A P- and T-invariant Characterization of Product Form and Decomposition in Stochastic Petri Nets

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Abstract

Structural product form and decomposition results for stochastic Petri nets are surveyed, unified and extended. The contribution is threefold. First, the literature on structural results for product form over the number of tokens at the places is surveyed and rephrased completely in terms of T-invariants. Second, based on the underlying concept of group-local-balance, the product form results for stochastic Petri nets are demarcated and an intuitive explanation is provided of these results based on T-invariants, only. Third, a decomposition result is provided that is completely formulated in terms of both T-invariants *and* P-invariants.

Keywords: Stochastic Petri net, Product form, Decomposition, T-invariant, P-invariant.

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1 Introduction

Competition over resources is an important issue in many practical systems. Examples of such systems are computer systems, telecommunication networks, flexible manufacturing systems and hospitals, which typically consist of many departments and serve a wide variety of patient types. Pathways of patients are generally stochastic and various patient flows share different resources, of which operating rooms and diagnostic testing facilities are the most apparent. Typical questions arising are identification of bottlenecks, achievable throughput and maximization of resource utilization. Therefore, performance analysis is an important issue in the design and implementation of such real life systems.

Several approaches exist for performance analysis of complex systems, such as discrete-event simulation, numerical approximations or exact analytical results. Obtaining analytical results has two main advantages. First, it provides vital insight in the qualitative behavior of involved systems, so that the key characteristics of a system can be detected. In particular, qualitative results related to the structure of the system are often of great importance. Second, it enables efficient computation of relevant performance measures. In many theoretical and practical studies of performance models involving stochastic effects, the statistical distribution of items (customers, jobs, etc.) over places (workstations, queues, etc.) is of great interest, since various of performance measures can be computed from this distribution.

Three main formalisms exist for obtaining analytical closed form results for networks: queueing networks, stochastic process algebras and stochastic Petri nets. The selection of a specific formalism when studying a system preferably depends on the characteristics under investigation. queueing networks are most suitable when the queueing structure at different locations in the network is the key aspect of the system. When a system consists of building blocks of different processes that are composed into a network, stochastic process algebras may be preferred. Stochastic Petri nets are appropriate when the flow of items and information through the network is the main feature of the system. When a specific formalism is applied, all network characteristics and all results are preferably formulated in the semantics of that formalism. In this paper we focus on Stochastic Petri nets, since we are interested in the interaction of flows within the system, such as naturally occurring in hospital environments. All results are formulated in terms of the Petri net structure given by the P- and T-invariants, the central concepts in Petri Nets.

Composition and decomposition of closed form results contribute to less computational effort requirements and greater understanding of network behavior and performance. It allows studying a system by analyzing the characteristics of separate components. In this paper, we study closed form results for the equilibrium distribution of the number of tokens at the places of a stochastic Petri net and the decomposition of this equilibrium distribution into several components corresponding to subnets of the stochastic Petri net. Exact analytical results for the distribution of the number of items at places in performance models are in general very difficult to obtain. One of the most important analytical results for the equilibrium distribution describing the number of items at places in a performance model is the so-called *product form* equilibrium distribution found for a fairly wide class of theoretical queueing models. However, practical performance models seldom satisfy the product form conditions. Still, results obtained via the theoretical product form distributions are used for practical problems since these results are found to be robust, that is models which violate the product form conditions are often found to behave in a way very similar to a product form counterpart. The obvious advantages of these product form distributions are their simplicity, since the network behavior is captured in closed form in only a limited set of parameters. This makes product form solutions easy and powerful to use for computational issues as well as for theoretical reflections for performance models involving congestion. Another important advantage of product form solutions is that it enables to break-down the analysis of a network in the analysis of separate components of the network.

It is widely believed that a form of *local balance* is the common element for all performance models with a product form equilibrium distribution. In this paper, *group-local-balance* will shown to be the concept identifying that the equilibrium distribution of a stochastic Petri net is of product-form nature. Boucherie and Van Dijk [6] presented the group-local-balance concept as the basis for the analysis of batch routing queueing networks. This paper provides a translation of these results into Petri net terminology. The results on the Markov chain level will then provide the foundation to discuss and further investigate structural Petri net implications. We survey the various structural results that are known for stochastic Petri nets with a product form equilibrium distribution over the number of tokens at the places ([4, 5, 12, 15, 22, 28]). The product form results for stochastic Petri nets known from literature will shown to be unified by group-local-balance, as it forms the connecting principle between these results and the results known for batch routing queueing networks ([6, 31]). The results are derived and presented step-by-step to provide an intuitive understanding of the Petri net structure underlying the product form results.

The first structural product form results for stochastic Petri nets were presented by Henderson et. al. [28]. These results are based on the assumption that a positive solution exists for a linear set of equations similar to the traffic equations for queueing networks. It will be shown that group-local-balance implies a positive solution to this linear set of equations, known as the *routing chain*, to exist. A characterization of the structure of the Petri net that is necessary and sufficient for the existence of a positive solution to the routing chain was provided by Boucherie and Sereno [4]. We show that this characterization implies that group-local-balance requires the stochastic Petri net to be an $\mathcal{S}\Pi$-net, a stochastic Petri net in which each transition is covered by a minimal support T-invariant [22]. Taking group-local-balance as starting point enables us to provide additional structural implications and a more intuitive explanation of the known results. By formulating every result in terms of the Petri net structure given by the T-invariants, we also provide structural insights for results known at an algebraic level.

Finally, from the detailed understanding of the structure behind product results, we are able to establish a decomposition result. This decomposition result is a generalization of the results obtained by Frosch and Natarajan [19, 20] for closed synchronized systems of stochastic sequential processes, a class of Petri nets in which state machines are synchronized via buffer places. The decomposition result is completely formulated in terms of P- and T-invariants. Similar to buffer places, we define conflict places, which are places that are shared by different minimal closed support T-invariants. Using the P-invariants to assign conflict places as surplus places, places that can be omitted in characterizing the marking of the Petri net, we obtain an algorithmic procedure to verify whether product form holds and for decomposition of the stochastic Petri net into subnets. These subnets correspond to one or more common input bag classes, equivalence classes of T-invariants of the stochastic Petri nets that share an input bag.

Statement of contribution. Our contribution is threefold:

1. We survey the various structural results that are known for stochastic Petri nets with a product form equilibrium distribution over the number of tokens at the places and rephrases all these results in terms of T-invariants.

2. We unify and extend the product form results for stochastic Petri nets by showing that *group-local-balance* can be identified as the concept underlying all these structural results and we provide additional structural implications and an intuitive explanation of the known and new results, all based on T-invariants only.

3. We provide a decomposition result that is completely formulated in terms of both P- and T-invariants and its derivatives as defined in the paper: common input bag classes, conflict places and surplus places.

Outline. This paper is organized as follows. In Section 2, a detailed literature survey of product form results and decomposition is provided. For insight and self-containedness, a thorough introduction into the (stochastic) Petri net formalism is provided in Section 3. In Section 4, product form results for batch routing queueing networks based on the group-local-balance concept are translated into Petri net terminology. These results, presented on the Markov chain level, provide the basis for Section 5, in which structural Petri net implications are discussed. This section is concluded by an algorithm to verify whether a specific stochastic Petri net possesses a product form equilibrium distribution, and if so, to construct this product form. Section 6 presents the new decomposition result and is ended with an algorithm by which all possible decompositions of a product form stochastic Petri can be generated. In the closing Section 7, the results are summarized and directions for future research are discussed.

2 Literature

Product form results exist on different levels. In the classical product form result the equilibrium distribution of a network can be expressed as a product over the nodes of the network. In this section we provide a survey of such results for queueing networks, stochastic process algebras and stochastic Petri nets in Section 2.1-2.3. A more general product form result is when the equilibrium distribution of a network is a (normalized) product over the marginal distribution of subnets. A survey of such decomposition results will be provided in Section 2.4.

2.1 Product form results for queueing networks

For queueing networks an important analytical result is the product form equilibrium distribution for the number of customers at the stations. The basis of the development of product form literature is given by Jackson [39]. Jackson's product form states that the equilibrium distribution of the queueing network is the product of the marginal distributions at the stations of the queueing network. Product form results for closed queueing networks, networks in which a fixed number of customers is present, were obtained by Gordon and Newell [21]. The results of Jackson [39] and Gordon and Newell [21] were proven on the basis of global balance.

The concept of partial balance as the basis of product form was introduced in [58, 59]. These results were generalized to Kelly-Whittle networks (see e.g. [40, 60]), networks with job-types and various service disciplines (see e.g. [1, 36, 54]) and to batch routing (see e.g. [6, 29, 31]) and discrete-time networks (see e.g. [14]). A different approach for obtaining product form equilibrium distributions is based on the notion of quasi-reversibility (see e.g. [11, 40, 48]).

2.2 Product form results for stochastic Petri nets

For stochastic Petri nets, the first product form results for the number of tokens at the places were obtained by Lazar and Robertazzi [43] for the class of stochastic Petri nets consisting of 'linear task sequences', a number of tasks that must be executed consecutively. Since these first results, considerable extensions have been derived by several authors. In a series of papers, Henderson et al. [28, 30, 32] translated and extended product form results for batch routing queueing networks to stochastic Petri nets, which are equivalent to batch routing queueing networks at the level of the underlying stochastic process.

The starting point for the analysis of product form stochastic Petri nets is the assumption that a solution exists for the 'routing chain', a set of linear equations similar to the traffic equations for queueing networks. The product form results for stochastic Petri nets obtained in [28, 30, 32] were based on the assumption that a positive solution exists for the routing chain. Necessary conditions for such a solution to exist were provided in Henderson et al. [28].

A full characterization of the structure of stochastic Petri nets necessary and sufficient for the existence of a positive solution for the routing chain was obtained in [4, 15]: all transitions of the Petri net should be covered by 'closed support T-invariants'. This new type of T-invariant was also introduced in [4, 15] and is a T-invariant that closely resembles the 'task sequences' used by Lazar and Robertazzi [43]. As such, the existence of a solution for the routing chain was completely characterized on the basis of the structure of the Petri net. This class of stochastic Petri nets was later denoted as $\mathcal{S}\Pi$-nets by Haddad et al [22].

For an $\mathcal{S}\Pi$-net, Coleman et al. [13] were the first to formulate an additional requirement sufficient for product form in stochastic Petri net by a numerical condition on the transition rates. Haddad et al. [22] established a characterization of $\mathcal{S}\Pi$-nets possessing a product form solution nets irrespective of the values of the transition rates and label these $\mathcal{S}\Pi$-nets as $\mathcal{S}\Pi^2$-nets. The conditions of Coleman et al. [13] and Haddad et al. [22] are algebraic conditions which lack intuition in terms of Petri net structure. *The present paper unifies these results by the concept of group-local-balance and extends these results by formulating all product form results in terms of T-invariants.*

2.3 Product form for stochastic process algebras

The stochastic process algebras formalism is was build upon the classical process algebras during the 1990s to include actions requiring a random time. The principle of process algebras is that complex systems are defined by a composed collection of agents who execute actions, which may or may not be concurrent. Various different languages of stochastic process algebras were introduced. Although most product form results are formulated in the paradigm of *Performance Evaluation Process Algebra* (PEPA), defined by Hillston in [33], the results can easily be generalized to any of the other stochastic process algebras. A comprehensive survey of product form results for stochastic process algebras can be found in the PhD thesis Marin [45]. Marin distinguishes between various types of product form results: models based on reversibility (e.g. [34]), models based on quasi-reversibility (e.g. [27]), models based on the product form results for stochastic Petri nets by Henderson et. al [28] and Coleman et al. [13] (e.g. [52]) and models based on the *Reversed Compound Agent Theorem*(RCAT) theorem and its extensions (e.g. [23, 24, 25, 26]). In addition, models based on the cooperating Markov chains of the form presented by Boucherie in [3] are distinguished (e.g. [26, 35]).

2.4 Decomposition

A network can be decomposed if its stationary distribution factorizes into the stationary distributions of the nodes of which the network is comprised; the network is then of product form. Apart from the theoretical interest, decomposition results are also of substantial practical importance: finding the stationary distribution of an entire network usually requires an enormous computational effort, whereas the stationary distribution of a single node can be found relatively easily. The first, and perhaps most famous, decomposition results for queueing networks have been reported by Jackson [39]: the classical Jackson product form result. Decomposition of networks into subnetworks have been a topic of research for queueing networks. Two streams of literature have been developed in parallel: results based on partial balance (e.g. [7, 9, 10, 37, 41]) and results based on quasi-reversibility (e.g. [2, 8, 55, 57]). Recently, in a setting of general stochastic processes, these results have been unified and extended in [11, 38].

For stochastic Petri nets decomposition results were initialized by Lazar and Robertazzi [44] for connected subnets of task sequences and extended by Boucherie [3] in the framework of competing Markov chains. Frosch and Natarajan [19, 20] derived product form results for so-called closed synchronized systems of stochastic sequential processes, a class of Petri nets in which state machines are synchronized via buffer places. The results in these references may also be interpreted as composition results since the networks are essentially obtained by composing subnets in to a larger net, similar to the composition structure of stochastic process algebras. As such, no procedure is provided in the literature to algorithmically characterize subnets in a given stochastic Petri net and to verify whether product form holds. *In this paper, decomposition results will be presented based on the structure of a Petri net formulated exclusively in terms of P- and T-invariants.*

3 Preliminaries

The aim of this section is to provide a general introduction into the formal Petri net language and the Petri net concepts that will be relevant for the analysis in subsequent sections. First, basic definitions of Petri nets and stochastic Petri nets are presented. Next, structural and behavioral properties are introduced. Also, some results derived from these properties of a Petri net that will be used in subsequent sections are listed.

3.1 Petri nets

Definitions, properties and results will be presented schematically to provide the reader a convenient reference to the numerous concepts. More elaborate overviews of definitions, properties and results can be found in the survey of Murata [49] and the book of Peterson [50].

3.1.1 Definitions

Definition 3.1 (**Petri net**). A Petri net is a weighted bipartite graph with nodes being either places or transitions and is defined by the 4-tuple $\mathcal{PN} = (P, T, I, O)$, where

- $P = \{p_1, \ldots, p_N\}$ is a finite set of places,
- $T = \{t_1, \ldots, t_M\}$ is a finite set of transitions,

- $I, O : P \times T \rightarrow \mathbb{N}$ are the input and output functions identifying the relation between the places and the transitions.

Definition 3.2 (**Marking**). A *marking* $\boldsymbol{m} = (m(n),\ n = 1, \ldots, N)$ of a Petri net is a vector in $\mathbb{N}_0^N$, where $m(n)$ represents the number of *tokens* at place p_n.

Definition 3.3 (**Marked Petri net**). A marked Petri net is a Petri net defined by the 5-tuple $(\mathcal{PN}, \boldsymbol{m}_0) = (P, T, I, O, \boldsymbol{m}_0)$, where $\boldsymbol{m}_0$ is the initial marking.

Definition 3.4 (**Input bag - Output bag**). $I(\cdot,\cdot)$ and $O(\cdot,\cdot)$ give the vectors $\boldsymbol{I}(t) = (I(t_1), \ldots, I(t_N))$ and $\boldsymbol{O}(t) = (O(t_1), \ldots, O(t_N))$, where $I_n(t) = I(p_n, t)$, and $O_n(t) = O(p_n, t)$. The vectors $\boldsymbol{I}(t)$ and $\boldsymbol{O}(t)$ are called the *input* and *output bags* of transition $t \in T$, respectively representing the number of tokens needed at the places to fire transition t, and the number of tokens released to the places after firing transition t.

Definition 3.5 (**Transition enabling and firing**). A necessary and sufficient condition for transition t to be *enabled* in marking $\boldsymbol{m}$ is that $m(n) \geq I_n(t)$. When transition t *fires*, then the next state of the Petri net is $\boldsymbol{m}' = \boldsymbol{m} - \boldsymbol{I}(t) + \boldsymbol{O}(t)$. Symbolically this is denoted as $\boldsymbol{m}[t > \boldsymbol{m}'$.

Definition 3.6 (**Firing sequence**). A finite sequence of transitions $\sigma = t_{\sigma_1} t_{\sigma_2} \cdots t_{\sigma_k}$ is a finite *firing sequence* of the Petri net if there exists a sequence of markings $\boldsymbol{m}_{\sigma_1}, \ldots, \boldsymbol{m}_{\sigma_k}$ for which $\boldsymbol{m}_{\sigma_i}[t_{\sigma_1} > \boldsymbol{m}_{\sigma_{i+1}},\ i = 1, \ldots, k-1$. Symbolically this will be denoted as $\boldsymbol{m}[\sigma > \boldsymbol{m}'$.

Definition 3.7 (**Incidence matrix**). The *incidence matrix* $\boldsymbol{A}$ with entries $A(p,t) = O(p,t) - I(p,t)$ describes the change in the number of tokens in place p when transition t fires.

Definition 3.8 (**Firing count vector**). A vector $\bar{\sigma}$ is the *firing count vector* of the firing sequence σ if $\bar{\sigma}(t)$ equals the number of times transition t occurs in the firing sequence σ.

Definition 3.9 (**State equation**). If $\boldsymbol{m}_0[\sigma > \boldsymbol{m}$, then $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{A}\bar{\sigma}$. This equation is referred to as the *state equation* for the Petri net.

Definition 3.10 (**Closed set**). For $\mathcal{T} \subseteq T$ define $\mathcal{R}(\mathcal{T})$, the set of input and output bags for the transitions in $\mathcal{T}$, as $\mathcal{R}(\mathcal{T}) = \bigcup_{t \in \mathcal{T}} \{\boldsymbol{I}(t) \cup \boldsymbol{O}(t)\}$. $\mathcal{R}(\mathcal{T})$ is a closed set if for all $\boldsymbol{g} \in \mathcal{R}(\mathcal{T})$ there exist $t, t' \in \mathcal{T}$ such that $\boldsymbol{g} = \boldsymbol{I}(t)$, as well as $\boldsymbol{g} = \boldsymbol{O}(t')$, that is if each output bag is also an input bag, and each input bag is also an output bag for a transition in $\mathcal{T}$.

3.1.2 Properties

Two types of properties are distinguished. Properties which depend on the initial marking are referred to as *behavioral* and those which are independent on the initial marking as *structural.* Behavioral and structural properties will respectively be marked by the labels [B] and [S].

Definition 3.11 (**Reachability [S]**). A marking $\boldsymbol{m}'$ is *reachable* from marking $\boldsymbol{m}_0$ if a firing sequence σ exists such that $\boldsymbol{m}_0[\sigma > \boldsymbol{m}'$.

Definition 3.12 (**Reachability set [B]**). The *reachability set* $\mathcal{M}(\mathcal{PN}, \boldsymbol{m}_0)$ is a subset of $\mathbb{N}^N$ and gives all reachable markings of the Petri net with initial making $\boldsymbol{m}_0$.

Definition 3.13 (***T*-invariant [S]**)**.** A vector $\boldsymbol{x} \in \mathbb{N}_0^M$ is a *T-invariant* if $\boldsymbol{x} \neq 0$, and $\boldsymbol{A}\boldsymbol{x} = 0$. From the state equation we obtain that a T-invariant represents a firing sequence that brings a marking back to itself (Murata [49]). So T-invariants define potential cycles in the reachability set.

Definition 3.14 (***P*-invariant [S]**)**.** A vector $\boldsymbol{y} \in \mathbb{N}_0^N$ is a *P-invariant* (sometimes called S-invariant) if $\boldsymbol{y} \neq 0$, and $\boldsymbol{y}\boldsymbol{A} = 0$. P-invariants correspond to the conservation of tokens in subsets of places. A P-invariant identifies a set of places such that the weighted sum of the number of tokens distributed over these places remains constant for all markings in the reachability set.

Definition 3.15 (**Support [S]**)**.** The *support* of a T-invariant $\boldsymbol{x}$ or P-invariant $\boldsymbol{y}$ is the set of transitions or places respectively corresponding to non-zero entries of $\boldsymbol{x}$ and $\boldsymbol{y}$, and are denoted by $\|\boldsymbol{x}\|$ and $\|\boldsymbol{y}\|$, i.e., $\|\boldsymbol{x}\| = \{t \in \mathcal{T} \mid x(t) > 0\}$ and $\|\boldsymbol{y}\| = \{p \in \mathcal{P} \mid y(p) > 0\}$.

Definitions 3.16 and 3.17 are stated in terms of T-invariants. The definitions are analogous for P-invariants.

Definition 3.16 (**Minimal invariant [S]**)**.** A T-invariant is minimal if no subset of the support is the support of some other T-invariant, i.e., $\boldsymbol{x}$ is a *minimal T-invariant* if there is no other T-invariant $\boldsymbol{x}'$ such that $x'(t) \leq x(t)$ for all t.

