_1 + r_0 \xi_0,$$

so, in view of (20),

$$\xi_i = \xi_0 Q_i(1), \quad i \in S,$$

and hence

$$\tau_i = 1 - (1 - \tau_0) Q_i(1), \quad i \in S. \tag{29}$$

Since $\{\tau_i\}$ constitutes the minimal non-negative solution of (29) (see Feller [7, p. 403]), we must have $\tau_i = 1$ for all $i \in S$ if $Q_\infty(1) \equiv \lim_{i\to\infty} Q_i(1) = \infty$, whereas $\tau_i = 1 - Q_i(1)/Q_\infty(1)$ otherwise. We can formulate these results slightly more efficiently as follows.

Theorem 6 For any initial state $i \in S$ absorption is certain if and only if

$$\sum_{j=0}^{\infty} \frac{1}{p_j \pi_j} \sum_{k=0}^{j} \kappa_k \pi_k = \infty, \tag{30}$$

otherwise the absorption probabilities satisfy

$$\tau_i = 1 - \frac{Q_i(1)}{Q_\infty(1)} < 1, \quad i \in S. \tag{31}$$

Proof It follows immediately from (24) that $Q_\infty(1) = \infty$, and hence $\tau_i = 1$, if (30) is satisfied. Conversely, let us define

$$\beta_j \equiv \frac{1}{p_j \pi_j} \sum_{k=0}^{j} \kappa_k \pi_k, \quad j \geq 0,$$

and assume that $\sum \beta_j$ converges. Again using (24) we have

$$Q_{n+1}(1) = Q_n(1) + \frac{1}{p_n \pi_n} \sum_{k=0}^{n} \kappa_k \pi_k Q_k(1) \leq Q_n(1)(1 + \beta_n), \quad n \geq 0,$$

since, by (24) again, $Q_k(1)$ is increasing in k. It follows that

$$Q_{n+1}(1) \leq \prod_{j=0}^{n} (1 + \beta_j), \quad n \geq 0.$$

But $\prod(1 + \beta_j)$ and $\sum \beta_j$ converge together, so we must have $Q_\infty(1) < \infty$. The theorem now follows by the statement preceding the theorem. □

5 Quasi-stationary distributions

In what follows we will tacitly assume (30), so that absorption at ∂ is certain.

If $P \equiv (P_{ij})$ is the matrix of 1-step transition probabilities of the birth-death process with killing $\mathcal{X}$, then the solution of the system of equations (1) is readily seen to be given by

$$\mu_j = \mu_0 \pi_j Q_j(x), \quad j \in S, \tag{32}$$

where μ_0 is some constant. Thus Theorem 1 tells us that, to obtain all quasi-stationary distributions for $\mathcal{X}$, we have to find out for which values of x, $0 < x < 1$ the numbers μ_j of (32) constitute a proper distribution with an appropriate choice of μ_0. Clearly, two conditions have to be satisfied. First, since the components of a quasi-stationary distribution are strictly positive, we must have $Q_j(x) > 0$ for all j and hence, by (27), $x \geq \rho$. More problematical is the second requirement, namely that the sum

$$\sum_{j \in S} \pi_j Q_j(x) \tag{33}$$

be finite. But in any case we have proven the first part of the following theorem.

Theorem 7 Let $\mathcal{X}$ be a birth-death process with killing such that absorption at ∂ is certain. Then the following hold:

(i) If $\rho = 1$ there is no quasi-stationary distribution for $\mathcal{X}$. If $\rho < 1$ then $\mu \equiv (\mu_j,\ j \in S)$ is a quasi-stationary distribution for $\mathcal{X}$ if and only if there is a real number x, $\rho \leq x < 1$, such that $\mu_j = \mu_j(x)$, $j \in S$, where

$$\mu_j(x) \equiv \mu_0(x) \pi_j Q_j(x), \quad j \in S, \tag{34}$$

and $\mu_0(x)^{-1} \equiv \sum_{j \in S} \pi_j Q_j(x) < \infty$.

