High-order accurate discontinuous Galerkin method for the indefinite time-harmonic Maxwell equations

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Abstract

We introduce a high-order accurate discontinuous Galerkin (DG) method for the indefinite frequency-domain Maxwell equations in three spatial dimensions. The novelty of the method lies in the way the numerical flux is computed. Instead of using the more popular local discontinuous Galerkin (LDG) or interior-penalty discontinuous Galerkin (IP-DG) numerical fluxes, we opt for a formulation which makes use of the local lifting operator. This allows us to choose a penalty parameter that is independent of the mesh size and the polynomial order. Moreover, we use a hierarchic construction of H(curl)-conforming basis functions, the first-order version of which correspond to the second family of Nédélec elements. We also provide a priori error bounds for our formulation, and carry out three-dimensional numerical experiments to validate the theoretical results.

Keywords: numerical flux with local lifting operator; IP-DG method; Maxwell equations; H(curl)-conforming vector elements.

1 Introduction

The difficulties of solving the Maxwell equations usually lie in the complexity of the geometry, the presence of material discontinuities and the fact that the curl operator has a large kernel. Moreover, the unknown fields in the Maxwell equations have special geometric characteristics. These are most pronounced in the three-dimensional version of the equations, and manifest themselves in the de Rham diagram; see e.g. [8, 21, 29]. However, many of the popular numerical discretisation techniques do not satisfy the de Rham diagram at the discrete level, and often contaminate the numerical solution by producing spurious modes. One notable exception are curl-conforming finite-element methods, which are special vector-valued polynomials that mimic the geometric properties of the electromagnetic fields at the discrete level. Based on the concept introduced by Whitney

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in the context of algebraic topology [39], they were proposed for the Maxwell system by Nédélec and Bossavit [7, 30, 31]. A hierarchic construction of high-order basis functions that satisfy the same properties are given in [1] for tetrahedral meshes and in [34] for more general three-dimensional meshes. The fact that these functions preserve the geometric properties of the Maxwell equations has motivated many to study the Maxwell system and its numerical discretisation in the framework of differential geometry [9, 21].

However, such elements suffer from a couple of practical hurdles. In particular, although they are capable of handling complex geometrical features and material discontinuities, implementation becomes increasingly difficult when high-order basis functions are used. Furthermore, extending the approach to non-conforming meshes–where the local polynomial order can vary between elements and hanging nodes can be present–poses considerable difficulties.

One attractive alternative is the discontinuous Galerkin (DG) finite element method. It can handle non-conforming meshes relatively easily and the implementation of high-order basis functions is also comparatively straightforward. Research in the field of DG methods has been very active in the past ten years or so; see the recent books [14] and [20] and references therein. In the context of the Maxwell equations, a nodal approach was developed in [18], and further studied in [19]. This approach had originally been based on Lax-Friedrichs type numerical fluxes, and was later applied to the local discontinuous Galerkin method [37]. In the meantime, various DG discretisations of the low-frequency Maxwell equations [23, 24] as well as the high-frequency Maxwell equations [22, 12, 11] have also been extensively studied. The question of spurious modes in DG discretisations has been addressed in [12, 37, 11] for conforming meshes and, more recently, in [13] for two-dimensional non-conforming meshes.

In this work, we investigate the time-harmonic Maxwell equations in a lossless medium with inhomogeneous boundary conditions, i.e. find the (scaled) electric field $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{x})$ that satisfies

$$\begin{aligned} \nabla \times \frac{1}{\mu_{\mathrm{r}}} \nabla \times \boldsymbol{E} - k^2 \varepsilon_{\mathrm{r}} \boldsymbol{E} = \boldsymbol{J} \quad & \text{in } \Omega, \\ \boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{g} \quad & \text{on } \Gamma, \end{aligned} \tag{1}$$

where Ω is an open bounded Lipschitz polyhedron on $\mathbb{R}^3$ with boundary $\Gamma = \partial\Omega$ and outward normal unit vector $\boldsymbol{n}$. The right-hand side $\boldsymbol{J}$ is the external source and k is the (real-valued) wave number with the assumption that k^2 is not a Maxwell eigenvalue. Throughout this article the (relative) permittivity and the (relative) permeability correspond to vacuum (or dry air). That is, we set $\varepsilon_{\mathrm{r}} = 1$ and $\mu_{\mathrm{r}} = 1$.

The most important new feature of the high-order DG method discussed here is the flux formulations we apply. In three-dimensional (or, indeed, in any dimensional) computations of the Maxwell equations the most widely used numerical fluxes are the Lax-Friedrichs flux, the LDG flux and the IP flux. See [20] for an overview. While we also study the computational performance of the IP-DG method, the focus of this article is on a numerical flux which makes use of the local lifting operator. This formulation was originally introduced in [10] and further analysed–together with a large number of other flux choices–in [4]. It has yielded promising results in the discretisation of the Laplace operator–most typically for applications in fluid dynamics [28].

We derive a priori error bounds for the discretisation we introduce. Our analysis proceeds along the lines of [22], and is therefore restricted to the case of smooth material coefficients. However, we believe that the analysis in [12], which covers discontinuous materials, can be extended to the DG method presented here. Our theoretical results demonstrate the main advantage of the formulation. Namely, that it allows us to use a stabilising parameter that is independent of both the mesh size and the polynomial order. This is especially important in three-dimensional computations, where there are still relatively few experiments available to help us tune a mesh-dependent parameter, such as that in the IP-DG method.

For our DG discretisation we use a hierarchic construction of $H(\text{curl})$-conforming basis functions [1, 34], which satisfy the global de Rham diagram in the continuous finite element setting. However, because of the discontinuous nature of the method discussed here, we cannot expect our discretisation to be globally curl-conforming and to satisfy the global de Rham diagram. Nevertheless, we believe that the use of $H(\text{curl})$-conforming basis function is beneficial, since it entails that the average across any face is also $H(\text{curl})$-conforming. Furthermore, the local lifting operator is approximated by the same local polynomial basis as the unknown field.

We implement the basis functions up to order five. In principle, it is possible to increase the order further, but implementation in three dimensions is hindered by a number of practical difficulties. First, high-order (i.e. $p > 9$) quadrature rules for tetrahedra are still sub-optimal and computationally expensive, making the assembly a lengthy procedure. Second, iterative solvers for indefinite linear systems are known to converge slowly. This property is exacerbated by the use of very high-order $H(\text{curl})$-conforming basis functions, as finding suitable preconditioners then becomes more of challenge.

The outline of this article is as follows. We define the tessellation and function spaces in Section 2, derive the DG discretisation for (1) in Section 3, and provide a priori error bounds in Section 4. The issue of preconditioning is very briefly addressed in Section 5. We verify and compare the numerical methods on both convex and concave domains in Section 6. Finally, in Section 7, we conclude and provide an outlook.

2 Tessellation and function spaces

We consider a tessellation $\mathcal{T}_h$ that partitions the polyhedral domain $\Omega \subset \mathbb{R}^3$ into a set of tetrahedra $\{K\}$. Throughout the article we assume that the mesh is shape-regular and that each tetrahedron is straight-sided. The notations $\mathcal{F}_h$, $\mathcal{F}_h^i$ and $\mathcal{F}_h^b$ stand respectively for the set of all faces $\{F\}$, the set of all internal faces, and the set of all boundary faces. For a bounded domain $D \subset \mathbb{R}^d$, $d = 2, 3$, we denote by $H^s(D)$ the standard Sobolev space of functions with regularity exponent $s \geq 0$ and norm $\|\cdot\|_{s,D}$. When $D = \Omega$, we write $\|\cdot\|_s$. On the computational domain Ω, we introduce the space

$$H(\text{curl}; \Omega) := \left\{ \boldsymbol{u} \in \left[L^2(\Omega)\right]^3 : \nabla \times \boldsymbol{u} \in \left[L^2(\Omega)\right]^3 \right\},$$

with the norm $\|\boldsymbol{u}\|_{\text{curl}}^2 = \|\boldsymbol{u}\|_0^2 + \|\nabla \times \boldsymbol{u}\|_0^2$. Let $H_0(\text{curl}; \Omega)$ denote the subspace of $H(\text{curl}; \Omega)$ of functions with zero tangential trace. We will also use the notation $(\cdot, \cdot)_D$ for the standard inner product in $[L^2(D)]^3$,

$$(\boldsymbol{u}, \boldsymbol{v})_D = \int_D \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}V,$$

and the operator ∇_h for the elementwise application of $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)^T$.

We now introduce the finite element space associated with the tessellation $\mathcal{T}_h$. Let $\mathcal{P}_p(K)$ be the space of polynomials of degree at most $p \geq 1$ on $K \in \mathcal{T}_h$. Over each element K the $H(\mathrm{curl})$-conforming polynomial space is defined as

$$Q^p = \left\{ \boldsymbol{u} \in [\mathcal{P}_p(K)]^3 \,;\; \boldsymbol{u}_T|_{s_i} \in [\mathcal{P}_p(s_i)]^2 \,;\; \boldsymbol{u} \cdot \boldsymbol{\tau}_j|_{e_j} \in \mathcal{P}_p(e_j) \right\}, \tag{2}$$

where s_i, $i = 1, 2, 3, 4$ are the faces of the element; e_j, $j = 1, 2, 3, 4, 5, 6$ are the edges of the element; $\boldsymbol{u}_T$ is the tangential component of $\boldsymbol{u}$; and $\boldsymbol{\tau}_j$ is the directed tangential vector on edge e_j. We define the space Σ_h^p as

$$\Sigma_h^p := \left\{ \boldsymbol{\sigma} \in [L^2(\Omega)]^3 \;\middle|\; \boldsymbol{\sigma}|_K \in Q^p, \, \forall K \in \mathcal{T}_h \right\}.$$

Consider an interface $F \in \mathcal{F}_h$ between element K^L and element K^R, and let $\boldsymbol{n}^L$ and $\boldsymbol{n}^R$ represent their respective outward pointing normal vectors. We define the tangential jump and the average of the quantity $\boldsymbol{u}$ across interface F as

$$[\![\boldsymbol{u}]\!]_T = \boldsymbol{n}^L \times \boldsymbol{u}^L + \boldsymbol{n}^R \times \boldsymbol{u}^R \quad \text{and} \quad \{\!\{\boldsymbol{u}\}\!\} = \left(\boldsymbol{u}^L + \boldsymbol{u}^R\right)/2,$$

respectively. Here $\boldsymbol{u}^L$ and $\boldsymbol{u}^R$ are the values of the trace of $\boldsymbol{u}$ at ∂K^L and ∂K^R, respectively. At the boundary Γ, we set $\{\!\{\boldsymbol{u}\}\!\} = \boldsymbol{u}$ and $[\![\boldsymbol{u}]\!]_T = \boldsymbol{n} \times \boldsymbol{u}$. In case we only need the average of the tangential components, we use the notation $\{\!\{\boldsymbol{u}\}\!\}_T$.

For the analysis in Section 4, we also define the DG norm

$$\|\boldsymbol{v}\|_{\mathrm{DG}} = (\|\boldsymbol{v}\|_0^2 + \|\nabla_h \times \boldsymbol{v}\|_0^2 + \|\mathtt{h}^{-\frac{1}{2}} \, [\![\boldsymbol{v}]\!]_T \, \|_{0,\mathcal{F}_h}^2)^{\frac{1}{2}},$$

where $\|\cdot\|_{0,\mathcal{F}_h}$ denotes the the $L^2(\mathcal{F})$ norm, and $\mathtt{h}(\boldsymbol{x}) = h_F$, which is the diameter of face F containing $\boldsymbol{x}$. Similarly, h_K denotes the diameter of element K. Note that the shape-regular property of the mesh implies that there is a positive constant C_d independent of the mesh size such that for all faces F and the associated elements K^R and K^L we have

$$h_F \leq C_d \min\{h_{K^L}, h_{K^R}\}. \tag{3}$$

To derive the DG formulations (in the next section) we first need to introduce global lifting operators for $\boldsymbol{u} \in \Sigma_h^p$. The global lifting operator $\mathcal{L} : [L^2(\mathcal{F}_h^i)]^3 \to \Sigma_h^p$ is defined as

$$(\mathcal{L}(\boldsymbol{u}), \boldsymbol{v})_\Omega = \int_{\mathcal{F}_h^i} \boldsymbol{u} \cdot [\![\boldsymbol{v}]\!]_T \, \mathrm{d}A, \quad \forall \boldsymbol{v} \in \Sigma_h^p, \tag{4}$$

and the global lifting operator $\mathcal{R}(\boldsymbol{u}) : [L^2(\mathcal{F}_h)]^3 \to \Sigma_h^p$ as

$$(\mathcal{R}(\boldsymbol{u}), \boldsymbol{v})_\Omega = \int_{\mathcal{F}_h} \boldsymbol{u} \cdot \{\!\{\boldsymbol{v}\}\!\} \, \mathrm{d}A, \quad \forall \boldsymbol{v} \in \Sigma_h^p. \tag{5}$$

For a given face $F \in \mathcal{F}_h$, we will also need the local lifting operator $\mathcal{R}_F(\boldsymbol{u}) : [L^2(F)]^3 \rightarrow \Sigma_h^p$, defined as

$$(\mathcal{R}_F(\boldsymbol{u}), \boldsymbol{v})_\Omega = \int_F \boldsymbol{u} \cdot \{\!\{\boldsymbol{v}\}\!\} \,\mathrm{d}A, \quad \forall \boldsymbol{v} \in \Sigma_h^p. \tag{6}$$

Note that $\mathcal{R}_F(\boldsymbol{u})$ vanishes outside the elements connected to the face F so that for a given element $K \in \mathcal{T}_h$ we have the relation

$$\mathcal{R}(\boldsymbol{u}) = \sum_{F \in \mathcal{F}_h} \mathcal{R}_F(\boldsymbol{u}), \quad \forall \boldsymbol{u} \in \left[L^2(\mathcal{F}_h)\right]^3. \tag{7}$$

3 Discontinuous Galerkin discretisation

We now derive the DG formulation for (1). We first provide a general bilinear form where the choice of the numerical flux is not yet specified. Then we consider two different definitions of the numerical flux, each of which results in a symmetric algebraic system.