Definition 3.17 (**Minimal support invariant [S]**)**.** A support is minimal if no proper nonempty subset of the support is also a support of a T-invariant. An invariant with minimal support is a *minimal support invariant.*

Definition 3.18 (**Closed *T*-invariant [S]**)**.** A T-invariant is *closed* if the set of input and output bags for the transitions in its support, $\mathcal{R}(\|\boldsymbol{x}\|)$, is a closed set.

Definition 3.19 (**Minimal closed support *T*-invariant [S]**)**.** A T-invariant is a *minimal closed support T-invariant* if it is closed and has minimal support.

Definition 3.20 (**Liveness [B]**)**.** A transition is $t \in T$ is *live* if no matter what marking has been reached from $\boldsymbol{m}_0$ it is possible to ultimately fire transition t again. A Petri net is *live* under initial marking $\boldsymbol{m}_0$ if every transition is live under $\boldsymbol{m}_0$. An extensive discussion of liveness and related concepts is given in Murata [49].

Definition 3.21 (**Structural liveness [S]**)**.** A Petri net is *structurally live* if there exists an initial marking $\boldsymbol{m}_0$ for which the net is live.

Definition 3.22 (**Home state [B]**)**.** A marking $\boldsymbol{m}$ is a *home state* if for each marking in $\boldsymbol{m}' \in \mathcal{M}(\mathcal{PN}, \boldsymbol{m}_0)$, $\boldsymbol{m}$ is reachable from $\boldsymbol{m}'$, i.e., $\forall \boldsymbol{m}' \in \mathcal{M}(\mathcal{PN}, \boldsymbol{m}_0) : \boldsymbol{m} \in \mathcal{M}(\mathcal{PN}, \boldsymbol{m}')$.

Definition 3.23 (**Boundedness [B]**)**.** A Petri net is *k-bounded* or simply *bounded* if the number of tokens in each place does not exceed a finite number k for any marking in the reachability set $\mathcal{M}(\mathcal{PN}, \boldsymbol{m}_0)$.

Definition 3.24 (**Structural Boundedness [S]**)**.** A Petri net is *structurally bounded* if it is bounded for all initial markings.

Definition 3.25 (**Conservative [S]**)**.** A Petri net is *conservative* if there exists a positive integer

$y(p)$ for every place p such that the weighted sum of tokens $\boldsymbol{ym} = \boldsymbol{ym}_0$, for every marking $\boldsymbol{m}$ in the reachability set $\mathcal{M}(\mathcal{PN}, \boldsymbol{m}_0)$.

Definition 3.26 (Consistent [S]). A Petri net is *consistent* if there exists a marking $\boldsymbol{m}_0$ and a firing sequence σ from $\boldsymbol{m}_0$ back to $\boldsymbol{m}_0$ such that every transition occurs at least once in σ.

3.1.3 Results

Result 3.27 (Murata [49])**.** A Petri net is conservative if and only if it is covered by P-invariants, that is for all $p \in P$ there exists a P-invariant $\boldsymbol{y}$ such that $p \in \|\boldsymbol{y}\|$.

Result 3.28 (Murata [49])**.** A conservative Petri net is structurally bounded. As a consequence, a Petri net that is covered by P-invariants is structurally bounded.

Result 3.29 (Murata [49])**.** A Petri net is consistent if and only if it is covered by T-invariants, that is for all $t \in T$ there exists a T-invariant $\boldsymbol{x}$ such that $t \in \|\boldsymbol{x}\|$.

Result 3.30 (Murata [49])**.** A live Petri net is consistent. As a consequence, a live Petri net is covered by T-invariants.

Result 3.31 (Murata [49])**.** A structurally bounded and structurally live Petri net is both conservative and consistent. As a consequence, a structurally bounded and structurally live Petri net is both covered by P-invariants and T-invariants.

Result 3.32 (Memmi and Roucairol [47])**.** There is a unique minimal T-invariant corresponding to a minimal support (*minimal support T-invariant*). Let $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k$ denote the minimal support T-invariants. Any T-invariant $\boldsymbol{x}$ can be written as a linear combination of minimal support T-invariants:

$$\boldsymbol{x} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}^i$$

where $\lambda_i \in \mathbb{Q}^+$, $i = 1, \ldots, k$. The equivalent result holds for P-invariants.

Remark 3.33. Two remarks with respect to the decomposition result 3.32 of Memmi and Roucairol can be made. First, since the elements of minimal invariants are required to be non-negative, the minimal support invariants may be linearly dependent, so that there may exist more invariants than the dimension of the null space. Second, for the decomposition to be in minimal support invariants it is essential that the weight factors λ_i are allowed to be rational numbers. If one restricts to integral weight factors, additional invariants may need to be added to the set of minimal support T-invariants to obtain a decomposition result. An extensive discussion on different decomposition results is provided by Krückeberg and Jaxy [42]. In this reference, also efficient algorithms are presented to obtain the sets of minimal T- and P-invariants from the incidence matrix $\boldsymbol{A}$.

Result 3.34 (Boucherie and Sereno [5])**.** A T-invariant $\boldsymbol{x}$ is a minimal closed support T-invariant if the firing sequence of $\boldsymbol{x}$ is *linear*, that is for each $t \in \|\boldsymbol{x}\|$ there is a unique $t' \in \|\boldsymbol{x}\|$ such that $\boldsymbol{O}(t) = \boldsymbol{I}(t')$. As a consequence $x_i \leq 1$, $i = 1, \ldots, M$. Conversely, if the firing sequence of a T-invariant $\boldsymbol{x}$ is linear, then $\boldsymbol{x}$ is a closed support T-invariant.

3.2 Stochastic Petri nets

Definition 3.35 (Stochastic Petri net). A stochastic Petri net is a Petri net defined by the 5-tuple $\mathcal{SPN} = (P, T, I, O, Q)$, where (P, T, I, O) is a Petri net, and $Q = (q(t_1), \ldots, q(t_M))$ is a set of exponential firing rates associated with the set of transitions $T = \{t_1, \ldots, t_M\}$. Distributions associated with different transitions are independent. The firing execution policy of the stochastic Petri net is the race model with age memory.

Definition 3.36 (Marked stochastic Petri net). A marked stochastic Petri net is a stochastic Petri net defined by the 6-tuple $(\mathcal{SPN}, \boldsymbol{m}_0) = (P, T, I, O, Q, \boldsymbol{m}_o)$, where $\boldsymbol{m}_0$ is the initial marking.

Definition 3.37 ($\mathcal{S}$Π-net). A Π-net is a Petri net in which all transitions $t \in T$ are covered by minimal closed support T-invariants, that is for all $t \in T$ there exists an $i \in \{1, \ldots, k\}$ such that $t \in \|\boldsymbol{x}^i\|$ and $\|\boldsymbol{x}^i\|$ is a closed set. An $\mathcal{S}$Π-net is a stochastic Π-net.

There exist various firing execution policies for stochastic Petri nets. For an extensive discussion on these policies, see [46]. We assume that the firing execution policy follows a race model with age memory. The race model with age memory states that whenever a change of marking enables a transition that was not previously enabled since its last firing, this transition samples a firing delay from its associated distribution and sets a timer at that value. While the transition is enabled the timer is decreased and while the transition is disabled the countdown is paused. When the timer reaches zero the transition fires. For exponentially distributed firing times, due to the memoryless property of the exponential distribution, the time until firing of a the transition that was disabled and has become enabled again, is again exponentially distributed with the same mean. Since the minimum of two exponential random variables is exponentially distributed, the time until the first transition fires in marking $\boldsymbol{m}$ is also exponentially distributed.

As a consequence of the exponential firing times, the stochastic process describing the evolution of the Petri net is a time-homogeneous continuous-time Markov chain $\mathbf{X}$ at state space $\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$. Denote the transition rates of $\mathbf{X}$ by $Q = (q(\boldsymbol{m}, \boldsymbol{m}'), \boldsymbol{m}, \boldsymbol{m}' \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0))$. To avoid anomalies, we assume the process is regular, that is, at most finitely many transitions can fire in finite time ([56], Chapter 2). It will be assumed that each transition of the Markov chain representing the Petri net is due to exactly one transition $t \in T$ that fires. Note that the firing of multiple transitions can be incorporated by adding extra transitions representing the combination of several transitions that fire with suitable firing rates.

The evolution of the Markov chain describing the stochastic Petri net is as follows. A transition t in marking $\boldsymbol{m}$ can be enabled only if $\boldsymbol{m} - \boldsymbol{I}(t) \in \mathbb{N}_0^N$. Furthermore, we will allow multiple transitions to have the same enabling condition, i.e., for $t_i \neq t_j$ it is allowed that $\boldsymbol{I}(t_i) = \boldsymbol{I}(t_j)$. Of course, the output bag will not be the same, otherwise these two transitions could be represented by only one. The rate

$$q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{m} - \boldsymbol{I}(t)) \tag{1}$$

is associated with transition t bringing $\boldsymbol{m}$ to $\boldsymbol{m}' = \boldsymbol{m} - \boldsymbol{I}(t) + \boldsymbol{O}(t)$. Note that a transition from marking $\boldsymbol{m}$ to marking $\boldsymbol{m} - \boldsymbol{I}(t) + \boldsymbol{O}(t)$ may occur due to other transitions too. The total transition rate from marking $\boldsymbol{m}$ to marking $\boldsymbol{m}'$ is therefore

$$q(\boldsymbol{m}, \boldsymbol{m}') = \sum_{\{\boldsymbol{n} \in \mathbb{N}_0^N,\ t \in T:\ \boldsymbol{n} + \boldsymbol{I}(t) = \boldsymbol{m},\ \boldsymbol{n} + \boldsymbol{O}(t) = \boldsymbol{m}'\}} q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{n}). \tag{2}$$

When analyzing the Markov chain $\mathbf{X}$ describing the behavior of a stochastic Petri net, it will be convenient to consider the equivalent semi-Markov description. In the semi-Markov description all transitions, say $t_{i_1}, \ldots, t_{i_k}$ with identical input bag $\boldsymbol{I}(t_{i_1})$ are amalgamated into a single transition t_i with firing rate

$$q(t_i; \boldsymbol{m} - \boldsymbol{I}(t_{i_j})) = \sum_{j=1}^{k} q(\boldsymbol{I}(t_{i_j}), \boldsymbol{O}(t_{i_j}); \boldsymbol{m} - \boldsymbol{I}(t_{i_j})). \tag{3}$$

The output bag of this new transition is probabilistic. The probability that output bag $\boldsymbol{O}(t_{i_j})$ occurs is determined by the original firing rates:

$$p(\boldsymbol{I}(t_i), \boldsymbol{O}(t_{i_j}); \boldsymbol{m} - \boldsymbol{I}(t_{i_j})) = \frac{q(\boldsymbol{I}(t_{i_j}), \boldsymbol{O}(t_{i_j}); \boldsymbol{m} - \boldsymbol{I}(t_{i_j}))}{q(t_i; \boldsymbol{m} - \boldsymbol{I}(t_{i_j}))}, \tag{4}$$

so that

$$q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{m} - \boldsymbol{I}(t)) = q(t; \boldsymbol{m} - \boldsymbol{I}(t)) p(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{m} - \boldsymbol{I}(t)). \tag{5}$$

The advantage of the semi-Markov description of the Markov chain $\mathbf{X}$ is that it establishes a unique relation between an input bag and a transition. In the following we will use both formulations interchangeably. When analyzing the structural properties of a stochastic Petri net it will be convenient to have a unique relation between an input bag and an output bag. Note that the equivalence of the Markov and the semi-Markov description is due to the memoryless property of the exponential firing rates.

We are interested in calculating the steady-state behavior of the continuous-time Markov chain $\mathbf{X}$ modelling the marked stochastic Petri net $(\mathcal{SPN}, \boldsymbol{m}_0)$. From standard Markov theory we know that $\mathbf{X}$ is irreducible and positive recurrent if and only if a unique collection of positive numbers $\pi = (\pi(\boldsymbol{m}), \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0))$ summing to unity, exists satisfying the *global balance equations*,

$$\sum_{\boldsymbol{m}' \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)} \{\pi(\boldsymbol{m}) q(\boldsymbol{m}, \boldsymbol{m}') - \pi(\boldsymbol{m}') q(\boldsymbol{m}', \boldsymbol{m})\} = 0 \,, \qquad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0). \tag{6}$$

This $\pi = (\pi(\boldsymbol{m}), \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0))$ is called the *equilibrium distribution*.

As the Markov chain is chosen such that it describes the evolution of the stochastic Petri net under consideration, irreducibility and positive recurrence properties necessary to obtain a unique equilibrium distribution for the Markov chain should preferably be characterized directly from the Petri net structure.

The state space of a Markov chain $\mathbf{X}$ partitions in communicating classes [51]. As we are interested in the steady state behavior of $\mathbf{X}$ we can analyze the process at each class separately. Moreover, we are not interested in transient classes, as transient states will vanish in the equilibrium distribution of the stochastic Petri net. Thus, we will focus on stochastic Petri nets of which the corresponding Markov chain $\mathbf{X}$ is irreducible.

To prevent the presence of transient classes, we are restricted to Petri nets that are live and therefore covered by T-invariants. If the Petri net is live and has a home state, then $\mathbf{X}$ is irreducible. (Note that irreducibility of the Markov chain is called reversibility in the Petri net literature [49]. The notion of reversibility for Petri nets should not be confused with the notion of reversibility for Markov chains [40]).

If the reachability set is finite, positive recurrence follows from irreducibility. Otherwise, for $\mathbf{X}$ to be stable additional assumptions on the transition rates are required to ensure that the rate at which tokens are created is smaller then the rate at which they are destroyed. This problem is for example addressed in [18]. To avoid non-regularity, we restrict our attention to stochastic Petri nets with a finite reachability set, thus to structurally bounded nets. By Result 3.31, for a live net to be structurally bounded, the net must be covered by P-invariants.

A live Petri net is structurally live. A complete characterization of structural liveness for a general Petri net is unknown [49]. Liveness and boundedness are not related to the existence of a home state [49]. It is beyond the scope of this paper to provide a complete overview for general Petri nets (see [16] and [49] for elaborate discussions). For $\mathcal{S}\Pi$-nets (see Definition 3.37), in Theorem 5.7 we will provide a complete characterization of structurally liveness and existence of a home state. Note that also in this case, for a specific initial marking liveness still needs to be checked, which may be a cumbersome problem (see Haddad et al. [22] for some exploratory results).

4 The Markov Chain and Group-local-balance

In this section, we first analyze the Markov chain $\mathbf{X}$ of an $\mathcal{SPN}$. Boucherie and Van Dijk [6] presented the group-local-balance concept as the basis for the analysis of product form batch routing queueing networks. Here, we translate the definitions and results of Boucherie and Van Dijk into Petri net terminology. It is showed that group-local-balance allows us to calculate the steady state distribution of an $\mathcal{SPN}$. This will serve as the foundation to investigate the structural Petri net implications of group-local-balance in Section 5.

Inserting (2) into the global balance equations (6) yields that a distribution π at $\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$ is the unique equilibrium distribution if for all $\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$:

$$\sum_{\{\boldsymbol{n},\, t,\, t' \in T : \boldsymbol{n}+\boldsymbol{I}(t)=\boldsymbol{n}+\boldsymbol{O}(t')=\boldsymbol{m}\}} \{\pi(\boldsymbol{m}) q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{n}) - \pi(\boldsymbol{n} + \boldsymbol{I}(t')) q(\boldsymbol{I}(t'), \boldsymbol{O}(t'); \boldsymbol{n})\} = 0. \tag{7}$$

A distribution satisfying these equations for fixed combinations of residual marking $\boldsymbol{n}$ and input bag $\boldsymbol{I}(t)$ is the unique equilibrium distribution. This form of *local balance* is introduced in [6] as *group-local-balance*.

Definition 4.1 (Group-local-balance). A measure ϕ satisfies *group-local-balance* (GLB) if, for all fixed residual markings $\boldsymbol{n}$ and for all fixed input bags $\boldsymbol{I}(t)$, such that $\boldsymbol{n}+\boldsymbol{I}(t) \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$:

$$\sum_{\{t' \in T:\, \boldsymbol{I}(t')=\boldsymbol{I}(t)\}} \phi(\boldsymbol{n} + \boldsymbol{I}(t')) q(\boldsymbol{I}(t'), \boldsymbol{O}(t'); \boldsymbol{n}) = \sum_{\{t' \in T:\, \boldsymbol{O}(t')=\boldsymbol{I}(t)\}} \phi(\boldsymbol{n} + \boldsymbol{I}(t')) q(\boldsymbol{I}(t'), \boldsymbol{O}(t'); \boldsymbol{n}). \tag{8}$$

Summation of the group-local-balance equations over all $\boldsymbol{n}, \boldsymbol{I}(t)$ such that $\boldsymbol{n} + \boldsymbol{I}(t) = \boldsymbol{m}$ gives the global balance equations. The Markov chain $\mathbf{X}$ has the GLB-property if the equilibrium distribution π satisfies (8).

GLB expresses that under a given residual marking the rate at which input bag $\boldsymbol{I}(t)$ is absorbed is balanced by the rate at which exactly $\boldsymbol{I}(t)$ is formed. Obviously, the group-local-balance equations (8) are generally more restrictive than the global balance equations (7). GLB requires that $\boldsymbol{I}(t)$ is an output bag of a transition t'. Also, GLB requires that the output bag of a transition t, is an input bag for another transition t'.

Lemma 4.2. If the Markov chain $\mathbf{X}$ of an $\mathcal{SPN}$ satisfies GLB, then $\mathcal{R}(T)$ is a closed set.

Proof. From the group-local-balance equations (8) it is seen that if $\boldsymbol{I}(t)$ is an input bag of a transition that is enabled in an arbitrary marking $\boldsymbol{m}$, then, if GLB holds, $\boldsymbol{I}(t)$ must also be an output bag of a transition t'. If there is no such transition t', the left hand side of (8) would be positive while the right hand side is zero, which contradicts GLB.

Similarly, if $\boldsymbol{O}(t')$ is an output bag of a transition that is enabled in an arbitrary marking $\boldsymbol{m}$, then, if GLB holds, $\boldsymbol{O}(t')$ must also be an input bag of a transition t. If there is no such transition t, the right hand side of (8) would be positive while the left hand side is zero, which contradicts GLB. □

Following [6], let us introduce the concepts of the *local state space* and the *local irreducible sets*. For a fixed $\boldsymbol{n}$ the local state space $V(\boldsymbol{n})$ is the state space of the Markov chain with transition rates $q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{n})$ restricted to $\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$. So $V(\boldsymbol{n})$ consists of all states $\boldsymbol{n} + \boldsymbol{I}(t)$ and $\boldsymbol{n} + \boldsymbol{O}(t)$, for which $q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{n}) > 0$. Let $V_i(\boldsymbol{n})$ denote the local irreducible sets in $V(\boldsymbol{n})$ with respect to the Markov chain with transition rates $q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{n})$ for fixed $\boldsymbol{n}$. A state $\boldsymbol{m}$ may be element of different local state spaces $V(\boldsymbol{n})$, so that transitions from one local state space to another are possible. It is not uncommon that $V(\boldsymbol{n})$ consists of multiple local irreducible sets $V_i(\boldsymbol{n}), i \in \{1, \ldots, k(\boldsymbol{n})\}$, which is shown in [6] via an example. In addition, it is shown that if a Markov chain satisfies GLB, the local state spaces $V(\boldsymbol{n})$ consist only of irreducible sets.

Lemma 4.3 ([6])**.** If the equilibrium distribution π satisfies GLB, then for any $\boldsymbol{n}$ it must be that

$$V(\boldsymbol{n}) = \bigcup_{i=1}^{k(\boldsymbol{n})} V_i(\boldsymbol{n}). \tag{9}$$

Proof. Provided in the appendix for completeness. □

From Lemma 4.2 and Lemma 4.3 it follows that, if the Markov chain $\mathbf{X}$ of an $\mathcal{SPN}$ net has the GLB property, then for any fixed $\boldsymbol{n}$ for which $V(\boldsymbol{n}) \neq \emptyset$ and $i \in \{1, \ldots, k(\boldsymbol{n})\}$ the following set of equations has a unique positive solution up to a multiplicative constant:

$$x(\boldsymbol{n}; \boldsymbol{I}(t)) \sum_{t' \in T} q(\boldsymbol{I}(t), \boldsymbol{I}(t'); \boldsymbol{n}) = \sum_{t' \in T} x(\boldsymbol{n}; \boldsymbol{I}(t'))q(\boldsymbol{I}(t'), \boldsymbol{I}(t); \boldsymbol{n}), \quad \boldsymbol{n} + \boldsymbol{I}(t) \in V_i(\boldsymbol{n}) \tag{10}$$

These local solutions can be used to characterize the equilibrium distribution π. To this end, an additional process with transition rate $\bar{q}$ is defined. In this $\bar{q}$-process every transition that has positive rate in the original process has positive rate too, and in addition, the reversed transitions have positive rate. The transition rates of the $\bar{q}$-process are expressed as a function of the local solutions $x(\boldsymbol{n}; \boldsymbol{I}(t))$. The newly defined $\bar{q}$-process will be the key in obtaining the equilibrium distribution π.