(ii) If $(\mu_j(x),\ j \in S)$ constitutes a quasi-stationary distribution, then

$$\sum_{j \in S} \kappa_j \pi_j Q_j(x) = (1 - x) \sum_{j \in S} \pi_j Q_j(x). \tag{35}$$

(iii) If $(\mu_j(x),\ j \in S)$ constitutes a quasi-stationary distribution, then also $(\mu_j(y),\ j \in S)$ is a quasi-stationary distribution for all y in the interval $\rho \leq y \leq x$.

Proof The second statement is a consequence of the equivalence of statements (i) and (iii) in Theorem 1. To prove (iii) we observe from (28) that if $(\mu_j(x),\ j \in S)$, constitutes a quasi-stationary distribution, then $\sum \pi_j Q_j(y) \leq \sum \pi_j Q_j(x) < \infty$, and so $(\mu_j(y),\ j \in S)$ is a quasi-stationary distribution, for all y in the interval $\rho \leq y \leq x$. □

We note that as a consequence of this theorem there must be a ρ-invariant quasi-stationary distribution, and so, by the Corollary to Theorem 4, we must have $\rho = \rho_\partial$, if a quasi-stationary distribution exists.

The next result gives a sufficient condition for $\mu(x) \equiv (\mu_j(x),\ j \in S)$, to constitute a quasi-stationary distribution for *each* value of x in the interval $\rho \leq x < 1$. It is a generalization of [4, Theorem 4.2], which concerns pure birth-death processes.

Theorem 8 Let $\mathcal{X}$ be a birth-death process with killing for which absorption at ∂ is certain and $\kappa_i > 0$ for only finitely many states $i \in S$. If $\rho < 1$ then $(\mu_j(x),\ j \in S)$ constitutes a quasi-stationary distribution for each x in the interval $\rho \leq x < 1$.

Proof Since $\kappa_i > 0$ for only finitely many states $i \in S$ we have,

$$\sum_{i=0}^{\infty} \frac{1}{p_j \pi_j} = \infty,$$

by Theorem 6. Now suppose that $\rho \leq x < 1$, and the sum in (33) diverges. Then,

$$\sum_{i=0}^{j} (\kappa_i - 1 + x)\pi_i Q_i(x) \to -\infty \quad \text{as} \quad j \to \infty,$$

so that, by (23), $Q_i(x)$ must be negative for i sufficiently large. This, however, contradicts (27) and (26). Hence the sum in (33) must be finite for each x in the interval $\rho \leq x < 1$, and so, by the previous theorem, with each such x a quasi-stationary distribution can be associated in the manner indicated. □

When infinitely many killing probabilities are positive the situation is quite different. In fact, there may be 0, 1 or infinitely many quasi-stationary distributions when $\rho < 1$ and absorption is certain. Moreover, even if there are infinitely many quasi-stationary distributions, $(\mu_j(x),\ j \in S)$ need not be a quasi-stationary distribution for all x in the interval $\rho \leq x < 1$. We give examples of each type of behaviour.

First, we construct a process which is such that an x-invariant, and hence quasi-stationary, distribution exists if and only if $\rho \leq x < a$ for some $a < 1$. Indeed, let $\mathcal{X}$ be a birth-death process with killing with birth, death, self-transition and killing probabilities $p_i,\ q_{i+1}, r_i$, and $\kappa_i,\ i \in S$, respectively, 1-step transition matrix P and decay parameter ρ. We allow $\kappa_i = 0$ for all i. Next choose $0 < \kappa < 1$ and let $\tilde{\mathcal{X}}$ be the birth-death process with killing with 1-step transition probabilities

$$\tilde{p}_i \equiv (1-\kappa)p_i,\ \tilde{q}_{i+1} \equiv (1-\kappa)q_{i+1},\ \tilde{r}_i \equiv (1-\kappa)r_i, \quad i \in S,$$