3.1 Derivation of the bilinear form

The derivation follows the same lines as the one in [35] for the Laplace operator. However, this time it is carried out for the curl-curl operator. We also refer to [4] for a unified analysis on DG methods for elliptic problems.

We first introduce the auxiliary variable $\boldsymbol{q} \in [L^2(\Omega)]^3$ so that, instead of (1), we can consider the first-order system

$$\begin{aligned} &\nabla \times \boldsymbol{q} - k^2 \boldsymbol{E} = \boldsymbol{J} \quad \text{in } \Omega, \\ &\boldsymbol{q} = \nabla \times \boldsymbol{E} \quad \text{in } \Omega, \\ &\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{g} \quad \text{on } \Gamma. \end{aligned} \tag{8}$$

From here we follow the standard DG approach (given, for example, in [4] or [35] for elliptic operators): *a)* integrate both equations in (8) by parts; *b)* in the element boundary integrals substitute the numerical fluxes $\boldsymbol{q}_h^*$ and $\boldsymbol{E}_h^*$ for their original counterparts; *c)* and finally integrate again the second equation in (8) by parts. Then we seek the pair $(\boldsymbol{E}_h, \boldsymbol{q}_h)$ such that for all test functions $(\boldsymbol{\phi}, \boldsymbol{\pi}) \in \boldsymbol{\Sigma}_h^p \times \boldsymbol{\Sigma}_h^p$:

$$(\boldsymbol{q}_h, \nabla_h \times \boldsymbol{\phi})_\Omega - k^2 (\boldsymbol{E}_h, \boldsymbol{\phi})_\Omega + \sum_{K \in \mathcal{T}_h} (\boldsymbol{n} \times \boldsymbol{q}_h^*, \boldsymbol{\phi})_{\partial K} = (\boldsymbol{J}, \boldsymbol{\phi})_\Omega, \tag{9}$$

$$(\boldsymbol{q}_h, \boldsymbol{\pi})_\Omega = (\nabla_h \times \boldsymbol{E}_h, \boldsymbol{\pi})_\Omega + \sum_{K \in \mathcal{T}_h} (\boldsymbol{n} \times (\boldsymbol{E}_h^* - \boldsymbol{E}_h), \boldsymbol{\pi})_{\partial K}. \tag{10}$$

Before we proceed, we make use of the following result: for any given $\boldsymbol{u}, \boldsymbol{v} \in \Sigma_h^p$, the identity

$$\sum_{K \in \mathcal{T}_h} (\boldsymbol{n} \times \boldsymbol{u}, \boldsymbol{v})_{\partial K} = - \int_{\mathcal{F}_h^i} \{\!\{\boldsymbol{u}\}\!\} \cdot [\![\boldsymbol{v}]\!]_T \,\mathrm{d}A + \int_{\mathcal{F}_h^i} \{\!\{\boldsymbol{v}\}\!\} \cdot [\![\boldsymbol{u}]\!]_T \,\mathrm{d}A + \int_{\mathcal{F}_h^b} (\boldsymbol{n} \times \boldsymbol{u}) \cdot \boldsymbol{v} \,\mathrm{d}A \tag{11}$$

holds. Combine this with (9) and (10) to obtain

$$(\boldsymbol{q}_h, \nabla_h \times \boldsymbol{\phi})_\Omega - k^2 (\boldsymbol{E}_h, \boldsymbol{\phi})_\Omega - \int_{\mathcal{F}_h^i} \{\!\{\boldsymbol{q}_h^*\}\!\} \cdot [\![\boldsymbol{\phi}]\!]_T \, \mathrm{d}A + \int_{\mathcal{F}_h^i} \{\!\{\boldsymbol{\phi}\}\!\} \cdot [\![\boldsymbol{q}_h^*]\!]_T \, \mathrm{d}A + \int_{\mathcal{F}_h^b} (\boldsymbol{n} \times \boldsymbol{q}_h^*) \cdot \boldsymbol{\phi} \, \mathrm{d}A = (\boldsymbol{J}, \boldsymbol{\phi})_\Omega \quad (12)$$

and

$$(\boldsymbol{q}_h, \boldsymbol{\pi})_\Omega = (\nabla_h \times \boldsymbol{E}_h, \boldsymbol{\pi})_\Omega - \int_{\mathcal{F}_h^i} \{\!\{\boldsymbol{E}_h^* - \boldsymbol{E}_h\}\!\} \cdot [\![\boldsymbol{\pi}]\!]_T \, \mathrm{d}A + \int_{\mathcal{F}_h^i} \{\!\{\boldsymbol{\pi}\}\!\} \cdot [\![\boldsymbol{E}_h^* - \boldsymbol{E}_h]\!]_T \, \mathrm{d}A + \int_{\mathcal{F}_h^b} (\boldsymbol{n} \times (\boldsymbol{E}_h^* - \boldsymbol{E}_h)) \cdot \boldsymbol{\pi} \, \mathrm{d}A. \quad (13)$$

We can use the lifting operators to express–and thus eliminate–the auxiliary variable $\boldsymbol{q}_h$ as a function of $\boldsymbol{E}_h$. From (13) and from the definition of the lifting operators (4) and (5), it follows that

$$\boldsymbol{q}_h = \nabla_h \times \boldsymbol{E}_h - \mathcal{L}(\{\!\{\boldsymbol{E}_h^* - \boldsymbol{E}_h\}\!\}) + \mathcal{R}([\![\boldsymbol{E}_h^* - \boldsymbol{E}_h]\!]_T). \quad (14)$$

Here we have also used the boundary definition of $[\![\cdot]\!]_T$. Substituting (14) into (12) and applying (10) result in the weak form

$$\begin{aligned}
\mathcal{B}(\boldsymbol{E}_h, \boldsymbol{\phi}) &:= (\nabla_h \times \boldsymbol{E}_h, \nabla_h \times \boldsymbol{\phi})_\Omega - k^2 (\boldsymbol{E}_h, \boldsymbol{\phi})_\Omega \\
&- \int_{\mathcal{F}_h^i} \{\!\{\boldsymbol{E}_h^* - \boldsymbol{E}_h\}\!\} \cdot [\![\nabla_h \times \boldsymbol{\phi}]\!]_T \, \mathrm{d}A + \int_{\mathcal{F}_h^i} [\![\boldsymbol{E}_h^* - \boldsymbol{E}_h]\!]_T \cdot \{\!\{\nabla_h \times \boldsymbol{\phi}\}\!\} \, \mathrm{d}A \\
&- \int_{\mathcal{F}_h^i} \{\!\{\boldsymbol{q}_h^*\}\!\} \cdot [\![\boldsymbol{\phi}]\!]_T \, \mathrm{d}A + \int_{\mathcal{F}_h^i} [\![\boldsymbol{q}_h^*]\!]_T \cdot \{\!\{\boldsymbol{\phi}\}\!\} \, \mathrm{d}A \\
&+ \int_{\mathcal{F}_h^b} (\boldsymbol{n} \times (\boldsymbol{E}_h^* - \boldsymbol{E}_h)) \cdot (\nabla_h \times \boldsymbol{\phi}) \, \mathrm{d}A - \int_{\mathcal{F}_h^b} \boldsymbol{q}_h^* \cdot (\boldsymbol{n} \times \boldsymbol{\phi}) \, \mathrm{d}A = (\boldsymbol{J}, \boldsymbol{\phi})_\Omega \, . \quad (15)
\end{aligned}$$

This is the general primal formulation where one still has freedom to make the choices about the numerical fluxes $\boldsymbol{E}_h^*$ and $\boldsymbol{q}_h^*$ that are most suitable for the problem. An overview of different fluxes for the Poisson equation is given in [4].

3.2 Numerical fluxes

At this point, we specify the numerical fluxes $\boldsymbol{E}_h^*$ and $\boldsymbol{q}_h^*$ in (15). We investigate two different formulations, one of which results in the IP-DG formulation that was thoroughly analysed in [22]. The other is similar to the stabilised central flux, except that in the stabilisation term we use the local lifting operator (6). Note that in both cases the numerical fluxes are consistent, i.e. $\forall \boldsymbol{E}, \boldsymbol{q} \in H(\mathrm{curl}, \Omega)$ the relations $\{\!\{\boldsymbol{E}\}\!\}_T = \boldsymbol{n} \times \boldsymbol{E}$, $\{\!\{\boldsymbol{q}\}\!\}_T = \boldsymbol{n} \times \boldsymbol{q}_h$, $[\![\boldsymbol{E}]\!]_T = \boldsymbol{0}$ and $[\![\boldsymbol{q}]\!]_T = \boldsymbol{0}$ hold. The consistency of the DG formulation with the numerical flux of Brezzi et al. [10] is discussed in Appendix A.

3.2.1 Interior-penalty flux

First, we define the numerical fluxes so that they correspond to the IP flux,

$$\begin{aligned} &\boldsymbol{E}_h^* = \{\!\{\boldsymbol{E}_h\}\!\}, \quad \boldsymbol{q}_h^* = \{\!\{\nabla_h \times \boldsymbol{E}_h\}\!\} - \tau \, [\![\boldsymbol{E}_h]\!]_T, \quad \text{if} \quad F \in \mathcal{F}_h^i, \\ &\boldsymbol{n} \times \boldsymbol{E}_h^* = \boldsymbol{g}, \quad \boldsymbol{q}_h^* = \nabla_h \times \boldsymbol{E}_h - \tau \, (\boldsymbol{n} \times \boldsymbol{E}_h) + \tau \boldsymbol{g}, \quad \text{if} \quad F \in \mathcal{F}_h^b, \end{aligned} \tag{16}$$

with τ being the penalty parameter. We can now transform the following face integrals as

$$\begin{aligned} &\int_{\mathcal{F}_h^i} [\![\boldsymbol{E}_h^* - \boldsymbol{E}_h]\!]_T \cdot \{\!\{\nabla_h \times \boldsymbol{\phi}\}\!\} \, \mathrm{d}A = -\int_{\mathcal{F}_h^i} [\![\boldsymbol{E}_h]\!]_T \cdot \{\!\{\nabla_h \times \boldsymbol{\phi}\}\!\} \, \mathrm{d}A, \\ &\int_{\mathcal{F}_h^b} (\boldsymbol{n} \times (\boldsymbol{E}_h^* - \boldsymbol{E}_h)) \cdot (\nabla_h \times \boldsymbol{\phi}) \, \mathrm{d}A = \int_{\mathcal{F}_h^b} (\boldsymbol{g} - \boldsymbol{n} \times \boldsymbol{E}_h) \cdot (\nabla_h \times \boldsymbol{\phi}) \, \mathrm{d}A, \\ &\int_{\mathcal{F}_h^i} \{\!\{\boldsymbol{q}_h^*\}\!\} \cdot [\![\boldsymbol{\phi}]\!]_T \, \mathrm{d}A = \int_{\mathcal{F}_h^i} \{\!\{\nabla_h \times \boldsymbol{E}_h\}\!\} \cdot [\![\boldsymbol{\phi}]\!]_T \, \mathrm{d}A - \int_{\mathcal{F}_h^i} \tau \, [\![\boldsymbol{E}_h]\!]_T \cdot [\![\boldsymbol{\phi}]\!]_T \, \mathrm{d}A, \\ &\int_{\mathcal{F}_h^b} (\boldsymbol{n} \times \boldsymbol{q}_h^*) \cdot \boldsymbol{\phi} \, \mathrm{d}A = -\int_{\mathcal{F}_h^b} (\nabla_h \times \boldsymbol{E}_h) \cdot (\boldsymbol{n} \times \boldsymbol{\phi}) \, \mathrm{d}A \\ &\qquad\qquad + \int_{\mathcal{F}_h^b} \tau \, (\boldsymbol{n} \times \boldsymbol{E}_h) \cdot (\boldsymbol{n} \times \boldsymbol{\phi}) \, \mathrm{d}A - \int_{\mathcal{F}_h^b} \tau \boldsymbol{g} \cdot (\boldsymbol{n} \times \boldsymbol{\phi}) \, \mathrm{d}A, \end{aligned}$$

while the other face integrals are zero. If we plug these back to (15), we have the IP-DG method for the time-harmonic Maxwell equations,

$$\begin{aligned} &\mathcal{B}_h^{ip}(\boldsymbol{E}_h, \boldsymbol{\phi}) := (\nabla_h \times \boldsymbol{E}_h, \nabla_h \times \boldsymbol{\phi})_\Omega - k^2 \, (\boldsymbol{E}_h, \boldsymbol{\phi})_\Omega - \int_{\mathcal{F}_h} [\![\boldsymbol{E}_h]\!]_T \cdot \{\!\{\nabla_h \times \boldsymbol{\phi}\}\!\} \, \mathrm{d}A \\ &- \int_{\mathcal{F}_h} \{\!\{\nabla_h \times \boldsymbol{E}_h\}\!\} \cdot [\![\boldsymbol{\phi}]\!]_T \, \mathrm{d}A + \int_{\mathcal{F}_h} \tau \, [\![\boldsymbol{E}]\!]_T \cdot [\![\boldsymbol{\phi}]\!]_T \, \mathrm{d}A \\ &\qquad = (\boldsymbol{J}, \boldsymbol{\phi})_\Omega - \int_{\mathcal{F}_h^b} \boldsymbol{g} \cdot (\nabla_h \times \boldsymbol{\phi}) \, \mathrm{d}A + \int_{\mathcal{F}_h^b} \tau \boldsymbol{g} \cdot (\boldsymbol{n} \times \boldsymbol{\phi}) \, \mathrm{d}A. \end{aligned} \tag{17}$$

Note that in the left-hand side we no longer distinguish explicitly between internal and boundary faces. This is permissible thanks to the definitions of the average and the tangential jump at the boundary.