Definition 4.4 (**$\bar{q}$-process**)**.** If for any fixed $\boldsymbol{n}$ for which $V(\boldsymbol{n}) \neq \emptyset$, for $i \in \{1, \ldots, k(\boldsymbol{n})\}$ the system (10) has a unique positive solution $\{x(\boldsymbol{I}(t); \boldsymbol{n}) \mid \boldsymbol{n} + \boldsymbol{I}(t) \in V_i(\boldsymbol{n})\}$ up to a multiplicative constant, then the following process, called the *$\bar{q}$-process*, can be defined.

For any $\boldsymbol{n}, i \in \{1, \ldots, k(\boldsymbol{n})\}$, and $\boldsymbol{n} + \boldsymbol{I}(t), \boldsymbol{n} + \boldsymbol{I}(t') \in V_i(\boldsymbol{n})$, for which $q(\boldsymbol{I}(t), \boldsymbol{I}(t'); \boldsymbol{n}) > 0$ or $q(\boldsymbol{I}(t'), \boldsymbol{I}(t); \boldsymbol{n}) > 0$

$$\frac{\bar{q}(\boldsymbol{I}(t), \boldsymbol{I}(t'); \boldsymbol{n})}{\bar{q}(\boldsymbol{I}(t'), \boldsymbol{I}(t); \boldsymbol{n})} = \frac{x(\boldsymbol{I}(t'), \boldsymbol{n})}{x(\boldsymbol{I}(t), \boldsymbol{n})}, \tag{11}$$

and otherwise

$$\bar{q}(\boldsymbol{I}(t), \boldsymbol{I}(t'); \boldsymbol{n}) = 0.$$

The transition rates $\bar{q}$ are uniquely defined up to a multiplicative constant at each of the local irreducible sets $V_i(\boldsymbol{n})$. Therefore, the ratios of the $\bar{q}$ are unique. Only these ratios will be used in the theory below. Note that for any Markov chain $\mathbf{X}$ at $\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$ that satisfies the equations (10) the $\bar{q}$-process can be defined. However, such a Markov chain does not necessarily satisfy the GLB property. To point out in when this relation does hold, [6] introduces the concept of strong reversibility.

Definition 4.5 (**Strong reversibility**)**.** The $\bar{q}$-process is *strongly reversible* at $\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$ if for all $\boldsymbol{n}$ for which $V(\boldsymbol{n}) \neq \emptyset$ and $i \in \{1, \dots, k(\boldsymbol{n})\}$, the equilibrium distribution $\bar{\pi}$ satisfies

$$\bar{\pi}(\boldsymbol{n} + \boldsymbol{I}(t))\bar{q}(\boldsymbol{I}(t), \boldsymbol{I}(t'); \boldsymbol{n}) = \bar{\pi}(\boldsymbol{n} + \boldsymbol{I}(t'))\bar{q}(\boldsymbol{I}(t'), \boldsymbol{I}(t); \boldsymbol{n}), \quad \boldsymbol{n} + \boldsymbol{I}(t), \boldsymbol{n} + \boldsymbol{I}(t') \in V_i(\boldsymbol{n}). \tag{12}$$

By definition, the $\bar{q}$-process is reversible at the local state spaces $V(\boldsymbol{n})$. As in the original Markov chain, in the $\bar{q}$-process transitions between different local state spaces $V(\boldsymbol{n})$ and $V(\boldsymbol{n'})$ are generally possible. Strong reversibility expresses that reversibility not only applies with respect to the local solutions $x(\boldsymbol{I}(t); \boldsymbol{n})$, but also with respect to a global solution of the $\bar{q}$-process $\bar{\pi}(\boldsymbol{n}+\boldsymbol{I}(t))$.

The following theorem relates the equilibrium distribution of the original Markov chain $\mathbf{X}$ to the equilibrium distribution of the $\bar{q}$-process. It shows that the $\bar{q}$-process can be exploited to calculate the equilibrium distribution π.

Theorem 4.6 ([6])**.** The equilibrium distribution of a Markov chain $\mathbf{X}$ at $\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$ satisfies GLB if and only if the $\bar{q}$-process is defined and is strongly reversible at $\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$. Moreover, with $\bar{\pi}$ its equilibrium distribution, for all $\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$

$$\pi(\boldsymbol{m}) = \bar{\pi}(\boldsymbol{m}). \tag{13}$$

Moreover, the equilibrium distribution π satisfies GLB if and only if for an arbitrary reference state $\boldsymbol{m}_0$, and all $\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$

$$\pi(\boldsymbol{m}) = \pi(\boldsymbol{m}_0) \prod_{k=0}^{s} \frac{\bar{q}(\boldsymbol{I}(t_k), \boldsymbol{I}(t'_k); \boldsymbol{n}_k)}{\bar{q}(\boldsymbol{I}(t'_k), \boldsymbol{I}(t_k); \boldsymbol{n}_k)}, \tag{14}$$

for all firing sequences of the form

$$\begin{aligned} \boldsymbol{m}_0 = \boldsymbol{n}_0 + I(t_0) \to \boldsymbol{n}_0 + I(t'_0) = \boldsymbol{n}_1 + \boldsymbol{I}(t_1) \to \boldsymbol{n}_1 + I(t'_1) = \ldots \to \\ \ldots = \boldsymbol{n}_s + \boldsymbol{I}(t_s) \to \boldsymbol{n}_s + \boldsymbol{I}(t'_s) = \boldsymbol{n}_{s+1} + \boldsymbol{I}(t_{s+1}) = \boldsymbol{m}. \end{aligned} \tag{15}$$

such that the denominator of (14) is positive.

Proof. Provided in the appendix for completeness. □

The following corollary provides the relation between the equilibrium distribution π and the local solutions $x(\boldsymbol{n}; \boldsymbol{I}(t))$.

Corollary 4.7. The equilibrium distribution π satisfies GLB if and only if for $\boldsymbol{n}, \boldsymbol{I}(t)$ and $\boldsymbol{I}(t')$ such that $\boldsymbol{n} + \boldsymbol{I}(t), \boldsymbol{n} + \boldsymbol{I}(t') \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$, for which $q(\boldsymbol{I}(t), \boldsymbol{I}(t'); \boldsymbol{n}) > 0$

$$\frac{\pi(\boldsymbol{n} + \boldsymbol{I}(t))}{\pi(\boldsymbol{n} + \boldsymbol{I}(t'))} = \frac{x(\boldsymbol{I}(t); \boldsymbol{n})}{x(\boldsymbol{I}(t'); \boldsymbol{n})}. \tag{16}$$

Note that (16) is a condition for $\boldsymbol{n}$, $\boldsymbol{I}(t)$ and $\boldsymbol{I}(t')$ such that $\boldsymbol{n}+\boldsymbol{I}(t)$ and $\boldsymbol{n}+\boldsymbol{I}(t')$ are within a single local irreducible set $V_i(\boldsymbol{n})$, and it relates the ratio $x(\boldsymbol{I}(t);\boldsymbol{n})/x(\boldsymbol{I}(t');\boldsymbol{n})$ to the ratio $\pi(\boldsymbol{n}+\boldsymbol{I}(t))/\pi(\boldsymbol{n}+\boldsymbol{I}(t'))$. For a firing sequence from marking $\boldsymbol{m}$ to $\boldsymbol{m}'$ that traverses multiple local irreducible sets $V_j(\boldsymbol{n}_j)$, $j=1,\dots,s$, for each transition in this firing sequence (16) is imposed. The latter implies that if there exist multiple firing sequences from $\boldsymbol{m}$ to $\boldsymbol{m}'$ additional restrictions on the ratios $\bar{q}(\boldsymbol{I}(t_k),\boldsymbol{I}(t'_k);\boldsymbol{n}_k)/\bar{q}(\boldsymbol{I}(t'_k),\boldsymbol{I}(t_k);\boldsymbol{n}_k)$ in (14) are implied to obtain consistency in the ratio $\pi(\boldsymbol{m})/\pi(\boldsymbol{m}')$ in (14). In Section 5, the impact of these conditions at the Petri net level will be studied in detail.

This section has described results on the Markov chain level. Reversibility of the $\bar{q}$-process provides a way to 'build' the solution $\bar{\pi}(\boldsymbol{m})$, following any path to $\boldsymbol{m}$ from the initial marking $\boldsymbol{m}_0$. To understand and exploit the results on the Petri net level, in the next section, we will investigate the translation of these characteristics to the stochastic Petri nets and in particular present the implications for the stochastic Petri net structure. The key ingredients of that analysis will be the local irreducible sets and ratio condition of Corollary 4.7.

5 The Stochastic Petri Net and Group-local-balance

In this section, we will show that stochastic Petri nets with marking-independent firing rates for which group-local-balance holds have a steady state distribution that is a product over the places of the network. Therefore, we are interested in the necessary and sufficient structural properties of Petri nets that are required to obtain group-local-balance.

The first structural condition was already presented in Lemma 4.2: the set of input and output bags $\mathcal{R}(T)$ is a closed set. In Section 5.1, this condition is extended to 'each transition has to be covered by a minimal closed support T-invariant', i.e., the $\mathcal{SPN}$ has to be an $\mathcal{S}\Pi$-net. To this end, it is shown that the local irreducible sets defined in Section 4 are sets of minimal closed support T-invariants. Section 5.2 shows that an $\mathcal{S}\Pi$-net does not necessarily has a product form solution. The additional relation between states can be found by tracing closed support T-invariants. This observation forms the key to formulate the additional requirements to obtain a characterization of product form stochastic Petri nets. Section 5.3 identifies the structural characteristics of $\mathcal{S}\Pi$-nets for which a product form equilibrium distribution can be concluded without considering the numerical values of the transition rates and nets for which these values have to satisfy specific conditions. This subsection is concluded with an algorithm to verify whether a specific $\mathcal{SPN}$ possesses a product form equilibrium distribution, and if so, to construct this product form. Section 5.4 provides several insightful examples of product form $\mathcal{SPN}$s.

The Markov chain $\mathbf{X}$ at state space $\mathcal{M}(\mathcal{SPN},\boldsymbol{m}_0)$ modelling the Petri net with marking-independent firing rates has transition rates

$$q(\boldsymbol{I}(t),\boldsymbol{O}(t);\boldsymbol{m}-\boldsymbol{I}(t))=\mu(t)\mathbb{1}_{(m(n)\geq I_n(t),\,n=1,\dots,N)} \tag{17}$$

Observe that for the nets with transition rates (17) the condition $m(n)\geq I_n(t)$, $n=1,\dots,N$, is necessary and sufficient for transition t to be enabled in marking $\boldsymbol{m}$. It will sometimes be convenient to amalgamate transitions with the same input bag into a single transition with a probabilistic output bag, thus focussing on the Markov jump structure of the stochastic Petri net (see (3)-(5)).

5.1 The routing chain and minimal closed support T-invariants

Under marking independent transition rates the equations (10) are equivalent for all $\boldsymbol{n} + \boldsymbol{I}(t) \in V_i(\boldsymbol{n})$, which can be seen from inserting (17) in (10), for all $\boldsymbol{n} + \boldsymbol{I}(t) \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$:

$$\begin{aligned} x(\boldsymbol{I}(t); \boldsymbol{n}) \sum_{t' \in T} \mu(t) p(\boldsymbol{I}(t), \boldsymbol{I}(t')) \mathbb{1}_{(m(n) \geq I_n(t)),\, n=1,\dots,N)} \\ = \\ \sum_{t' \in T} x(\boldsymbol{I}(t'); \boldsymbol{n}) \mu(t) p(\boldsymbol{I}(t'), \boldsymbol{I}(t)) \mathbb{1}_{(m(n) \geq I_n(t')),\, n=1,\dots,N)} \end{aligned} \tag{18}$$

Considering (18) for all residual markings $\boldsymbol{n}$ and input bags $\boldsymbol{I}(t)$ and local irreducible sets $V_i(\boldsymbol{n})$ such that $\boldsymbol{n} + \boldsymbol{I}(t) \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$, exposes that the set of equations of the form (18) only differ in the local irreducible sets $V_i(\boldsymbol{n})$ ($i \in 1, \dots, k(\boldsymbol{n})$) being enabled or disabled. Therefore, if the equilibrium distribution π satisfies GLB, then for each $\boldsymbol{n} + \boldsymbol{I}(t) \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$ equation (18) has a unique positive solution $x(\boldsymbol{I}(t); \boldsymbol{n}) := y(\boldsymbol{I}(t))$.

This implies that we can find a positive solution to the global balance equations of a Markov chain which is defined by Henderson et al. as the *routing chain* [28]. Define the Markov chain $\mathbf{Y} = (Y(t), t \geq 0)$ at finite state space $S = \{\boldsymbol{I}(t), t \in T\}$ with transition rates $q_{\mathbf{Y}}(\boldsymbol{I}(t), \boldsymbol{I}(t')) = \mu(t) p(\boldsymbol{I}(t), \boldsymbol{I}(t'))$. The global balance equations for $\mathbf{Y}$ are, for $t \in T$,

$$\sum_{t' \in T} \{ y(\boldsymbol{I}(t)) \mu(t) p(\boldsymbol{I}(t), \boldsymbol{I}(t')) - y(\boldsymbol{I}(t')) \mu(t') p(\boldsymbol{I}(t'), \boldsymbol{I}(t)) \} = 0. \tag{19}$$

These global balance equations for Markov chain $\mathbf{Y}$ are state independent versions of the group-local-balance equations (10). The definition of the routing chain relies on the condition that $\mathcal{R}(T)$ is closed set. Otherwise $p(\boldsymbol{I}(t), \boldsymbol{I}(t'))$ may be zero for all $t' \in T$ since without the condition of closedness $\boldsymbol{O}(t)$ need not be an input bag for some transition t'.

Remark 5.1. From the closedness of $\mathcal{R}(T)$ we obtain that for each transition t there exists a transition t' such that $\boldsymbol{O}(t) = \boldsymbol{I}(t')$, and the first summation in the routing chain is equivalent to $\sum_{\{t' \in T\colon \boldsymbol{O}(t) = \boldsymbol{I}(t')\}} y(\boldsymbol{I}(t)) \mu(t) p(\boldsymbol{I}(t), \boldsymbol{O}(t))$. Obviously, the second summation is equivalent to $\sum_{\{t' \in T\colon \boldsymbol{O}(t') = \boldsymbol{I}(t)\}} y(\boldsymbol{I}(t')) \mu(t') p(\boldsymbol{I}(t'), \boldsymbol{O}(t'))$, which shows that the routing chain do not exclude any transitions depositing or consuming $\boldsymbol{I}(t)$. $\square$

Observe that GLB cannot hold if no positive solution for the routing chain can be found. Therefore, in the following, we first investigate the structural conditions under which a positive solution for the routing chain exists. The condition that $\mathcal{R}(T)$ is a closed set is necessary for a solution $\mathbf{Y}$ to exist. This condition is exactly the condition that Henderson et al. impose in Corollary 1 of [28] on the $\mathcal{SPN}$s they consider. In their further analysis, they assume a positive solution for the routing chain exists; an assumption which is usually made in the literature. The following example, taken from [4], shows that the closedness of $\mathcal{R}(T)$ is not a sufficient condition for GLB to hold.

Example 5.2. Consider the $\mathcal{SPN}$ depicted in Figure 1. $\boldsymbol{I}(t_1) = (1,0,1,0), \boldsymbol{I}(t_2) = (1,1,0,0), \boldsymbol{I}(t_3) = (1,1,0,0), \boldsymbol{I}(t_4) = (0,1,0,1), \boldsymbol{I}(t_5) = (0,0,1,1)$ and $\boldsymbol{O}(t_1) = (0,1,0,1)$, $\boldsymbol{O}(t_2) = (1,0,1,0)$, $\boldsymbol{O}(t_3) = (0,0,1,1)$, $\boldsymbol{O}(t_4) = (1,0,1,0)$, $\boldsymbol{O}(t_5) = (1,1,0,0)$, which shows that $\mathcal{R}(T)$ is a closed set. Amalgamating transitions t_2 and t_3 into a single transition, the state space of the routing chain is

$$S = \{\boldsymbol{I}(t_1), \boldsymbol{I}(t_2), \boldsymbol{I}(t_4), \boldsymbol{I}(t_5)\}$$

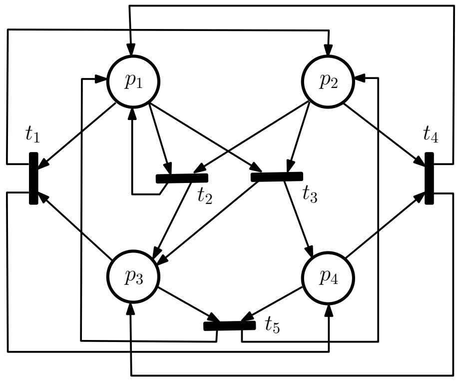

Figure 1: Petri net for which $\mathcal{R}(T)$ is a closed set.

and the solution for the routing chain (19) is (up to a multiplicative constant)

$$y(\boldsymbol{I}(t_1)) = 1/\mu_1, y(\boldsymbol{I}(t_4)) = 1/\mu_4, y(\boldsymbol{I}(t_2)) = y(\boldsymbol{I}(t_3)) = y(\boldsymbol{I}(t_5)) = 0$$

which shows that closedness of $\mathcal{R}(T)$ is not sufficient for a *positive* solution for the the routing chain. □

In Example 5.2, $\mathbf{Y}$ does not partition in irreducible classes, since $S_1 = \{\boldsymbol{I}(t_2), \boldsymbol{I}(t_4), \boldsymbol{I}(t_5)\}$ is a transient class. Boucherie and Sereno [5] present a necessary and sufficient condition: for an $\mathcal{SPN}$ a positive solution for the routing chain exists if and only if all transitions $t \in T$ are covered by minimal closed support T-invariants, i.e., it is an $\mathcal{S}\Pi$-net. They prove this by showing that only in this case the state space of the Markov chain $\mathbf{Y}$ partitions in irreducible sets.

Obviously, the condition of the $\mathcal{SPN}$ to be an $\mathcal{S}\Pi$-net implies that $\mathcal{R}(T)$ is a closed set. In addition to the closedness condition, in an $\mathcal{S}\Pi$-net transitions t, s with $\boldsymbol{O}(t) = \boldsymbol{I}(s)$ are elements of the support of a single minimal closed support T-invariant. Returning to example 5.2 illustrates this essential extension.

Example 5.2 revisited. From the incidence matrix

$$\boldsymbol{A} = \begin{pmatrix} -1 & 0 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 & -1 \end{pmatrix}$$

we obtain that this net has 3 minimal support T-invariants: $\boldsymbol{x}^1 = (10010)$, $\boldsymbol{x}^2 = (00101)$, $\boldsymbol{x}^3 = (12001)$, of which $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ have closed support, but $\boldsymbol{x}^3$ does not have closed support. Since transition t_2 is contained in $\|\boldsymbol{x}^3\|$ only, t_2 is not covered by a minimal closed support T-invariant, which contradicts the definition of an $\mathcal{S}\Pi$-net. This explains why no positive solution for the routing chain exists. □

Observe that the essential characteristic of an $\mathcal{S}\Pi$-net is that all transitions are contained in a *closed* support T-invariant. The condition that all transitions are covered by minimal support

T-invariants (closed or not closed) is a natural assumption if we are interested in the equilibrium or stationary distribution of a stochastic Petri net (see Section 3.2).

To obtain the partitioning of $\mathbf{Y}$ into irreducible classes, we first provide a decomposition of the transitions of the Petri net into equivalence classes based on the characterization of minimal closed support T-invariants that are connected by having an input bag in common. By this equivalence class decomposition, the global balance equations of the routing chain (19) decompose into disjoint sets of equations, one set of equations for each equivalence class of connected T-invariants. The equivalence relation is defined by analogy with a similar equivalence relation introduced in Frosch and Natarajan [20] for cyclic state machines.

Assume that the minimal support T-invariants $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^h$ are numbered such that $ClT \stackrel{\text{def}}{=} \{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k\}$ is the set of minimal closed support T-invariants ($k \leq h$).

Definition 5.3 (**Common input bag relation**). Let $\boldsymbol{x}$, $\boldsymbol{x}' \in ClT$. We say that $\boldsymbol{x}$, $\boldsymbol{x}'$ are in common input bag relation (notation: $\boldsymbol{x}\ CI\ \boldsymbol{x}'$) if there exist $t \in \|\boldsymbol{x}\|$, $t' \in \|\boldsymbol{x}'\|$ such that $\boldsymbol{I}(t) = \boldsymbol{I}(t')$. The relation CI^* is the transitive closure of CI[1].

