A Bayesian analysis of the mixed labelling phenomenon in two-target tracking

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Abstract

In mulit-target tracking and labelling (MTTL), mixed labelling corresponds to a situation where there is ambiguity in labelling, i.e. in the assignment of labels to locations (where a "location" here means simply an unlabelled single-target state. The phenomenon is well-known in literature, and known to occur in the situation where targets move in close proximity to each other and afterwards separate.

The occurrence of mixed labelling has been empirically observed using particle filter implementations of the Bayesian MTTL recursion. In this memorandum, we will instead demonstrate the occurrence of mixed labelling (in the situation of closely spaced targets) using only the Bayesian recursion itself, for a scenario containing two targets and no target births or deaths. We will also show how mixed labelling generally persists after the targets become well-separated, and how mixed labelling might not happen when the unlabelled single-target state contains non-kinematic quantities.

Notation conventions

An upper-case letter (like X) denotes a vector-valued random variable, and its lower-case counterpart (like x) denotes, as usual, a particular realization. An upper-case bold-faced letter (like $\mathbf{X}$) denotes a finite set-valued random variable, and its lower-case counterpart denotes the corresponding realization.

I. The MTTL Bayesian recursion in the two-target case

Consider the mathematical formulation of the Bayesian MTTL problem in [1]. Assuming that there are two targets, that the number of targets is known, and that there are no target births or deaths, at some time k, let $\mathbf{X}_k = \left\{ X_k^{(1)}, X_k^{(2)} \right\}$ be a Random Finite Set (RFS) describing the labelled states, $\mathbf{S}_k = \left\{ S_k^{(1)}, S_k^{(2)} \right\}$ be a RFS describing the unlabelled states (locations), and $\mathbf{L}_k = \left\{ L_k^{(1)}, L_k^{(2)} \right\}$ be a RFS describing the labels. Without loss of generality, we also assume that a target's label is either A or B.

First, let us define

$$
\begin{aligned}
p_{AB}\left(s_k^{(1)}, s_k^{(2)}\right) &\triangleq p_l\left(\left\{ \begin{bmatrix} s_k^{(1)} \\ A \end{bmatrix}, \begin{bmatrix} s_k^{(2)} \\ B \end{bmatrix} \right\} \middle| \left\{ s_k^{(1)}, s_k^{(2)} \right\} \right), \\
p_{BA}\left(s_k^{(1)}, s_k^{(2)}\right) &\triangleq p_l\left(\left\{ \begin{bmatrix} s_k^{(1)} \\ B \end{bmatrix}, \begin{bmatrix} s_k^{(2)} \\ A \end{bmatrix} \right\} \middle| \left\{ s_k^{(1)}, s_k^{(2)} \right\} \right), \\
p_{AB}\left(s_{k-1}^{(1)}, s_{k-1}^{(2)}\right) &\triangleq p_l\left(\left\{ \begin{bmatrix} s_{k-1}^{(1)} \\ A \end{bmatrix}, \begin{bmatrix} s_{k-1}^{(2)} \\ B \end{bmatrix} \right\} \middle| \left\{ s_{k-1}^{(1)}, s_{k-1}^{(2)} \right\} \right), \\
p_{BA}\left(s_{k-1}^{(1)}, s_{k-1}^{(2)}\right) &\triangleq p_l\left(\left\{ \begin{bmatrix} s_{k-1}^{(1)} \\ B \end{bmatrix}, \begin{bmatrix} s_{k-1}^{(2)} \\ A \end{bmatrix} \right\} \middle| \left\{ s_{k-1}^{(1)}, s_{k-1}^{(2)} \right\} \right)
\end{aligned}
$$

where $p_l(\cdot|\cdot)$ denotes the *labelling probability* (see [1]). From [1], we know that

$$
p_l(\mathbf{x}_k|\mathbf{s}_k) = \frac{\int f(\mathbf{x}_k|\mathbf{x}_{k-1}) f(\mathbf{x}_{k-1}|Z^{k-1}) \delta\mathbf{x}_{k-1}}{f(\mathbf{s}_k|Z^{k-1})} \tag{1}
$$

and hence

$$
\begin{aligned}
& p_{AB}\left(s_k^{(1)}, s_k^{(2)}\right) \\
&= \frac{1}{f\left(\left\{ s_k^{(1)}, s_k^{(2)} \right\} \middle| Z^{k-1} \right)} \int f\left(\left\{ \begin{bmatrix} s_k^{(1)} \\ A \end{bmatrix}, \begin{bmatrix} s_k^{(2)} \\ B \end{bmatrix} \right\} \middle| \mathbf{x}_{k-1} \right) f\left(\mathbf{x}_{k-1} \middle| Z^{k-1} \right) \delta\mathbf{x}_{k-1} \\
&= \frac{1}{f\left(\left\{ s_k^{(1)}, s_k^{(2)} \right\} \middle| Z^{k-1} \right)} \left(p_{AB}^{AB}\left(s_k^{(1)}, s_k^{(2)}\right) + p_{AB}^{BA}\left(s_k^{(1)}, s_k^{(2)}\right) \right)
\end{aligned} \tag{2}
$$