3.2.2 Numerical flux of Brezzi et al.

As a next step, we define the numerical fluxes in the manner of Brezzi et al. [10]:

$$\begin{aligned} &\boldsymbol{E}_h^* = \{\!\{\boldsymbol{E}_h\}\!\}, \quad \boldsymbol{q}_h^* = \{\!\{\boldsymbol{q}_h\}\!\} - \alpha_{\mathcal{R}}([\![\boldsymbol{E}_h]\!]_T), \quad \text{if} \quad F \in \mathcal{F}_h^i, \\ &\boldsymbol{n} \times \boldsymbol{E}_h^* = \boldsymbol{g}, \quad \boldsymbol{q}_h^* = \boldsymbol{q}_h - \alpha_{\mathcal{R}}(\boldsymbol{n} \times \boldsymbol{E}_h) + \alpha_{\mathcal{R}}(\boldsymbol{g}), \quad \text{if} \quad F \in \mathcal{F}_h^b. \end{aligned} \tag{18}$$

where $\alpha_{\mathcal{R}}(\boldsymbol{u}) = \eta_F \left\{\!\!\left\{ \mathcal{R}_F(\boldsymbol{u}_h) \right\}\!\!\right\}$ for $F \in \mathcal{F}_h$ and $\eta_F \in \mathbb{R}^+$. Following the same line of argument as before and using (14), the bilinear form (15) now transforms as

$$
\begin{aligned}
&\mathcal{B}_h^{br}(\boldsymbol{E}_h, \boldsymbol{\phi}) := (\nabla_h \times \boldsymbol{E}_h, \nabla_h \times \boldsymbol{\phi})_\Omega - k^2 (\boldsymbol{E}_h, \boldsymbol{\phi})_\Omega \\
&- \int_{\mathcal{F}_h} [\![\boldsymbol{E}_h]\!]_T \cdot \{\!\{\nabla_h \times \boldsymbol{\phi}\}\!\} \, \mathrm{d}A - \int_{\mathcal{F}_h} \{\!\{\nabla_h \times \boldsymbol{E}_h\}\!\} \cdot [\![\boldsymbol{\phi}]\!]_T \, \mathrm{d}A \\
&- \int_{\mathcal{F}_h} \{\!\{\mathcal{R}([\![\boldsymbol{E}_h^* - \boldsymbol{E}_h]\!]_T)\}\!\} \cdot [\![\boldsymbol{\phi}]\!]_T \, \mathrm{d}A + \sum_{F \in \mathcal{F}_h} \int_F \eta_F \{\!\{\mathcal{R}_F([\![\boldsymbol{E}_h]\!]_T)\}\!\} \cdot [\![\boldsymbol{\phi}]\!]_T \, \mathrm{d}A \\
&\qquad + \int_{\mathcal{F}_h^b} \boldsymbol{g} \cdot (\nabla_h \times \boldsymbol{\phi}) \, \mathrm{d}A - \sum_{F \in \mathcal{F}_h^b} \int_F \eta_F \mathcal{R}_F(\boldsymbol{g}) \cdot (\boldsymbol{n} \times \boldsymbol{\phi}) \, \mathrm{d}A. \qquad (19)
\end{aligned}
$$

We can now use the relation

$$
\begin{aligned}
&\int_{\mathcal{F}_h} \{\!\{\mathcal{R}([\![\boldsymbol{E}_h^* - \boldsymbol{E}_h]\!]_T)\}\!\} \cdot [\![\boldsymbol{\phi}]\!]_T \, \mathrm{d}A = (\mathcal{R}([\![\boldsymbol{E}_h^* - \boldsymbol{E}_h]\!]_T), \mathcal{R}([\![\boldsymbol{\phi}]\!]_T))_\Omega \\
&\approx n_f \sum_{F \in \mathcal{F}_h} (\mathcal{R}_F([\![\boldsymbol{E}_h^* - \boldsymbol{E}_h]\!]_T), \mathcal{R}_F([\![\boldsymbol{\phi}]\!]_T))_\Omega \\
&= -n_f \sum_{F \in \mathcal{F}_h^i} (\mathcal{R}_F([\![\boldsymbol{E}_h]\!]_T), \mathcal{R}_F([\![\boldsymbol{\phi}]\!]_T))_\Omega + n_f \sum_{F \in \mathcal{F}_h^b} (\mathcal{R}_F([\![\boldsymbol{g} - \boldsymbol{E}_h]\!]_T), \mathcal{R}_F([\![\boldsymbol{\phi}]\!]_T))_\Omega \\
&\qquad = -n_f \sum_{F \in \mathcal{F}_h} (\mathcal{R}_F([\![\boldsymbol{E}_h]\!]_T), \mathcal{R}_F([\![\boldsymbol{\phi}]\!]_T))_\Omega + n_f \sum_{F \in \mathcal{F}_h^b} (\mathcal{R}_F([\![\boldsymbol{g}]\!]_T), \mathcal{R}_F([\![\boldsymbol{\phi}]\!]_T))_\Omega \, ,
\end{aligned}
$$

where n_f is the number of faces of an element.

Let us introduce the bilinear form $\mathcal{B}_h^{br} : \Sigma_h^p \times \Sigma_h^p \to \mathbb{R}$ and the linear form $\mathcal{J}_h^{br} : \Sigma_h^p \to \mathbb{R}$ as

$$
\begin{aligned}
\mathcal{B}_h^{br}(\boldsymbol{E}_h, \boldsymbol{\phi}) &= (\nabla_h \times \boldsymbol{E}_h, \nabla_h \times \boldsymbol{\phi})_\Omega - k^2 (\boldsymbol{E}_h, \boldsymbol{\phi})_\Omega - \int_{\mathcal{F}_h} [\![\boldsymbol{E}_h]\!]_T \cdot \{\!\{\nabla_h \times \boldsymbol{\phi}\}\!\} \, \mathrm{d}A \\
&- \int_{\mathcal{F}_h} \{\!\{\nabla_h \times \boldsymbol{E}_h\}\!\} \cdot [\![\boldsymbol{\phi}]\!]_T \, \mathrm{d}A + \sum_{F \in \mathcal{F}_h} (\eta_F + n_f) (\mathcal{R}_F([\![\boldsymbol{E}]\!]_T), \mathcal{R}_F([\![\boldsymbol{\phi}]\!]_T))_\Omega \, , \qquad (20)
\end{aligned}
$$

and

$$
\mathcal{J}_h^{br}(\boldsymbol{\phi}) = (\boldsymbol{J}, \boldsymbol{\phi})_\Omega - \int_{\mathcal{F}_h^b} \boldsymbol{g} \cdot (\nabla_h \times \boldsymbol{\phi}) \, \mathrm{d}A + \sum_{F \in \mathcal{F}_h^b} (\eta_F + n_f) (\mathcal{R}_F(\boldsymbol{g}), \mathcal{R}_F(\boldsymbol{n} \times \boldsymbol{\phi}))_\Omega \, , \qquad (21)
$$

respectively, then the discrete formulation for the time-harmonic Maxwell equations can be written as follows. Find $\boldsymbol{E}_h \in \Sigma_h^p$ such that for all $\boldsymbol{\phi} \in \Sigma_h^p$ the relation

$$
\mathcal{B}_h^{br}(\boldsymbol{E}_h, \boldsymbol{\phi}) = \mathcal{J}_h^{br}(\boldsymbol{\phi}) \qquad (22)
$$

is satisfied.

4 A priori error bounds

Following the lines of [22] we prove that the discontinuous Galerkin discretisation in Section 3 results in an optimal convergence rate of the numerical solution with respect to the classical DG norm. For simplicity, we restrict the analysis to homogeneous boundary conditions.

A key ingredient in the error estimation formula is the following Gårding inequality.

Lemma 1 *There exists a constant η, independent of the discretisation parameter $h = \max_{K\in\mathcal{T}_h} \operatorname{diam} K$ and the wave number k, such that for all $\boldsymbol{v} \in \Sigma_h^p$ we have the following inequality*

$$\mathcal{B}_h^{br}(\boldsymbol{v},\boldsymbol{v}) \geq \beta^2 \|\boldsymbol{v}\|_{\mathrm{DG}}^2 - (k^2+\beta^2)\|\boldsymbol{v}\|_0^2, \tag{23}$$

with $\beta^2 = \min\{\frac{1}{2}, \frac{1}{\tilde{C}_{\mathrm{inv}}}\}$ and $\tilde{C}_{\mathrm{inv}}$ determined by inequality (30).

Proof: The right hand side of (23) can be rewritten as

$$\beta^2(\|\nabla_h \times \boldsymbol{v}\|_0^2 + \|\mathtt{h}^{-\frac{1}{2}} [\![\boldsymbol{v}]\!]_T \|_{0,\mathcal{F}_h}^2) - k^2\|\boldsymbol{v}\|_0^2.$$

Therefore, using (20) we have to prove that

$$\begin{aligned} &\|\nabla_h \times \boldsymbol{v}\|_0^2 - 2\int_{\mathcal{F}_h} [\![\boldsymbol{v}]\!]_T \cdot \{\!\{\nabla_h \times \boldsymbol{v}\}\!\}\ \mathrm{d}A + \sum_{F\in\mathcal{F}_h} (n_f + \eta_F)\|\mathcal{R}_F([\![\boldsymbol{v}]\!]_T)\|_0^2 \\ &\geq \beta^2(\|\nabla_h \times \boldsymbol{v}\|_0^2 + \|\mathtt{h}^{-\frac{1}{2}} [\![\boldsymbol{v}]\!]_T \|_{0,\mathcal{F}_h}^2). \end{aligned} \tag{24}$$

The second term on the left hand side can be estimated with any positive C_1 as

$$\begin{aligned} 4\int_{\mathcal{F}_h} [\![\boldsymbol{v}]\!]_T \cdot \{\!\{\nabla_h \times \boldsymbol{v}\}\!\}\ \mathrm{d}A &= \int_{\mathcal{F}_h} 2\cdot 2\mathtt{h}^{-\frac{1}{2}} C_1^{-1} [\![\boldsymbol{v}]\!]_T \cdot \mathtt{h}^{\frac{1}{2}} C_1 \{\!\{\nabla_h \times \boldsymbol{v}\}\!\}\ \mathrm{d}A \\ &\leq \int_{\mathcal{F}_h} 4\mathtt{h}^{-1}C_1^{-2}|\, [\![\boldsymbol{v}]\!]_T\,|^2 + \mathtt{h}C_1^2|\,\{\!\{\nabla_h \times \boldsymbol{v}\}\!\}\,|^2\ \mathrm{d}A \\ &\leq 4C_1^{-2}\|\mathtt{h}^{-\frac{1}{2}} [\![\boldsymbol{v}]\!]_T \|_{0,\mathcal{F}_h}^2 + C_1^2\sum_{F\in\mathcal{F}_h} h_F\|\,\{\!\{\nabla_h \times \boldsymbol{v}\}\!\}\,\|_{0,F}^2. \end{aligned} \tag{25}$$

Next, we use in Σ_h^p the inverse inequality

$$\|\boldsymbol{w}\|_{0,\partial K}^2 \leq C_{\mathrm{inv}}(h_K)^{-1}\|\boldsymbol{w}\|_{0,K}^2, \tag{26}$$

which is a straightforward consequence of the local inverse inequality (Lemma 1.38) and the trace theorem (B.52) in [14]. The constant C_{inv} does not depend on the mesh size, as we have a shape-regular mesh. Applying (26) in the second term on the right hand side of (25) for an arbitrary face F associated with elements K^L and K^R and using (3) we obtain

$$\begin{aligned} h_F C_1^2\|\,\{\!\{\nabla_h \times \boldsymbol{v}\}\!\}\,\|_{0,F}^2 &\leq \frac{1}{2}h_F C_1^2(\|\nabla_h \times \boldsymbol{v}^L\|_{0,F}^2 + \|\nabla_h \times \boldsymbol{v}^R\|_{0,F}^2) \\ &\leq \frac{1}{2}C_d\min\{h_{K^L}, h_{K^R}\}C_1^2(\|\nabla_h \times \boldsymbol{v}^L\|_{0,F}^2 + \|\nabla_h \times \boldsymbol{v}^R\|_{0,F}^2) \\ &\leq \frac{1}{2}C_1^2 C_{\mathrm{inv}}C_d(\|\nabla_h \times \boldsymbol{v}\|_{0,K^L}^2 + \|\nabla_h \times \boldsymbol{v}\|_{0,K^R}^2). \end{aligned} \tag{27}$$

A summation of these inequalities over all faces gives that

$$\sum_{F\in\mathcal{F}_h} h_F C_1^2 \| \{\!\{\nabla_h \times \boldsymbol{v}\}\!\} \|_{0,F}^2 \leq \|\nabla_h \times \boldsymbol{v}\|_0^2, \tag{28}$$

with the choice

$$C_1^2 = \frac{2}{C_{\text{inv}} C_d n_f}. \tag{29}$$

The estimate preceeding (3.52) in [14] gives that for all $\boldsymbol{v} \in \Sigma_h^p$

$$\bar{C}_{\text{inv}} \sum_{F\in\mathcal{F}_h} \|\mathcal{R}_F([\![\boldsymbol{v}]\!]_T)\|_0^2 \leq \|\mathtt{h}^{-\frac{1}{2}} [\![\boldsymbol{v}]\!]_T \|_{0,\mathcal{F}_h}^2 \leq \tilde{C}_{\text{inv}} \sum_{F\in\mathcal{F}_h} \|\mathcal{R}_F([\![\boldsymbol{v}]\!]_T)\|_0^2 \tag{30}$$

with the constants $\bar{C}_{\text{inv}}$ and $\tilde{C}_{\text{inv}}$ independent of h. Therefore, using (29) we obtain

$$\begin{aligned} C_1^{-2} \|\mathtt{h}^{-\frac{1}{2}} [\![\boldsymbol{v}]\!]_T \|_{0,\mathcal{F}_h}^2 &= \frac{n_f C_d C_{\text{inv}}}{2} \|\mathtt{h}^{-\frac{1}{2}} [\![\boldsymbol{v}]\!]_T \|_{0,\mathcal{F}_h}^2 \\ &\leq \frac{n_f C_d C_{\text{inv}}}{2} \tilde{C}_{\text{inv}} \sum_{F\in\mathcal{F}_h} \|\mathcal{R}_F([\![\boldsymbol{v}]\!]_T)\|_0^2. \end{aligned} \tag{31}$$