Definition 5.4 (**Common Input Bag Class**). The common input bag class $CI(\boldsymbol{x})$ is the equivalence class of $\boldsymbol{x} \in ClT$, that is $CI(\boldsymbol{x}) = \{\boldsymbol{x}' | \boldsymbol{x}\ CI^*\ \boldsymbol{x}'\}$.

The common input bag relation characterizes the irreducible sets of the routing chain. The equivalence classes partition ClT: each $\boldsymbol{x} \in ClT$ belongs to exactly one equivalence class. Let $\boldsymbol{x} \in ClT$ with equivalence class $CI(\boldsymbol{x})$. Define $S(\boldsymbol{x}) \subset S$, the input bags corresponding to $CI(\boldsymbol{x})$, as

$$S(\boldsymbol{x}) = \{\boldsymbol{I}(t) \mid \exists\ \boldsymbol{x}' \in CI(\boldsymbol{x}) \text{ such that } x'(t) > 0\}.$$

Boucherie and Sereno [5] show that the partitioning of ClT into equivalence classes $\{CI(\boldsymbol{x})\}_{\boldsymbol{x} \in ClT}$ induces a partition $\{S(\boldsymbol{x})\}_{\boldsymbol{x} \in ClT}$ of S into irreducible sets of the Markov chain $\mathbf{Y}$ if and only if all transitions are covered by minimal closed support T-invariants.

Theorem 5.5. ([5]) For the stochastic Petri net $\mathcal{SPN}$ a positive solution for the routing chain (19) exists if and only if $\mathcal{SPN}$ is an $\mathcal{S}\Pi$-net.

Proof. For a complete proof, see [5]. As it provides insight, here we present the intuition for the proof. The equations (19) are the global balance equations of $\mathbf{Y}$ at state space S. Therefore it is sufficient to prove that the condition that each transition is covered by a minimal closed support T-invariant is necessary and sufficient for the partition of S into irreducible sets $\{S(\boldsymbol{x})\}_{\boldsymbol{x} \in ClT}$.

First $S(\boldsymbol{x}') = S(\boldsymbol{x})$ if $CI(\boldsymbol{x}') = CI(\boldsymbol{x})$, and $S(\boldsymbol{x}') \cap S(\boldsymbol{x}) = \emptyset$ if $CI(\boldsymbol{x}') \cap CI(\boldsymbol{x}) = \emptyset$. Second, by the definition of $S(x)$, the input bags $\boldsymbol{I}(t)$ in a set $S(\boldsymbol{x})$ are communicating states. Third, since every transition is covered by a minimal closed support T-invariant, each transition is contained in a set $S(\boldsymbol{x}) \in S$. As a consequence, $\{S(\boldsymbol{x})\}_{\boldsymbol{x} \in ClT}$ forms a partition of S into irreducible sets.

Conversely, assume that an invariant measure exists to the marking independent traffic equations. The existence of this invariant measure implies that S is partitioned in irreducible sets and immediately implies that for all $t \in T$, $\exists\ t' \in T$ such that $\boldsymbol{O}(t) = \boldsymbol{I}(t')$. Furthermore, in an irreducible set all states communicate. For this two reasons all cyclic firing sequences within an

[1]The transitive closure of a relation is defined as follows: if $\boldsymbol{x}$, $\boldsymbol{x}'$, $\boldsymbol{x}'' \in ClT$, and $\boldsymbol{x}\ CI\ \boldsymbol{x}'$, $\boldsymbol{x}'\ CI\ \boldsymbol{x}''$, then we define $\boldsymbol{x}\ CI^*\ \boldsymbol{x}'$, $\boldsymbol{x}'\ CI^*\ \boldsymbol{x}''$, and $\boldsymbol{x}\ CI^*\ \boldsymbol{x}''$. This reflects the property that we can go from $\boldsymbol{x}$ to $\boldsymbol{x}''$ via $\boldsymbol{x}'$. This makes the common input bag relation CI^* an equivalence relation on ClT.

irreducible set form closed support T-invariants, and each state is contained in at least one such cyclic firing sequences.

From Result 3.32 we obtain that each support of an invariant can be decomposed into a union of minimal supports which implies that all transitions are covered by a minimal closed support T-invariants. □

In the next corollary, Theorem 5.5 is expanded to the reachability set level. A proof is omitted, as it follows exactly the lines as the proof of Theorem 5.5.

Corollary 5.6. For an $S\Pi$-net, there is a one-to-one mapping between the partitioning of S into irreducible sets $\{S(\boldsymbol{x})\}_{\boldsymbol{x}\in ClT}$ that is induced by the partitioning of ClT into equivalence classes $\{CI(\boldsymbol{x})\}_{\boldsymbol{x}\in ClT}$ and the partitioning of local state spaces $V(\boldsymbol{n})$ into the local irreducible sets $V_i(\boldsymbol{n})$.

We now have the results to show, as announced in Section 3.2, that $\mathcal{S}\Pi$-nets are structurally live and have a home state.

Theorem 5.7. The marked Π-net $\mathcal{PN} = (P, T, I, O, \boldsymbol{m}_0)$ underlying a marked $\mathcal{S}\Pi$-net $(\mathcal{SPN}, \boldsymbol{m}_0)$ has home state $\boldsymbol{m}_0$ and is structurally live.

Proof. Consider the marked Π-net $(\mathcal{PN}, \boldsymbol{m}_0)$ underlying $(\mathcal{SPN}, \boldsymbol{m}_0)$.

(1) For $\boldsymbol{x} \in ClT$, let $\mathcal{T}(t, \boldsymbol{x}) = \{t' \in T \mid \exists \boldsymbol{x}' \in CI(\boldsymbol{x}) \text{ with } t' \in \|\boldsymbol{x}'\| \text{ such that } t \in \|\boldsymbol{x}\|\}$. Assume that $\boldsymbol{m} \in \mathcal{M}(\mathcal{PN}, \boldsymbol{m}_0)$ is such that $t \in T$ is enabled. Such $\boldsymbol{m}$ exists, otherwise remove t. Then for all $t' \in \mathcal{T}(t, \boldsymbol{x})$, there exists an $\boldsymbol{m}' \in \mathcal{M}(\mathcal{PN}, \boldsymbol{m})$ such that t' is enabled in marking $\boldsymbol{m}'$. The firing sequence σ from $\boldsymbol{m}$ to $\boldsymbol{m}'$ can be constructed such that it contains transitions from $\mathcal{T}(t, \boldsymbol{x})$ only. To see this, observe that for t to be enabled in $\boldsymbol{m}$ it must be that $\boldsymbol{m} - \boldsymbol{I}(t) \in \mathbb{N}_0^N$ (the enabling condition). Let $t \in \|\boldsymbol{x}\|$, $\boldsymbol{x} \in ClT$, and $t' \in \mathcal{T}(t, \boldsymbol{x})$. Then there exists an $\boldsymbol{x}' \in ClT$ such that $t' \in \|\boldsymbol{x}'\|$ and $\boldsymbol{x}\ CI^*\ \boldsymbol{x}'$. As a consequence, there exists a firing sequence from t to t', say $\sigma = t_{\sigma_0} t_{\sigma_1} \cdots t_{\sigma_k} t_{\sigma_{k+1}}$, $t = t_{\sigma_0}$, $t' = t_{\sigma_{k+1}}$, such that $\boldsymbol{O}(t_{\sigma_i}) = \boldsymbol{I}(t_{\sigma_{i+1}})$, $i = 0, \ldots, k$. The corresponding sequence of markings is $\boldsymbol{m}[t_{\sigma_0} > \boldsymbol{m}_{\sigma_1}[t_{\sigma_1} > \cdots \boldsymbol{m}_{\sigma_k}[t_{\sigma_k} > \boldsymbol{m}_{\sigma_{k+1}}$, where $\boldsymbol{m}_{\sigma_1} = \boldsymbol{m} - \boldsymbol{I}(t_{\sigma_0}) + \boldsymbol{O}(t_{\sigma_0})$, $\boldsymbol{m}_{\sigma_2} = \boldsymbol{m}_{\sigma_1} - \boldsymbol{I}(t_{\sigma_1}) + \boldsymbol{O}(t_{\sigma_1}) = \boldsymbol{m} - \boldsymbol{I}(t_{\sigma_0}) + \boldsymbol{O}(t_{\sigma_1}), \ldots, \boldsymbol{m}_{\sigma_k} = \boldsymbol{m} - \boldsymbol{I}(t_{\sigma_0}) + \boldsymbol{O}(t_{\sigma_{k-1}})$, $\boldsymbol{m}_{\sigma_{k+1}} = \boldsymbol{m} - \boldsymbol{I}(t_{\sigma_0}) + \boldsymbol{O}(t_{\sigma_k})$. Since $\boldsymbol{O}(t_{\sigma_k}) = \boldsymbol{I}(t_{\sigma_{k+1}})$ we have that t' is enabled in $\boldsymbol{m}_{\sigma_{k+1}}$ if and only if t is enabled in $\boldsymbol{m}$. Following the same reasoning, for every marking $\boldsymbol{m}''$ on the path $\boldsymbol{m}[\sigma > \boldsymbol{m}'$, if another transition s is fired, a path can be constructed back to $\boldsymbol{m}''$ using the transitions from a closed support T-invariant of which s is an element. This establishes that if a transition t is enabled then all $t' \in \mathcal{T}(t, \boldsymbol{x})$ are live.

(2) For all $\boldsymbol{m} \in \mathcal{M}(\mathcal{PN}, \boldsymbol{m}_0)$ there is a firing sequence σ such that $\boldsymbol{m}_0[\sigma > \boldsymbol{m}$. By induction on the length ℓ of this firing sequence we prove that there is a firing sequence σ' such that $\boldsymbol{m}[\sigma' > \boldsymbol{m}_0$.

$\ell = 1$. Let $\sigma = t$, then $\boldsymbol{m} = \boldsymbol{m}_0 - \boldsymbol{I}(t) + \boldsymbol{O}(t)$. $\mathcal{PN}$ being a Π-net implies that there exists an $\boldsymbol{x} \in ClT$ such that $t \in \|\boldsymbol{x}\|$. Let $\sigma_{\boldsymbol{x}}$ be the unique linear firing sequence of $\boldsymbol{x}$ (Result 3.34), say $\sigma_{\boldsymbol{x}} = t_{\sigma_{\boldsymbol{x},1}} t_{\sigma_{\boldsymbol{x},2}} \cdots t_{\sigma_{\boldsymbol{x},k}}$. Without loss of generality, assume that $t = t_{\sigma_{\boldsymbol{x},1}}$. Similar to the construction above, if t is enabled then $\sigma_{\boldsymbol{x}}$ is enabled, and for $\sigma' = t_{\sigma_{\boldsymbol{x},2}} \cdots t_{\sigma_{\boldsymbol{x},k}}$ we have $\boldsymbol{m}[\sigma' > \boldsymbol{m}_0$.

Assume that for any firing sequence δ of length k such that $\boldsymbol{m}_0[\delta > \boldsymbol{m}_\delta$ there is a firing sequence δ' such that $\boldsymbol{m}_\delta[\delta' > \boldsymbol{m}_0$. Let $\ell = k+1$ and $\sigma = t_{\sigma_1} t_{\sigma_2} \cdots t_{\sigma_k} t_{\sigma_{k+1}}$ such that $\boldsymbol{m}_0[\sigma > \boldsymbol{m}$. Let $\delta = t_{\sigma_1} t_{\sigma_2} \cdots t_{\sigma_k}$, $\boldsymbol{m}_0[\delta > \boldsymbol{m}_\delta$. It is sufficient to prove that there exists a firing sequence ν such that $\boldsymbol{m}[\nu > \boldsymbol{m}_\delta$. To this end, observe that there exists an $\boldsymbol{x} \in ClT$ such that $t_{\sigma_{k+1}} \in \|\boldsymbol{x}\|$. By the construction used for $\ell = 1$ we have that $\boldsymbol{x} = t_{\sigma_{k+1}} \nu$ defines ν. (Note that this is true

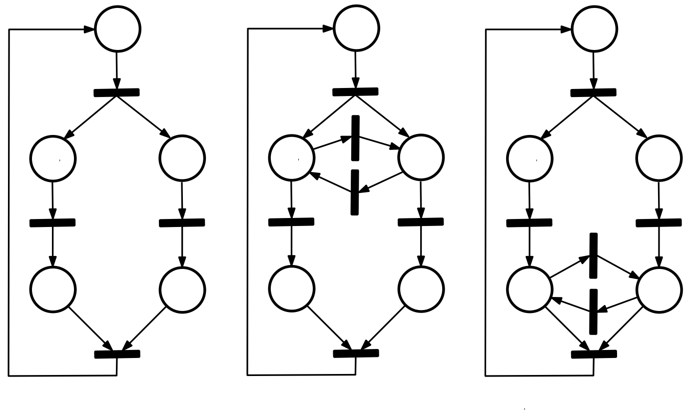

Figure 2: a. Figure 2: b. Figure 2: c.

because $\boldsymbol{x}$ has closed support.) Now $\sigma' = \nu\delta'$, completing the induction step. As a consequence, $\boldsymbol{m}_0$ is a home state.

(3) Let $\boldsymbol{m}_0$ be such that at least one transition in each equivalence class $\mathcal{T}(t, \boldsymbol{x})$ is enabled. Result (2) shows that $\boldsymbol{m}_0$ is a home state, and result (1) implies that all transitions are live. This shows that the untimed Petri net is structurally live. □

Theorem 5.7 shows that an $\mathcal{S}\Pi$-net not only guarantees a positive solution for the global balance equations for the routing chain (19), but for live initial markings also for the global balance equations (6) for the Markov chain $\mathbf{X}$ of the stochastic Petri net. If the net is covered by P-invariants, it is structurally bounded (Result 3.31). Positive recurrence then follows and thus a positive solution solution summing to unity exists. Furthermore, Theorem 5.7 shows that there exists an initial marking for which the net is live. The proof indicates that if each common input bag is initially marked, the net is live. If not each common input bag is initially marked, checking liveness may be cumbersome (see Haddad et al. [22]).

Remark 5.8. When the equilibrium behaviour of stochastic Petri nets is of interest, a natural condition is that all transitions are covered by minimal support T-invariants. For bounded nets this condition is necessary for liveness (see Result 3.31). If this condition is not satisfied, there exists a transition, say t_0, that is enabled in a reachable marking $\boldsymbol{m}$, and $\boldsymbol{x}(t_0) = 0$ for all minimal support T-invariants (if t_0 is never enabled, then we can delete t_0 from T). Let t_0 fire in marking $\boldsymbol{m}$. Then there exists no firing sequence from $\boldsymbol{m} - \boldsymbol{I}(t_0) + \boldsymbol{O}(t_0)$ back to $\boldsymbol{m}$ (otherwise t_0 would be contained in a T-invariant). Thus $\boldsymbol{m}$ is a transient state and does not appear in the equilibrium description of the stochastic Petri net. As a consequence, both $\boldsymbol{m}$ and t_0 can be deleted from the equilibrium description of the Petri net.

As can be seen from the Petri net of Figure 2b, the condition that all transitions are covered by T-invariants is necessary, but not sufficient for liveness of the Petri net. For liveness additional conditions are required.

An $\mathcal{S}\Pi$-net does guarantee structural liveness of the Petri net. As can be seen from Figure 2a, and 2c, the condition of an $\mathcal{SPN}$ being an $\mathcal{S}\Pi$-net is sufficient, but not necessary. Comparison of Figure 2b, and 2c, however, shows that the property of liveness is cumbersome since Petri nets that are almost identical may show completely different behaviour. Therefore, a characterization of liveness for $\mathcal{S}\Pi$-nets is of interest on its own. □

5.2 Group-local-balance and product form

In Section 5.1, we have first seen that if GLB holds, a positive solution to the routing chain (19) and thus to the local balance equations (10) is guaranteed. Second, a positive solution to the routing chain exists if and only if the stochastic Petri net is an $\mathcal{S}\Pi$-net. In this section, we investigate the equivalence of GLB and a product form solution over the places of the Petri net. As can be seen from Corollary 4.7, a positive solution to the routing chain does not yet imply GLB and thus a product from solution. The additional condition to be satisfied is also formulated in this section, of which the structural implications are discussed in Section 5.3.

From Corollary 4.7 we obtain the key idea that under GLB the marking independent solution $y(\cdot)$ of the routing chain can be translated into a marking dependent solution with the same properties. This is reflected by the ratio condition (16). For state independent firing rates this leads to the following theorem, which is similar to Theorem 1 of Henderson and Taylor [31].

Theorem 5.9. The equilibrium distribution π of an $\mathcal{SPN}$ with state independent firing rates satisfies GLB if and only if it is an $\mathcal{S}\Pi$-net and a function $\pi_y : \mathcal{M}(SPN, \boldsymbol{m}_0) \to \mathbb{R}^+$ exists such that for all $\boldsymbol{n} + \boldsymbol{I}(t) \in \mathcal{M}(SPN, \boldsymbol{m}_0)$, $t, t' \in T$ with $p(\boldsymbol{I}(t), \boldsymbol{I}(t')) > 0$,

$$\frac{\pi_y(\boldsymbol{n} + \boldsymbol{I}(t))}{\pi_y(\boldsymbol{n} + \boldsymbol{I}(t'))} = \frac{y(\boldsymbol{I}(t))}{y(\boldsymbol{I}(t'))} \tag{20}$$

and $\pi(\boldsymbol{m}) = B\pi_y(\boldsymbol{m}),\ \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$ with $B^{-1} = \sum_{\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)} \pi_y(\boldsymbol{m})$ is the unique equilibrium distribution of the Markov chain describing $\mathcal{SPN}$.

Proof. For an $\mathcal{S}\Pi$-net a solution y to the routing chain exists. From the analysis in Section 5.1 we know that $x(\boldsymbol{I}(t); \boldsymbol{n}) = y(\boldsymbol{I}(t))$ is a solution to the local balance equations (10). By Corollary 4.7, $\pi(\boldsymbol{m}) = B\pi_y(\boldsymbol{m}),\ \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$ with $B^{-1} = \sum_{\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)} \pi_y(\boldsymbol{m})$ is the unique equilibrium distribution of the Markov chain describing the $\mathcal{SPN}$ and π satisfies GLB. The reversed statement is concluded from Theorem 5.5 and inserting $x(\boldsymbol{I}(t); \boldsymbol{n}) = y(\boldsymbol{I}(t))$ in Corollary 4.7. □

Note that Condition (20) is a condition on y and *not* on the structure of the Petri net. If a solution $y(\cdot)$ for the routing chain is found, a function $\pi_y(\cdot)$ satisfying (20) cannot always be found without additional assumptions on the $\mathcal{SPN}$. Theorem 5.13 below provides a product form solution for π_y under additional conditions on the Petri net. To formulate and understand the structural characterization of the $\mathcal{SPN}$s guaranteeing the ratio condition (20), first Lemma 5.10 and 5.12 and Corollary 5.11 are presented.

Theorem 5.9 implies that the equilibrium distribution π of an $\mathcal{S}\Pi$-net with state independent firing rates satisfies GLB if and only if for an arbitrary reference state $\boldsymbol{m}_0$, and all $\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$

$$\pi(\boldsymbol{m}) = \pi(\boldsymbol{m}_0) \prod_{k=0}^{s} \frac{y(\boldsymbol{I}(t_k))}{y(\boldsymbol{I}(t'_k))}, \tag{21}$$

for all firing sequences of the form

$$\boldsymbol{m}_0 = \boldsymbol{n}_0 + I(t_0) \to \boldsymbol{n}_0 + I(t_0') = \boldsymbol{n}_1 + \boldsymbol{I}(t_1) \to \boldsymbol{n}_1 + I(t_1') = \ldots \to \\ \ldots = \boldsymbol{n}_s + \boldsymbol{I}(t_s) \to \boldsymbol{n}_s + \boldsymbol{I}(t_s') = \boldsymbol{n}_{s+1} + \boldsymbol{I}(t_{s+1}) = \boldsymbol{m}$$

This is seen by first observing that for state independent firing rates $x(\boldsymbol{I}(t); \boldsymbol{n}) = y(\boldsymbol{I}(t))$ is a solution of the local balance equations (10) and then substituting (11) in (14) of Theorem 4.6. Applying (21) to a cyclic firing sequence, so for $\boldsymbol{m}_0 = \boldsymbol{m}$, yields the following lemma.

