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On superadditivity and convexity for cooperative games with random payoffs

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On Superadditivity and Convexity for Cooperative Games with Random Payoffs

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Abstract

In this paper we study the relations between superadditivity and several types of convexity for cooperative games with random payoffs. The types of convexity considered are marginal convexity (all marginal vectors belong to the core), individual-merge convexity (any individual player is better off joining a larger coalition) and coalitional-merge convexity (any coalition of players is better off joining a larger coalition).

In particular, in this work we answer two open questions in the literature. The first question is whether a marginal convex game is always superadditive. In general, the answer is negative as is shown by two counterexamples. However, for some type of games marginal convexity does imply superadditivity.

The second question is whether individual-merge convexity implies coalitional-merge convexity for games with at least four players. An example of a four-player game that is individual-merge convex but not coalitional merge convex shows that this is not the case in general. But also here we show that for some type of games the implication does hold.

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Key Words: cooperative games, random payoffs, superadditivity, convexity

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1 Introduction

In this paper we study the relations between superadditivity and several types of convexity for cooperative games with random payoffs. These games are introduced in [6] and subsequently studied in [5] and [4]. The main characteristics of these games are that the payoff to a coalition is modeled as a random variable and that the players have preferences to compare any two payoffs. For more on these games we refer to [6].

The types of convexity considered are marginal convexity (all marginal vectors belong to the core), individual-merge convexity (any individual player is better off joining a larger coalition) and coalitional-merge convexity (any coalition of players is better off joining a larger coalition). These types arise from extensions of several equivalent formulation of convexity for TU games to cooperative games with random payoffs, but they are not equivalent. The same holds for the types of convexity for NTU games [1, 2, 7] and for stochastic cooperative games [3].

In particular, in this paper we answer the two open questions in [6]:

- Is a marginal convex game superadditive?
- Is an individual-merge convex game with at least four players coalitional-merge convex?

The answer to the first question is, in general, negative. We provide two counterexamples for different types of games. The first one is a three-person game with nonlinear preferences. The second one is a four-person game with a certain type of linear preferences. Further, we show that for some other type of linear preferences marginal convexity does imply superadditivity.

The second question follows from the fact that [6] shows that for games with at most three players individual-merge convexity implies coalitional-merge convexity. This implication fails for games with at least four players, as is shown by an example of a four player game that is individual-merge convex but not coalitional-merge convex. However, we show that for games with a certain type of linear preferences individual-merge convexity implies coalitional-merge convexity.

This paper is organized as follows. In section 2 we introduce the concepts and definitions needed. The relations between marginal convexity and superadditivity are studied in section 3. Section 4 continues with analysing the relations between individual-merge convexity and coalitional-merge convexity. All proofs are postponed to appendix A.

2 Preliminaries

We follow the notation of [6]. A cooperative game with random payoffs, or game in short, is a pair (N, α). N is the finite set of players and $\alpha = (\alpha_i)_{i \in N}$ contains a complete description of the players' preferences. This α will be specified below. A subset $S \subseteq N$ is called a coalition and its cardinality is denoted by $|S|$. This coalition gives rise to a subgame (S, α) of the original game.

Denote by $R(S)$ the stochastic payoff to coalition S. This payoff is a nonnegative stochastic variable with finite expectation. An allocation of $R(S)$ among the members in S is a vector of multiples of $R(S)$, for example $p \in \mathbb{R}^S$. Then player $i \in S$ receives $p_i R(S)$. Such an allocation is efficient if $p \in \Delta^*(S) = \{p \in \mathbb{R}^S | \sum_{i \in S} p_i = 1\}$. Let $\mathcal{A}$ be the set of coalitions with payoffs unequal to the payoff zero for sure.

The preference relation of player i is denoted by $\succsim_i$. We assume the following about this preference relation.

Assumption 1 ([6]) *For each player $i \in N$ there exist functions $f_S^i : \mathbb{R} \to \mathbb{R}$, for $S \in \mathcal{A}$, that are surjective, continuous and strictly monotone increasing, such that*

1. $f_S^i(t)R(S) \succsim_i f_T^i(t')R(T)$ *if and only if* $t \geq t'$,
2. $f_S^i(0) = 0$

for any coalitions S and T in $\mathcal{A}$.