Introducing the sum of (28) and (31) in (25) then gives

$$2\int_{\mathcal{F}_h} [\![\boldsymbol{v}]\!]_T \cdot \{\!\{\nabla_h \times \boldsymbol{v}\}\!\} \, \mathrm{d}A \leq \frac{1}{2}\|\nabla_h \times \boldsymbol{v}\|_0^2 + n_f C_d C_{\text{inv}} \tilde{C}_{\text{inv}} \sum_{F\in\mathcal{F}_h} \|\mathcal{R}_F([\![\boldsymbol{v}]\!]_T)\|_0^2, \tag{32}$$

and, using (30) we obtain

$$\begin{aligned} &\|\nabla_h \times \boldsymbol{v}\|_0^2 - 2\int_{\mathcal{F}_h} [\![\boldsymbol{v}]\!]_T \cdot \{\!\{\nabla_h \times \boldsymbol{v}\}\!\} \, \mathrm{d}A + \sum_{F\in\mathcal{F}_h} (n_f + \eta_F) \|\mathcal{R}_F([\![\boldsymbol{v}]\!]_T)\|_0^2 \\ &\geq \frac{1}{2}\|\nabla_h \times \boldsymbol{v}\|_0^2 + \sum_{F\in\mathcal{F}_h} (n_f + \eta_F - n_f C_d C_{\text{inv}} \tilde{C}_{\text{inv}}) \|\mathcal{R}_F([\![\boldsymbol{v}]\!]_T)\|_0^2 \\ &\geq \min_{F\in\mathcal{F}_h} \{\frac{1}{2}, n_f + \eta_F - n_f C_d C_{\text{inv}} \tilde{C}_{\text{inv}}\} (\|\nabla_h \times \boldsymbol{v}\|_0^2 + \sum_{F\in\mathcal{F}_h} \|\mathcal{R}_F([\![\boldsymbol{v}]\!]_T)\|_0^2) \\ &\geq \min_{F\in\mathcal{F}_h} \{\frac{1}{2}, n_f + \eta_F - n_f C_d C_{\text{inv}} \tilde{C}_{\text{inv}}\} (\|\nabla_h \times \boldsymbol{v}\|_0^2 + \frac{1}{\tilde{C}_{\text{inv}}} \|\mathtt{h}^{-\frac{1}{2}} [\![\boldsymbol{v}]\!]_T \|_{0,\mathcal{F}_h}^2). \end{aligned}$$

Inequality (24) is, therefore satisfied if

$$\min_{F\in\mathcal{F}_h} (n_f + \eta_F - n_f C_d C_{\text{inv}} \tilde{C}_{\text{inv}}) \geq \min\left\{\frac{1}{2}, \frac{1}{\tilde{C}_{\text{inv}}}\right\},$$

or, equivalently,

$$\eta \geq n_f (C_d C_{\text{inv}} \tilde{C}_{\text{inv}} - 1) + \min\left\{\frac{1}{2}, \frac{1}{\tilde{C}_{\text{inv}}}\right\} \tag{33}$$

with $\eta = \min_{F\in\mathcal{F}_h} \eta_F$. $\square$

Remark: The constant C_{inv} in (26) can be estimated using Theorem 4 in [38].

In the analysis we consider now the extended (cf. with (20)) bilinear form

$$\mathcal{B}_h : (H_0(\text{curl},\Omega) + \Sigma_h^p) \times (H_0(\text{curl},\Omega) + \Sigma_h^p) \to \mathbb{R},$$

which is given as

$$\begin{aligned}\mathcal{B}_h(\boldsymbol{u},\boldsymbol{v}) = (\nabla_h \times \boldsymbol{u}, \nabla_h \times \boldsymbol{v})_\Omega - k^2(\boldsymbol{u},\boldsymbol{v})_\Omega - \sum_{F\in\mathcal{F}_h} (\mathcal{R}_F([\![\boldsymbol{u}]\!]_T), \nabla_h \times \boldsymbol{v})_\Omega \\ + (\mathcal{R}_F([\![\boldsymbol{v}]\!]_T), \nabla_h \times \boldsymbol{u})_\Omega + \sum_{F\in\mathcal{F}_h} (n_f + \eta_F)(\mathcal{R}_F([\![\boldsymbol{u}]\!]_T), \mathcal{R}_F([\![\boldsymbol{v}]\!]_T))_\Omega\end{aligned}$$

and the linear form $\mathcal{J}_h : H_0(\text{curl},\Omega) + \Sigma_h^p \to \mathbb{R}$, defined as

$$\mathcal{J}_h(\boldsymbol{v}) = (\boldsymbol{J},\boldsymbol{v})_\Omega.$$

Lemma 2 *The bilinear form $\mathcal{B}_h$ is continuous on $(H_0(\text{curl},\Omega)+\Sigma_h^p)\times(H_0(\text{curl},\Omega)+\Sigma_h^p)$ with respect to the DG norm, i.e. the following inequality holds for all $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_h$ and $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_h$ with $\boldsymbol{u}_0, \boldsymbol{v}_0 \in H_0(\text{curl},\Omega)$ and $\boldsymbol{u}_h, \boldsymbol{v}_h \in \Sigma_h^p$:*

$$\mathcal{B}_h(\boldsymbol{u},\boldsymbol{v}) \le C\|\boldsymbol{u}\|_{\text{DG}}\|\boldsymbol{v}\|_{\text{DG}}. \tag{34}$$

Proof: Using the result of Lemma 4.8 in [22] it is sufficient to prove that

$$\sum_{F\in\mathcal{F}_h} |(\mathcal{R}_F([\![\boldsymbol{u}]\!]_T), \mathcal{R}_F([\![\boldsymbol{v}]\!]_T))_\Omega| \le C\|\boldsymbol{u}\|_{\text{DG}}\|\boldsymbol{v}\|_{\text{DG}}$$

with some mesh-independent constant C. Using the decomposition of $\boldsymbol{u}$ and $\boldsymbol{v}$, which implies that $[\![\boldsymbol{u}_0]\!]_T = [\![\boldsymbol{v}_0]\!]_T = 0$, the Schwarz inequality and the estimate in (30) we obtain

$$\begin{aligned}\sum_{F\in\mathcal{F}_h} |(\mathcal{R}_F([\![\boldsymbol{u}]\!]_T), \mathcal{R}_F([\![\boldsymbol{v}]\!]_T))_\Omega| &= \sum_{F\in\mathcal{F}_h} |(\mathcal{R}_F([\![\boldsymbol{u}_h]\!]_T), \mathcal{R}_F([\![\boldsymbol{v}_h]\!]_T))_\Omega| \\ &\le \sqrt{\sum_{F\in\mathcal{F}_h} \|\mathcal{R}_F([\![\boldsymbol{u}_h]\!]_T)\|_0^2}\sqrt{\sum_{F\in\mathcal{F}_h} \|\mathcal{R}_F([\![\boldsymbol{v}_h]\!]_T)\|_0^2} \\ &\le \bar{C}_{\text{inv}}^{-1}\|\mathrm{h}^{-\frac{1}{2}}\,[\![\boldsymbol{u}_h]\!]_T\,\|_{0,\mathcal{F}_h}\|\mathrm{h}^{-\frac{1}{2}}\,[\![\boldsymbol{v}_h]\!]_T\,\|_{0,\mathcal{F}_h} \\ &= C_{\text{inv}}^{-1}\|\mathrm{h}^{-\frac{1}{2}}\,[\![\boldsymbol{u}]\!]_T\,\|_{0,\mathcal{F}_h}\|\mathrm{h}^{-\frac{1}{2}}\,[\![\boldsymbol{v}]\!]_T\,\|_{0,\mathcal{F}_h} \\ &\le C_{\text{inv}}^{-1}\|\boldsymbol{u}\|_{\text{DG}}\|\boldsymbol{v}\|_{\text{DG}}. \quad \square\end{aligned}$$

Next, we will estimate the residual term corresponding to the bilinear form $\mathcal{B}_h$, which is defined for an arbitrary $\boldsymbol{v} \in \Sigma_h^p$, as

$$r_h(\boldsymbol{v}) = \mathcal{B}_h(\boldsymbol{E},\boldsymbol{v}) - \mathcal{J}_h(\boldsymbol{v}), \tag{35}$$

where $\boldsymbol{E}$ is the exact solution of (1). We can easily obtain an equivalent form

$$r_h(\boldsymbol{v}) = \mathcal{B}_h(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \Sigma_h^p. \tag{36}$$

Lemma 3 *Using the notation $\Pi_h : [L_2(\Omega)]^3 \to \Sigma_h^p$ for the L_2 projection to the finite element space we have for all $\boldsymbol{v} \in \Sigma_h^p$ the relation*

$$r_h(\boldsymbol{v}) = \int_{\mathcal{F}_h} [\![\boldsymbol{v}]\!]_T \cdot \{\!\{\nabla \times \boldsymbol{E} - \Pi_h(\nabla \times \boldsymbol{E})\}\!\} \, \mathrm{d}A. \tag{37}$$

If $\nabla \times \boldsymbol{E} \in [H^s(\Omega)]^3$ also holds for some s with $s > \frac{1}{2}$ then the following upper bound is valid for all $\boldsymbol{v} \in \Sigma_h^p$:

$$|r_h(\boldsymbol{v})| \le C \|\boldsymbol{v}\|_{DG} \sqrt{n_f \sum_{K \in \mathcal{T}_h} h_K^{2\min\{p,s\}} \|\nabla \times \boldsymbol{E}\|_{s,K}^2}, \tag{38}$$

where C does not depend on the mesh size.

Proof: First, we note that for $\boldsymbol{v} \in \Sigma_h^p \cap H_0(\mathrm{curl}, \Omega)$ we have $[\![\boldsymbol{v}]\!]_T = 0$ and therefore, the bilinear form $\mathcal{B}_h(\boldsymbol{E}, \boldsymbol{v})$ simplifies to

$$\mathcal{B}_h(\boldsymbol{E}, \boldsymbol{v}) = (\nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{v}) - k^2(\boldsymbol{E}, \boldsymbol{v}). \tag{39}$$

Using (35) and the weak formulation of the time-harmonic Maxwell equations we obtain that

$$r_h(\boldsymbol{v}) = \mathcal{B}_h(\boldsymbol{E}, \boldsymbol{v}) - \mathcal{J}_h(\boldsymbol{v}) = (\nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{v})_\Omega - k^2(\boldsymbol{E}, \boldsymbol{v})_\Omega - (\boldsymbol{J}, \boldsymbol{v}) = 0 \tag{40}$$

is valid for all $\boldsymbol{v} \in \Sigma_h^p \cap H_0(\mathrm{curl}, \Omega)$. This implies the orthogonality relation

$$\mathcal{B}_h(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \Sigma_h^p \cap H_0(\mathrm{curl}, \Omega).$$

The Green formula, the fact that the tangential components of $\nabla_h \times \boldsymbol{E}$ and $\{\!\{\nabla_h \times \boldsymbol{E}\}\!\}$ coincide, and the definition of the L_2 projection result in the following relation for an arbitrary $\boldsymbol{v} \in \Sigma_h^p$

$$\begin{aligned}
r_h(\boldsymbol{v}) &= \mathcal{B}_h(\boldsymbol{E}, \boldsymbol{v}) - \mathcal{J}_h(\boldsymbol{v}) \\
&= (\nabla \times \boldsymbol{E}, \nabla_h \times \boldsymbol{v})_\Omega - k^2(\boldsymbol{E}, \boldsymbol{v})_\Omega - \sum_{F \in \mathcal{F}_h} (\mathcal{R}_F([\![\boldsymbol{v}]\!]_T), \nabla \times \boldsymbol{E})_\Omega - (\boldsymbol{J}, \boldsymbol{v})_\Omega \\
&= (\nabla \times \nabla \times \boldsymbol{E}, \boldsymbol{v}) + \int_{\mathcal{F}_h} \nabla \times \boldsymbol{E} \cdot [\![\boldsymbol{v}]\!]_T \, \mathrm{d}A - k^2(\boldsymbol{E}, \boldsymbol{v}) \\
&- \sum_{F \in \mathcal{F}_h} (\mathcal{R}_F([\![\boldsymbol{v}]\!]_T), \nabla \times \boldsymbol{E})_\Omega - (\boldsymbol{J}, \boldsymbol{v})_\Omega \\
&= \int_{\mathcal{F}_h} \{\!\{\nabla \times \boldsymbol{E}\}\!\} \cdot [\![\boldsymbol{v}]\!]_T \, \mathrm{d}A - \sum_{F \in \mathcal{F}_h} (\mathcal{R}_F([\![\boldsymbol{v}]\!]_T), \Pi_h(\nabla \times \boldsymbol{E}))_\Omega \\
&= \int_{\mathcal{F}_h} [\![\boldsymbol{v}]\!]_T \cdot \{\!\{\nabla \times \boldsymbol{E} - \Pi_h(\nabla \times \boldsymbol{E})\}\!\} \, \mathrm{d}A,
\end{aligned} \tag{41}$$

which proves (37).

