
Department of Applied Mathematics
Faculty of EEMCS



University of Twente
The Netherlands

P.O. Box 217
7500 AE Enschede
The Netherlands

Phone: +31-53-4893400

Fax: +31-53-4893114

Email: memo@math.utwente.nl
www.math.utwente.nl/publications

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and birth-death processes with killing**

P. COOLEN-SCHRIJNER¹ AND

E.A. VAN DOORN

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¹Department of Mathematical Sciences, Durham University, Durham DH1 3LE, United Kingdom

ORTHOGONAL POLYNOMIALS ON \mathbb{R}^+ AND BIRTH-DEATH PROCESSES WITH KILLING

Pauline Coolen-Schrijner* and Erik A. van Doorn[†]

* Department of Mathematical Sciences
Durham University
Durham DH1 3LE, United Kingdom
E-mail: Pauline.Schrijner@durham.ac.uk

[†] Department of Applied Mathematics
University of Twente
P.O. Box 217, 7500 AE Enschede, The Netherlands
E-mail: e.a.vandoorn@utwente.nl

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Abstract. The purpose of this paper is to extend some results of Karlin and McGregor's and Chihara's concerning the three-terms recurrence relation for polynomials orthogonal with respect to a measure on the nonnegative real axis. Our findings are relevant for the analysis of a type of Markov chains known as birth-death processes with killing.

Keywords and phrases: Favard's theorem, Stieltjes moment problem, three-terms recurrence relation, birth-death process.

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1 Introduction and main results

Our point of departure will be the familiar three-terms recurrence relation for orthogonal polynomials. That is, we consider a sequence of monic polynomials $\{P_n(x)\}_{n=0}^\infty$ satisfying the recurrence relation

$$\begin{aligned} P_n(x) &= (x - c_n)P_{n-1}(x) - d_nP_{n-2}(x), \quad n > 1, \\ P_0(x) &= 1, \quad P_1(x) = x - c_1, \end{aligned} \tag{1}$$

where c_n is real and $d_n > 0$. Hence, by Favard's theorem (see, for example, Chihara [3]), there exists a measure ψ on \mathbb{R} with respect to which the polynomials $\{P_n(x)\}$ are orthogonal, that is,

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)\psi(dx) = k_n\delta_{nm}, \quad n, m \geq 0, \tag{2}$$

with $k_n > 0$. (Here and in what follows the term *measure* stands for probability measure with infinite support.) It is well known that, conversely, any sequence of polynomials $\{P_n(x)\}$ orthogonal with respect to a measure on \mathbb{R} satisfies a recurrence relation of the type (1) with c_n real and $d_n > 0$.

Clearly, whether or not the orthogonalizing measure ψ is uniquely determined by the recurrence relation (1), the moments

$$m_n := \int_{-\infty}^{\infty} x^n \psi(dx), \quad n \geq 0,$$

are uniquely determined by (1), and may be expressed recursively in terms of the parameters c_n and d_n . We will therefore refer to the problem of finding an orthogonalizing measure for the polynomials $\{P_n(x)\}$ as the *Hamburger moment problem (Hmp) associated with (1)*. Favard's theorem thus tells us that the Hmp associated with (1) has a solution.

The *Stieltjes moment problem (Smp) associated with (1)* is the problem of finding an orthogonalizing measure for $\{P_n(x)\}$ on $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$, if any exist. A necessary and sufficient condition for the Smp associated with (1) to have a solution is given in the next theorem, which is due to Karlin and McGregor [8] and Chihara [1] (see also [3, Theorem I.9.1]), but can be partly traced back to the work of Stieltjes [10].