Lemma 5.10. The equilibrium distribution π of an $\mathcal{S}\Pi$-net with state independent firing rates (17) satisfies GLB if and only if for each T-invariant $\boldsymbol{x} = (x_1, \ldots, x_M)$

$$\prod_{t=1}^{M} \left(\frac{y(\boldsymbol{I}(t))}{y(\boldsymbol{O}(t))} \right)^{x_t} = 1. \tag{22}$$

In Section 5.3, we will investigate which structural Petri net conditions Lemma 5.10 imposes. First, we will use Lemma 5.10 in showing that a solution π_y satisfying the ratio condition (20) must be a product form over the places of the network.

Following Coleman et al. [13], we introduce the row vector $\boldsymbol{C}(y)$, defined as $\boldsymbol{C}(y)_t = \log\left(y(\boldsymbol{I}(t))/y(\boldsymbol{O}(t))\right)$. As $y(\cdot)$ is determined up to a multiplicative constant, and $\boldsymbol{C}(y)$ is determined by the ratios of y's, the vector $\boldsymbol{C}(y)$ is unique, so that is can safely be denoted by $\boldsymbol{C}$. Taking logarithms on both sides in equation (22), Lemma 5.10 can now be reformulated as follows.

Corollary 5.11. The equilibrium distribution π of an $\mathcal{S}\Pi$-net with state independent firing rates (17) satisfies GLB if and only if $\boldsymbol{C}\boldsymbol{x} = 0$ for every T-invariant $\boldsymbol{x}$.

Coleman [12] presents the following equivalent statements.

Lemma 5.12 ([12]). The following statements are equivalent

(i) $\boldsymbol{C}\boldsymbol{x} = 0$ for each T-invariant $\boldsymbol{x}$

(ii) $Rank[\boldsymbol{A}] = Rank[\boldsymbol{A}|\boldsymbol{C}]$, where $[\boldsymbol{A}|\boldsymbol{C}]$ is the matrix augmented with the row vector $\boldsymbol{C}$.

(iii) Equation $z\boldsymbol{A} = \boldsymbol{C}$ has a solution z.

Proof. Provided in the appendix for completeness. □

The following key-result identifies the equivalence between GLB and a product form solution over the places of the network. The solution z of the condition (iii) is used to express the product form.

Theorem 5.13. Consider an $\mathcal{SPN}$ with state independent firing rates (17). The equilibrium distribution π satisfies GLB if and only if the $\mathcal{SPN}$ is an $\mathcal{S}\Pi$-net, $z\boldsymbol{A} = \boldsymbol{C}$ has a solution and π is a product form over the places of the network

$$\pi_y(\boldsymbol{m}) = \prod_{p=1}^{N} (f_p)^{m_p}, \qquad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0) \tag{23}$$

where $f_p = e^{-z_p}$ and $\pi(\boldsymbol{m}) = B\pi_y(\boldsymbol{m})$ with $B^{-1} = \sum_{\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)} \pi_y(\boldsymbol{m})$.

Proof. Under GLB, by Corollary 5.11, $\boldsymbol{Cx} = 0$ for each minimal support T-invariant. This implies by lemma 5.12 that the equation $z\boldsymbol{A} = \boldsymbol{C}$ has a solution. Thus we obtain for each transition $t \in T$

$$\sum_{p=1}^{N} z_p A(p,t) = \log\left(\frac{y(\boldsymbol{I}(t))}{y(\boldsymbol{O}(t))}\right). \tag{24}$$

Taking exponentials gives

$$\prod_{p=1}^{N} e^{z_p A(p,t)} = \left(\frac{y(\boldsymbol{I}(t))}{y(\boldsymbol{O}(t))}\right).$$

By Theorem 5.9, we then have for all $\boldsymbol{n} + \boldsymbol{I}(t) \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$, $t, t' \in T$ with $p(\boldsymbol{I}(t), \boldsymbol{I}(t')) > 0$

$$\frac{\pi_y(\boldsymbol{n} + \boldsymbol{I}(t))}{\pi_y(\boldsymbol{n} + \boldsymbol{I}(t'))} = \frac{y(\boldsymbol{I}(t))}{y(\boldsymbol{I}(t'))} = \prod_{p=1}^{N} e^{z_p A(p,t)}.$$

By (21), for all markings $\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$, $\pi(\boldsymbol{m})$ can be expressed in the reference state $\boldsymbol{m}_0$

$$\begin{aligned}
\pi(\boldsymbol{m}) &= \pi(\boldsymbol{m}_0) \prod_{k=0}^{s} \prod_{p=1}^{N} e^{z_i A(i,t_k)} = \pi(\boldsymbol{m}_0) \prod_{p=1}^{N} e^{z_p (m_0(p) - m(p))} \\
&= \pi(\boldsymbol{m}_0) \left\{\prod_{p=1}^{N} e^{z_p m_0(p)}\right\} \left\{\prod_{p=1}^{N} e^{-z_p m(p)}\right\} = B \prod_{p=1}^{N} (f_p)^{m(p)} = B\pi_y(\boldsymbol{m}).
\end{aligned}$$

Conversely, if an $S\Pi$-net has an equilibrium distribution $\pi(\boldsymbol{m}) = B \prod_{p=1}^{N} f_p^{m(p)}$, then GLB is satisfied, since for a $S\Pi$-net the GLB equations (8) reduce to

$$\pi(\boldsymbol{n} + \boldsymbol{I}(t)) \sum_{t' \in T} q(\boldsymbol{I}(t), \boldsymbol{I}(t'); \boldsymbol{n}) = \sum_{t' \in T} \pi(\boldsymbol{n} + \boldsymbol{I}(t')) q(\boldsymbol{I}(t'), \boldsymbol{I}(t); \boldsymbol{n}) \tag{25}$$

for all $\boldsymbol{n}, \boldsymbol{I}(t)$ such that $\boldsymbol{n} + \boldsymbol{I}(t) \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$. Substituting $\pi(\boldsymbol{m}) = B \prod_{p=1}^{N} f_p^{m(p)}$ into (25) and dividing by $B \prod_{p=1}^{N} f_p^{n_p}$ yields

$$\prod_{p=1}^{N} f_p^{(\boldsymbol{I}_p(t))} \sum_{t' \in T} \mu(t) p(\boldsymbol{I}(t), \boldsymbol{I}(t')) = \sum_{t' \in T} \prod_{p=1}^{N} f_p^{(\boldsymbol{I}_p(t'))} \mu(t') p(\boldsymbol{I}(t'), \boldsymbol{I}(t))$$

We recognize the routing chain equations (19). The solution $y(\cdot)$ to the routing chain is unique. So for the GLB-equations to be verified, it remains to show that, for all $t \in T$

$$\prod_{p=1}^{N} f_p^{(\boldsymbol{I}_p(t))} = y(\boldsymbol{I}(t)). \tag{26}$$

To this end, note that by the definition of the f_p's

$$\log\left(\frac{y(\boldsymbol{I}(t))}{y(\boldsymbol{O}(t))}\right) = \sum_{p=1}^{N} A(p,t) z_p = \sum_{p=1}^{N} \boldsymbol{I}_p(t) \log(f_p) - \boldsymbol{O}_p(t) \log(f_p) = \sum_{p=1}^{N} \log\left(\frac{f_p^{(\boldsymbol{I}_p(t))}}{f_p^{(\boldsymbol{O}_p(t))}}\right)$$

and thus

$$\frac{y(\boldsymbol{I}(t))}{y(\boldsymbol{O}(t))} = \prod_{p=1}^{N} \frac{f_p^{(\boldsymbol{I}_p(t))}}{f_p^{(\boldsymbol{O}_p(t))}},$$

which shows that (26) is satisfied. □

Under the condition that a solution to the routing chain exists, equivalence of condition (ii) of Lemma (5.12) and product form π_y satisfying (20), was obtained by Coleman et al. [13]. The solution z of the alternative condition (iii) was used to express the explicit solution of the product form. The contribution of Theorem 5.13 is the explicit relation between GLB and product form.

Theorem 5.13 characterizes product forms for $\mathcal{SPN}$s based on the incidence matrix. The product form (23) is of the Jackson-type since it is a product over the places similar to the result of Jackson [39]. Note that the Petri nets are substantially more complex than Jackson networks. The product form distribution (23) contains one term for each token in the Petri net. Therefore, under GLB the only dependence between tokens lies in the normalising constant, as is the case in closed Jackson networks. Observe that Theorem 5.13 does not state that an arbitrary $\mathcal{SPN}$ with product form equilibrium distribution satisfies GLB.

Remark 5.14. Each T-invariant can be written as a linear combination of minimal support T-invariants (result 3.32). Therefore, it can readily be seen that in Lemma 5.10, Corollary 5.11 and Lemma 5.12 the statement 'for each T-invariant', can be replaced by 'for each minimal support T-invariant'. This observation will be convenient when studying the structural implications of the results presented in this section.

5.3 Structural implications of product form $\mathcal{SPN}$s

In this section, we study the structural implication of Theorem 5.13 on the Petri net. The condition $Rank[\boldsymbol{A}] = Rank[\boldsymbol{A}|\boldsymbol{C}]$ was presented in Coleman et al. [13] as a necessary and sufficient condition for product form. Three comments can be placed regarding their results: (1) they assumed a solution of the routing chain to exist, (2) the condition $Rank[\boldsymbol{A}] = Rank[\boldsymbol{A}|\boldsymbol{C}]$ generally depends on the numerical values of the transition rates, and (3) $Rank[\boldsymbol{A}] = Rank[\boldsymbol{A}|\boldsymbol{C}]$ is a technical condition without intuitive interpretation.

The first comment is addressed in Theorem 5.5; for a solution of the routing chain to exist the Petri net must be an $S\Pi$-net. The second comment was already observed by Coleman et al. [13], where it is shown that in some cases conditions on the numerical values of the firing rates must be imposed and in some cases not. To this end, Haddad et al. [22] introduced $S\Pi^2$-nets, a subclass of $S\Pi$-nets that have product form irrespective of the numerical values of the firing rates. However, this characterization of $S\Pi^2$-nets does not intuitively explain why no restrictions on the numerical values of the firing rates are imposed. The structural implications of the product form results of Theorem 5.13, are based on the *minimal support* T-invariants (see Remark 5.14). First, we will show that $S\Pi$-nets in which all minimal support T-invariants are minimal *closed* support T-invariants have product form without additional conditions on the firing rates. Second, we will show that this characterization exactly corresponds to the definition of $S\Pi^2$-nets provided by Haddad et al. [22]. Third, via this characterization in terms of the minimal support T-invariants we are able to provide an explanation in terms of T-invariants of the condition $Rank[\boldsymbol{A}] = Rank[\boldsymbol{A}|\boldsymbol{C}]$ of the $\mathcal{SPN}$. The condition is shown to be required only for $S\Pi$-nets that are not $S\Pi^2$-nets.

Theorem 5.15. For an $\mathcal{SPN}$, (22) is satisfied for each minimal *closed* support T-invariant $\boldsymbol{x}$. For an $S\Pi$-net in which each minimal support T-invariant is a minimal *closed* support T-invariant, the equivalent conditions (i)-(iii) of Lemma 5.12 are satisfied.

Proof. The firing sequence of a minimal closed support T-invariant is linear (Result 3.34). Thus, $x_t \leq 1,\ t = 1, \ldots, T$, and within this T-invariant every output bag is an input bag of a unique next transition. Therefore, in (22) the denominator of each fraction $y(\boldsymbol{I}(t))/y(\boldsymbol{O}(t))$ is cancelled by the nominator of the fraction of the subsequent transition in this T-invariant. As a consequence, conditions (i)-(iii) of Lemma 5.12 are satisfied irrespective of of the numerical values of the firing rates. □

By means of Theorem 5.15, in the case that there exists a minimal T-invariant that is *not* closed, additional conditions are required on the numerical values of the firing rates to ensure a product form solution. Below, we will provide an intuitive explanation of these additional conditions. First, the definition of $S\Pi^2$-nets, as introduced by Haddad et al. [22], is presented.

Definition 5.16 ($S\Pi^2$-net)**.** A Π^2-net is a Π-net such that for every $\boldsymbol{g} \in \mathcal{R}(\mathcal{T})$, there is an $\boldsymbol{a}_g \in \mathbb{Q}^N$ such that

$$\boldsymbol{a}_g \boldsymbol{A} = \boldsymbol{b}_g$$

in which for $p = 1, \ldots, N$

$$\boldsymbol{b}_g(p) = \begin{cases} -1 & \text{if } \boldsymbol{g} = \boldsymbol{I}(t), \\ 1 & \text{if } \boldsymbol{g} = \boldsymbol{O}(t), \\ 0 & \text{otherwise} \end{cases}$$

An $S\Pi^2$-net is a stochastic Π^2-net.

Although not defined as such by Haddad et al. [22], and not recognized before, the characterization of an $S\Pi^2$-net can be provided via the the minimal support T-invariants of the $\mathcal{S}\Pi$-net, as is shown in the next theorem.

Theorem 5.17. An $S\Pi$-net is an $S\Pi^2$-net if and only if *all* minimal support T-invariants are minimal *closed* support T-invariants.

Proof. Consider an $\mathcal{S}\Pi$-net. We must show that $\boldsymbol{a}_g \boldsymbol{A} = \boldsymbol{b}_g$ has a solution if and only if all minimal support T-invariants are minimal closed support T-invariant. First observe that $\boldsymbol{a}_g \boldsymbol{A} = \boldsymbol{b}_g$ has a solution if and only if the row vector $\boldsymbol{b}_g$ is a linear combination of the rows of $\boldsymbol{A}$, i.e., $\boldsymbol{b}_g \boldsymbol{x} = 0$ for all $\boldsymbol{x} \ni \boldsymbol{A}\boldsymbol{x} = 0$, that is $\boldsymbol{b}_g \boldsymbol{x} = 0$ for all T-invariants. Second, if a solution $\boldsymbol{a}_g$ exists, it is rational since $\boldsymbol{A}$ is an integer matrix and $\boldsymbol{b}_g$ an integer vector.

Now, assume that all minimal support T-invariants are minimal closed support. Consider a minimal closed support T-invariants $\boldsymbol{x}$ and a bag $\boldsymbol{g} \in \mathcal{R}(T) \ni (\boldsymbol{O}(t_i) = \boldsymbol{I}(t_j))$, then $\boldsymbol{b}_g \boldsymbol{x} = x_{t_i} - x_{t_j}$, since the firing sequence of $\boldsymbol{x}$ is linear (Result 3.34). Either $\boldsymbol{g}$ is both an input bag and an input bag of transitions in the firing sequence of $\boldsymbol{x}$ (i.e., $x_{t_i} = x_{t_j} = 1$), or $\boldsymbol{g}$ is neither an input bag nor an output bag of any transition in the firing sequence of $\boldsymbol{x}$ (i.e., $x_{t_i} = x_{t_j} = 0$). By assumption all minimal support T-invariants are minimal closed support, which completes the first part of the proof.

Conversely, if there is a minimal support T-invariant $\boldsymbol{x}$ of which the support is not closed, then $\exists \boldsymbol{g} \in \mathcal{R}(T), t \in \|\boldsymbol{x}\|$, such that $\boldsymbol{b}$ is the output of t, but there is no $t' \in \|\boldsymbol{x}\|$ such that $\boldsymbol{g}$ is the input bag of t'. For such $\boldsymbol{x}$ we have $\boldsymbol{b}_g \boldsymbol{x} \neq 0$ and this completes the proof of the second part. □

Corollary 5.18. For an $S\Pi^2$-net the equivalent conditions (i)-(iii) of Lemma 5.12 are satisfied irrespective of the firing rates. Therefore, GLB and a product form solution of the form (23) can be verified without checking one of these conditions.

Proof. By Theorem 5.15 and Theorem 5.17, for an $S\Pi^2$-net the equivalent conditions (i)-(iii) of Lemma 5.12 are satisfied irrespective of the transition rates. Applying Theorem 5.13 concludes the proof. □

Theorem 5.13 states that the equilibrium distribution of an $\mathcal{S}\Pi$-net is characterized by the solution of the routing chain $y(\cdot)$, characterized by the probability flow through classes of minimal closed support T-invariants. In $S\Pi$-nets, all transitions are covered by minimal closed support T-invariants. Therefore, every minimal support T-invariant that is not closed support is build up by transitions of different minimal closed support T-invariants. The conditions (i)-(iii) of Lemma 5.12 imply that the probability flow through a minimal non-closed support T-invariant should be harmonized with the probability flow imposed by the minimal closed support T-invariants. Examples 5.21 and 5.22 in the next subsection will provide an illustration.

From the results presented above, it is clear that characterization of product form results for $\mathcal{SPN}$s with transition rates (17) can be done at the structural level. The steps that have to be performed to this end are summarized in the following algorithm.

Algorithm 5.19 (Structural characterization of product form)**.**

Step 1. *Obtain the incidence matrix $\boldsymbol{A}$ of the $\mathcal{SPN}$ and compute the minimal support T-invariants $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^h$ and the minimal support P-invariants $\boldsymbol{y}^1, \ldots, \boldsymbol{y}^j$.*

Step 2. *Obtain the minimal closed support T-invariants from the minimal support T-invariants, and renumber the T-invariants such that $\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k\}$ is the set of minimal closed support T-invariants ($k \leq h$).*

Step 3. *Verify that all transitions are covered by minimal closed support T-invariants and minimal support P-invariants. If not: stop, we cannot conclude a product form equilibrium distribution, else: go to step 4.*

Step 4. *Determine from $\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k\}$ the set of common input bag classes $\{CI(\boldsymbol{x}^1), \ldots CI(\boldsymbol{x}^l)\}$. Compute per common input bag class i the solution to the routing chain $y^i(\cdot)$. If all minimal support T-invariants are minimal closed support T-invariants, i.e., $k = h$, then proceed to step 6, else go to step 5.*

Step 5. *Determine $\boldsymbol{C}$ and verify that $\boldsymbol{C}\boldsymbol{x}^i = 0$, for the minimal non-closed support T-invariants $\boldsymbol{x}^{k+1}, \ldots, \boldsymbol{x}^h$. If not: stop, the $\mathcal{SPN}$ does not have a product form equilibrium distribution, else go to step 6.*

Step 6. *Solve $z\boldsymbol{A} = \boldsymbol{C}$. The equilibrium distribution is $\pi(\boldsymbol{m}) = B\pi_y(\boldsymbol{m})$ with π_y given in (23).*

5.4 Examples of product form $\mathcal{SPN}$s

This section presents some examples illustrating the structural characterization of product form presented above. First, in Example 5.20 we present an example of an $S\Pi^2$-net. Examples 5.21 and 5.22 present $S\Pi$-nets that are not $S\Pi^2$-nets, so that a product form equilibrium distribution can

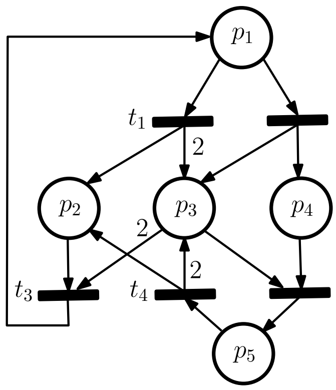

Figure 3: a.

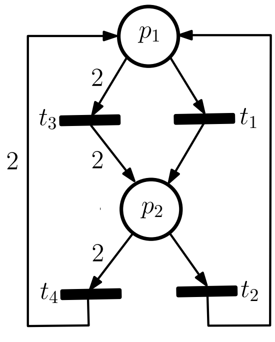

Figure 3: b.

only be concluded for a specific choice of the firing rates. Finally, in Example 5.23, we illustrate the importance of the boundedness assumption, by presenting a net that may not possess an equilibrium distribution, due to a possibly unbounded number of tokens. Examples 5.20, 5.21 and 5.23 are obtained from [4].

Example 5.20. Consider the $\mathcal{SPN}$ depicted in Figure 3a and execute the steps of the algorithm of Section 5.3.

Step 1-3. From the incidence matrix

$$\boldsymbol{A} = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 2 & 1 & -2 & 2 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

we obtain that this net has two minimal support T-invariants $\boldsymbol{x}^1 = (10100)$, $\boldsymbol{x}^2 = (01111)$, which are both minimal closed support T-invariants, and two minimal support P-invariants $\boldsymbol{y}^1 = (11011)$, $\boldsymbol{y}^2 = (20112)$. $\mathcal{SPN}$ is covered by both minimal support T-invariants and P-invariants.