If $\boldsymbol{v} \in [H^s(K)]^3$ with $s > \frac{1}{2}$ we use the interpolation estimate, see Theorem 1.7 in [2],

$$\|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_{0,K}^2 + h_K \|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_{0,\partial K}^2 \le C_K h_K^{2\min\{p,s\}} \|\boldsymbol{v}\|_{s,K}^2, \tag{42}$$

where the constant C_K is independent of the mesh size h and the polynomial order $p \in \Sigma_h^p$. Using (41), the Schwarz inequality and elementwise summation of (42) give that

$$\begin{aligned}
r_h(\boldsymbol{v}) &= \int_{\mathcal{F}_h} [\![\boldsymbol{v}]\!]_T \cdot \{\!\{\nabla \times \boldsymbol{E} - \Pi_h(\nabla \times \boldsymbol{E})\}\!\} \, \mathrm{d}A \\
&= \sqrt{\int_{\mathcal{F}_h} \mathtt{h}^{-1} |\, [\![\boldsymbol{v}]\!]_T \,|^2 \, \mathrm{d}A} \sqrt{\int_{\mathcal{F}_h} \mathtt{h} |\, \{\!\{\nabla \times \boldsymbol{E} - \Pi_h(\nabla \times \boldsymbol{E})\}\!\} \,|^2 \, \mathrm{d}A} \\
&\leq \|\boldsymbol{v}\|_{\mathrm{DG}} \sqrt{n_f \sum_{K \in \mathcal{T}_h} C_K h_K^{2\min\{p,s\}} \|\nabla \times \boldsymbol{E}\|_{s,K}^2}.
\end{aligned}$$

Using the shape-regular property of the mesh we obtain the upper bound as stated in (38). □

Remark: Using the mesh parameter h, the upper bound (38) can be rewritten as

$$|r_h(\boldsymbol{v})| \leq C h^{\min\{p,s\}} \|\boldsymbol{v}\|_{\mathrm{DG}} \|\nabla_h \times \boldsymbol{E}\|_s. \tag{43}$$

For the error estimation we split the computational error into three parts.

Lemma 4 *Let η satisfy the assumption in Lemma 1. Then the computational error of the solution $\boldsymbol{E}_h \in \Sigma_h^p$ of (1) can be estimated as*

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathrm{DG}} \leq C \left(\inf_{\boldsymbol{v} \in \Sigma_h^p} \|\boldsymbol{E} - \boldsymbol{v}\|_{\mathrm{DG}} + \sup_{\boldsymbol{0} \neq \boldsymbol{v} \in \Sigma_h^p} \frac{r_h(\boldsymbol{v})}{\|\boldsymbol{v}\|_{\mathrm{DG}}} + \sup_{\boldsymbol{0} \neq \boldsymbol{v} \in \Sigma_h^p} \frac{(\boldsymbol{v}, \boldsymbol{E} - \boldsymbol{E}_h)_\Omega}{\|\boldsymbol{v}\|_{\mathrm{DG}}} \right), \tag{44}$$

where the constant C does not depend on the mesh size.

Proof: Using the triangle inequality and the Gårding inequality stated in Lemma 1 we obtain that for an arbitrary $\boldsymbol{v} \in \Sigma_h^p$

$$\begin{aligned}
\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathrm{DG}} &\leq \|\boldsymbol{E} - \boldsymbol{v}\|_{\mathrm{DG}} + \|\boldsymbol{E}_h - \boldsymbol{v}\|_{\mathrm{DG}} \\
&\leq \|\boldsymbol{E} - \boldsymbol{v}\|_{\mathrm{DG}} + \frac{1}{\beta^2} \sup_{0 \neq \boldsymbol{w} \in \Sigma_h^p} \frac{B_h(\boldsymbol{E}_h - \boldsymbol{v}, \boldsymbol{w})}{\|\boldsymbol{w}\|_{\mathrm{DG}}} + \frac{k^2 + \beta^2}{\beta^2} \sup_{0 \neq \boldsymbol{w} \in \Sigma_h^p} \frac{(\boldsymbol{E}_h - \boldsymbol{v}, \boldsymbol{w})}{\|\boldsymbol{w}\|_{\mathrm{DG}}} \\
&\leq \|\boldsymbol{E} - \boldsymbol{v}\|_{\mathrm{DG}} + \frac{1}{\beta^2} \sup_{0 \neq \boldsymbol{w} \in \Sigma_h^p} \frac{B_h(\boldsymbol{E}_h - \boldsymbol{E}, \boldsymbol{w})}{\|\boldsymbol{w}\|_{\mathrm{DG}}} + \frac{1}{\beta^2} \sup_{0 \neq \boldsymbol{w} \in \Sigma_h^p} \frac{B_h(\boldsymbol{E} - \boldsymbol{v}, \boldsymbol{w})}{\|\boldsymbol{w}\|_{\mathrm{DG}}} \\
&+ \frac{k^2 + \beta^2}{\beta^2} \sup_{0 \neq \boldsymbol{w} \in \Sigma_h^p} \frac{(\boldsymbol{E}_h - \boldsymbol{E}, \boldsymbol{w})}{\|\boldsymbol{w}\|_{\mathrm{DG}}} + \frac{k^2 + \beta^2}{\beta^2} \sup_{0 \neq \boldsymbol{w} \in \Sigma_h^p} \frac{(\boldsymbol{E} - \boldsymbol{v}, \boldsymbol{w})}{\|\boldsymbol{w}\|_{\mathrm{DG}}} \\
&\leq \|\boldsymbol{E} - \boldsymbol{v}\|_{\mathrm{DG}} + \frac{1}{\beta^2} \sup_{0 \neq \boldsymbol{w} \in \Sigma_h^p} \frac{r_h(\boldsymbol{w})}{\|\boldsymbol{w}\|_{\mathrm{DG}}} + \frac{C}{\beta^2} \|\boldsymbol{E} - \boldsymbol{v}\|_{\mathrm{DG}} \\
&+ \frac{k^2 + \beta^2}{\beta^2} \sup_{0 \neq \boldsymbol{w} \in \Sigma_h^p} \frac{(\boldsymbol{E}_h - \boldsymbol{E}, \boldsymbol{w})}{\|\boldsymbol{w}\|_{\mathrm{DG}}} + \frac{k^2 + \beta^2}{\beta^2} \|\boldsymbol{E} - \boldsymbol{v}\|_{\mathrm{DG}} \\
&= (1 + \frac{C}{\beta^2} + \frac{k^2 + \beta^2}{\beta^2}) \|\boldsymbol{E} - \boldsymbol{v}\|_{\mathrm{DG}} + \frac{1}{\beta^2} \sup_{0 \neq \boldsymbol{w} \in \Sigma_h^p} \frac{r_h(\boldsymbol{w})}{\|\boldsymbol{w}\|_{\mathrm{DG}}} \\
&+ \frac{k^2 + \beta^2}{\beta^2} \sup_{0 \neq \boldsymbol{w} \in \Sigma_h^p} \frac{(\boldsymbol{E}_h - \boldsymbol{E}, \boldsymbol{w})}{\|\boldsymbol{w}\|_{\mathrm{DG}}}.
\end{aligned}$$

Taking the infimum over all $\boldsymbol{v} \in \Sigma_h^p$ we obtain the statement in the lemma. □

Using smoothness assumptions to the exact solution of (1), we can now prove the main result of this section.

Theorem 1 *Assume that η satisfies the condition in Lemma 1 and for some parameter $s > \frac{1}{2}$ the exact solution of* (1) *satisfies*

$$\boldsymbol{E} \in H^s(\Omega) \qquad \text{and} \qquad \nabla \times \boldsymbol{E} \in H^s(\Omega).$$

Then for a mesh size h sufficiently small, we have the following optimal error bound

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{DG} \leq Ch^{\min\{p,s\}} \left(\|\boldsymbol{E}\|_{s,0} + \|\nabla \times \boldsymbol{E}\|_{s,0} \right). \tag{45}$$

Proof: We have to estimate the last two terms on the right hand side of (44). For the estimation of the third term we refer to Proposition 5.2 in [22]. Note that in the proof therein the bilinear form $\tilde{a}$ is investigated on $(H_0(\text{curl}, \Omega) + \Sigma_h^p) \times (H_0(\text{curl}, \Omega) + \Sigma_h^p)$, which coincides with $\mathcal{B}_h$ on the same domain. Therefore, the same estimation holds, i.e.

$$\sup_{0 \neq \boldsymbol{v} \in \Sigma_h^p} \frac{(\boldsymbol{v}, \boldsymbol{E} - \boldsymbol{E}_h)_\Omega}{\|\boldsymbol{v}\|_{DG}} \leq Ch^s \|\boldsymbol{E} - \boldsymbol{E}_h\|_{\text{DG}}. \tag{46}$$

For the estimation of the term $\inf_{\boldsymbol{v} \in \Sigma_h^p} \|\boldsymbol{E} - \boldsymbol{v}\|_{DG}$ we take the Nédélec interpolant $\boldsymbol{v} = \Pi_{N,h} \boldsymbol{E}$, which satisfies the relation

$$\|\boldsymbol{E} - \Pi_{N,h}\boldsymbol{E}\|_{\text{DG}} = \|\boldsymbol{E} - \Pi_{N,h}\boldsymbol{E}\|_{\text{curl}} \leq Ch^{\min\{p,s\}} (\|\boldsymbol{E}\|_s + \|\nabla \times \boldsymbol{E}\|_s). \tag{47}$$

See Theorem 5.41 and Theorem 8.15 in [29] for the first and second family of Nédélec elements. Using (46), (47) and the result in (43) we obtain the estimate in the theorem. □

Remarks:

1. This result guarantees the same convergence rate as Theorem 3.2 in [22]. The profound difference between the IP method (discussed in [22]) and the approach analysed here is that we do not have to use mesh dependent constants in the bilinear form. More precisely, η does not depend on h, which results in the uniform stability of the solution.

2. The upper bound is not sharp if the ratio $\frac{\max_{K \in \mathcal{T}_h} h_K}{\min_{K \in \mathcal{T}_h} h_K}$ is large. In those cases one should use the result of Lemma 3 and the elementwise interpolation estimate corresponding to (47).

3. According to (46) the mesh size should satisfy the inequality $Ch^s < 1$.

4. For an extensive study on the smoothness assumption in Theorem 1 we refer to [3].

5 A note on preconditioning

The discretisations (17) and (20) result in linear algebraic systems. In this section we simplify notation and write the linear system as

$$Ax = b, \tag{48}$$

which can now denote either of the linear systems that result from the discretisations (17) and (20), respectively. The system (48) can be solved by a direct method or an iterative method, typically one of the numerous Krylov subspace methods. Direct methods are reliable but not well suited for large-scale problems because one also needs to store the matrix fill-in during the solution process. On the other hand, efficient iterative methods require a good preconditioner. Since matrix A is symmetric but not definite, finding an efficient preconditioner is a non-trivial task. See for example [5, 6, 15, 33, 36].

The approach we adopt here is that of shifted preconditioners for second-order derivative operators. Let S be the matrix that corresponds to the discretisation of the curl-curl operator, and let M be the discretisation of the second term in (20) (and also in (17)). One can then rewrite (48) as

$$(S - M)x = b. \tag{49}$$

We use preconditioners that take the form

$$P = S_B + \gamma M_B, \tag{50}$$

where S_B consists of N_K-by-N_K diagonal blocks of S, with N_K being the local number of degrees of freedom in element K. The positive real number γ is the shift parameter and we determine its value experimentally. The matrix M_B is either taken to be the identity matrix I or chosen to be M itself. Finally, we compute the Cholesky factorisation of P and use its resulting matrices to apply two-sided preconditioning to the linear system (48).

In principle, it is possible to choose S_B to be S, and this is often the case when S represents the discrete counterpart of the Laplacian. However, in the present situation the Cholesky factorisation turns out to be computationally very expensive if $S_B = S$.

The approach we have just presented here is similar to the one proposed in [36] for the discrete Helmholtz operator. But the analysis in that work is not directly applicable to the DG discretisations discussed here. Instead, we rely on computational evidence to verify the approach. On the whole, we have found that, if γ is suitably chosen, the preconditioners speed up the convergence of the MINRES method in terms of iteration steps as well as CPU time. The optimal value of γ depends slightly on the mesh-size h and more considerably on the polynomial order of the approximation p. We will briefly illustrate this behaviour in Section 6.3.

6 Numerical experiments

We now provide three-dimensional numerical examples to verify and demonstrate the DG discretisations of (1). For all computations, $H(\text{curl})$-conforming vector-valued basis functions are used [1, 34]. We also mention here that the first six of the basis functions constitute the first-order first-family of Nédélec elements [30]. The first twelve of the basis functions used here are not the same as those that form the first-order second-family of Nédélec elements. However, they span exactly the same space and have exactly the same properties as those defined in [31].

In the IP-DG discretisation, the interior-penalty parameter in (17) is often defined as

$$\tau = C_{\mathrm{IP}} \frac{p^2}{h_F}, \tag{51}$$

where $C_{\mathrm{IP}} = 10$, p is the polynomial order, and h_F is the diameter of the face $F \in \mathcal{F}_h$. The value of the constant C_{IP} is based on numerical experiments. It has been demonstrated for a range of problems that this choice of C_{IP} is sufficiently large to guarantee stability. See [25] and references therein. We have found, however, that in the case of three-dimensional computations for the Maxwell system, this penalty is not large enough. So instead of (51), we define the above parameter as

$$\tau = C_{\mathrm{IP}} \frac{(p+1)^2}{h_F},$$

with $C_{\mathrm{IP}} = 10$. See also the recent article [13], where the same penalty term is used for computing Maxwell eigenvalues on two-dimensional domains.

The parameter η_F (20), resulting from the stabilisation term α_R in (18), is set to

$$\eta_F = 20,$$

but the results do not significantly depend on this choice, as long as η_F satisfies condition (33).

All numerical computations have been carried out in the framework of *hp*GEM [32], a software environment for DG discretisations suitable for a variety of physical problems.