and

$$\tilde{\kappa}_i = \kappa + (1-\kappa)\kappa_i, \quad i \in S,$$

and 1-step transition matrix $\tilde{P}$. One might interpret κ as the killing probability in each state due to some new phenomenon, while the 1-step transition probabilities of $\tilde{\mathcal{X}}$, conditional on non-occurrence of this new phenomenon, equal those of $\mathcal{X}$. Obviously, the n-step transition probabilities of $\tilde{\mathcal{X}}$ and $\mathcal{X}$ are related as

$$\tilde{P}_{ij}(n) = (1-\kappa)^n P_{ij}(n), \quad i, j \in S,\ n \geq 0,$$

whence the decay parameter of $\tilde{\mathcal{X}}$ satisfies $\tilde{\rho} = (1-\kappa)\rho$. It is evident from (1) that an x-invariant distribution for P is an $(1-\kappa)x$-invariant distribution for $\tilde{P}$, and vice versa. Now, if we choose $\mathcal{X}$ such that it satisfies the conditions of, say, Theorem 8 (so that for each x in the interval $\rho \leq x < 1$ there exists an x-invariant, and hence quasi-stationary, distribution for P), then for each $\tilde{x}$ in the interval $\tilde{\rho} \leq \tilde{x} < 1-\kappa$ there exists an $\tilde{x}$-invariant, and hence quasi-stationary, distribution for $\tilde{P}$, but there are no $\tilde{x}$-invariant quasi-stationary distributions for $\tilde{P}$ with $\tilde{x} \geq 1-\kappa$, since an x-invariant distribution for P must have $x < 1$. Thus $\tilde{\mathcal{X}}$ has the required property, with $a = 1-\kappa$.

If, in the setting above, $\mathcal{X}$ is positive recurrent, then $\rho = 1$ and there is exactly one 1-invariant distribution, namely the equilibrium distribution of $\mathcal{X}$.

As a consequence $\tilde{\rho} = 1-\kappa$, and $\tilde{\mathcal{X}}$ has exactly one quasi-stationary distribution, which is $\tilde{\rho}$-invariant. Another setting in which there is precisely one quasi-stationary distribution is obtained by taking $p < 1/2$ in the example in the next section (see Theorem 10).

Finally, by taking $p = 1/2$ in the example in the next section we see that it is possible to have certain absorption and $\rho < 1$, but *no* quasi-stationary distribution at all.

6 Example

Interesting cases arise if $\kappa_i > 0$ for infinitely many states i, while κ_i is not constant. We will analyse a simple example satisfying these requirements, namely the process $\mathcal{X}$ with birth, death, self-transition and killing probabilities

$$p_i \equiv p, \; q_i \equiv q\mathbb{I}_{\{i>0\}}, \; r_i \equiv (1-p)\mathbb{I}_{\{i=0\}} \;\text{ and }\; \kappa_i \equiv \kappa\mathbb{I}_{\{i>0\}}, \quad i \in S, \tag{36}$$

respectively, where $p > 0$, $q > 0$ and $\kappa > 0$ are such that $p + q + \kappa = 1$, and $\mathbb{I}_E$ denotes the indicator function of an event E. The continuous-time counterpart of this process has been studied in [6], which enables us to translate some pertinent results from that paper into the discrete-time setting at hand. Our aim is to determine ρ, ρ_∂ and all quasi-stationary distributions.

It is easily seen that (30) is satisfied, so that absorption is certain. To calculate ρ we employ the representation (26), so we have to study the measure ψ with respect to which the polynomials $\{Q_n\}$ are orthogonal. By (20) these polynomials satisfy the recurrence relation

$$\begin{aligned} &xQ_n(x) = qQ_{n-1}(x) + pQ_{n+1}(x), \quad n > 0, \\ &Q_0(x) = 1, pQ_1(x) = x - 1 + p, \end{aligned} \tag{37}$$