Theorem 1 (Karlin and McGregor [8], Chihara [1]) For the sequence of polynomials $\{P_n(x)\}$ satisfying the recurrence relation (1) to be orthogonal with respect to a measure on \mathbb{R}^+ , it is necessary and sufficient that there exist numbers $\alpha_n > 0$ and $\beta_{n+1} > 0$, $n \geq 1$, such that

$$c_1 = \alpha_1 \quad \text{and} \quad c_n = \alpha_n + \beta_n, \quad d_n = \alpha_{n-1}\beta_n, \quad n > 1. \quad (3)$$

It has been observed by Karlin and McGregor [8] and Chihara [1] (see also the Corollary to [3, Theorem I.9.1]) that (3) may actually be relaxed to

$$c_1 \geq \alpha_1 \quad \text{and} \quad c_n = \alpha_n + \beta_n, \quad d_n = \alpha_{n-1}\beta_n, \quad n > 1. \quad (4)$$

But we may weaken condition (3) even further, as appears from the next result.

Theorem 2 For the sequence of polynomials $\{P_n(x)\}$ satisfying the recurrence relation (1) to be orthogonal with respect to a measure on \mathbb{R}^+ , it is necessary and sufficient that there exist numbers $\alpha_n > 0$ and $\beta_{n+1} > 0$, $n \geq 1$, such that

$$c_1 \geq \alpha_1 \quad \text{and} \quad c_n \geq \alpha_n + \beta_n, \quad d_n = \alpha_{n-1}\beta_n, \quad n > 1. \quad (5)$$

Proof The necessity of the condition follows immediately from Theorem 1. To prove the sufficiency, suppose there are positive numbers α_n and β_{n+1} , $n \geq 1$, such that (5) is satisfied. We can then recursively define the quantities

$$\tilde{\alpha}_1 := c_1 \quad \text{and} \quad \tilde{\beta}_n := (\alpha_{n-1}/\tilde{\alpha}_{n-1})\beta_n, \quad \tilde{\alpha}_n := c_n - \tilde{\beta}_n, \quad n > 1, \quad (6)$$

and note, by induction, that

$$0 < \tilde{\beta}_n \leq \beta_n, \quad \tilde{\alpha}_n \geq c_n - \beta_n \geq \alpha_n > 0, \quad n > 1. \quad (7)$$

So we have constructed positive quantities $\tilde{\alpha}_n$ and $\tilde{\beta}_{n+1}$, $n \geq 1$, such that

$$\begin{aligned} P_n(x) &= (x - \tilde{\alpha}_n - \tilde{\beta}_n)P_{n-1}(x) - \tilde{\alpha}_{n-1}\tilde{\beta}_n P_{n-2}(x), \quad n > 1, \\ P_0(x) &= 1, \quad P_1(x) = x - \tilde{\alpha}_1. \end{aligned} \quad (8)$$

Theorem 1 now tells us that the Smp associated with (1) has a solution. \square

Remark The condition in Theorem 1 may be formulated alternatively using the concept of a *chain sequence* (cf. [3, Theorem I.9.2]). T.S. Chihara (personal communication) has observed that this alternative formulation, together with Wall's comparison test for chain sequences (see [3, Theorem III.5.7]) immediately leads to the result of Theorem 2.

We know from [8] and [1] that there exists an orthogonalizing measure ψ on \mathbb{R}^+ for $\{P_n(x)\}$ with a finite moment of order -1 , that is,

$$\psi(\{0\}) = 0 \quad \text{and} \quad \int_{(0,\infty)} \frac{\psi(dx)}{x} < \infty, \quad (9)$$

if and only if there exist positive numbers α_n and β_{n+1} , $n \geq 1$, satisfying (4) with the inequality being strict. Also this condition can be relaxed, as shown in our second main result.

Theorem 3 There exists an orthogonalizing measure on \mathbb{R}^+ for $\{P_n(x)\}$ with a finite moment of order -1 if and only if there exist positive numbers α_n and β_{n+1} , $n \geq 1$, satisfying (5) with at least one of the inequalities being strict.

The necessity of the condition in this theorem follows from the result preceding the theorem. The sufficiency of the condition will be proven in Section 3, where also some remarks are made about how much freedom we have in choosing α_n and β_n successively when c_n and d_n are given.