This assumption implies that there exists a unique number $\alpha_i \in \mathbb{R}$ such that $\alpha_i R(T) \sim_i p_i R(S)$. The function $\alpha_i : \mathcal{A} \times \mathcal{A} \times \mathbb{R} \to \mathbb{R}$, $\alpha_i(T, S, p_i)$, is defined to take this unique value. The use of this function is restricted to $i \in S \subseteq T$. The condition $S \subseteq T$ is not a real restriction because $\alpha_i(S, T, p_i)$ is the inverse of $\alpha_i(T, S, p_i)$. We extend the domain of α_i by defining $\alpha_i(T, S, 0) = 1$ and $\alpha_i(T, T, q_i) = 1$ if $T \notin \mathcal{A}$. Two properties are $\alpha_i(S, S, p_i) = 1$ and $\alpha_i(T, S, \alpha_i(S, U, p_i)) = \alpha_i(T, U, p_i)$ for $p_i \in \mathbb{R}$ and $S, T, U \in \mathcal{A}$. If $\alpha_i(T, S, p_i) = p_i \alpha_i(T, S, 1)$ for all T and S then we say that α_i is linear. Then, $\alpha_i(S, U, 1)\alpha_i(T, S, 1) = \alpha_i(T, U, 1)$. If all α_i are identical, that is, $\alpha_i(T, S, p_i) = \alpha_j(T, S, p_i)$ for all $i, j \in N$, then we write $\alpha_\star(T, S, p_i)$ instead of $\alpha_i(T, S, p_i)$. $I(S, \alpha)$ is the set of all individual rational allocations for coalition S, that is, all allocations p for which $p_i \geq \alpha_i(S, \{i\}, 1)$ for all $i \in S$. The set undom(S) contains all the allocations for coalition N that cannot be dominated by coalition S:

$$\text{undom}(S) = \{p \in \mathbb{R}^N | \nexists q \in \Delta^*(S) : \alpha_i(N, S, q_i) > p_i \text{ for all } i \in S\}.$$

The core $C(N, \alpha)$ of the game (N, α) consists of the efficient allocations for coalition N that are not dominated by any coalition S:

$$C(N, \alpha) = \{p \in \Delta^*(N) | p \in \text{undom}(S) \text{ for all coalitions } S\}.$$

Before we can define marginal vectors, the following assumption is needed.

Assumption 2 ([6]) *If $T \notin \mathcal{A}$ for some coalition T then $S \notin \mathcal{A}$ for all $S \subseteq T$.*

Let σ be a bijection of the players in N then $\sigma(i)$ denotes the player that is at position i. For three-person games we may write $\sigma = (\sigma(1), \sigma(2), \sigma(3))$. Let $S_i^\sigma = \{\sigma(k) | k \leq i\}$ be the set of the first i players according to σ. The marginal contribution of the ith player according to the bijection σ in the game (N, α) is

$$y_{\sigma(i)}^\sigma(\alpha) = 1 - \sum_{k=1}^{i-1} \alpha_{\sigma(k)}(S_i^\sigma, S_k^\sigma, y_{\sigma(k)}^\sigma(\alpha))$$

with the convention $y_{\sigma(1)}^\sigma(\alpha) = 1$. Now the marginal vector $m^\sigma(\alpha)$ is defined by

$$m_{\sigma(i)}^\sigma(\alpha) = \alpha_{\sigma(i)}(N, S_i^\sigma, y_{\sigma(i)}^\sigma(\alpha)).$$

A marginal vector for the subgame (S, α) is denoted by $m^\sigma(S, \alpha)$.