which, by the transformation

$$S_n(x) \equiv (-1)^n \left(\sqrt{\frac{p}{q}}\right)^n Q_n(-2x\sqrt{pq}), \quad n \geq 0, \tag{38}$$

reduces to

$$\begin{aligned} &S_n(x) = 2xS_{n-1}(x) - S_{n-2}(x), \quad n > 1, \\ &S_1(x) = 2x + \gamma, \quad S_0(x) = 1, \end{aligned} \tag{39}$$

where

$$\gamma \equiv \frac{1-p}{\sqrt{pq}}. \tag{40}$$

The polynomials $\{S_n\}$ can be represented as

$$S_n(x) = U_n(x) + \gamma U_{n-1}(x), \quad n \geq 1, \tag{41}$$

where $\{U_n\}$ denote the *Chebysev polynomials of the second kind*. The latter satisfy the recurrence

$$\begin{aligned} &U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n > 1, \\ &U_1(x) = 2x, \quad U_0(x) = 1, \end{aligned} \tag{42}$$

and may be represented as

$$U_n(x) = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}}, \quad x = \frac{1}{2}(z + z^{-1}), \quad n \geq 0. \tag{43}$$

It will be useful to observe that

$$U_n(x) = (-1)^n U_n(-x), \quad n \geq 0. \tag{44}$$

After suitably transforming the orthogonalizing measure for $\{S_n\}$, given in Chihara [1, p. 205], we conclude that the polynomials $\{Q_n\}$ are orthogonal with respect to a measure which consists of a positive density on the interval $(-2\sqrt{pq}, 2\sqrt{pq})$, and, if $p + \sqrt{pq} < 1$, a point mass at $1 - p + pq/(1-p) = (\gamma + \gamma^{-1})\sqrt{pq} > 2\sqrt{pq}$. It thus follows from (26) that

$$\rho = \begin{cases} 2\sqrt{pq} & \text{if} \quad p + \sqrt{pq} \geq 1 \\ 1 - p + pq/(1-p) & \text{if} \quad p + \sqrt{pq} < 1. \end{cases} \tag{45}$$

To obtain ρ_∂ we argue as follows. Let G_α denote a geometrically distributed random variable on $\{1, 2, \ldots\}$ with mean α^{-1}, and A a random variable representing the first-passage time from state 1 to state 0 in a simple random walk on the integers with probabilities $p/(p+q)$ and $q/(p+q)$ of making a jump to the right and to the left, respectively. (If $p > q$ the distribution of A is defective.) A little reflection then shows that, when the initial state is 0, the absorption time T of the process $\mathcal{X}$ may be represented as

$$T = G_p + G_\kappa \mathbb{I}_{\{G_\kappa \leq A\}} + (A + T^*)\mathbb{I}_{\{G_\kappa > A\}},$$

where T and T^* are independent but identically distributed. Denoting the probability generating function of T when the initial state is 0 by $\tilde{\tau}_0(s)$, and parallelling the argument in [6, Section 6] we obtain after some algebra (recall that $p + q + \kappa = 1$)

$$\tilde{\tau}_0(s) \equiv \mathbb{E}_0[s^T] = \frac{s^2(1-p-q)(1-p-(1-p)\tilde{a}((p+q)s))}{(1-(p+q)s)(1-ps-(1-p)s\tilde{a}((p+q)s))}, \tag{46}$$

where $\tilde{a}(s) \equiv \mathbb{E}[s^A]$. It is well known (see, for instance, Feller [7, p. 351] that

$$\tilde{a}(s) = \frac{p+q}{2ps}\left(1 - \sqrt{1 - 4pqs^2/(p+q)^2}\right),$$

which, upon substitution in (46) and some algebra again, leads to

$$\tilde{\tau}_0(s) = \frac{1-p-q}{2(1-p)} \, \frac{2p(1-p-q)s^2 + (1-s)(1-\sqrt{1-4pqs^2})}{(1-(p+q)s)(1-(1-p+pq/(1-p))s)}. \tag{47}$$