The remainder of this paper is organized as follows. Section 2 contains preliminary results, and in Section 4 we discuss criteria for determinacy, that is, for the existence of a unique solution, of the Smp and Hmp associated with (1) when the parameters satisfy (5). In Section 5 we briefly discuss the relevance of orthogonal polynomials defined by the recurrence relation (1) with parameters satisfying (5) for the analysis of a type of Markov chain known as *birth-death process with killing*.

Some of the results mentioned in this paper are quoted from the recent papers [5], [6] and [7], which deal with birth-death processes with killing. It is the purpose of this note to collect these results, and to elaborate on them from the perspective of orthogonal polynomials.

2 Preliminaries

In this section we will assume that the parameters c_n and d_n in the recurrence relation (1) satisfy the condition of Theorem 2. Letting

$$\gamma_1 := c_1 - \alpha_1 \quad \text{and} \quad \gamma_n := c_n - \alpha_n - \beta_n, \quad n > 1, \quad (10)$$

we then have $\gamma_n \geq 0$, and the recurrence relation (1) can be written as

$$\begin{aligned} P_n(x) &= (x - \alpha_n - \beta_n - \gamma_n)P_{n-1}(x) - \alpha_{n-1}\beta_n P_{n-2}(x), \quad n > 1, \\ P_0(x) &= 1, \quad P_1(x) = x - \alpha_1 - \gamma_1. \end{aligned} \quad (11)$$

Next defining

$$Q_0(x) := 1 \quad \text{and} \quad Q_n(x) := (-1)^n (\alpha_1 \alpha_2 \dots \alpha_n)^{-1} P_n(x), \quad n \geq 1, \quad (12)$$

the polynomials $Q_n(x)$, $n \geq 0$, are seen to satisfy

$$\begin{aligned} \alpha_n Q_n(x) &= (\alpha_n + \beta_n + \gamma_n - x)Q_{n-1}(x) - \beta_n Q_{n-2}(x), \quad n > 1, \\ Q_0(x) &= 1, \quad \alpha_1 Q_1(x) = \alpha_1 + \gamma_1 - x, \end{aligned} \quad (13)$$

in which we recognize the type of recurrence relation introduced in [6] in the setting of birth-death processes with killing (about which more in Section 5).

Letting

$$\rho_1 := 1 \quad \text{and} \quad \rho_n := \frac{\alpha_1 \alpha_2 \dots \alpha_{n-1}}{\beta_2 \beta_3 \dots \beta_n}, \quad n > 1, \quad (14)$$

it follows readily with induction from (13) (see also [5]) that

$$\alpha_n \rho_n (Q_n(x) - Q_{n-1}(x)) = \sum_{j=1}^n (\gamma_j - x) \rho_j Q_{j-1}(x), \quad n > 0. \quad (15)$$

Hence we have

$$Q_n(x) = 1 + \sum_{k=1}^n \frac{1}{\alpha_k \rho_k} \sum_{j=1}^k (\gamma_j - x) \rho_j Q_{j-1}(x), \quad n > 0, \quad (16)$$

and in particular

$$Q_0(0) = 1 \quad \text{and} \quad Q_n(0) = 1 + \sum_{k=1}^n \frac{1}{\alpha_k \rho_k} \sum_{j=1}^k \gamma_j \rho_j Q_{j-1}(0), \quad n > 0. \quad (17)$$

So $Q_n(0) \geq 1$, and $Q_n(0)$ is non-decreasing, while

$$Q_n(0) = 1 \quad \text{for all } n \geq 1 \iff \gamma_n = 0 \quad \text{for all } n \geq 1. \quad (18)$$

Moreover, we know from [7, Lemma 1] that

$$Q_\infty(0) := \lim_{n \rightarrow \infty} Q_n(0) = \infty \iff \sum_{n=1}^{\infty} \frac{1}{\alpha_n \rho_n} \sum_{j=1}^n \gamma_j \rho_j = \infty. \quad (19)$$

Another useful observation from (15) is given in the next lemma.