The game (N, α) is superadditive if for all coalitions $S, T \subseteq N$ with $S \cap T = \emptyset$ and for all allocations $p \in I(S, \alpha)$ and $q \in I(T, \alpha)$ there exists an allocation $r \in \Delta^*(S \cup T)$ for the joint coalition such that all players are weakly better off:

$$\begin{cases} r_i \geq \alpha_i(S \cup T, S, p_i) & i \in S, \\ r_i \geq \alpha_i(S \cup T, T, q_i) & i \in T. \end{cases} \tag{1}$$

An equivalent condition is

$$\sum_{i \in S} \alpha_i(S \cup T, S, p_i) + \sum_{i \in T} \alpha_i(S \cup T, T, q_i) \leq 1. \tag{2}$$

A game (N, α) is coalitional-merge convex if for all $U \subseteq N$, for all $S \subset T \subseteq N \setminus U$, for all $p \in I(S, \alpha)$, for all $q \in I(T, \alpha)$ and for all $r \in I(S \cup U, \alpha)$ such that $r_i \geq \alpha_i(S \cup U, S, p_i)$ for all $i \in S$ there exists an allocation $s \in \Delta^*(T \cup U)$ such that

$$\begin{cases} s_i \geq \alpha_i(T \cup U, T, q_i) & \text{for all } i \in T, \\ s_i \geq \alpha_i(T \cup U, S \cup U, r_i) & \text{for all } i \in U, \end{cases}$$

or equivalently,

$$\sum_{i \in T} \alpha_i(T \cup U, T, q_i) + \sum_{i \in U} \alpha_i(T \cup U, S \cup U, r_i) \leq 1. \tag{3}$$

A game is individual-merge convex if for all $i \in N$, for all $S \subset T \subseteq N \setminus \{i\}$, for all $p \in I(S, \alpha)$, for all $q \in I(T, \alpha)$ and for all $r \in I(S \cup \{i\}, \alpha)$ such that $r_j \geq \alpha_j(S \cup \{i\}, S, p_j)$ for all $j \in S$ there exists an allocation $s \in \Delta^*(T \cup \{i\})$ such that

$$\begin{cases} s_j \geq \alpha_j(T \cup \{i\}, T, q_j) & \text{for all } j \in T, \\ s_i \geq \alpha_i(T \cup \{i\}, S \cup \{i\}, r_i) & , \end{cases}$$

or equivalently,

$$\sum_{j\in T}\alpha_j(T\cup\{i\},T,q_j)+\alpha_i(T\cup\{i\},S\cup\{i\},r_i)\leq 1.$$

Finally, a game is marginal convex if all marginal vectors belong to the core. It was shown in [6] that a coalitional-merge convex game is individual-merge convex and an individual-merge convex game is marginal convex.

Theorem 3 ([6]) *If $I(S,\alpha)=\emptyset$ for some coalition S then the game (N,α) is not marginal convex.*

Some useful properties in a special setting.

Lemma 4 *If all α_i are linear and identical in the game (N,α) then*

1. $\sum_{i=1}^{t} m^{\sigma}_{\sigma(i)}(\alpha)=\alpha_\star(N,S^{\sigma}_t,1)$,
2. $undom(S)=\{p\in\mathbb{R}^N\,|\,\sum_{i\in S}p_i\geq\alpha_\star(N,S,1)\}$.

All proofs can be found in appendix A.

3 Marginal convexity and superadditivity

In this section we show that there is in general no relation between marginal convexity and superadditivity for cooperative games with random payoffs. This is in contrast to the result for TU games: a TU game is superadditive if all its marginal vectors belong to the core, that is, it is convex. Further, we present two classes of games for which marginal convexity implies superadditivity.

The example below shows that superadditive games need not be marginal convex, as is the case for TU games.

Example 5 *Consider the 3-player game (N,α) in which all the α_i are linear and identical such that*

S	$\{i\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$\alpha_i(N,S,1)$	0	1/5	2/5	7/10

Let $S,T\subseteq N$ with $S\cap T=\emptyset$. The superadditivity condition (1) is satisfied if both S and T are 1-player coalitions. Let $S=\{1\}$ and $T=\{2,3\}$. Take $p_1=1\in I(S,\alpha)$, $(q_2,q_3)\in I(T,\alpha)$ and $(r_1,r_2,r_3)=(3/10,7q_2/10,7q_3/10)$. For player 1 we have

$$r_1\geq\alpha_1(N,\{1\},p_1)\Leftrightarrow 3/10\geq 0$$

and for player $i \in \{2,3\}$

$$r_i \geq \alpha_i(N,\{2,3\},q_i) \Leftrightarrow 7q_i/10 \geq 7q_i/10$$

The reader may check that this game is superadditive.