An explicit formula for the absorption-time distribution when the initial state is 0 may be obtained by inverting this formula. But we are interested only in ρ_∂, which is the reciprocal of the radius of convergence of $\tilde{\tau}_0(s)$. Since the branch points of $\tilde{\tau}_0(s)$ at $\pm(4pq)^{-1/2}$ are always larger in absolute value than the pole at $(p+q)^{-1}$ it follows that $\rho_\partial = p + q$ or $\rho_\partial = 1 - p + pq/(1-p)$, whichever is larger. Noting that $p + q \leq 1 - p + pq/(1-p)$ if and only if $p \leq 1/2$, and collecting all our results we conclude the following.

Theorem 9 The process $\mathcal{X}$ with transition probabilities (36) has rates of convergence ρ and ρ_∂ given by

$$\rho = \rho_\partial = 1 - p + \frac{pq}{1-p} \quad \text{if} \quad p \leq 1/2,$$

$$\rho = 1 - p + \frac{pq}{1-p} < \rho_\partial = p + q \quad \text{if} \quad \sqrt{pq} < 1 - p < 1/2,$$

and

$$\rho = 2\sqrt{pq} < \rho_\partial = p + q \quad \text{if} \quad p + \sqrt{pq} \geq 1 \text{ (and hence } p > 1/2).$$

By the Corollary to Theorem 4 a quasi-stationary distribution can exist only if $\rho = \rho_\partial$ (and hence $p+\sqrt{pq} < 1$), so, by the preceding result, $p \leq 1/2$ is necessary for the existence of a quasi-stationary distribution. In view of Theorem 7 we next have to verify convergence of $\sum \pi_n Q_n(\rho)$ in order to establish the existence of a quasi-stationary distribution. To this end we note that, by (21) and (36),

$$\pi_n = \left(\frac{p}{q}\right)^n, \quad n \geq 0, \tag{48}$$

so that, by (38), (40) and Theorem 9,

$$\pi_n Q_n(\rho) = (-1)^n \left(\sqrt{\frac{p}{q}}\right)^n S_n\left(-\frac{1}{2}(\gamma + \gamma^{-1})\right), \quad n \geq 0, \tag{49}$$

if $p + \sqrt{pq} < 1$. Hence, it follows after some algebra from (41) – (44) that

$$\pi_n Q_n(\rho) = \left(\frac{p}{1-p}\right)^n, \quad p + \sqrt{pq} < 1,\ n \geq 0. \tag{50}$$

Thus we see that if $p \leq 1/2$ (and hence $p + \sqrt{pq} < 1$), then the sum $\sum \pi_n Q_n(\rho)$ converges, and hence a quasi-stationary distribution exists, if and only $p < 1/2$.

We finally wish to establish whether there exists precisely one, or an infinite family of quasi-stationary distributions when $p < 1/2$. So we have to investigate whether $\sum \pi_n Q_n(x)$ converges for $x > \rho$. We note from (38), (41) and (44) that

$$\pi_n Q_n(x) = \left(\sqrt{\frac{p}{q}}\right)^n \left(U_n\left(\frac{x}{2\sqrt{pq}}\right) - \gamma U_{n-1}\left(\frac{x}{2\sqrt{pq}}\right)\right), \quad n \geq 0. \tag{51}$$

Assuming $x > \rho$ $(= 1 - p + pq/(1-p) > 2\sqrt{pq})$, there is a unique $z > \gamma > 1$ such that

$$\frac{x}{2\sqrt{pq}} = \frac{1}{2}\left(z + z^{-1}\right), \tag{52}$$

so that, by (43) and (51),

$$\pi_n Q_n(x) = \left(\sqrt{\frac{p}{q}}\right)^n \left\{\frac{z^n(z-\gamma) - z^{-n}(z^{-1}-\gamma)}{z - z^{-1}}\right\}, \quad n \geq 0. \tag{53}$$

But since

$$z > \gamma = \frac{1-p}{\sqrt{pq}} > \frac{q}{\sqrt{pq}} = \sqrt{\frac{q}{p}},$$

it follows that $\sum \pi_n Q_n(x) = \infty$ for all $x > \rho$. We conclude the following.