Lemma 1 If $\gamma_n > 0$ for some $n \geq 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n \rho_n Q_{n-1}(0) Q_n(0)} < \infty.$$

Proof Suppose $\gamma_m > 0$. Then (15) implies that we can write, at least for $n \geq m$,

$$\frac{1}{\alpha_n \rho_n Q_{n-1}(0) Q_n(0)} = \frac{1}{\sum_{j=1}^n \gamma_j \rho_j Q_{j-1}(0)} \left(\frac{1}{Q_{n-1}(0)} - \frac{1}{Q_n(0)} \right),$$

and hence

$$\frac{1}{\alpha_n \rho_n Q_{n-1}(0) Q_n(0)} \leq \frac{1}{\gamma_m \rho_m Q_{m-1}(0)} \left(\frac{1}{Q_{n-1}(0)} - \frac{1}{Q_n(0)} \right), \quad n \geq m.$$

It follows that

$$\sum_{n=m}^{\infty} \frac{1}{\alpha_n \rho_n Q_{n-1}(0) Q_n(0)} \leq \frac{1}{\gamma_m \rho_m Q_{m-1}(0)} \left(\frac{1}{Q_{m-1}(0)} - \frac{1}{Q_\infty(0)} \right) < \infty,$$

proving our claim. \square

Next, let $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ be as in (6). We define $\tilde{Q}_n(x)$ and $\tilde{\rho}_n$ by analogy with (12) and (14), respectively, and note that the polynomials $\tilde{Q}_n(x)$, $n \geq 0$, satisfy the recurrence relation

$$\begin{aligned} \tilde{\alpha}_n \tilde{Q}_n(x) &= (\tilde{\alpha}_n + \tilde{\beta}_n - x) \tilde{Q}_{n-1}(x) - \tilde{\beta}_n \tilde{Q}_{n-2}(x), \quad n > 1, \\ \tilde{Q}_0(x) &= 1, \quad \tilde{\alpha}_1 \tilde{Q}_1(x) = \tilde{\alpha}_1 - x. \end{aligned} \quad (20)$$

With (6) it is easy to establish the relations

$$\sqrt{\tilde{\rho}_n} \tilde{Q}_{n-1}(x) = \sqrt{\rho_n} Q_{n-1}(x), \quad n \geq 1, \quad (21)$$

and

$$\tilde{\alpha}_n \tilde{\rho}_n \tilde{Q}_{n-1}(x) \tilde{Q}_n(x) = \alpha_n \rho_n Q_{n-1}(x) Q_n(x), \quad n \geq 1. \quad (22)$$

Moreover, it is evident from (20) that $\tilde{Q}_n(0) = 1$ for all $n \geq 0$, so (21) and (22) imply

$$\tilde{\rho}_n = \rho_n Q_{n-1}^2(0) \quad \text{and} \quad \tilde{\alpha}_n \tilde{\rho}_n = \alpha_n \rho_n Q_{n-1}(0) Q_n(0), \quad n \geq 1. \quad (23)$$

The result of Lemma 1 can therefore be stated as

$$\gamma_n > 0 \text{ for some } n \geq 1 \implies \sum_{n=1}^{\infty} \frac{1}{\tilde{\alpha}_n \tilde{\rho}_n} < \infty. \quad (24)$$

We have now collected sufficient information about the sequences $\{Q_n(x)\}$ and $\{\tilde{Q}_n(x)\}$ to continue with the proof of Theorem 3 and discuss some related issues.

3 Proof of Theorem 3 and related issues

It remains to be proven that the condition in Theorem 3 is sufficient. So we suppose that the parameters c_n and d_n satisfy the condition of Theorem 2 (as in the previous section), and, in addition, that at least one of the inequalities in (5) is strict. That is, defining γ_n as in (10), we have

$$\gamma_n > 0 \text{ for some } n \geq 1. \quad (25)$$

We define the parameters $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ as before.

Since the condition in Theorem 2 is fulfilled there exists an orthogonalizing measure ψ for $\{P_n(x)\}$ on \mathbb{R}^+ . In what follows ψ will always denote either the unique solution of the Smp associated with (1), or, if a unique solution does not exist, the one known as *minimal* solution (in the terminology of Karlin and McGregor [8]) or *natural* solution (in the terminology of Chihara [2]). The latter may be characterised as the solution that is supported by the limit points of the zeros of the polynomials $P_n(x)$, $n \geq 1$.