This game is not marginal convex. Take $\sigma = (3,1,2)$. *Then* $y_3^\sigma = 1$, $y_1^\sigma = 1$, $y_2^\sigma = 3/5$, *and so* $m^\sigma(\alpha) = (2/5, 3/5, 0)$. *This marginal vector does not belong to the core since it is dominated by coalition* $\{2,3\}$, $m^\sigma(\alpha) \notin undom(\{2,3\})$: *let* $q = (10/11, 1/11)$ *then* $q \in \Delta^*(\{2,3\})$ *and* $\alpha_i(N,\{2,3\},q_i) = 7q_i/10 > m_i^\sigma(\alpha)$ *for* $i = 2,3$.

The following example shows a three-person marginal convex game that is not superadditive. This is in contrast to the result for TU games, namely, convex TU games are superadditive.

Example 6 *Consider the game* (N,α) *with* $N = \{1,2,3\}$,

$$\alpha_1(T,S,p_1) = \begin{cases} 0, & |S| = 1, \\ p_1, & |S| = |T| > 1, \\ 4p_1/5, & S = \{1,2\},\ T = N, \\ \sqrt[10]{p_1/20}, & S = \{1,3\},\ T = N,\ p \geq 0, \\ p_1/2, & S = \{1,3\},\ T = N,\ p < 0. \end{cases}$$

and

$$\alpha_i(T,S,p_i) = \begin{cases} 0, & |S| = 1, \\ p_i, & |S| = |T| > 1, \\ p_i/2, & S = \{1,i\},\ T = N, \\ p_i/5, & S = \{2,3\},\ T = N, \end{cases}$$

for $i = 2,3$. *The six marginal vectors of this game are listed in the table below*

σ	$(1,2,3),\ (1,3,2)$	$(2,1,3),\ (2,3,1)$	$(3,1,2)$	$(3,2,1)$
$m^\sigma(\alpha)$	$(0,1/2,1/2)$	$(4/5,0,1/5)$	$(\sqrt[10]{1/20}, 1-\sqrt[10]{1/20}, 0)$	$(4/5,1/5,0)$

and they all belong to the core

$$C(N,\alpha) = \{p \in \Delta^*(N) | p_i \geq 0,\ i \in N;\ 5p_1 + 8p_2 \geq 4;\ 20(p_1)^{10} + 2p_3 \geq 1;\ 5p_2 + 5p_3 \geq 1\}.$$

Hence, the game is marginal convex. Nevertheless, superadditivity is not satisfied. For this, let $S = \{3\}$ *and* $T = \{1,2\}$. *Take* $p_3 = 1 \in I(\{3\},\alpha)$ *and* $(q_1,q_2) = (1/8, 7/8) \in I(\{1,2\},\alpha)$. *Then*

$$\alpha_1(N,\{1,2\},q_1) + \alpha_2(N,\{1,2\},q_2) + \alpha_3(N,\{3\},p_3) = 1/\sqrt[10]{160} + 7/16 + 0 > 1,$$

which contradicts (2). We conclude that the game is not superadditive.

S	$\alpha_1(N,S,1)$	S	$\alpha_2(N,S,1)$	S	$\alpha_3(N,S,1)$	S	$\alpha_4(N,S,1)$
$\{1\}$	15/100	$\{2\}$	0/100	$\{3\}$	0/100	$\{4\}$	0/100
$\{1,2\}$	16/100	$\{1,2\}$	16/100	$\{1,3\}$	16/100	$\{1,4\}$	17/100
$\{1,3\}$	16/100	$\{2,3\}$	16/100	$\{2,3\}$	16/100	$\{2,4\}$	16/100
$\{1,4\}$	17/100	$\{2,4\}$	16/100	$\{3,4\}$	17/100	$\{3,4\}$	17/100
$\{1,2,3\}$	30/100	$\{1,2,3\}$	35/100	$\{1,2,3\}$	35/100	$\{1,2,4\}$	35/100
$\{1,2,4\}$	30/100	$\{1,2,4\}$	40/100	$\{1,3,4\}$	40/100	$\{1,3,4\}$	40/100
$\{1,3,4\}$	30/100	$\{2,3,4\}$	90/100	$\{2,3,4\}$	40/100	$\{2,3,4\}$	40/100

Table 1: This table lists the diverse values of $\alpha_i(N,S,1)$ for the game in example 8.