6.1 Example 1: smooth solution

As a first example, we consider the Maxwell equations (1) with $k^2 = 1$ in the domain $\Omega = (0,1)^3$ and assume the boundary to be a perfect electric conductor (PEC), i.e. $\boldsymbol{g} = \boldsymbol{0}$ in (1). The source term is given as

$$\boldsymbol{J}(x,y,z) = \left(2\pi^2 - 1\right) \begin{pmatrix} \sin(\pi y)\sin(\pi z) \\ \sin(\pi z)\sin(\pi x) \\ \sin(\pi x)\sin(\pi y) \end{pmatrix}, \tag{52}$$

so we have the exact solution

$$\boldsymbol{E}(x,y,z) = \begin{pmatrix} \sin(\pi y)\sin(\pi z) \\ \sin(\pi z)\sin(\pi x) \\ \sin(\pi x)\sin(\pi y) \end{pmatrix}. \tag{53}$$

The computations are performed on two different sequences of meshes. The first are highly structured meshes and constructed as follows. The domain $\Omega = (0,1)^3$ is divided into $n \times n \times n$ number of congruent subcubes, with integer $n = 2^m$ and nonnegative integer m. We then divide each of these subcubes into five tetrahedra, four of which are congruent and have volume one-sixth of the original cube. The fifth has volume one-third of the original cube. Although the mesh is not uniform, this has proved to be a simple and convenient way of measuring convergence, as each time we refine the mesh, the maximum of the face diameter h_F will be exactly half of that of the previous mesh. The convergence results on structured meshes are shown in Table I for the IP-DG method and in Table III for the method of Brezzi et al.

We have also run the same example on a sequence of unstructured meshes. The meshes were generated by CentaurSoft (`http://www.centaursoft.com`), a package suitable for generating a variety of hybrid meshes with complex geometries. In this sequence of meshes, we begin with a coarse mesh of 54 tetrahedra. Then we divide each tetrahedron into eight smaller tetrahedra to get the next (finer) mesh. The convergence results on structured meshes are depicted in Table II for the IP-DG discretisation and in Table IV for the method of Brezzi et al.

Based on our analysis in Section 4 and on the analysis on [22], the optimal convergence rate for this example is $\mathcal{O}(h^{p+1})$ in the $L^2(\Omega)$-norm and $\mathcal{O}(h^p)$ in the DG norm. We can see that, for both methods on structured meshes, optimal convergence rate is achieved in the $L^2(\Omega)$-norm, and higher-than-optimal convergence rates are observed in the DG norm. On unstructured meshes, we only have an estimated convergence rate with $h \sim N_{\mathrm{el}}^{-\frac{1}{3}}$. The convergence rates are slightly suboptimal, in part because we have to estimate the rates of convergence, and in part because we are still in the pre-asymptotic regime.

6.2 Example 2: Fichera cube

As a second example, we investigate the DG discretisations (17) and (22) of the Maxwell equations (1) with non-smooth solution. To this end, we consider the domain $\Omega = (-1,1)^3 \setminus [-1,0]^3$, and select $\boldsymbol{J}$ and the non-homogeneous boundary conditions so that the analytical solution is given by

$$\boldsymbol{E} = \nabla\phi(r), \quad \text{with} \quad \phi(r) = \mathrm{e}^r \sin(r), \tag{54}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. The analytical solution contains a singularity at the re-entrant corner located at the origin of Ω. As a result, $\boldsymbol{E}$ lies in the Sobolev space $[H^{1-\epsilon}(\Omega)]^3$, $\epsilon > 0$. Again, based on our analysis in Section 4 and on the analysis of [22], the theoretically predicted asymptotical convergence rate for (54) is $\mathcal{O}(h^{\min(1,p+1)})$ in the $L^2(\Omega)$-norm and $\mathcal{O}(h^{\min(1,p)})$ in the DG norm.

We compute the numerical solutions for a sequence of globally refined unstructured meshes. The global refinement is, of course, far from being optimal for singular solutions, but it is nonetheless suitable for verifying the theoretical results. The convergence rates are again estimated with $h \sim N_{\mathrm{el}}^{-\frac{1}{3}}$.

We show the errors of the IP-DG approximations for (54) in Table V. The errors of the discretisation with the local lifting operator are depicted in Table VI. We can expect first-order convergence in both norms, which is what we approximately observe. Precise convergence rates are not achieved because the meshes are not subsequently refined (i.e. we do not use the previous mesh to construct the next), and also because they are still in the pre-asymptotic regime. Nevertheless, we see that the error, as predicted, is determined by the Sobolev coefficient and not by the order of the approximating polynomials.

6.3 Performance of the iterative solver

To illustrate the effect of the preconditioning technique briefly described in Section 5, we consider two cases of Example 1 from Section 6.1. One we call $\texttt{mesh}_4^1$ and it consists of 27648 elements with polynomial order $p = 1$. The other, $\texttt{mesh}_3^2$, consists of 3456 elements with polynomial order $p = 2$. In Table VII we show the relative residual and the computational work after 10000 iterations for different values of γ. We can see that with the correct value of γ, the convergence rate can be improved significantly. Let us assume, for example, that we want to achieve a tolerance of $\texttt{tol} = 10^{-6}$ for the given problem with $\texttt{mesh}_4^1$. With $\gamma = 10^6$ it takes 15324 iterations and 4990s to reach that tolerance. By contrast, the relative residual for the system without preconditioning is still only 0.0085 after 40000 iterations and 6097s. Finally, we note that there is a limit to how far the parameter γ can be increased without compromising the solution of the discrete weak formulation(s). That limit is around 10^{10}–10^{11}, and may thus mean that for high-order approximation we need to resort to a less-than-optimal value for γ.

7 Concluding remarks and outlook

We have introduced and analysed a discontinuous Galerkin method for the indefinite time-harmonic Maxwell equations in three dimensions. The novelty of this approach is twofold. First, we make use of a local lifting operator to compute the (stabilising) numerical flux. This approach is much in the spirit of [10], and it has been applied to the Laplace operator in a number of articles since. It also allows us to choose the constant in the flux formulation independent of the mesh size and the polynomial order. Second, we use $H(\text{curl})$-conforming vector-valued functions to build the local polynomial basis, which is a very natural choice for the Maxwell equations and is widely used in $H(\text{curl})$-conforming finite element discretisations [29].

We have presented a couple of numerical experiments which have demonstrated that the method converges at an optimal rate on both structured and unstructured meshes. We have also carried out the same experiments using the IP-DG flux with the same basis functions. Optimal convergence rate is, too, observed on both types of meshes. However, for the IP-DG method the penalty parameter depends on the mesh size as well as on the polynomial order. Furthermore, we found that we need a slightly larger penalisation than the empirical value given in [22, 25], which is mostly based on two-dimensional experiments. See also [13].

It is also clear that a number of questions have remained unanswered. The most obvious one concerns the spectral properties of the method. In [12] the authors provided a general framework for studying the spectral correctness of DG methods (see also [11, 13]). The method introduced in this article fits into that framework, and the study of its spectral properties is currently under way.

Another important step would be to provide a posteriori error indicator for the discretisation, either in the form of explicit a posteriori error analysis [25], or in the context of implicit a posteriori error estimation [26, 16].

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A Appendix: consistency

In order to show consistency for the bilinear forms resulting from the flux formulation (18), we begin with the general primal formulation

$$
\begin{aligned}
&\mathcal{B}_h(\boldsymbol{E}_h, \boldsymbol{\phi}) = (\nabla_h \times \boldsymbol{E}_h, \nabla_h \times \boldsymbol{\phi})_\Omega - k^2 (\boldsymbol{E}_h, \boldsymbol{\phi})_\Omega \\
&- \int_{\mathcal{F}_h^i} \{\!\{\boldsymbol{E}_h^* - \boldsymbol{E}_h\}\!\} \cdot [\![\nabla_h \times \boldsymbol{\phi}]\!]_T \, \mathrm{d}A + \int_{\mathcal{F}_h^i} [\![\boldsymbol{E}_h^* - \boldsymbol{E}_h]\!]_T \cdot \{\!\{\nabla_h \times \boldsymbol{\phi}\}\!\} \, \mathrm{d}A \\
&- \int_{\mathcal{F}_h^i} \{\!\{\boldsymbol{q}_h^*\}\!\} \cdot [\![\boldsymbol{\phi}]\!]_T \, \mathrm{d}A + \int_{\mathcal{F}_h^i} [\![\boldsymbol{q}_h^*]\!]_T \cdot \{\!\{\boldsymbol{\phi}\}\!\} \, \mathrm{d}A \\
&\qquad + \int_{\mathcal{F}_h^b} (\boldsymbol{n} \times (\boldsymbol{E}_h^* - \boldsymbol{E}_h)) \cdot (\nabla_h \times \boldsymbol{\phi}) \, \mathrm{d}A + \int_{\mathcal{F}_h^b} (\boldsymbol{n} \times \boldsymbol{q}_h^*) \cdot \boldsymbol{\phi} \, \mathrm{d}A.
\end{aligned}
$$

Using the identity

$$
\begin{aligned}
&(\nabla_h \times \boldsymbol{E}_h, \nabla_h \times \boldsymbol{\phi})_\Omega = (\nabla_h \times \nabla_h \times \boldsymbol{E}_h, \boldsymbol{\phi})_\Omega \\
&- \int_{\mathcal{F}_h^i} \{\!\{\boldsymbol{\phi}\}\!\} \cdot [\![\nabla_h \times \boldsymbol{E}_h]\!]_T \, \mathrm{d}A + \int_{\mathcal{F}_h^i} \{\!\{\nabla_h \times \boldsymbol{E}_h\}\!\} \cdot [\![\boldsymbol{\phi}]\!]_T \, \mathrm{d}A \\
&\qquad + \int_{\mathcal{F}_h^b} (\boldsymbol{n} \times \boldsymbol{\phi}) \cdot (\nabla_h \times \boldsymbol{E}_h) \, \mathrm{d}A,
\end{aligned}
$$

we have the equivalent formulation

$$
\begin{aligned}
&\mathcal{B}_h(\boldsymbol{E}_h, \boldsymbol{\phi}) = (\nabla_h \times \nabla_h \times \boldsymbol{E}_h, \boldsymbol{\phi})_\Omega - k^2 (\boldsymbol{E}_h, \boldsymbol{\phi})_\Omega \\
&- \int_{\mathcal{F}_h^i} \{\!\{\boldsymbol{E}_h^* - \boldsymbol{E}_h\}\!\} \cdot [\![\nabla_h \times \boldsymbol{\phi}]\!]_T \, \mathrm{d}A + \int_{\mathcal{F}_h^i} [\![\boldsymbol{E}_h^* - \boldsymbol{E}_h]\!]_T \cdot \{\!\{\nabla_h \times \boldsymbol{\phi}\}\!\} \, \mathrm{d}A \\
&- \int_{\mathcal{F}_h^i} \{\!\{\boldsymbol{q}_h^* - \nabla_h \times \boldsymbol{E}_h\}\!\} \cdot [\![\boldsymbol{\phi}]\!]_T \, \mathrm{d}A + \int_{\mathcal{F}_h^i} [\![\boldsymbol{q}_h^* - \nabla_h \times \boldsymbol{E}_h]\!]_T \cdot \{\!\{\boldsymbol{\phi}\}\!\} \, \mathrm{d}A \\
&\qquad + \int_{\mathcal{F}_h^b} (\boldsymbol{n} \times (\boldsymbol{E}_h^* - \boldsymbol{E}_h)) \cdot (\nabla_h \times \boldsymbol{\phi}) \, \mathrm{d}A + \int_{\mathcal{F}_h^b} (\boldsymbol{n} \times (\boldsymbol{q}_h^* - \nabla_h \times \boldsymbol{E}_h)) \cdot \boldsymbol{\phi} \, \mathrm{d}A.
\end{aligned}
$$

We now insert the exact solution $\boldsymbol{E}$ into the bilinear form to obtain

$$\begin{aligned}
\mathcal{B}_h(\boldsymbol{E},\boldsymbol{\phi}) &= (\nabla\times\nabla\times\boldsymbol{E},\boldsymbol{\phi})_\Omega - k^2(\boldsymbol{E},\boldsymbol{\phi})_\Omega \\
&- \int_{\mathcal{F}_h^i} (\{\!\{\boldsymbol{E}_h^*\}\!\} - \boldsymbol{E})\cdot[\![\nabla_h\times\boldsymbol{\phi}]\!]_T\,\mathrm{d}A + \int_{\mathcal{F}_h^i} [\![\boldsymbol{E}_h^*]\!]_T\cdot\{\!\{\nabla_h\times\boldsymbol{\phi}\}\!\}\,\mathrm{d}A \\
&- \int_{\mathcal{F}_h^i} (\{\!\{\boldsymbol{q}_h^*\}\!\} - \nabla_h\times\boldsymbol{E})\cdot[\![\boldsymbol{\phi}]\!]_T\,\mathrm{d}A + \int_{\mathcal{F}_h^i} [\![\boldsymbol{q}_h^*]\!]_T\cdot\{\!\{\boldsymbol{\phi}\}\!\}\,\mathrm{d}A \\
&+ \int_{\mathcal{F}_h^b} (\boldsymbol{n}\times(\boldsymbol{E}_h^*-\boldsymbol{E}))\cdot(\nabla_h\times\boldsymbol{\phi})\,\mathrm{d}A + \int_{\mathcal{F}_h^b} (\boldsymbol{n}\times(\boldsymbol{q}_h^*-\nabla_h\times\boldsymbol{E}))\cdot\boldsymbol{\phi}\,\mathrm{d}A,
\end{aligned}$$

since $\{\!\{\boldsymbol{E}\}\!\}_T = \boldsymbol{E}_T$, $\{\!\{\nabla_h\times\boldsymbol{E}\}\!\}_T = (\nabla_h\times\boldsymbol{E})_T$ and $[\![\boldsymbol{E}]\!]_T = [\![\nabla_h\times\boldsymbol{E}]\!]_T = 0$. Note that we only require the continuity of the tangential component of the exact solution $\boldsymbol{E}$. This is correct because we take the inner product of the fields with tangential jumps so the normal components do not have a contribution. It is easy to see that the numerical flux $\boldsymbol{E}_h^*$ is consistent (in either form) at the internal faces, that is, $\{\!\{\boldsymbol{E}_h^*\}\!\}_T = \boldsymbol{E}_T$ and $[\![\boldsymbol{E}_h^*]\!]_T = \boldsymbol{0}$. The bilinear form then further simplifies as

$$\begin{aligned}
\mathcal{B}_h(\boldsymbol{E},\boldsymbol{\phi}) &= (\nabla\times\nabla\times\boldsymbol{E},\boldsymbol{\phi})_\Omega - k^2(\boldsymbol{E},\boldsymbol{\phi})_\Omega \\
&- \int_{\mathcal{F}_h^i} (\{\!\{\boldsymbol{q}_h^*\}\!\} - \nabla_h\times\boldsymbol{E})\cdot[\![\boldsymbol{\phi}]\!]_T\,\mathrm{d}A + \int_{\mathcal{F}_h^i} [\![\boldsymbol{q}_h^*]\!]_T\cdot\{\!\{\boldsymbol{\phi}\}\!\}\,\mathrm{d}A \\
&+ \int_{\mathcal{F}_h^b} (\boldsymbol{n}\times(\boldsymbol{E}_h^*-\boldsymbol{E}))\cdot(\nabla_h\times\boldsymbol{\phi})\,\mathrm{d}A + \int_{\mathcal{F}_h^b} (\boldsymbol{n}\times(\boldsymbol{q}_h^*-\nabla_h\times\boldsymbol{E}))\cdot\boldsymbol{\phi}\,\mathrm{d}A.
\end{aligned}$$