Step 4. Since the T-invariants share $\boldsymbol{I}(t_1)$ they are in common input bag relation, which implies that the routing chain has one irreducible set:

$$S = \{\boldsymbol{I}(t_1), \boldsymbol{I}(t_3), \boldsymbol{I}(t_4), \boldsymbol{I}(t_5)\} \;\; (\boldsymbol{I}(t_1) = \boldsymbol{I}(t_2)).$$

Amalgamate transition t_1 and t_2 into a single transition t_{12} with $\mu(t_{12}) = \mu(t_1) + \mu(t_2)$, $p(\boldsymbol{I}(t_1), \boldsymbol{O}(t_1)) = \mu(t_1)/\mu(t_{12})$ and $p(\boldsymbol{I}(t_1), \boldsymbol{O}(t_2)) = \mu(t_2)/\mu(t_{12})$. The solution of the routing chain is (up to normalisation):

$$y(\boldsymbol{I}(t_1))\mu(t_{12}) = y(\boldsymbol{I}(t_3))\mu(t_3) = 1, \quad y(\boldsymbol{I}(t_4))\mu(t_4) = y(\boldsymbol{I}(t_5))\mu(t_5) = p(\boldsymbol{I}(t_1), \boldsymbol{O}(t_2)).$$

The $\mathcal{SPN}$ is an $S\Pi^2$-net, so we may proceed to step 6.

Step 6. The vector $\boldsymbol{C}$ is obtained from the solution of the routing chain:

$$\boldsymbol{C} = \left(\log\left[\frac{\mu(t_3)}{\mu(t_{12})}\right], \log\left[\frac{\mu(t_5)}{\mu(t_2)}\right], \log\left[\frac{\mu(t_{12})}{\mu(t_3)}\right], \log\left[\frac{\mu(t_2)\mu(t_3)}{\mu(t_{12})\mu(t_4)}\right], \log\left[\frac{\mu(t_4)}{\mu(t_5)}\right]\right).$$

A solution z of $z\boldsymbol{A} = \boldsymbol{C}$ is:

$$z_1 = 0,\ z_2 = \log\left(\frac{\mu(t_3)}{\mu(t_{12})}\right),\ z_3 = 0,\ z_4 = \log\left(\frac{\mu(t_5)}{\mu(t_2)}\right),\ z_5 = \log\left(\frac{\mu(t_4)}{\mu(t_2)}\right)$$

and the equilibrium distribution is

$$\pi(\boldsymbol{m}) = B\left(\frac{\mu(t_{12})}{\mu(t_3)}\right)^{m(2)}\left(\frac{\mu(t_2)}{\mu(t_5)}\right)^{m(4)}\left(\frac{\mu(t_2)}{\mu(t_4)}\right)^{m(5)}$$

at reachability set

$$\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0) = \{\boldsymbol{m} : \boldsymbol{y}^1(\boldsymbol{m} - \boldsymbol{m}_0) = 0,\ \boldsymbol{y}^2(\boldsymbol{m} - \boldsymbol{m}_0) = 0\},$$

where $\boldsymbol{y}^1 = (11011)$, $\boldsymbol{y}^2 = (20112)$ are the two minimal support P-invariants of the net. □

Example 5.21. Consider the $\mathcal{SPN}$ depicted in Figure 3b. This is an example of an $S\Pi$-net which is not an $S\Pi^2$-net so that additional conditions on the firing rates have to be satisfied.

Step 1-3. This $\mathcal{SPN}$ has incidence matrix

$$\boldsymbol{A} = \begin{pmatrix} -1 & 1 & -2 & 2 \\ 1 & -1 & 2 & -2 \end{pmatrix}.$$

Observe that each transition is covered by the minimal closed support T-invariants $\boldsymbol{x}^1 = (1100)$, $\boldsymbol{x}^2 = (0011)$, but that $\boldsymbol{x}^3 = (2001)$ and $\boldsymbol{x}^4 = (0210)$ are also minimal support T-invariants that do not have closed support. $\mathcal{SPN}$ is covered by its one minimal support P-invariant $\boldsymbol{y}^1 = (11)$.

Step 4. The routing chain has two irreducible sets $S(\boldsymbol{x}^1) = \{\boldsymbol{I}(t_1), \boldsymbol{I}(t_2)\}$, and $S(\boldsymbol{x}^2) = \{\boldsymbol{I}(t_3), \boldsymbol{I}(t_4)\}$. The solution of the routing chain is:

$$\frac{y^1(\boldsymbol{I}(t_2))}{y^1(\boldsymbol{I}(t_1))} = \frac{\mu(t_1)}{\mu(t_2)},\quad \frac{y^2(\boldsymbol{I}(t_4))}{y^2(\boldsymbol{I}(t_3))} = \frac{\mu(t_3)}{\mu(t_4)},$$

with corresponding vector $\boldsymbol{C}$

$$\boldsymbol{C} = \left(\log\left[\frac{\mu(t_2)}{\mu(t_1)}\right], \log\left[\frac{\mu(t_1)}{\mu(t_2)}\right], \log\left[\frac{\mu(t_4)}{\mu(t_3)}\right], \log\left[\frac{\mu(t_3)}{\mu(t_4)}\right]\right).$$

Step 5. $\boldsymbol{C}\boldsymbol{x}^i = 0$ for the minimal non-closed support T-invariants $\boldsymbol{x}^3 = (2001)$ and $\boldsymbol{x}^4 = (0210)$, if $2C_1 + C_4 = 0$ and $2C_2 + C_3 = 0$, thus if

$$\left(\frac{\mu(t_2)}{\mu(t_1)}\right)^2 = \frac{\mu(t_4)}{\mu(t_3)}. \tag{27}$$

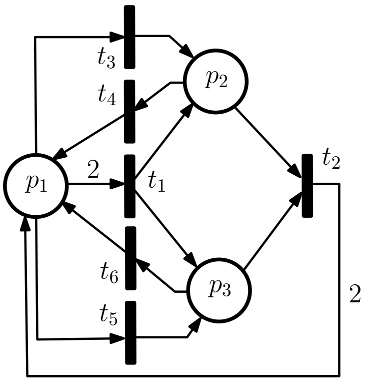

Figure 4: a.

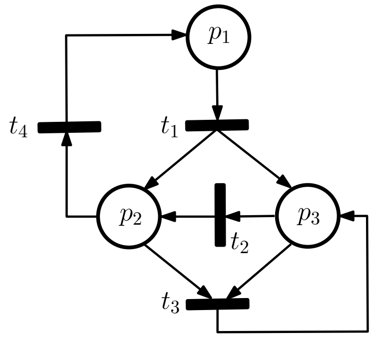

Figure 4: b.

Step 6. If (27) is satisfied, this $\mathcal{SPN}$ has an equilibrium distribution

$$\pi(\boldsymbol{m}) = B\left(\frac{\mu(t_2)}{\mu(t_1)}\right)^{\boldsymbol{m}(1)},$$

at reachability set

$$\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0) = \{\boldsymbol{m} : \boldsymbol{m}(1) + \boldsymbol{m}(2) = \boldsymbol{m}_0(1) + \boldsymbol{m}_0(2)\}.$$

This example provides insight in the intuition for the conditions of Lemma 5.12. As can be seen from Figure 3b, there are two possibilities for the movement of two tokens from place 1 to place 2. In the first case (via t_1) the tokens jump one after the other, in the second case (via t_3) the tokens jump simultaneously. The probability flow for these two possibilities must be the same. This is reflected in the condition (27) on the firing rates: two transitions with rate $\mu(t_1)$ must be proportional to one transition at rate $\mu(t_3)$. □

Example 5.22. Consider the $\mathcal{SPN}$ of Figure 4a. This example indicates that minimal *non-closed* support T-invariants can also exist in $\mathcal{S}\Pi$-nets of which in the minimal support T-invariants no transition fires more than once, i.e., $\boldsymbol{x}_t \leq 1, \forall t \in T$ is not sufficient for a T-invariant to be closed support.

Step 1-3. The minimal closed support T-invariants are $\boldsymbol{x}^1 = (110000), \boldsymbol{x}^2 = (001100)$ and $\boldsymbol{x}^3 = (000011)$ and the minimal non-closed support T-invariants $\boldsymbol{x}^4 = (100101)$ and $\boldsymbol{x}^5 = (011010)$. $\mathcal{SPN}$ is covered by its one minimal support P-invariant $\boldsymbol{y}^1 = (111)$.

Step 4-6. This $\mathcal{SPN}$ has a product form equilibrium distribution if $C_1 = C_3 + C_5$ and $C_2 = C_4 + C_6$, so if

$$\frac{\mu(t_2)}{\mu(t_1)} = \frac{\mu(t_4)}{\mu(t_3)}\frac{\mu(t_6)}{\mu(t_5)}.$$

□

Example 5.23. Consider the $\mathcal{SPN}$ of Figure 4b.

Step 1-3. The net has one T-invariant $\boldsymbol{x} = (1111)$ covering all transitions, and $\boldsymbol{x}$ has closed support. It has no P-invariants.

Note that without additional conditions the algorithm stops here. Yet we proceed to provide an illustration of such conditions that prevents the creation of an unbounded number of tokens.

Step 4. The solution of the routing chain is (up to a multiplicative constant)

$$y(\boldsymbol{I}(t_1)) = 1/\mu(t_1), \quad y(\boldsymbol{I}(t_2)) = 1/\mu(t_2), \quad y(\boldsymbol{I}(t_3)) = 1/\mu(t_3), \quad y(\boldsymbol{I}(t_4)) = 1/\mu(t_4),$$

Step 6. The $\mathcal{SPN}$ has an invariant measure

$$\pi_y(\boldsymbol{m}) = \left(\frac{\mu(t_3)\mu(t_4)}{\mu(t_1)\mu(t_2)}\right)^{\boldsymbol{m}(1)} \left(\frac{\mu(t_3)}{\mu(t_2)}\right)^{\boldsymbol{m}(2)} \left(\frac{\mu(t_4)}{\mu(t_2)}\right)^{\boldsymbol{m}(3)}.$$

From Figure 4b we can see that the number of tokens in the net is unbounded (repetitive firing of transitions t_1 and t_4 increases the number of tokens by 1), but that for every marking a firing sequence to $\boldsymbol{m}_0 = (100)$ exists. Under the *additional* conditions $\mu(t_3)\mu(t_4) < \mu(t_1)\mu(t_2)$, $\mu(t_3) < \mu(t_2)$, $\mu(t_4) < \mu(t_2)$ the $\mathcal{SPN}$ has an equilibrium distribution

$$\pi(\boldsymbol{m}) = B\pi_y(\boldsymbol{m}), \quad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0) = \mathbb{N}_0^3 \setminus \{0\}.$$

□

6 Decomposing the Stochastic Petri Net

The analysis of the previous sections enables us to formulate a new decomposition result. This result uses the T- and P-invariants to decompose an $\mathcal{SPN}$ in subnets, consisting of one or more common input bag classes as defined in section 5. It is a generalization of the decomposition result formulated by Frosch and Natarajan [20] for Closed Synchronized Systems of Stochastic Sequential Processes (CS) that consist of state machines connected by so-called buffer places. By removing these buffer places from the network, the equilibrium (product-form) distribution of a CS is shown to be a product over the product-form equilibrium distributions of the separate state machines. A CS is obtained by starting from separate networks and linking these by buffer places, so that the buffer places are defined beforehand. Therefore, the result of Frosch and Natarajan is more a composition result, than a decomposition result. This section generalizes the results of Frosch and Natarajan to decomposition results for product form $\mathcal{S}\Pi$-nets.

Starting from an arbitrary $\mathcal{S}\Pi$-net and decomposing this into subnetworks so that the equilibrium distribution is a product over product forms of the subnetworks requires the identification of conflict places, the generalization of buffer places, and sufficient conditions for decomposition. In this section, we will provide an identification of the conflict places, and formulate these sufficient conditions. The subnetworks in our decomposition result will be synchronized by places that are shared by different CI-classes, and the product form equilibrium distribution will shown to be a product over the invariant measures of the subnetworks defined by the CI-classes.

The outline of this section is as follows. First, Theorem 6.6 identifies $\mathcal{S}\Pi$-nets, which can be decomposed in subnets that each correspond to a unique common input bag class. Second, Theorem 6.6 expands the class of decomposable nets to $\mathcal{S}\Pi$-nets that can be decomposed into subnets each corresponding to one or more common input bag classes. Finally, an algorithm is presented by which all possible decompositions of an $\mathcal{S}\Pi$-net are generated.

The *sufficient place set*, introduced by Florin and Natkin [17], will be essential in our decomposition result. The places not contained in the sufficient place set will be the places at which we will decompose the $\mathcal{SPN}$. We define this complementary set of places as the *surplus place set.*

Definition 6.1 (**Sufficient place set - Surplus place set**)**.** A subset of places $\mathcal{P}^{suf} \subseteq P$ is a *sufficient place set* if the marking of each place in $\mathcal{P}^{suf}$ provides sufficient information to define uniquely the marking of all places. A subset of places $\mathcal{P}^{sur} \subseteq P$ is a *surplus place set* if the subset of places $P \setminus \mathcal{P}^{sur}$ is a sufficient place set. A place contained in a surplus place set will be referred to as a *surplus place.*

There may be solutions to the matrix equation $z\boldsymbol{A} = \boldsymbol{C}$ in Theorem 5.13 with $z_p = 0$ for some places p. Such a place has $f_p = 1$ and no term involving place p appears in the product form (23). The following lemma shows that such places are uniquely related to places contained in a surplus place set.

Lemma 6.2. Assume a solution to the matrix equation $z\boldsymbol{A} = \boldsymbol{C}$ exists. Then, there exists a solution to $z\boldsymbol{A} = \boldsymbol{C}$, where $z_p = 0$, for all $p \in \mathcal{P}'$ ($\mathcal{P}' \subseteq P$) if and only if $\mathcal{P}'$ is a surplus place set.

Proof. Consider a surplus set $\mathcal{P}'$. The marking changes of the places $p \in \mathcal{P}'$ in any firing sequence starting from $\boldsymbol{m}_0$ can be characterized by the marking change of the places $p \in \mathcal{P}^{suf} = P \setminus \mathcal{P}'$, which is characterized by the row vectors of $\boldsymbol{A}$ corresponding to the places $p \in \mathcal{P}^{suf}$. This implies that the row vectors $\boldsymbol{A}_p$ of $\boldsymbol{A}$ corresponding to the places $p \in \mathcal{P}'$ can be written as the linear combination $\sum_{\{p\in\mathcal{P}^{suf}\}} \alpha_p \boldsymbol{A}_p$, with $\alpha_p \in \mathbb{Q}$. (Note that α_p is rational since $\boldsymbol{A}$ is an integer matrix). Therefore, under the assumption that a solution z to $z\boldsymbol{A} = \boldsymbol{C}$ exists, there exists a solution where $z_p = 0, \forall p \in \mathcal{P}'$.

Conversely, consider a set places $\mathcal{P}' \subseteq P$. If there exists a solution to $z\boldsymbol{A} = \boldsymbol{C}$, where $z_p = 0$, $\forall p \in \mathcal{P}'$, we have $\boldsymbol{C} = \sum_{\{p\in P\setminus\mathcal{P}'\}} z_p \boldsymbol{A}_p$. This implies that for each $p \in \mathcal{P}'$, $\boldsymbol{A}_p$ can be written as the linear combination $\sum_{\{p\in P\setminus\mathcal{P}'\}} \alpha_p \boldsymbol{A}_p$, with $\alpha_p \in \mathbb{Q}$. Therefore, the marking changes of the places $p \in \mathcal{P}'$ in any firing sequence starting from $\boldsymbol{m}_0$ can be characterized by the marking change of the places $p \in P\setminus\mathcal{P}'$, which is characterized by the row vectors of $\boldsymbol{A}$ corresponding to the places $p \in P \setminus \mathcal{P}'$. As a consequence, $\mathcal{P}'$ is a surplus place set. □

For a given $\mathcal{SPN}$, the sufficient place set (and the corresponding surplus place set) is in general not unique. Since P-invariants characterize a constant weighted marking over a subset of places (see Definition 3.14), surplus places can be characterized from the P-invariants of the $\mathcal{SPN}$. Later in this section, we will show how to find all sufficient place sets. At this point, let us first provide the minimal number of places a sufficient place set must contain. This number was already expressed (and defined as the *dimension of the marking process*) by Florin and Natkin [17], but without proof. For completeness a proof is included here.

Lemma 6.3 ([17])**.** For each sufficient place set $\mathcal{P}^{suf}$:

$$\left|\mathcal{P}^{suf}\right| \geq N - dim(Ker(A^T)).$$

Proof. The number of linearly independent minimal support P-invariants is $dim(Ker(A^T))$. Recall that this number can be smaller than the number of minimal support P-invariants (see Remark 3.33). From each additional linearly independent P-invariant an additional surplus place can be selected. As such, the maximum number of places in a surplus place set is $dim(Ker(A^T))$, and thus the number of places in a sufficient place set must exceed $N - dim(Ker(A^T))$. □

For each common input bag class $CI(\boldsymbol{x})$, denote the set of places that are elements of the closed support T-invariants in $CI(\boldsymbol{x})$ by $P(CI(\boldsymbol{x}))$:

$$P(CI(\boldsymbol{x})) = \{p \in P \mid \exists \boldsymbol{x} \in CI(x) \wedge \exists t \in \|\boldsymbol{x}\| \text{ with } I_p(t) \geq 0\}.$$

Firing of transitions of T-invariants of different CI-classes interacts and conflicts in the places that are shared among the common input bag classes. Focussing on such places will enable us to formulate decomposition results. Therefore, we formally define *conflict places* and the set of all conflicting places among all common input bag classes.

Definition 6.4 (**Conflict place - Conflict place set**)**.** Let $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ be minimal closed support T-invariants such that $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are not in common input bag relation, i.e., $CI(\boldsymbol{x}^1) \neq CI(\boldsymbol{x}^2)$. Let p be a place that is an element of both $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$, i.e., $p \in \left(P(CI(\boldsymbol{x}^1)) \cap P(CI(\boldsymbol{x}^2))\right)$. Then p is called a *conflict place* of $CI(\boldsymbol{x}^1)$ and $CI(\boldsymbol{x}^2)$. The *conflict place set* is the subset $\mathcal{P}^{con} \subseteq P$, of places that are a conflict place between any two common input bag classes:

$$\mathcal{P}^{con} = \left\{p \in P \mid p \in \left(P(CI(\boldsymbol{x}^i)) \cap P(CI(\boldsymbol{x}^j))\right), \forall i, j \text{ with } CI(\boldsymbol{x}^i) \neq CI(\boldsymbol{x}^j)\right\}.$$

Our decomposition result will be obtained by removing conflict places. Therefore, the following lemma will be of importance.

Lemma 6.5. If in an $\mathcal{S}\Pi$-net $\mathcal{SPN}$ the places and all arcs incident to all the places $p \in \mathcal{P} \subset P$ can be removed so that no complete input bag is removed, then the remaining net is an $\mathcal{S}\Pi$-net, possibly consisting of several strictly separated components.

Proof. Remove from $\mathcal{SPN}$ the place $p' \in \mathcal{P}$ and the arcs incident to this place. Place p' is not a complete input bag, since by removing all places $p \in \mathcal{P}$ no complete input bag is removed. Denote the remaining net by $\mathcal{SPN}'$. $\mathcal{SPN}'$ only differs from $\mathcal{SPN}$ in the transitions incident to place p. We need to show that these transitions are still covered by minimal closed support T-invariants. Consider the set of minimal closed support T-invariants in $\mathcal{SPN}$ that visit place p, i.e., $\{\boldsymbol{x} \mid \exists t \in \|\boldsymbol{x}\| \text{ with } \boldsymbol{I}_p(t) \geq 0 \vee \boldsymbol{O}_p(t) \geq 0\}$. Now consider the consecutive transitions $t, t' \in \|\boldsymbol{x}\|$ for which $\boldsymbol{O}(t) = \boldsymbol{I}(t')$ and $\boldsymbol{O}_p(t) \geq 0$ in the original net $\mathcal{SPN}$. In the net $\mathcal{SPN}$, $\boldsymbol{O}(t) = \boldsymbol{I}(t')$ still holds, since both in $\boldsymbol{O}(t)$ and $\boldsymbol{I}(t')$ place p is removed. Therefore, each minimal closed support T-invariant $\boldsymbol{x}$ in $\mathcal{SPN}$ is still a minimal closed support T-invariant in $\mathcal{SPN}'$. Since it may be that for two minimal closed support T-invariants $\boldsymbol{x}^1, \boldsymbol{x}^2$ that visit place p, place p is the only conflict place of $CI(\boldsymbol{x}^1)$ and $CI(\boldsymbol{x}^2)$, i.e., $CI(\boldsymbol{x}^1) \cap CI(\boldsymbol{x}^2) = p$, $\mathcal{SPN}'$ may consist of two strictly separate $\mathcal{S}\Pi$-nets. The proof is completed by repeating this argument until all places $p \in \mathcal{P}$ are removed. □

At this point, we are ready to state the first decomposition result. Let $\{CI(\boldsymbol{x}^1), \ldots CI(\boldsymbol{x}^l)\}$ be the set of common input bag classes.