In the example above player 1 has nonlinear preferences. If a three-person game has linear α_i then marginal convexity implies superadditivity.

Theorem 7 *A marginal convex game G is superadditive if it is a three-person game with linear α_i.*

This result does not hold for games with at least four players.

Example 8 *Consider the game (N,α) with four players, $N = \{1,2,3,4\}$, and linear α_i. The various values of $\alpha_i(N,S,1)$ are listed in table 1. Numerous calculations show that all 24 marginal vectors belong to the core of the game.*

Next, let $S = \{1\}$ and $T = \{2,3,4\}$ and consider the allocations $p_1 = 1 \in I(S,\alpha)$ and $(q_2,q_3,q_4) = (1,0,0) \in I(T,\alpha)$. Then $S \cup T = N$ and

$$\alpha_1(N,S,p_1) + \sum_{i\in T}\alpha_i(N,T,q_i) = 15/100 + 90/100 + 0 + 0 > 1,$$

which contradicts (2). We conclude that this game is marginal convex but not superadditive.

If we impose the restriction that all α_i are linear and identical then marginal convex games with an arbitrary number of players are superadditive.

Theorem 9 *A marginal convex game G is superadditive if all α_i are linear and identical.*

4 Individual and coalitional merge convexity

In this section we show that an individual-merge convex game need not be coalitional-merge convex. An example with linear α_i is given below. However, as shown further on, if all α_i are linear and identical then individual-merge convexity implies coalitional-merge convexity.

S	$\alpha_1(N,S,1)$	S	$\alpha_2(N,S,1)$	S	$\alpha_3(N,S,1)$	S	$\alpha_4(N,S,1)$
$\{1\}$	2/100	$\{2\}$	2/100	$\{3\}$	2/100	$\{4\}$	1/100
$\{1,2\}$	26/100	$\{1,2\}$	18/100	$\{1,3\}$	15/100	$\{1,4\}$	15/100
$\{1,3\}$	16/100	$\{2,3\}$	15/100	$\{2,3\}$	15/100	$\{2,4\}$	15/100
$\{1,4\}$	16/100	$\{2,4\}$	12/100	$\{3,4\}$	12/100	$\{3,4\}$	12/100
$\{1,2,3\}$	40/100	$\{1,2,3\}$	38/100	$\{1,2,3\}$	36/100	$\{1,2,4\}$	35/100
$\{1,2,4\}$	41/100	$\{1,2,4\}$	37/100	$\{1,3,4\}$	58/100	$\{1,3,4\}$	80/100
$\{1,3,4\}$	38/100	$\{2,3,4\}$	34/100	$\{2,3,4\}$	35/100	$\{2,3,4\}$	35/100

Table 2: This table lists the diverse values of $\alpha_i(N,S,1)$ for the game in example 10.

Example 10 *Consider the game (N,α) with four players, $N = \{1,2,3,4\}$, and linear α_i. The various values of $\alpha_i(N,S,1)$ are listed in table 2. Numerous calculations show that this game is individual-merge convex.*

Next, let $S = \{1\}$, $T = \{1,2\}$ and $U = \{3,4\}$. Consider the allocations $p_1 = 1 \in I(S,\alpha)$, $(q_1,q_2) = (8/9,1/9) \in I(T,\alpha)$ and $(r_1,r_3,r_4) = (1/19,1/29,1-1/19-1/29) \in I(S\cup U,\alpha)$. Then $r_1 \geq \alpha_1(S\cup U,S,p_1)$, $T\cup U = N$ and

$$\sum_{j\in T}\alpha_j(N,T,q_j) + \sum_{i\in U}\alpha_i(N,S\cup U,r_i) = 52/225 + 1/50 + 1/50 + (1-1/19-1/29)4/5 > 1,$$

which contradicts (3). We conclude that this game is individual-merge convex but not coalitional-merge convex.