Theorem 10 For the process $\mathcal{X}$ with transition probabilities (36) there is no quasi-stationary distribution when $p \geq 1/2$, and precisely one quasi-stationary distribution $(c(p/(1-p))^j,\ j \in S)$ when $p < 1/2$, where $c \equiv (1-p)/(1-2p)$.

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References

[1] T.S. Chihara, *An Introduction to Orthogonal Polynomials.* Gordon and Breach, New York, 1978.

[2] J.N. Darroch and E. Seneta, On quasi-stationary distributions in absorbing discrete-time finite Markov chains. *J. Appl. Probab.* **2** (1965) 88-100.

[3] D.J. Daley, Quasi-stationary behaviour of a left-continuous random walk. *Ann. Math. Statist.* **40** 532-539.

[4] E.A. van Doorn and P. Schrijner, Geometric ergodicity and quasi-stationarity in discrete-time birth-death processes. *J. Austral. Math. Soc. (B)* **37** (1995) 121-144.

[5] E.A. van Doorn and P. Schrijner, Ratio limits and limiting conditional distributions for discrete-time birth-death processes. *J. Math. Anal. Appl.* **190** (1995) 263-284.

[6] E.A. van Doorn and A.I. Zeifman, Extinction probability in a birth-death process with killing. *J. Appl. Probab.* **42** (2005), to appear.

[7] W. Feller, *An Introduction to Probability Theory and Its Applications, Vol. I,* rev. ed. Wiley, New York, 1970.

[8] P.A. Ferrari, H. Kesten and S. Martínez, R-positivity, quasi-stationary distributions and ratio limit theorems for a class of probabilistic automata. *Ann. Appl. Probab.* **6** (1996) 577-616.

[9] P.A. Ferrari, H. Kesten, S. Martínez and S. Picco, Existence of quasistationary distributions. A renewal dynamical approach. *Ann. Probab.* **23** (1995) 501-521.

[10] S.D. Jacka and G.O. Roberts, Weak convergence of conditioned processes on a countable state space. *J. Appl. Probab.* **32** (1995) 902-916.

[11] S. Karlin and J.L. McGregor, Random walks. *Illinois J. Math.* **3** (1959) 66-81.

[12] H. Kesten, A ratio limit theorem for (sub)Markov chains on $\{0, 1, 2, \ldots\}$ with bounded jumps. *Adv. Appl. Probab.* **27** (1995) 652-691.

[13] M. Kijima, On the existence of quasi-stationary distributions in denumerable R-transient Markov chains. *J. Appl. Probab.* **29** (1992) 21-36. Correction. *J. Appl. Probab.* **30** (1993), 496.

[14] M. Kijima, Quasi-limiting distributions of Markov chains that are skip-free to the left in continuous time. *J. Appl. Probab.* **30** (1993) 509-517.

[15] J.F.C. Kingman, The exponential decay of Markov transition probabilities. *Proc. London Math. Soc.* (3rd Series) **13** (1963) 337-358.

[16] J.A. Moler, F. Plo and M. San Miguel, Minimal quasi-stationary distribution under null R-recurrence. *Test* **9** (2000) 455-470.

[17] M.G. Nair and P.K. Pollett, On the relationship between μ-invariant measures and quasi-stationary distributions for continuous-time Markov chains. *Adv. Appl. Probab.* **25** (1993) 82-102.

[18] A.G. Pakes, The Galton-Watson process with killing. *Math. Biosci.* **69** (1984) 171-188.

[19] E. Seneta and D. Vere-Jones, On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states. *J. Appl. Probab.* **3** (1966) 403-434.

[20] D. Vere-Jones, Ergodic properties of nonnegative matrices - I. *Pacific J. Math.* **22** (1967) 361-386.