Theorem 6.6. Consider a product form $\mathcal{S}\Pi$-net. If there exists a surplus place set $\mathcal{P}^{sur}$ and corresponding sufficient place set $\mathcal{P}^{suf}$, such that

1. all conflicting places are contained in the surplus place set, i.e., $\mathcal{P}^{con} \subseteq \mathcal{P}^{sur}$, and
2. there is no transition for which the complete input bag is contained in the conflict place set, i.e., $\nexists t \in T$ for which $\{p \in P \mid \boldsymbol{I}_p(t) \geq 0\} \subseteq \mathcal{P}^{con}$,

then

- removing all places $p \in \mathcal{P}^{con}$ and all arcs incident to the places $p \in \mathcal{P}^{con}$ yields ℓ product form $\mathcal{S}\Pi$-nets: $\mathcal{SPN}^1, \ldots, \mathcal{SPN}^\ell$, where each $\mathcal{SPN}^i$ corresponds to a common input bag class $CI(\boldsymbol{x}^i)$,
- the equilibrium distribution π is a product over the invariant measures of the subnets:

$$\pi(\boldsymbol{m}) = B \prod_{i=1}^{\ell} \pi_y^{\mathcal{SPN}(\boldsymbol{x}^i)}(\boldsymbol{m}^i), \quad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0),$$

  where $\boldsymbol{m}^i$ is the submarking in places that belong to subnet $\mathcal{SPN}^i$, $\pi_y^{\mathcal{SPN}(\boldsymbol{x}^i)}(\boldsymbol{m}^i)$ is the invariant measure of subnet $\mathcal{SPN}^i$ with

$$\pi_y^{\mathcal{SPN}(\boldsymbol{x}^i)}(\boldsymbol{m}^i) = \prod_{\{p \in P(CI(\boldsymbol{x}^i)) \setminus \mathcal{P}^{con}\}} f_p^{m_p}$$

  and B is a normalizing constant such that $B^{-1} = \sum_{\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)} \pi_y(\boldsymbol{m})$.

Proof. In an $\mathcal{S}\Pi$-net each transition belongs to a unique CI-class. When all conflict places are removed, each remaining place belongs to a unique CI-class. In addition, when all arcs connected to conflict places are removed, each connection between places and transition of different CI-classes is removed. Therefore, one obtains ℓ subnets $\mathcal{SPN}^1, \ldots, \mathcal{SPN}^\ell$. By Lemma 6.5, these subnets are again $\mathcal{S}\Pi$-nets.

For the second part, by Lemma 6.2, for $\mathcal{SPN}$ there exists a solution to $z\boldsymbol{A} = \boldsymbol{C}$, in which $z_p = 0, \forall p \in \mathcal{P}^{con}$. The Product Form solution (23) can thus be rewritten as

$$\pi_y(\boldsymbol{m}) = \prod_{i=1}^{\ell} \left\{ \prod_{\{p \in P(CI(\boldsymbol{x}^i)) \setminus \mathcal{P}^{con}\}} f_p^{m_p} \right\}.$$

We are left to show that the f_p values are the same for the subnets as for the original net. This can be seen as follows. Introduce matrix $\boldsymbol{A}'$, which is the modified incidence matrix $\boldsymbol{A}$ so that the rows corresponding to the places of the conflict place set are set to zero, i.e., $\boldsymbol{a}_p = 0$ for all $p \in \mathcal{P}^{con}$. Then we have $z\boldsymbol{A} = z\boldsymbol{A}'$. The system of equations $z\boldsymbol{A}' = \boldsymbol{C}$ can be permuted such

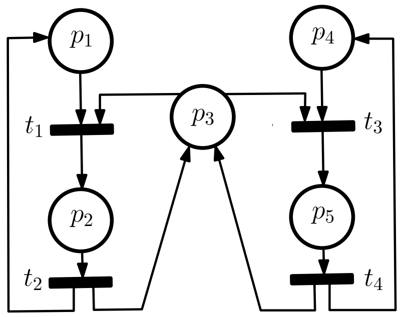

Figure 5: An $\mathcal{SPN}$ decomposing into all CI-classes.

that that the conflict places are grouped and the places of each $CI(\boldsymbol{x}^i)$ class are grouped:

$$\widetilde{z}\widetilde{\boldsymbol{A}'} = \widetilde{z} \begin{pmatrix} \boldsymbol{A}^1 & 0 & \cdots & 0 \\ 0 & \boldsymbol{A}^2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & \cdots & 0 & \boldsymbol{A}^\ell \\ 0 & \cdots & \cdots & 0 \end{pmatrix} = \widetilde{\boldsymbol{C}} = \begin{pmatrix} \boldsymbol{C}^1 & \cdots & \boldsymbol{C}^\ell \end{pmatrix}.$$

The proof is concluded by observing that the matrices $\boldsymbol{A}^i$ and vectors $\boldsymbol{C}^i, i = 1, \ldots, \ell$ correspond exactly to the incidence matrices and the $\boldsymbol{C}$-vectors of the subnets $\mathcal{SPN}^1, \ldots, \mathcal{SPN}^\ell$. □

To illustrate Theorem 6.6 and its connection to the decomposition result of Frosch and Natarajan [20], an example is provided in which Theorem 6.6 is applied for a CS.

Example 6.7. Consider the Petri net depicted in Figure 5. From the incidence matrix

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

we obtain that this net has two T-invariants $\boldsymbol{x}^1 = (1100)$ and $\boldsymbol{x}^2 = (0011)$ and three minimal support P-invariants $\boldsymbol{y}^1 = (11000)$, $\boldsymbol{y}^2 = (00011)$ and $\boldsymbol{y}^3 = (01101)$, which are linearly independent. The number of places in a sufficient place set is thus $N - 3 = 2$. The two minimal support T-invariants both have a closed support, so that it is an $\mathcal{S}\Pi^2$-net, and $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are not in common input bag relation, so that we have common input bag classes $CI(\boldsymbol{x}^1)$ and $CI(\boldsymbol{x}^2)$, with one conflict place p_3.

Consider the sufficient place set $\mathcal{P}^{suf} = \{p_1, p_4\}$, with corresponding surplus place set $\mathcal{P}^{sur} = \{p_2, p_3, p_5\}$. Then, both conditions of Theorem 6.6 are satisfied, and by removing place p_3 the net decomposes into two subnets: $\mathcal{SPN}^1$ related to $CI(\boldsymbol{x}^1)$ and $\mathcal{SPN}^2$ related to $CI(\boldsymbol{x}^2)$, with invariant measures

$$\pi_y^{\mathcal{SPN}^1}(\boldsymbol{m}^1) = \left(\frac{\mu_2}{\mu_1}\right)^{m_1} \quad \text{and} \quad \pi_y^{\mathcal{SPN}^2}(\boldsymbol{m}^2) = \left(\frac{\mu_4}{\mu_3}\right)^{m_4}.$$

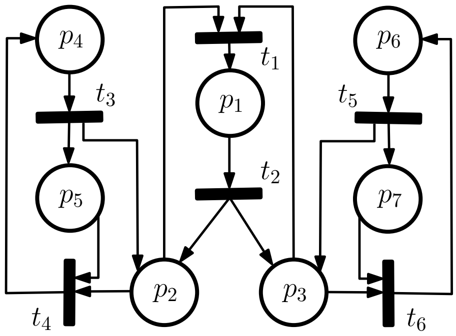

Figure 6: An $\mathcal{SPN}$ that decomposes into two components.

The equilibrium distribution of $\mathcal{SPN}$ is

$$\pi(\boldsymbol{m}) = B\pi_y^{\mathcal{SPN}^1}(\boldsymbol{m}^1)\pi_y^{\mathcal{SPN}^2}(\boldsymbol{m}^2), \qquad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0).$$

□

When decomposing according to Theorem 6.6, it is not allowed that the conflict place set of an $\mathcal{SPN}$ is such that a complete input bag is contained in this set. Otherwise, at least one of the minimal closed support T-invariants would be removed. This is illustrated in the following example.

Example 6.8. Consider the Petri net depicted in Figure 6. From the incidence matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

we obtain that this net has three T-invariants $\boldsymbol{x}^1 = (110000)$, $\boldsymbol{x}^2 = (001100)$ and $\boldsymbol{x}^3 = (00001)$ and four minimal support P-invariants $\boldsymbol{y}^1 = (0001100)$, $\boldsymbol{y}^2 = (0000011)$ and $\boldsymbol{y}^3 = (1101000)$ $\boldsymbol{y}^4 = (1010010)$, which are linearly independent. The number of places in a minimal sufficient place set is thus $N - 4 = 3$. The three minimal support T-invariants all have a closed support, so that it is an $\mathcal{S}\Pi^2$-net, and $\boldsymbol{x}^1$, $\boldsymbol{x}^2$ and $\boldsymbol{x}^3$ are not in common input bag relation, so that we have common input bag classes $CI(\boldsymbol{x}^1) = \{\boldsymbol{x}^1\}$, $CI(\boldsymbol{x}^2) = \{\boldsymbol{x}^2\}$ and $CI(\boldsymbol{x}^3) = \{\boldsymbol{x}^3\}$. The conflict place set is $\mathcal{P}^{con} = \{p_2, p_3\}$. The complete input bag of transition t_1 is contained in the conflict set, so that Theorem 6.6 can not be applied. □

The previous example shows that not all conflict places can be removed. However, the connection of common input bag class $CI(\boldsymbol{x}^3)$ with the rest of the network is such that it can be decomposed from the network, which suggests a less restrictive decomposition result in which only part of the common input bag classes are decomposed. This result will be presented in Theorem

6.9. Note that for a given sufficient place set $\mathcal{P}^{suf}$ and corresponding surplus place set $\mathcal{P}^{sur}$, $\mathcal{P}' = (\mathcal{P}^{suf} \cup p)$ with $p \in \mathcal{P}^{sur}$ is also a sufficient place set.

Theorem 6.9. Consider a product form $\mathcal{SPN}$ and a surplus place set P^{sur} with corresponding sufficient place set P^{suf}. If there is no transition for which the complete input bag is contained in the intersection of the surplus place set and conflict place set, i.e., $\mathcal{P} = \{p \in P \mid p \in (\mathcal{P}^{con} \cap \mathcal{P}^{sur})\}$ and $\nexists t \in T$ for which $\{p \in P \mid I_p(t) \geq 0\} \subseteq \mathcal{P}$, then

- removing all places $p \in \mathcal{P}$ and all arcs incident to the places $p \in \mathcal{P}$ yields s product form $\mathcal{S}\Pi$-nets: $\mathcal{SPN}^1, \ldots, \mathcal{SPN}^s$; each $\mathcal{SPN}^i$ corresponding of one or more connected common input bag classes.
- the equilibrium distribution π of $\mathcal{SPN}$ is a product over the invariant measures of the subnets:

  $$\pi(\boldsymbol{m}) = B \prod_{i=1}^{s} \pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i), \quad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0),$$

  where $\boldsymbol{m}^i$ is the submarking in places that belong to subnet $\mathcal{SPN}^i$, $\pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i)$ is the invariant measure of subnet $\mathcal{SPN}^i$ with

  $$\pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i) = \prod_{\{p \in \cap_{j=1}^{J} P(CI^i(\boldsymbol{x}^j)) \setminus \mathcal{P}^{con}\}} f_p^{m_p}, \tag{28}$$

  where $CI^i(\boldsymbol{x}^j), j = 1, \ldots, J^i$, denote the J^i common input bag classes contained in subnet $\mathcal{SPN}^i$, and B is a normalizing constant such that $B^{-1} = \sum_{\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)} \pi_y(\boldsymbol{m})$.

Proof. The proof is analogous to the proof of Theorem 6.6. Since in this case not necessarily all conflict places are removed, common input bag classes that share a conflict place that is not contained in $\mathcal{P}$ are contained in the same subnet $\mathcal{SPN}^i$. □

Example 6.8 revisited. Let us return to Example 6.8. Now choose $\mathcal{P}^{sur} = \{p_3, p_5, p_7\}$, so that $(\mathcal{P}^{sur} \cap \mathcal{P}^{con}) = \{p_3\}$. By Theorem 6.9 the network decomposes into $\mathcal{SPN}^1 = \{CI(\boldsymbol{x}^1), CI(\boldsymbol{x}^2)\}$ and $\mathcal{SPN}^2 = \{CI(\boldsymbol{x}^3)\}$, with invariant measures

$$\pi_y^{\mathcal{SPN}^1}(\boldsymbol{m}^1) = \mu_1^{m_1} \mu_2^{m_2} \left(\frac{\mu_4}{\mu_3}\right)^{m_4} \quad \text{and} \quad \pi_y^{\mathcal{SPN}^2}(\boldsymbol{m}^2) = \left(\frac{\mu_6}{\mu_5}\right)^{m_6}.$$

The equilibrium distribution of $\mathcal{SPN}$ is

$$\pi(\boldsymbol{m}) = B \pi_y^{\mathcal{SPN}^1}(\boldsymbol{m}^1) \pi_y^{\mathcal{SPN}^2}(\boldsymbol{m}^2) \qquad , \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$$

□

Remark 6.10. In Frosch and Natarajan [20], the connection of the state machines is such that the state machines are synchronized by the buffer places in such a way that the transitions of the state machines are expanded with arcs to the buffer places so that only minimal closed support T-invariants are formed from the T-invariants of the state machines. As a consequence, a CS is an $\mathcal{S}\Pi^2$-net. Frosch and Natarajan did not yet mention the concept of a minimal closed support T-invariants, and thus not yet observed CS's to be $\mathcal{S}\Pi^2$-nets. Our decomposition results are not restricted to connected state machines and not restricted to $\mathcal{S}\Pi^2$-nets. This will also be illustrated by the following Example 6.11. □

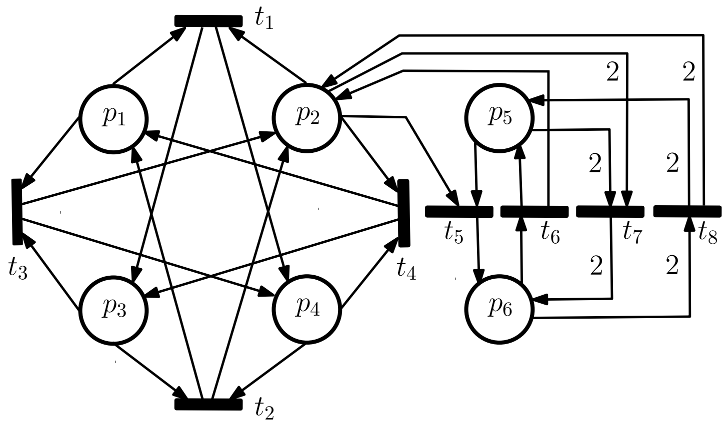

Figure 7: A decomposable $\mathcal{SPN}$ which neither a CS nor an $\mathcal{S}\Pi^2$-net.

Example 6.11. Consider the stochastic Petri net $\mathcal{SPN}$ depicted in Figure 7. This is an example of an $\mathcal{S}\Pi$-net, which is neither a CS, the class of decomposable $\mathcal{SPN}$s defined by Frosch and Natarajan [20], nor an $\mathcal{S}\Pi^2$-net. From the incidence matrix

$$A = \begin{pmatrix} -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & -1 & 1 & -2 & 2 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 & -2 \end{pmatrix}$$

we obtain that this net has six minimal support T-invariants $\boldsymbol{x}^1 = (11000000)$, $\boldsymbol{x}^2 = (00110000)$, $\boldsymbol{x}^3 = (00001100)$, $\boldsymbol{x}^4 = (00000011)$, $\boldsymbol{x}^5 = (00000110)$ and $\boldsymbol{x}^6 = (00001001)$, of which $\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3$ and $\boldsymbol{x}^4$ have a closed support. It has three minimal support P-invariants $\boldsymbol{y}^1 = (100100)$, $\boldsymbol{y}^2 = (011001)$ and $\boldsymbol{y}^3 = (000011)$, which are linearly independent. The number of places in a sufficient place set is thus $N - 3 = 3$.

The minimal closed support T-invariants $\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3, \boldsymbol{x}^4$ are not in common input bag relation, so that we have common input bag classes $CI(\boldsymbol{x}^1), CI(\boldsymbol{x}^2), CI(\boldsymbol{x}^3)$ and $CI(\boldsymbol{x}^4)$ with conflict place set $\mathcal{P}^{con} = \{p_1, p_2, p_3, p_4, p_5, p_6\}$. Since $\mathcal{SPN}$ is not an $\mathcal{S}\Pi^2$-net, for product form an additional condition on the numerical values of the transition rates is imposed, which is $(\mu_5/\mu_6)^2 = \mu_7/\mu_8$.

$CI(\boldsymbol{x}^1)$ and $CI(\boldsymbol{x}^2)$ cannot be disconnected according to Theorem 6.9, since it would require removal of a complete output bag. The same holds for $CI(\boldsymbol{x}^3)$ and $CI(\boldsymbol{x}^4)$. Therefore, consider the surplus place set $\mathcal{P}^{sur} = \{p_1, p_2\}$, with corresponding sufficient place set $\mathcal{P}^{suf} = \{p_3, p_4, p_5, p_6\}$. Then the conditions of Theorem 6.6 are satisfied, and by removing places p_1 and p_2 the net decomposes in two subnets: $\mathcal{SPN}^1$ related to $CI(\boldsymbol{x}^1)$ and $CI(\boldsymbol{x}^2)$, and $\mathcal{SPN}^2$ related to $CI(\boldsymbol{x}^3)$ and $CI(\boldsymbol{x}^4)$, with invariant measures

$$\pi_y^{\mathcal{SPN}^1}(\boldsymbol{m}^1) = \left(\frac{\mu_1}{\mu_2}\right)^{m_3} \left(\frac{\mu_3}{\mu_4}\right)^{m_4} \quad \text{and} \quad \pi_y^{\mathcal{SPN}^2}(\boldsymbol{m}^2) = \left(\frac{1}{\mu_5}\right)^{m_5} \left(\frac{1}{\mu_6}\right)^{m_6}.$$

The equilibrium distribution of $\mathcal{SPN}$ is

$$\pi(\boldsymbol{m}) = B\pi_y^{\mathcal{SPN}^1}(\boldsymbol{m}^1)\pi_y^{\mathcal{SPN}^2}(\boldsymbol{m}^2), \qquad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0).$$

To conclude, observe that an example of a decomposable $\mathcal{S}\Pi$-net which is not a CS, but which is an $\mathcal{S}\Pi^2$-net would be $\mathcal{SPN}$ without transitions t_7 and t_8. □

Since a sufficient place set is in general not unique, the decomposition according to Theorem 6.9 is not unique. For instance, in Example 6.8, a decomposition in $\mathcal{SPN}^1 = \{CI(\boldsymbol{x}^1), CI(\boldsymbol{x}^3)\}$, $\mathcal{SPN}^2 = \{CI(\boldsymbol{x}^2)\}$ is possible too. Following our decomposition approach each choice for a sufficient place set either removes a complete input bag or implies a decomposition according to Theorem 6.9. Different sufficient place sets may imply the same decomposition. To effectively exploit our decomposition result, we will develop a procedure to find all sufficient place sets of an $\mathcal{SPN}$. Therefore, Lemma 6.12 is presented which enables the identification of all surplus, and thus all sufficient, place sets.

Lemma 6.12. Let $\{\boldsymbol{y}^1, \ldots, \boldsymbol{y}^p\}$ be the set of minimal support P-invariants and $\{\bar{\boldsymbol{y}}^1, \ldots, \bar{\boldsymbol{y}}^r\}$ a basis of this set composed of elements of $\{\boldsymbol{y}^1, \ldots, \boldsymbol{y}^p\}$. Consider the set

$$\mathcal{A} = \left\{ \cup_{i=1}^r \{p_i\} \mid \{p_i\} \in \|\bar{\boldsymbol{y}}^i\|, i = 1, \ldots, r \right\}.$$

Then, each place set $\mathcal{P} \in A$ is a surplus place set. The set of all surplus place sets is given by:

$$\mathcal{B} = \{\mathcal{P}' \mid \mathcal{P}' \subseteq \mathcal{P}, \mathcal{P} \in \mathcal{A}\}.$$

Proof. Consider the set of linearly independent equations $\bar{\boldsymbol{y}}^i\boldsymbol{m} = \bar{\boldsymbol{y}}^i\boldsymbol{m}_0, i = 1, \ldots, r$. In equation i, $i = 1, \ldots, r$, the marking of place $\{p_i\} \in \|\bar{\boldsymbol{y}}^i\|$, say, can be expressed in the marking of the places $\|\bar{y}^i\| \setminus \{p_i\}$. Thus, $\cup_{i=1}^r \{p_i\}$ is a surplus place set. As a consequence, all elements of $\mathcal{A}$ are surplus place sets, and therefore all elements of $\mathcal{B}$ are surplus place sets.