This example shows that linear α_i are not sufficient for individual-merge convex games to imply coalitional-merge convexity. The next theorem provides a first step to show that linear and identical α_i are sufficient.

Theorem 11 *A marginal convex game G is coalitional-merge convex if all α_i are linear and identical.*

Since any individual-merge convex game is marginal convex, the result below follows immediately.

Corollary 12 *An individual-merge convex game G is coalitional-merge convex if all α_i are linear and identical.*

A Proofs

Proof of Lemma 4.

Let (N,α) be a game with linear and identical α_i. The proof of item 1 is by induction. The statement is correct for $t=1$ by definition of marginal vectors. Now assume that the statement holds for some $t \geq 1$. Then

$$\begin{aligned}
\sum_{i=1}^{t+1} m^{\sigma}_{\sigma(i)}(\alpha) &= \sum_{i=1}^{t} m^{\sigma}_{\sigma(i)}(\alpha) + m^{\sigma}_{\sigma(t+1)}(\alpha) \\
&= \alpha_\star(N, S^{\sigma}_t, 1) + y^{\sigma}_{\sigma(t+1)}(\alpha)\alpha_\star(N, S^{\sigma}_{t+1}, 1) \\
&= \alpha_\star(N, S^{\sigma}_t, 1) + (1 - \sum_{k=1}^{t} \alpha_\star(S^{\sigma}_{t+1}, S^{\sigma}_k, y^{\sigma}_{\sigma(k)}))\alpha_\star(N, S^{\sigma}_{t+1}, 1) \\
&= \alpha_\star(N, S^{\sigma}_t, 1) + \alpha_\star(N, S^{\sigma}_{t+1}, 1) - \sum_{k=1}^{t} \alpha_\star(N, S^{\sigma}_k, y^{\sigma}_{\sigma(k)}) \\
&= \alpha_\star(N, S^{\sigma}_t, 1) + \alpha_\star(N, S^{\sigma}_{t+1}, 1) - \sum_{k=1}^{t} m^{\sigma}_{\sigma(k)}(\alpha) \\
&= \alpha_\star(N, S^{\sigma}_{t+1}, 1),
\end{aligned}$$

where induction is used in the first and last equality.

Finally,

$$\begin{aligned}
\text{undom}(S) &= \{p \in \mathbb{R}^N \mid \nexists q \in \Delta^*(S) : q_i\alpha_\star(N, S, 1) > p_i \text{ for all } i \in S\} \\
&= \{p \in \mathbb{R}^N \mid \sum_{i \in S} p_i \geq \alpha_\star(N, S, 1)\},
\end{aligned}$$

which shows item 2. □

Proof of Theorem 7.

Let $G = (N,\alpha)$ be a marginal convex game with linear α_i. Let $S, T \subseteq N$, $S \cap T = \emptyset$. If $|S| = |T| = 1$ then the superadditivity condition holds because of theorem 3.

Without loss of generality assume now that $|S| = 1$ and $|T| = 2$, that is, $S = \{i\}$ and $T = \{j,k\}$. Take $p \in I(S,\alpha)$, that is, $p_i = 1$. According to theorem 3 there exists $q \in I(T,\alpha)$. All elements in $I(T,\alpha)$ are linear combinations of the two extreme values $q^a = (\alpha_j(T,\{j\},1), 1-\alpha_j(T,\{j\},1))$ and $q^b = (1-\alpha_k(T,\{k\},1), \alpha_k(T,\{k\},1))$. Substitution of q^a in (2) gives

$$\begin{aligned}
&\alpha_i(N,\{i\},1) + \alpha_j(N,\{j\},1) + (1-\alpha_j(T,\{j\},1))\alpha_k(N,T,1) \\
&= \alpha_i(N,\{i\},1) + m^{(j,k,i)}_j(\alpha) + m^{(j,k,i)}_k(\alpha) \\
&= \alpha_i(N,\{i\},1) + 1 - m^{(j,k,i)}_i(\alpha) \\
&\leq 1.
\end{aligned}$$

The inequality follows from $m^{(j,k,i)} \in \text{undom}(\{i\})$. Similarly, q^b satisfies (2). Therefore, this inequality holds for all $q \in I(T,\alpha)$. We conclude that the game is superadditive. □

Proof of Theorem 9.