Recall that

$$\boldsymbol{q}_h = \nabla\times\boldsymbol{E}_h - \mathcal{L}(\{\!\{\boldsymbol{E}_h^*-\boldsymbol{E}_h\}\!\}) + \mathcal{R}([\![\boldsymbol{E}_h^*-\boldsymbol{E}_h]\!]_T).$$

If we replace $\boldsymbol{E}_h$ with $\boldsymbol{E}$, we obtain

$$[\![\boldsymbol{E}_h^*-\boldsymbol{E}]\!]_T = [\![\{\!\{\boldsymbol{E}\}\!\}-\boldsymbol{E}]\!]_T = 0,$$

and

$$(\mathcal{L}(\{\!\{\boldsymbol{E}_h^*-\boldsymbol{E}\}\!\},\boldsymbol{\phi}))_\Omega = \int_{\mathcal{F}_h^i} \{\!\{\boldsymbol{E}_h^*-\boldsymbol{E}\}\!\}\cdot[\![\boldsymbol{\phi}]\!]_T\,\mathrm{d}A = \int_{\mathcal{F}_h^i} \{\!\{\boldsymbol{E}_h^*-\boldsymbol{E}\}\!\}_T\cdot[\![\boldsymbol{\phi}]\!]_T\,\mathrm{d}A = 0,$$

since $\{\!\{\boldsymbol{E}_h^*\}\!\}_T = \boldsymbol{E}_T$. Thus $\{\!\{\boldsymbol{q}_h^*\}\!\} = \{\!\{\nabla\times\boldsymbol{E}\}\!\}$ at $\mathcal{F}_h^i$ and $\{\!\{\boldsymbol{q}_h^*\}\!\} = \nabla\times\boldsymbol{E}$ at $\mathcal{F}_h^b$. We also have $[\![\boldsymbol{q}_h^*]\!]_T = \boldsymbol{0}$, from which we obtain

$$\begin{aligned}
\mathcal{B}_h(\boldsymbol{E},\boldsymbol{\phi}) &= (\nabla\times\nabla\times\boldsymbol{E},\boldsymbol{\phi})_\Omega - k^2(\boldsymbol{E},\boldsymbol{\phi})_\Omega \\
&- \int_{\mathcal{F}_h^i} (\{\!\{\nabla\times\boldsymbol{E}\}\!\} - \nabla\times\boldsymbol{E})\cdot[\![\boldsymbol{\phi}]\!]_T\,\mathrm{d}A + \int_{\mathcal{F}_h^b} (\boldsymbol{n}\times(\boldsymbol{E}_h^*-\boldsymbol{E}))\cdot(\nabla_h\times\boldsymbol{\phi})\,\mathrm{d}A \\
&+ \int_{\mathcal{F}_h^b} (\boldsymbol{n}\times(\boldsymbol{q}_h^*-\nabla\times\boldsymbol{E}))\cdot\boldsymbol{\phi}\,\mathrm{d}A = (\nabla\times\nabla\times\boldsymbol{E},\boldsymbol{\phi})_\Omega - k^2(\boldsymbol{E},\boldsymbol{\phi})_\Omega,
\end{aligned}$$

thanks to the fact that

$$(\{\!\{\nabla \times \boldsymbol{E}\}\!\} - \nabla \times \boldsymbol{E}) \cdot [\![\boldsymbol{\phi}]\!]_T = (\{\!\{\nabla \times \boldsymbol{E}\}\!\}_T - (\nabla \times \boldsymbol{E})_T) \cdot [\![\boldsymbol{\phi}]\!]_T = 0,$$

and to the conditions

$$\boldsymbol{n} \times \boldsymbol{E}_h^* = \boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{g} \quad \text{and} \quad \boldsymbol{q}_h^* = \nabla \times \boldsymbol{E} \quad \text{at} \quad \mathcal{F}_h^b.$$

B Appendix: construction of the linear system

In this appendix we discuss some of the implementation details of the DG discretisations introduced in Section 3. We use the hierarchic construction of $H(\text{curl})$-conforming basis functions from [1] (see also [34]). The complete set of hierarchic basis functions $\{\boldsymbol{\psi}_i^K\}$ that satisfy the discrete de Rham diagram (i.e. L^2-, H^1-, $H(\text{curl})$- and $H(\text{div})$-conforming basis functions) can also be found in [1] and [34]. The basis functions are usually defined for a reference element $\hat{K} \in \mathbb{R}^3$, which in our case is chosen to be given by the vertices

$$\boldsymbol{v}_1 = (0,0,0)\,, \quad \boldsymbol{v}_2 = (1,0,0)\,, \quad \boldsymbol{v}_3 = (0,1,0)\,, \quad \boldsymbol{v}_4 = (0,0,1)\,.$$

Then the basis functions need be transformed from the reference element to each physical mesh element $K \in \mathcal{T}_h$. We refer to [29] for the details of the $H(\text{curl})$ transformation rules.

We can now express the unknown field with the polynomial expansion in each element as

$$\boldsymbol{E}_h^K(\boldsymbol{x}) = \sum_{j=1}^{N_p} E_j^K \boldsymbol{\psi}_j^K(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in K. \tag{55}$$

The entries of the elemental stiffness matrix $\mathbf{S}_K$ and the elemental mass matrix $\mathbf{M}_K$ can be expressed as

$$\mathbf{S}_{ij}^K = \int_K (\nabla_h \times \boldsymbol{\psi}_i) \cdot (\nabla_h \times \boldsymbol{\psi}_j)\,\mathrm{d}V \quad \text{and} \quad \mathbf{M}_{ij}^K = k^2 \int_K \boldsymbol{\psi}_i \cdot \boldsymbol{\psi}_j\,\mathrm{d}V,$$

respectively. Here, the indices run from $i, j = 1, \ldots, N_K$, with N_K being the number of degrees of freedom in element K.

As for the face contributions $F \in \mathcal{F}_h$ in (17) and (20), we need to consider values in the two elements K^L and K^R which are connected through face F. So we (abuse the notation slightly and) define the matrices $\mathbf{D}^{LR}$, $\mathbf{G}^{LR}$ and $\mathbf{H}^{LR}$ as

$$\begin{aligned}
\mathbf{D}_{ij}^{LR} &= \int_F \boldsymbol{\psi}_i^L \cdot \left(\boldsymbol{n}^R \times \boldsymbol{\psi}_j^R\right)\mathrm{d}A,\\
\mathbf{G}_{ij}^{LR} &= \int_F \left(\nabla_h \times \boldsymbol{\psi}_i^L\right) \cdot \left(\boldsymbol{n} \times \boldsymbol{\psi}_j^R\right)\mathrm{d}A,\\
\mathbf{H}_{ij}^{LR} &= \tau \int_F \left(\boldsymbol{n} \times \boldsymbol{\psi}_i^L\right) \cdot \left(\boldsymbol{n} \times \boldsymbol{\psi}_j^R\right)\mathrm{d}A.
\end{aligned}$$

The indices i and j now run between 1 and N_L and between 1 and N_R, respectively, with N_L and N_R being the number of degrees of freedom in element K_L and K_R. Note that the face matrices are 'sparse' as many of the basis functions' tangential components vanish at a given interface. This is especially true for higher order elements.

Exploiting the appropriate transformation rules [29], the above integrals can be computed on the reference domain $\hat{K}$ (and on the reference face $\hat{F} \in \mathbb{R}^2$) by means of Gauss cubature rules. One way to define Gauss cubatures on a tetrahedra is to compute them for the cube and 'collapse' the cubature points (and the associated weights) into the tetrahedron. However, this procedure turns out to be very expensive for higher-order discretisations. Instead, we are making use of the so-called *economical* Gauss cubatures [34], which have been derived for polynomials up to order $p \leq 9$. The construction of these points and weights is based on topological symmetries within the tetrahedron, and is considerable more complicated for orders $p > 9$. Since we implement basis functions up to polynomial degree five, the highest order cubature rule we need (to compute the entries of the mass matrix, for example) is $p = 10$. Table VIII shows the number of cubature points needed to integrate polynomials up to order $p \leq 13$. (The table is taken from [34] and we are not aware of any improvements on the cubature rules since.) We can immediately see that numerical integration over a tetrahedron becomes increasingly costly, which practically prohibits the use of very high-order polynomials for three-dimensional problems. This hurdle can be partially circumvented by using nodal-based polynomial bases. See [17, 27, 20] for example.

We now focus on computing the lifting operators in the last term of (20). We approximate the local lifting operator $\mathcal{R}_F$ in (6) by using the same basis as for the discretisation of $\boldsymbol{E}_h$,

$$\mathcal{R}_F^K([\![\boldsymbol{E}_h]\!]_T)(\boldsymbol{x}) = \sum_{j=1}^{N_p} R_j^{K,F} \boldsymbol{\psi}_j^K(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in K.$$

Since $\mathcal{R}_F$ is only nonzero in the two elements K_L and K_R which are connected to the face F, we have

$$\int_{K_L} \boldsymbol{\phi}^L \cdot \mathcal{R}_F^L([\![\boldsymbol{E}_h]\!]_T)\,\mathrm{d}V + \int_{K_R} \boldsymbol{\phi}^R \cdot \mathcal{R}_F^R([\![\boldsymbol{E}_h]\!]_T)\,\mathrm{d}V = \frac{1}{2}\int_F \left(\boldsymbol{\phi}^L + \boldsymbol{\phi}^R\right) \cdot \left(\boldsymbol{n}^L \times \boldsymbol{E}_h^L + \boldsymbol{n}^R \times \boldsymbol{E}_h^R\right) \mathrm{d}A, \quad \forall \boldsymbol{\phi}^L, \boldsymbol{\phi}^R \in \boldsymbol{\Sigma}_h^p. \tag{56}$$

If we substitute these into (56) and use the fact that it must be satisfied for arbitrary test functions $\boldsymbol{\phi}^L$ and $\boldsymbol{\phi}^R$, we obtain the following matrix relations

$$\begin{aligned} \mathbf{M}^L R^L &= \frac{1}{2}\mathbf{D}^{LL} E^L + \frac{1}{2}\mathbf{D}^{LR} E^R, \\ \mathbf{M}^R R^R &= \frac{1}{2}\mathbf{D}^{RL} E^L + \frac{1}{2}\mathbf{D}^{RR} E^R, \end{aligned} \tag{57}$$

where

$$\mathbf{M}_{ij}^K = k^2 \int_K \boldsymbol{\psi}_i \cdot \boldsymbol{\psi}_j \,\mathrm{d}V \quad \text{and} \quad \mathbf{D}_{ij}^{LR} = \oint_F \boldsymbol{\psi}_i^L \cdot \left(\boldsymbol{n}^R \times \boldsymbol{\psi}_j^R\right) \mathrm{d}A.$$

Let us again use the definition of the local lifting operator (6) for a given face $F \in \mathcal{F}$ to recover

$$(\mathcal{R}_F([\![\boldsymbol{E}]\!]_T), \mathcal{R}_F([\![\boldsymbol{\phi}]\!]_T))_\Omega = \int_F [\![\boldsymbol{\phi}]\!]_T \cdot \{\!\{\mathcal{R}_F([\![\boldsymbol{E}_h]\!]_T)\}\!\} \, \mathrm{d}A =$$
$$\frac{1}{2}\int_F \left(\boldsymbol{n}^L \times \boldsymbol{\phi}^L + \boldsymbol{n}^R \times \boldsymbol{\phi}^R\right) \cdot \left(\mathcal{R}_F^L([\![\boldsymbol{E}_h]\!]_T) + \mathcal{R}_F^R([\![\boldsymbol{E}_h]\!]_T)\right) \mathrm{d}A, \quad (58)$$

which in turn can be approximated as

$$\frac{1}{2}\int_F \left(\boldsymbol{n}^L \times \boldsymbol{\phi}^L + \boldsymbol{n}^R \times \boldsymbol{\phi}^R\right) \cdot \left(\mathcal{R}_F^L([\![\boldsymbol{E}_h]\!]_T) + \mathcal{R}_F^R([\![\boldsymbol{E}_h]\!]_T)\right) \mathrm{d}A \approx$$
$$\frac{1}{4}\left(\mathbf{C}^{LL}\left(\mathbf{M}^L\right)^{-1}\mathbf{D}^{LL} + \mathbf{C}^{LR}\left(\mathbf{M}^R\right)^{-1}\mathbf{D}^{RL}\right) +$$
$$\frac{1}{4}\left(\mathbf{C}^{LL}\left(\mathbf{M}^L\right)^{-1}\mathbf{D}^{LR} + \mathbf{C}^{LR}\left(\mathbf{M}^R\right)^{-1}\mathbf{D}^{RR}\right) +$$
$$\frac{1}{4}\left(\mathbf{C}^{RL}\left(\mathbf{M}^L\right)^{-1}\mathbf{D}^{LL} + \mathbf{C}^{RR}\left(\mathbf{M}^R\right)^{-1}\mathbf{D}^{RL}\right) +$$
$$\frac{1}{4}\left(\mathbf{C}^{RL}\left(\mathbf{M}^L\right)^{-1}\mathbf{D}^{LR} + \mathbf{C}^{RR}\left(\mathbf{M}^R\right)^{-1}\mathbf{D}^{RR}\right).$$

We use this relation to compute the last term of (20). The construction of the other elemental matrices follow the standard procedure. Finally, an assembly procedure is performed to arrive at the linear system

$$\boldsymbol{\mathcal{A}} E_h = b_h, \quad (59)$$

where the matrix $\boldsymbol{\mathcal{A}}$ is symmetric but not definite.