From [8, Lemma 6 on p. 527] we know that $\psi(\{0\}) = 0$ and

$$\int_{(0,\infty)} \frac{\psi(dx)}{x} = \sum_{n=1}^{\infty} \frac{1}{\tilde{\alpha}_n \tilde{\rho}_n} \quad (26)$$

if and only if the sum in (26) is finite. By (24) the latter indeed holds under the prevailing condition (25), so ψ has a finite moment of order -1 . This concludes the proof of Theorem 3.

The following results will be needed in the second part of this section. Maintaining assumption (25), it follows by induction from the recurrence relation

(20) and (26) that

$$\int_{(0,\infty)} \tilde{Q}_{j-1}(x) \frac{\psi(dx)}{x} = \sum_{n=j}^{\infty} \frac{1}{\tilde{\alpha}_n \tilde{\rho}_n}, \quad j \geq 1,$$

and hence, in view of (21),

$$\int_{(0,\infty)} Q_{j-1}(x) \frac{\psi(dx)}{x} = \sqrt{\frac{\tilde{\rho}_j}{\rho_j}} \sum_{n=j}^{\infty} \frac{1}{\tilde{\alpha}_n \tilde{\rho}_n} > 0, \quad j \geq 1. \quad (27)$$

Moreover, it is shown in [6, Lemma 3] (under the implicit assumption (25)) that

$$\sum_{j=1}^{\infty} \gamma_j \rho_j \int_{(0,\infty)} Q_{j-1}(x) \frac{\psi(dx)}{x} = 1 - \frac{1}{Q_{\infty}(0)}. \quad (28)$$

Hence, it follows from (19) that

$$\sum_{j=1}^{\infty} \gamma_j \rho_j \int_{(0,\infty)} Q_{j-1}(x) \frac{\psi(dx)}{x} \leq 1, \quad (29)$$

with equality prevailing if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n \rho_n} \sum_{j=1}^n \gamma_j \rho_j = \infty. \quad (30)$$

We will now describe a procedure for successively choosing the parameters α_n and β_n of (5), given the parameters c_n and d_n of the recurrence relation (1).

First, by letting

$$\tilde{\alpha}_1 := c_1 \quad \text{and} \quad \tilde{\beta}_n := d_n / \tilde{\alpha}_{n-1}, \quad \tilde{\alpha}_n := c_n - \tilde{\beta}_n, \quad n \geq 2,$$

and checking whether $\tilde{\alpha}_n > 0$ for all $n \geq 1$, we can verify whether the polynomials $\{P_n(x)\}$ are orthogonal with respect to a measure on \mathbb{R}^+ . If this is the case we must check whether the right-hand side of (26) is finite. If not, the (minimal) orthogonalizing measure ψ does not have a finite moment of order -1 , and the only permissible choice for the parameters α_n and β_n is $\alpha_n := \tilde{\alpha}_n$ and $\beta_{n+1} := \tilde{\beta}_{n+1}$ (and hence $\gamma_n = 0$) for $n \geq 1$. If, however, the right-hand side of (26) is finite, so that ψ *does* have a finite moment of order -1 , we may proceed as follows. First, choose $\gamma_1 \geq 0$ such that

$$\gamma_1 \sum_{n=1}^{\infty} \frac{1}{\tilde{\alpha}_n \tilde{\rho}_n} \leq 1,$$

and let $\alpha_1 := c_1 - \gamma_1$. By [8, Lemma 1] we know that $0 < \alpha_1 \leq c_1$. Subsequently, in step $n = 2, 3, \dots$, let

$$\beta_n := d_n / \alpha_{n-1},$$

choose $\gamma_n \geq 0$ such that

$$\gamma_n \sqrt{\rho_n \tilde{\rho}_n} \sum_{k=n}^{\infty} \frac{1}{\tilde{\alpha}_k \tilde{\rho}_k} \leq 1 - \sum_{j=1}^{n-1} \gamma_j \sqrt{\rho_j \tilde{\rho}_j} \sum_{k=j}^{\infty} \frac{1}{\tilde{\alpha}_k \tilde{\rho}_k} \quad (31)$$