Let (N,α) be a marginal convex game with linear and identical α_i. Let $S, T \subseteq N$, $S \cap T = \emptyset$, and take the allocations $p \in I(S,\alpha)$ and $q \in I(T,\alpha)$. Then

$$\begin{aligned} \sum_{i \in S} \alpha_i(S \cup T, S, p_i) + \sum_{i \in T} \alpha_i(S \cup T, T, q_i) &= \sum_{i \in S} p_i \alpha_\star(S \cup T, S, 1) + \sum_{i \in T} q_i \alpha_\star(S \cup T, T, 1) \\ &= \alpha_\star(S \cup T, S, 1) + \alpha_\star(S \cup T, T, 1). \end{aligned}$$

The subgame $(S \cup T, \alpha)$ is also marginal convex [6, theorem 6]. Then $m^\sigma(S \cup T, \alpha) \in \text{undom}(S) \cap \text{undom}(T)$. Using item 2 in lemma 4, applied to the subgame $(S \cup T, \alpha)$,

$$\alpha_\star(S \cup T, S, 1) + \alpha_\star(S \cup T, T, 1) \leq \sum_{i \in S} m_i^\sigma(S \cup T, \alpha) + \sum_{i \in T} m_i^\sigma(S \cup T, \alpha) = 1.$$

We conclude that the game is superadditive. □

Proof of Theorem 11.

Let (N,α) be a marginal convex game with linear and identical α_i. Let $U \subseteq N$ and $S \subset T \subseteq N \backslash U$. Further, take $p \in I(S,\alpha)$, $q \in I(T,\alpha)$ and $r \in I(S \cup U, \alpha)$ such that $r_i \geq p_i \alpha_\star(S \cup U, S, 1)$ for all $i \in S$. Then

$$\sum_{i \in U} r_i = 1 - \sum_{i \in S} r_i \leq 1 - \sum_{i \in S} p_i \alpha_\star(S \cup U, S, 1) = 1 - \alpha_\star(S \cup U, S, 1)$$

and consequently

$$\begin{aligned} &\sum_{i \in U} r_i \alpha_\star(T \cup U, S \cup U, 1) \\ &\quad \leq \quad (1 - \alpha_\star(S \cup U, S, 1)) \alpha_\star(T \cup U, S \cup U, 1) \\ &\quad = \quad \alpha_\star(T \cup U, S \cup U, 1) - \alpha_\star(T \cup U, S, 1). \end{aligned} \tag{4}$$

The subgame $(T \cup U, \alpha)$ is also marginal convex [6, Theorem 6]. Let σ be such that the players in S go first, followed by the players in $T \setminus S$, then those in U and finally the remaining ones in $N \setminus (T \cup U)$. Then

$$\begin{aligned} \alpha_\star(T \cup U, S \cup U, 1) &\leq \sum_{i \in S \cup U} m_i^\sigma(T \cup U, \alpha) \\ &= \sum_{i \in S} m_i^\sigma(T \cup U, \alpha) + \sum_{i \in U} m_i^\sigma(T \cup U, \alpha) \\ &= \alpha_\star(T \cup U, S, 1) + \sum_{j=1}^{|T \cup U|} m_{\sigma(j)}^\sigma(T \cup U, \alpha) - \sum_{j=1}^{|T|} m_{\sigma(j)}^\sigma(T \cup U, \alpha) \\ &= \alpha_\star(T \cup U, S, 1) + \alpha_\star(T \cup U, T \cup U, 1) - \alpha_\star(T \cup U, T, 1) \\ &= \alpha_\star(T \cup U, S, 1) + 1 - \alpha_\star(T \cup U, T, 1), \end{aligned}$$

where lemma 4, applied to the subgame $(T \cup U, \alpha)$, is used. Combining this with equation (4) results in

$$\alpha_\star(T \cup U, T, 1) + \sum_{i \in U} r_i \alpha_\star(T \cup U, S \cup U, 1) \leq 1,$$

and so, (3) implies that the game is coalitional-merge convex. □

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