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Table I: *Example 1.* Convergence of the IP-DG method on structured meshes.

	$p=1$			
	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{DG}$	Order
$N_{\mathrm{el}}=5$	2.5854E-01	—	4.5133E-01	—
$N_{\mathrm{el}}=40$	2.5686E-01	0.01	3.9962E-01	0.18
$N_{\mathrm{el}}=320$	5.8863E-02	2.13	1.1723E-01	1.78
$N_{\mathrm{el}}=2560$	1.4605E-02	2.01	4.5535E-02	1.36
$N_{\mathrm{el}}=20480$	3.6754E-03	1.99	2.0669E-02	1.14
	$p=2$			
	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{DG}$	Order
$N_{\mathrm{el}}=5$	2.8524E-01	—	4.1467E-01	—
$N_{\mathrm{el}}=40$	3.1044E-02	3.20	5.0040E-02	3.05
$N_{\mathrm{el}}=320$	3.7101E-03	3.06	8.2802E-03	2.60
$N_{\mathrm{el}}=2560$	4.6444E-04	3.00	1.7224E-03	2.27
	$p=3$			
	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{DG}$	Order
$N_{\mathrm{el}}=5$	5.7244E-02	—	8.5302E-02	—
$N_{\mathrm{el}}=40$	4.5008E-03	3.67	7.1218E-03	3.58
$N_{\mathrm{el}}=320$	2.3366E-04	4.27	5.0151e-04	3.83
	$p=4$			
	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{DG}$	Order
$N_{\mathrm{el}}=5$	2.3057E-02	—	3.2834E-02	—
$N_{\mathrm{el}}=40$	5.3477E-04	5.43	8.1995E-04	5.32
$N_{\mathrm{el}}=320$	1.5714E-05	5.09	3.0315E-05	4.75
	$p=5$			
	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{DG}$	Order
$N_{\mathrm{el}}=5$	4.4752E-03	—	6.4666E-03	—
$N_{\mathrm{el}}=40$	1.4442E-04	4.95	2.0711E-04	4.96
$N_{\mathrm{el}}=320$	1.1092E-06	7.02	1.8604E-06	6.80

Table II: *Example 1.* Convergence of the IP-DG on unstructured meshes.

	$p = 1$			
	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{DG}$	Order
$N_{\text{el}} = 54$	2.2548E-01	—	3.6943E-01	—
$N_{\text{el}} = 432$	7.1925E-02	1.65	1.4363E-01	1.36
$N_{\text{el}} = 3456$	2.1031E-02	1.77	6.1771E-02	1.22
$N_{\text{el}} = 27648$	6.2947E-03	1.74	3.8283E-02	0.69
	$p = 2$			
	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{DG}$	Order
$N_{\text{el}} = 54$	3.0435E-02	—	4.9090E-02	—
$N_{\text{el}} = 432$	4.9945E-03	2.61	1.0397E-02	2.24
$N_{\text{el}} = 3456$	7.2720E-04	2.78	2.4843E-03	2.07
	$p = 3$			
	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{DG}$	Order
$N_{\text{el}} = 54$	4.8645E-03	—	7.9219E-03	—
$N_{\text{el}} = 432$	4.9752E-04	3.29	9.8238E-04	3.01
$N_{\text{el}} = 3456$	4.1326E-05	3.60	1.2622E-04	2.96
	$p = 4$			
	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{DG}$	Order
$N_{\text{el}} = 54$	5.4669E-04	—	8.2955E-04	—
$N_{\text{el}} = 432$	3.7641E-05	3.86	6.3357E-05	3.71
	$p = 5$			
	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{DG}$	Order
$N_{\text{el}} = 54$	1.4740E-04	—	2.1325E-04	—
$N_{\text{el}} = 432$	6.0287E-06	4.61	9.2191E-06	4.53

Table III: *Example 1.* Convergence of the method of Brezzi et al. on structured meshes.

	$p=1$			
	$\lVert\boldsymbol{E}-\boldsymbol{E}_h\rVert_0$	Order	$\lVert\boldsymbol{E}-\boldsymbol{E}_h\rVert_{DG}$	Order
$N_{\mathrm{el}}=5$	5.2216E-01	—	7.4201E-01	—
$N_{\mathrm{el}}=40$	3.0615E-01	0.77	4.3594E-01	0.77
$N_{\mathrm{el}}=320$	7.1871E-02	2.09	1.0625E-01	2.04
$N_{\mathrm{el}}=2560$	1.7673E-02	2.02	2.9920E-02	1.83
$N_{\mathrm{el}}=20480$	4.4003E-03	2.01	1.0473E-02	1.51
	$p=2$			
	$\lVert\boldsymbol{E}-\boldsymbol{E}_h\rVert_0$	Order	$\lVert\boldsymbol{E}-\boldsymbol{E}_h\rVert_{DG}$	Order
$N_{\mathrm{el}}=5$	3.0892E-01	—	4.3901E-01	—
$N_{\mathrm{el}}=40$	3.3887E-02	3.19	4.9367E-02	3.15
$N_{\mathrm{el}}=320$	4.0850E-03	3.05	6.7364E-03	2.87
$N_{\mathrm{el}}=2560$	5.0782E-04	3.01	1.1718E-03	2.52
	$p=3$			
	$\lVert\boldsymbol{E}-\boldsymbol{E}_h\rVert_0$	Order	$\lVert\boldsymbol{E}-\boldsymbol{E}_h\rVert_{DG}$	Order
$N_{\mathrm{el}}=5$	6.4391E-02	—	9.1864E-02	—
$N_{\mathrm{el}}=40$	4.7730E-03	3.75	6.9565E-03	3.72
$N_{\mathrm{el}}=320$	2.4716E-04	4.27	4.3197E-04	4.01
	$p=4$			
	$\lVert\boldsymbol{E}-\boldsymbol{E}_h\rVert_0$	Order	$\lVert\boldsymbol{E}-\boldsymbol{E}_h\rVert_{DG}$	Order
$N_{\mathrm{el}}=5$	2.3335E-02	—	3.3088E-02	—
$N_{\mathrm{el}}=40$	5.5087E-04	5.40	8.1681E-04	5.34
$N_{\mathrm{el}}=320$	1.6179E-05	5.09	2.8348E-05	4.85
	$p=5$			
	$\lVert\boldsymbol{E}-\boldsymbol{E}_h\rVert_0$	Order	$\lVert\boldsymbol{E}-\boldsymbol{E}_h\rVert_{DG}$	Order
$N_{\mathrm{el}}=5$	4.3251E-03	—	6.1734E-03	—
$N_{\mathrm{el}}=40$	1.4449E-04	4.90	2.0586E-04	4.91
$N_{\mathrm{el}}=320$	1.1041E-06	7.03	1.8247E-06	6.82

Table IV: *Example 1.* Convergence of the method of Brezzi et al. on unstructured meshes.

	$p=1$			
	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{DG}$	Order
$N_{\text{el}}=54$	2.9871E-01	—	4.2626E-01	—
$N_{\text{el}}=432$	9.4108E-02	1.67	1.3758E-01	1.63
$N_{\text{el}}=3456$	2.7543E-02	1.77	4.3294E-02	1.67
$N_{\text{el}}=27648$	8.3263E-03	1.73	1.5441E-02	1.49
	$p=2$			
	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{DG}$	Order
$N_{\text{el}}=54$	3.3293E-02	—	4.8203E-02	—
$N_{\text{el}}=432$	5.4652E-03	2.61	8.4958E-03	2.50
$N_{\text{el}}=3456$	7.9569E-04	2.78	1.5428E-03	2.46
	$p=3$			
	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{DG}$	Order
$N_{\text{el}}=54$	5.2936E-03	—	7.7574E-03	—
$N_{\text{el}}=432$	5.2925E-04	3.32	8.3911E-04	3.21
$N_{\text{el}}=3456$	4.3710E-05	3.60	8.7359E-05	3.26
	$p=4$			
	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{DG}$	Order
$N_{\text{el}}=54$	5.6374E-04	—	8.2022E-04	—
$N_{\text{el}}=432$	3.8520E-05	3.87	5.8694E-05	3.80
	$p=5$			
	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{DG}$	Order
$N_{\text{el}}=54$	1.4759E-04	—	2.1091E-04	—
$N_{\text{el}}=432$	6.0329E-06	4.61	8.8707E-06	4.57

Table V: *Example 2.* Convergence of the IP-DG method on unstructured meshes.

	$p = 1$			
	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{DG}$	Order
$N_{\text{el}} = 177$	1.5881E-01	—	2.3438E-01	—
$N_{\text{el}} = 478$	8.7179E-02	1.81	1.2750E-01	1.84
$N_{\text{el}} = 1532$	6.8334E-02	0.63	9.9882E-02	0.63
$N_{\text{el}} = 5856$	4.2021E-02	1.09	6.4586E-02	0.98
$N_{\text{el}} = 27084$	3.3663E-02	0.43	5.0961E-02	0.46
	$p = 2$			
	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{DG}$	Order
$N_{\text{el}} = 177$	5.7874E-02	—	9.1278E-02	—
$N_{\text{el}} = 478$	2.9493E-02	2.04	5.4558E-02	1.55
$N_{\text{el}} = 1532$	1.9221E-02	1.10	3.6505E-02	1.04
$N_{\text{el}} = 5856$	1.3413E-02	0.80	2.6045e-02	0.76
	$p = 3$			
	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{DG}$	Order
$N_{\text{el}} = 177$	2.2440E-02	—	4.0852E-02	—
$N_{\text{el}} = 478$	1.3614E-02	1.51	3.1392E-02	0.80
$N_{\text{el}} = 1532$	7.9415E-03	1.39	2.0326E-02	1.12
	$p = 4$			
	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{DG}$	Order
$N_{\text{el}} = 177$	1.1709E-02	—	2.2644E-02	—
$N_{\text{el}} = 478$	7.0465E-03	1.53	1.8140E-02	0.67

Table VI: *Example 2.* Convergence of the method of Brezzi et al. on unstructured meshes.

	$p=1$			
	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{DG}$	Order
$N_{\mathrm{el}}=177$	1.5629E-01	—	2.3634E-01	—
$N_{\mathrm{el}}=478$	8.6095E-02	1.80	1.3488E-01	1.69
$N_{\mathrm{el}}=1532$	6.8245E-02	0.60	1.0016E-01	0.77
$N_{\mathrm{el}}=5856$	4.1962E-02	1.09	6.5097E-02	0.96
$N_{\mathrm{el}}=27084$	3.3624E-02	0.43	5.1538E-02	0.46
	$p=2$			
	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{DG}$	Order
$N_{\mathrm{el}}=177$	5.8141E-02	—	9.1806E-02	—
$N_{\mathrm{el}}=478$	2.9484E-02	2.05	5.4598E-02	1.57
$N_{\mathrm{el}}=1532$	1.9231E-02	1.10	3.6324E-02	1.05
$N_{\mathrm{el}}=5856$	1.3387E-02	0.81	2.5930E-02	0.75
	$p=3$			
	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{DG}$	Order
$N_{\mathrm{el}}=177$	2.2312E-02	—	4.0874E-02	—
$N_{\mathrm{el}}=478$	1.3563E-02	1.50	3.1478E-02	0.79
$N_{\mathrm{el}}=1532$	7.9297E-03	1.38	2.0212E-02	1.14
	$p=4$			
	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0$	Order	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{DG}$	Order
$N_{\mathrm{el}}=177$	1.1736E-02	—	2.2663E-02	—
$N_{\mathrm{el}}=478$	7.0572E-03	1.54	1.8099E-02	0.68

Table VII: Computational cost of solving the discrete system with `MINRES`.

	$\mathtt{mesh}_4^1$		$\mathtt{mesh}_3^2$	
Preconditioner	Rel. residual	CPU time	Rel. residual	CPU time
None	3.9E-02	1516s	1.6E-02	933s
$\gamma = 0$	2.5E-04	3185s	8.2E-03	2110s
$\gamma = 10^2$	2.5E-04	3183s	8.2E-03	2110s
$\gamma = 10^4$	2.2E-04	3183s	7.4E-03	2107s
$\gamma = 10^6$	9.7E-06	3175s	8.9E-04	2107s
$\gamma = 10^8$	3.3E-04	3189s	3.6E-04	2105s
$\gamma = 10^{10}$	—	—	1.5E-05	2107s

Table VIII: Known or predicted minimum numbers and achieved numbers of cubature points for the Gauss integration rule over triangles and tetrahedra

	Triangles		Tetrahedra	
Poly. order	Min.	Achieved	Min.	Achieved
1	1	1	1	1
2	3	3	4	4
3	4	4	5	5
4	6	6	11	11
5	7	7	14	14
6	12	12	24	24
7	13	13	28	31
8	16	16	40	43
9	19	19	52	53
10	24	25	68	
11	27	27		126
12	33	33		
13	36	37		210