(recall (27) and (29)) and let $\alpha_n := c_n - \beta_n - \gamma_n$. Evidently, if equality prevails in (31) in step $n = m$, say, then we must have $\gamma_n = 0$ for all $n > m$. It remains to be shown that the values of α_n obtained by this procedure are positive. This, however, is a consequence of the following lemma.

Lemma 2 Let $\sum_{n=1}^{\infty} (\tilde{\alpha}_n \tilde{\rho}_n)^{-1} < \infty$, and suppose the polynomials $P_n(x)$, $n \geq 0$, satisfy the recurrence relation (11) with parameters $\alpha_n > 0$, $\beta_{n+1} > 0$ and $\gamma_n \geq 0$, such that $\gamma_n = 0$ for $n \geq m \geq 1$. Then the polynomials $P_n(x)$, $n \geq 0$, satisfy a recurrence relation of the type (11) with parameters $\alpha'_n > 0$, $\beta'_{n+1} > 0$ and $\gamma'_n \geq 0$, such that $\alpha'_n = \alpha_n$, $\beta'_{n+1} = \beta_{n+1}$ and $\gamma'_n = \gamma_n$ for $n < m$, $\gamma'_m > 0$, and $\gamma'_n = 0$ for $n > m$, if and only if

$$\sum_{j=1}^{m-1} \gamma_j \sqrt{\tilde{\rho}_j \rho_j} \sum_{k=j}^{\infty} \frac{1}{\tilde{\alpha}_k \tilde{\rho}_k} < 1. \quad (32)$$

Proof Adapting the proof of [8, Lemma 1] to the present setting leads to the conclusion that a necessary and sufficient condition for $\{P_n(x)\}$ to satisfy the requirements is given by

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n \rho_n} < \infty. \quad (33)$$

But, since $\gamma_j = 0$ for $j \geq m$, the conditions (32) and (33) are equivalent, in view of (27), (28) and (19). \square

4 Determinacy of the Smp and Hmp

We will assume again that the parameters c_n and d_n of the recurrence relation (1) satisfy the condition of Theorem 2, so, defining $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ as before, the polynomials $P_n(x)$, $n \geq 0$, satisfy the recurrence relation (8).

From Karlin and McGregor [8, Theorem 14] we know that the Smp associated with (8), and hence with (1), is *determined*, that is, has a unique solution, if and only if

$$\sum_{n=1}^{\infty} \left(\tilde{\rho}_n + \frac{1}{\tilde{\alpha}_n \tilde{\rho}_n} \right) = \infty, \quad (34)$$

while, by Karlin and McGregor [9, p. 391] and Chihara [4], the Hmp associated with (8), and hence with (1), is determined if and only if

$$\sum_{n=1}^{\infty} \tilde{\rho}_{n+1} \left(\sum_{k=1}^n \frac{1}{\tilde{\alpha}_k \tilde{\rho}_k} \right)^2 = \infty. \quad (35)$$

Now suppose the parametrization of the recurrence coefficients c_n and d_n is given by (11) rather than (8). The preceding results together with (23) and (24) then tell us the following.

Theorem 4 If $\gamma_n > 0$ for some $n \geq 1$, then the Hmp and Smp associated with (11) are both determined or are both indeterminate according as $\sum_{n=0}^{\infty} \rho_{n+1} Q_n^2(0)$ diverges or converges.

Remark An alternative proof for this theorem may be based on the observation that, by a theorem of Wall's mentioned just before Theorem 1 of [2], convergence of $\sum_{n=1}^{\infty} (\tilde{\alpha}_n \tilde{\rho}_n)^{-1}$ is equivalent to the condition in Theorem 2(A) of [2].