Now, consider an arbitrary place set $\mathcal{P} \notin \mathcal{B}$. Then, $\exists \{p_i\}, \{p_j\} \in \mathcal{P}$ and $\bar{\boldsymbol{y}}^k \in \{\bar{\boldsymbol{y}}^1, \ldots, \bar{\boldsymbol{y}}^r\}$ such that $\{p_i, p_j\} \subseteq \|\bar{\boldsymbol{y}}^k\|$ and $\nexists \bar{\boldsymbol{y}}^{k'} \in \{\bar{\boldsymbol{y}}^1, \ldots, \bar{\boldsymbol{y}}^r\}$ such that $\{p_i\} \in \|\bar{\boldsymbol{y}}^{k'}\|$ or $\{p_j\} \in \|\bar{\boldsymbol{y}}^{k'}\|$. As a consequence, the marking at places p_i and p_j is not uniquely characterized by the set of linearly independent equations $\bar{\boldsymbol{y}}^i\boldsymbol{m} = \bar{\boldsymbol{y}}^i\boldsymbol{m}_0, i = 1, \ldots, r$. Therefore, $\mathcal{P}$ is not a surplus place set, which concludes the proof. □

To conclude, we will present an algorithm that exploits Theorem 6.9 and Lemma 6.12 to find all possible decompositions. Before presenting the algorithm, a few observations are made. First, note that from Lemma 6.12 it follows that each place with a non-zero coefficient in one of the rows $\bar{\boldsymbol{y}}^i$ (i.e., $\{p\} \in \cup_{i=1}^r \|\bar{\boldsymbol{y}}^i\|$) can be selected as a surplus place. Second, note that Lemma 6.12 yield sufficient place sets that are not necessarily minimal sufficient place sets. Only if in each $\|\bar{\boldsymbol{y}}^i\|$ a different place p_i is selected, and $\mathcal{P}^{sur} := \cup_{i=1}^r \{p_i\}$ then $\mathcal{P}^{suf} = P \setminus \mathcal{P}^{sur}$ is a minimal sufficient place set. Third, observe that decomposition according to Theorem 6.9 is realized by assigning conflict places as a surplus place. As such, a surplus place set can only provide a decomposition if conflict places are contained. In the algorithm below, we find all decompositions by generating surplus place sets. To prevent the generation of surplus place set in which no surplus place is contained, we will intersect all $\|\bar{\boldsymbol{y}}^i\|$ with the conflict place set $\mathcal{P}^{con}$. Finally, each surplus place set that provides a decomposition, provides a specific decomposition. However, different surplus place sets may lead to the same decomposition if they have a corresponding intersection with the conflict place set.

Algorithm 6.13 (Generating all decompositions)**.**

Step 1. *Consider a product form $\mathcal{SPN}$. Determine from the set of common input bag classes $\{CI(\boldsymbol{x}^1), \ldots CI(\boldsymbol{x}^l)\}$, the set of conflict places:*

$\mathcal{P}^{con} = \left\{p \in P \mid p \in \left(P(CI(\boldsymbol{x}^i)) \cap P(CI(\boldsymbol{x}^j))\right), \forall i, j \text{ with } CI(\boldsymbol{x}^i) \neq CI(\boldsymbol{x}^j)\right\}.$

Step 2. *Obtain the set of minimal P-invariants $\{\boldsymbol{y}^1, \ldots, \boldsymbol{y}^p\}$ and a basis $\{\bar{\boldsymbol{y}}^1, \ldots, \bar{\boldsymbol{y}}^r\}$ composed of elements of $\{\boldsymbol{y}^1, \ldots, \boldsymbol{y}^p\}$.*

Step 3. *For $i = 1, \ldots, r$: obtain $\mathcal{P}^{con}_{\bar{\boldsymbol{y}}^i} = \left(\|\bar{\boldsymbol{y}}^i\| \cap \mathcal{P}^{con}\right)$.*

Step 4. *Obtain the Cartesian product of the sets $\{\emptyset, \mathcal{P}^{con}_{\bar{\boldsymbol{y}}^1}\}, \ldots, \{\emptyset, \mathcal{P}^{con}_{\bar{\boldsymbol{y}}^r}\}$.*

Step 5. *From the obtained set of place sets $\{\mathcal{P}^1, \ldots, \mathcal{P}^{\prod_{i=0}^{r}\left(\left|\mathcal{P}^{con}_{\bar{\boldsymbol{y}}^i}\right|+1\right)}\}$, remove duplicates and each set that contains a complete input bag.*

Step 6. *For each remaining surplus place set $\mathcal{P}^w$, solving $z\boldsymbol{A} = \boldsymbol{C}$ with $z_p = 0$ for $p \in \mathcal{P}^w$, yields a unique decomposition of the equilibrium distribution of $\mathcal{SPN}$:*

$\pi(\boldsymbol{m}) = B \prod_{i=1}^{s} \pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i)$, *with $\pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i)$ given in (28).*

Example 6.8 revisited. To illustrate the application of Algorithm 6.13 let us return to the simple and insightful Example 6.8 once more and execute the algorithm.

Step 1. The conflict place set is $\mathcal{P}^{con} = \{p_2, p_3\}$.

Step 2-3. The four linear independent minimal support P-invariants are $\bar{\boldsymbol{y}}^1 = (0001100)$, $\bar{\boldsymbol{y}}^2 = (0000011)$, $\bar{\boldsymbol{y}}^3 = (1100001)$ and $\bar{\boldsymbol{y}}^4 = (1011000)$, and these yield $\mathcal{P}^{con}_{\bar{\boldsymbol{y}}^1} = \emptyset$, $\mathcal{P}^{con}_{\bar{\boldsymbol{y}}^2} = \emptyset$, $\mathcal{P}^{con}_{\bar{\boldsymbol{y}}^3} = \{p_2\}$ and $\mathcal{P}^{con}_{\bar{\boldsymbol{y}}^4} = \{p_3\}$.

Step 4-5. The following sets are generated: $\mathcal{P}^1 = \{p_2\}, \mathcal{P}^2 = \{p_3\}$ and $\mathcal{P}^3 = \{p_2, p_3\}$. The obtained set of surplus place sets $\{\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3\}$ contains no duplicates. $\mathcal{P}^3$ is removed as it contains a complete input bag.

Step 6. The two possible decompositions both divide the $\mathcal{SPN}$ in two subnetworks such that

$$\pi(\boldsymbol{m}) = B\pi_y^{\mathcal{SPN}^1}(\boldsymbol{m}^1)\pi_y^{\mathcal{SPN}^2}(\boldsymbol{m}^2), \qquad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0),$$

where for the first surplus place set $\mathcal{P}^1$ the two subnetworks are $\mathcal{SPN}^1 = \{CI(\boldsymbol{x}^1, \boldsymbol{x}^3)\}$ and $\mathcal{SPN}^2 = \{CI(\boldsymbol{x}^2)\}$ and for the second surplus place set $\mathcal{P}^2$ these are $\mathcal{SPN}^1 = \{CE(\boldsymbol{x}^1, \boldsymbol{x}^2)\}$ and $\mathcal{SPN}^2 = \{CI(\boldsymbol{x}^3)\}$.

□

7 Discussion

Structural product form and decomposition results for stochastic Petri nets have been surveyed, unified and extended. Group-local-balance has been shown to be the unifying concept between known product form results for stochastic Petri net and has provided the ground to formulate necessary and sufficient structural conditions for product form and decomposition and to obtain a structural and intuitive explanation of these conditions, completely in terms of P- and T-invariants. Product form has been discussed in Section 5 and decomposition was the topic of Section 6. Below we provide an overview of this paper via a flowchart of the main results.

Theorem 4.6 opens the batch-routing queueing network literature for stochastic Petri nets as it provides the translation of product form results for batch routing queueing networks based on group-local-balance to stochastic Petri nets. Group-local-local balance implies that for product form a positive solution is required to the routing chain (19). Theorem 5.5 states that for a stochastic Petri net a positive solution for the routing chain exists if and only if it is an $\mathcal{S}\Pi$-net. Theorem 5.13 states that an $\mathcal{S}\Pi$-net has an equilibrium distribution that is a product form over the places of the network if and only if it satisfies group-local-balance. As such, Theorem 5.13 closes the cycle to batch-routing queueing networks. This brings us in the position to investigate the Petri net structure behind group-local-balance.

From Theorem 5.13 it appears that, in general, for group-local-balance to hold in an $\mathcal{S}\Pi$-net, an additional condition on the numerical values of the transition rates is required to be satisfied (see Lemma 5.12). Theorem 5.15 shows that for each minimal *closed* support T-invariant this numerical condition is satisfied irrespective of the numerical values of the transition rates. Therefore, for an $S\Pi$-net in which each minimal support T-invariant is a minimal *closed* support T-invariant, group-local-balance is satisfied, and thus product form holds.

In this way, we have have unified the key steps presented in literature with respect to structural results for product form stochastic Petri nets. Henderson et al. [28] introduced the routing chain. Assuming a solution to the routing chain to exist, they showed that if a closed form solution to ratio condition (20) on the solution of the routing chain can be found, this is the equilibrium distribution. Coleman et al. [13] identified the numerical condition, which is in this paper stated in Lemma 5.12, under which such a closed form solution exists and is of product form. We have shown that both the results of Henderson et al. and Coleman et al. can be explained to origin from group-local-balance. The last step was to unify Theorem 5.15 with the characterization by Haddad et al. [22] of rate-insensitive product form stochastic Petri nets. Their algebraic definition of $S\Pi^2$-nets, a subclass of $S\Pi$-nets, was in Theorem 5.17 shown to be equivalent with our characterization of rate-insensitive product form stochastic Petri nets; Theorem 5.17 states that an $S\Pi$-net is an $S\Pi^2$-net if and only if *all* minimal support T-invariants are minimal *closed* support T-invariants.

Product form results for network structures often allow for hierarchical composition and decomposition of subnetworks. When interested in global characteristics of a network it is convenient to decompose the network so that local characteristics can be investigated without considering the complete network in detail. Section 6 introduced decomposition results by which subnetworks can be identified in which a given product form stochastic Petri net can be decomposed. These subnetworks correspond to one or more common input bag classes, equivalence classes of minimal closed support T-invariants connected by having an input bag in common. Essential in achieving the decomposition is the notion of the sufficient place set of a Petri net, the set of places sufficient for uniquely characterizing the marking of a Petri net at all its places. A procedure to find all possible sufficient place sets of a Petri net from its P-invariants is provided in Lemma 6.12. The reciprocal of the sufficient place set is the surplus place set, places that can be omitted in characterizing the marking of the Petri net. Removing conflict places that can be assigned as a surplus place yields decomposition. The restriction is that no complete input bag may be removed. To be specific, Theorem 6.9 states that if a sufficient place set can be found so that there is no input bag of which all places are both surplus and conflict places, a product form stochastic Petri net decomposes into subnets each corresponding to one or more common input bag classes. The steps that have to be performed to verify and construct product form and to obtain all possible decompositions are summarized in Algorithms 5.19 and 6.13.

Observe that characterizing product form for a stochastic Petri net can be done completely in terms of its T-invariants, while decomposition of the network into subnetworks not only requires the T-invariants, but also its P-invariants.

Finally, observe that characterizing product form for a stochastic Petri net can be done completely in terms of its T-invariants, while decomposition of the network into subnetworks not only requires the T-invariants, but also its P-invariants. The results presented in this paper suggest several directions for future research. A first extension would be to include state dependent firing and enabling, similar to Henderson et al. [28], Boucherie and Sereno [5] and Haddad et al. [22]. Also, colouring of tokens such as included in Henderon and Taylor [32] can be incorporated in the model by enlarging the state space in a way very similar to the inclusion of multiple customer types in Markov chain models for product form queueing networks (e.g. Boucherie and van Dijk [6], Serfozo [53]). In addition, we have a particular interest in extending the decomposition results. First, decomposition results seem possible not by removing places, but by assigning conflict places to a unique common input bag class. Second, such a decomposition result may be an opening to a decomposition result in which a stochastic Petri net completely decomposes into its T-invariants. Finally, such exact decomposition results could provide a starting point for deriving approximate results for non-product form stochastic Petri nets which may also be useful in developing a method to algorithmically identify subnets in the framework of competing Markov chains as introduced in [3].

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Appendix

Proof Lemma 4.3. If $V(\boldsymbol{n}) = \emptyset$ then (9) is trivially fulfilled. If $V(\boldsymbol{n}) \neq \emptyset$ it is sufficient to prove that the Markov chain at $V(\boldsymbol{n})$, for fixed $\boldsymbol{n}$, has no transient states. By virtue of the GLB-property this local Markov chain has an equilibrium distribution $c\pi$ where

$$c^{-1} = \sum_{\{t \in T:\, q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{n}) > 0)\}} \pi(\boldsymbol{n} + \boldsymbol{I}(t))$$

As $\pi(\cdot) > 0$ each state is positive recurrent.

Proof Theorem 4.6. If π satisfies GLB then π is a solution of (10). And since we have shown that, if GLB holds, (10) has a unique solution, the $\bar{q}$-process is defined. For $\boldsymbol{n}+\boldsymbol{I}(t), \boldsymbol{n}+\boldsymbol{I}(t') \in V_i(\boldsymbol{n})$ with $\bar{q}(\boldsymbol{I}(t), \boldsymbol{I}(t'); \boldsymbol{n}) = 0$ (and thus $\bar{q}(\boldsymbol{I}(t'), \boldsymbol{I}(t); \boldsymbol{n}) = 0$) (12) is trivially satisfied. For $\boldsymbol{n}+\boldsymbol{I}(t), \boldsymbol{n}+\boldsymbol{I}(t') \in V_i(\boldsymbol{n})$ with $\bar{q}(\boldsymbol{I}(t), \boldsymbol{I}(t'); \boldsymbol{n}) > 0$ we have

$$\frac{x(\boldsymbol{I}(t), \boldsymbol{n})}{x(\boldsymbol{I}(t'), \boldsymbol{n})} = \frac{\bar{\pi}(\boldsymbol{n}+\boldsymbol{I}(t))}{\bar{\pi}(\boldsymbol{n}+\boldsymbol{I}(t'))} \tag{29}$$

so that by (11) for any $i \in \{1, \ldots, k(\boldsymbol{n})\}$, and $\boldsymbol{n}+\boldsymbol{I}(t), \boldsymbol{n}+\boldsymbol{I}(t') \in V_i(\boldsymbol{n})$ with $\bar{q}(\boldsymbol{I}(t), \boldsymbol{I}(t'); \boldsymbol{n}) > 0$

$$\frac{\bar{q}(\boldsymbol{I}(t), \boldsymbol{I}(t'); \boldsymbol{n})}{\bar{q}(\boldsymbol{I}(t'), \boldsymbol{I}(t); \boldsymbol{n})} = \frac{\bar{\pi}(\boldsymbol{n}+\boldsymbol{I}(t))}{\bar{\pi}(\boldsymbol{n}+\boldsymbol{I}(t'))}.$$

As $\bar{q}(\boldsymbol{I}(t), \boldsymbol{I}(t'); \boldsymbol{n}) = \bar{q}(\boldsymbol{I}(t'), \boldsymbol{I}(t); \boldsymbol{n}) = 0$ if $\boldsymbol{n}+\boldsymbol{I}(t), \boldsymbol{n}+\boldsymbol{I}(t')$ are not contained in the same local irreducible set $V_i(\boldsymbol{n})$, the $\bar{q}$-process strongly reversible at $\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$ with $\bar{\pi} = \pi$.

Conversely, if the $\bar{q}$-process is defined and is strongly reversible at $\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$ then (11) and (12) imply that

$$\frac{\bar{\pi}(\boldsymbol{n}+\boldsymbol{I}(t))}{\bar{\pi}(\boldsymbol{n}+\boldsymbol{I}(t'))} = \frac{\bar{q}(\boldsymbol{I}(t), \boldsymbol{I}(t'); \boldsymbol{n})}{\bar{q}(\boldsymbol{I}(t'), \boldsymbol{I}(t); \boldsymbol{n})} = \frac{x(\boldsymbol{I}(t'), \boldsymbol{n})}{x(\boldsymbol{I}(t), \boldsymbol{n})}.$$

Hence, the distribution $\pi(\cdot) = \bar{\pi}(\cdot)$ satisfies (10), which shows that π satisfies the GLB equations (7).

Let us now prove the second statement. By irreducibility a firing sequence between $\boldsymbol{m}_0$ and $\boldsymbol{m}$ exists. If π satisfies GLB, then by the first part of the theorem, the $\bar{q}$-process is strongly reversible. Then from (12)

$$\prod_{k=0}^{p} \frac{\bar{q}(\boldsymbol{I}(t_k), \boldsymbol{I}(t'_k); \boldsymbol{n}_k)}{\bar{q}(\boldsymbol{I}(t'_k), \boldsymbol{I}(t_k); \boldsymbol{n}_k)} = \prod_{k=0}^{p} \frac{\bar{\pi}(\boldsymbol{n}_k+\boldsymbol{I}(t'_k))}{\bar{\pi}(\boldsymbol{n}_k+\boldsymbol{I}(t_k))} = \prod_{k=0}^{p} \frac{\bar{\pi}(\boldsymbol{n}_{k+1}+\boldsymbol{I}(t_{k+1}))}{\bar{\pi}(\boldsymbol{n}_k+\boldsymbol{I}(t_k))} = \frac{\pi(\boldsymbol{m})}{\pi(\boldsymbol{m}_0)}.$$

Conversely, applying (14) for the firing sequences from $\boldsymbol{m}_0$ to $\boldsymbol{m}$ and $\boldsymbol{m}'$, with $\boldsymbol{m}' = \boldsymbol{n}_s+\boldsymbol{I}(t_{s-1}) = \boldsymbol{n}_s+\boldsymbol{I}(t_s) = \boldsymbol{m}$, strong reversibility and therefore GLB follows

$$\frac{\pi(\boldsymbol{n}_s+\boldsymbol{I}(t_{s-1}))}{\pi(\boldsymbol{n}_s+\boldsymbol{I}(t_s))} = \frac{\pi(\boldsymbol{m}_0)}{\pi(\boldsymbol{m}_0)} \prod_{k=0}^{s} \frac{\bar{q}(\boldsymbol{I}(t_k), \boldsymbol{I}(t'_k); \boldsymbol{n}_k)}{\bar{q}(\boldsymbol{I}(t'_k), \boldsymbol{I}(t_k); \boldsymbol{n}_k)} \prod_{k=0}^{s-1} \frac{\bar{q}(\boldsymbol{I}(t'_k), \boldsymbol{I}(t_k); \boldsymbol{n}_k)}{\bar{q}(\boldsymbol{I}(t_k), \boldsymbol{I}(t'_k); \boldsymbol{n}_k)} = \frac{\bar{q}(\boldsymbol{I}(t_s), \boldsymbol{I}(t_{s-1}); \boldsymbol{n}_s)}{\bar{q}(\boldsymbol{I}(t_{s-1}), \boldsymbol{I}(t_s); \boldsymbol{n}_s)}.$$

Proof Lemma 5.12.

(i) $\Rightarrow$ (ii) Assume (i) is true. This is, for each $\boldsymbol{x}$ such that $\boldsymbol{A}\boldsymbol{x} = 0$, also $\boldsymbol{C}\boldsymbol{x} = 0$. This implies that the kernel of $\boldsymbol{A}$ is a subspace of the kernel of $[\boldsymbol{A}|\boldsymbol{C}]$, which induces $dim(ker(\boldsymbol{A})) \leq dim(ker([\boldsymbol{A}|\boldsymbol{C}])$. Hence, $rank(\boldsymbol{A}) \geq Rank([\boldsymbol{A}|\boldsymbol{C}])$. Of course, since A is a submatrix of $[\boldsymbol{A}|\boldsymbol{C}]$, also $rank(\boldsymbol{A}) \leq rank([\boldsymbol{A}|\boldsymbol{C}])$. Combining these relations yields $Rank(\boldsymbol{A}) = Rank([\boldsymbol{A}|\boldsymbol{C}])$.

(ii) $\Rightarrow$ (iii) $Rank(\boldsymbol{A}) = Rank([\boldsymbol{A}|\boldsymbol{C}])$ implies that the row vector $\boldsymbol{C}$ can be written as a linear combination of the rows of $\boldsymbol{A}$, i.e., $z\boldsymbol{A} = \boldsymbol{C}$ has a solution.

(iii) $\Rightarrow$ (i) $z\boldsymbol{A} = \boldsymbol{C}$ has a solution means that the row vector $\boldsymbol{C}$ can be written as a linear combination of the rows of $\boldsymbol{A}$. For a T-invariant $\boldsymbol{A}\boldsymbol{x} = 0$. Combining these statements implies $\boldsymbol{C}\boldsymbol{x} = 0$.

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