Considering the recursive expression (17) for $Q_n(0)$, it does not seem possible to give a necessary and sufficient condition for convergence of $\sum_{n=0}^{\infty} \rho_{n+1} Q_n^2(0)$ explicitly in terms of the parameters α_n , β_n , and γ_n . Since $Q_n(0) \geq 1$, we can, however, formulate the following sufficient condition, which does not require assumption (25).

Corollary The Hmp (and hence the Smp) associated with (11) is determined if

$$\sum_{n=1}^{\infty} \rho_{n+1} \left(1 + \sum_{k=1}^n \frac{1}{\alpha_k \rho_k} \sum_{j=1}^k \gamma_j \rho_j \right)^2 = \infty. \quad (36)$$

5 Birth-death processes with killing

A birth-death process with killing is a Markov chain $\mathcal{X} := \{X(t), t \geq 0\}$ taking values in $S := \{0, 1, \dots\}$ with q -matrix $Q := (q_{ij}, i, j \in S)$ given by

$$\begin{aligned} q_{i,i+1} &= \lambda_i, & q_{i+1,i} &= \mu_{i+1}, & q_{ii} &= -(\lambda_i + \mu_i + \nu_i), \\ q_{ij} &= 0, & |i - j| &> 1, \end{aligned} \tag{37}$$

where $\lambda_i > 0$ and $\nu_i \geq 0$ for $i \geq 0$, $\mu_i > 0$ for $i > 0$, and $\mu_0 = 0$. The parameters λ_i and μ_i are the birth and death rates in state i , while ν_i may be regarded as the rate of absorption, or *killing*, into a fictitious state ∂ , say.

It has been shown in [6] that the transition probabilities

$$p_{ij}(t) := \Pr\{X(t) = j \mid X(0) = i\}, \quad t \geq 0, \quad i, j \in S,$$

of the process \mathcal{X} can be represented in the form

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-xt} R_i(x) R_j(x) \psi(dx), \quad t \geq 0, \quad i, j \in S, \tag{38}$$

where π_n , $n \geq 0$, are constants given by

$$\pi_0 := 1 \quad \text{and} \quad \pi_n := \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n}, \quad n > 0,$$

$R_n(x)$, $n \geq 0$, are polynomials defined by the recurrence relation

$$\begin{aligned} \lambda_n R_{n+1}(x) &= (\lambda_n + \mu_n + \nu_n - x) R_n(x) - \mu_n R_{n-1}(x), & n \geq 1, \\ \lambda_0 R_1(x) &= \lambda_0 + \nu_0 - x, & R_0(x) = 1, \end{aligned} \tag{39}$$

and ψ is a measure with respect to which the polynomials $\{R_n(x)\}$ are orthogonal. Making the identification

$$\lambda_n = \alpha_{n+1}, \quad \mu_{n+1} = \beta_{n+2} \quad \text{and} \quad \nu_n = \gamma_{n+1}, \quad n \geq 0, \tag{40}$$

we see that, for all $n \geq 0$, the constant π_n can be identified with the constant ρ_{n+1} of (14), and the polynomial $R_n(x)$ with the polynomial $Q_n(x)$ of (13).

It has been shown in [6, Theorem 4] that the transition probabilities (if they should satisfy both *backward* and *forward* Kolmogorov differential equations) are uniquely determined by the parameters (37) if and only if at least one of the conditions $\lim_{n \rightarrow \infty} R_n(0) = \infty$ or

$$\sum_{n=0}^{\infty} \left(\pi_n + \frac{1}{\lambda_n \pi_n} \right) = \infty \tag{41}$$

be satisfied. If we apply the identification (40) we see that, by (23), condition (41) is equivalent to (34), and hence determinacy of the associated Smp, if $\lim_{n \rightarrow \infty} R_n(0) < \infty$. In view of (19) and (28), we therefore conclude that the process is uniquely determined by its rates if and only if the associated Smp is determined or

$$\sum_{j=0}^{\infty} \nu_j \pi_j \int_{(0, \infty)} R_j(x) \frac{\psi(dx)}{x} = 1.$$

This generalizes [8, Theorem 15].

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