where, assuming without loss of generality that location $S_k^{(i)}$ assumes values in an Euclidean space $\mathbb{R}^n$, we have

$$\begin{aligned}
&p_{AB}^{AB}\left(s_k^{(1)}, s_k^{(2)}\right)\\
&= \int\int f\left(\left\{\begin{bmatrix} s_k^{(1)} \\ A\end{bmatrix}, \begin{bmatrix} s_k^{(2)} \\ B\end{bmatrix}\right\}\middle| \left\{\begin{bmatrix} s_{k-1}^{(1)} \\ A\end{bmatrix}, \begin{bmatrix} s_{k-1}^{(2)} \\ B\end{bmatrix}\right\}\right) f\left(\left\{\begin{bmatrix} s_{k-1}^{(1)} \\ A\end{bmatrix}, \begin{bmatrix} s_{k-1}^{(2)} \\ B\end{bmatrix}\right\}\middle| Z^{k-1}\right) ds_{k-1}^{(1)} ds_{k-1}^{(2)},\\
&p_{AB}^{BA}\left(s_k^{(1)}, s_k^{(2)}\right)\\
&= \int\int f\left(\left\{\begin{bmatrix} s_k^{(1)} \\ A\end{bmatrix}, \begin{bmatrix} s_k^{(2)} \\ B\end{bmatrix}\right\}\middle| \left\{\begin{bmatrix} s_{k-1}^{(1)} \\ B\end{bmatrix}, \begin{bmatrix} s_{k-1}^{(2)} \\ A\end{bmatrix}\right\}\right) f\left(\left\{\begin{bmatrix} s_{k-1}^{(1)} \\ B\end{bmatrix}, \begin{bmatrix} s_{k-1}^{(2)} \\ A\end{bmatrix}\right\}\middle| Z^{k-1}\right) ds_{k-1}^{(1)} ds_{k-1}^{(2)}.
\end{aligned}$$

Let us recall the formula of the state transition density for the case of no births/deaths (from [1]):

$$f(\mathbf{x}_k|\mathbf{x}_{k-1}) = \sum_{\theta\in\Theta_{t_k}} \prod_{i=1}^{t_k} p\left(s_k^{(i)}\middle| s_{k-1}^{(\theta(i))}\right) \delta_{l_k^{(i)} l_{k-1}^{(\theta(i))}} \tag{3}$$

where Θ_{t_k} is the set of all permutations on $\{1,\ldots,t_k\}$. From the definition of split density in [1]

$$\begin{aligned}
f\left(\left\{\begin{bmatrix} s_{k-1}^{(1)} \\ A\end{bmatrix}, \begin{bmatrix} s_{k-1}^{(2)} \\ B\end{bmatrix}\right\}\middle| Z^{k-1}\right) &= p_{AB}\left(s_{k-1}^{(1)}, s_{k-1}^{(2)}\right) f\left(\left\{s_{k-1}^{(1)}, s_{k-1}^{(2)}\right\}\middle| Z^{k-1}\right),\\
f\left(\left\{\begin{bmatrix} s_{k-1}^{(1)} \\ B\end{bmatrix}, \begin{bmatrix} s_{k-1}^{(2)} \\ A\end{bmatrix}\right\}\middle| Z^{k-1}\right) &= p_{BA}\left(s_{k-1}^{(1)}, s_{k-1}^{(2)}\right) f\left(\left\{s_{k-1}^{(1)}, s_{k-1}^{(2)}\right\}\middle| Z^{k-1}\right)
\end{aligned} \tag{4}$$

and using (3), we obtain

$$\begin{aligned}
p_{AB}^{AB}\left(s_k^{(1)}, s_k^{(2)}\right) &= \int\int p\left(s_k^{(1)}\middle| s_{k-1}^{(1)}\right) p\left(s_k^{(2)}\middle| s_{k-1}^{(2)}\right) p_{AB}\left(s_{k-1}^{(1)}, s_{k-1}^{(2)}\right) f\left(\left\{s_{k-1}^{(1)}, s_{k-1}^{(2)}\right\}\middle| Z^{k-1}\right) ds_{k-1}^{(1)} ds_{k-1}^{(2)}\\
p_{AB}^{BA}\left(s_k^{(1)}, s_k^{(2)}\right) &= \int\int p\left(s_k^{(1)}\middle| s_{k-1}^{(2)}\right) p\left(s_k^{(2)}\middle| s_{k-1}^{(1)}\right) p_{BA}\left(s_{k-1}^{(1)}, s_{k-1}^{(2)}\right) f\left(\left\{s_{k-1}^{(1)}, s_{k-1}^{(2)}\right\}\middle| Z^{k-1}\right) ds_{k-1}^{(1)} ds_{k-1}^{(2)}.
\end{aligned}$$

Analogously, we can show that

$$p_{BA}\left(s_k^{(1)}, s_k^{(2)}\right) = \frac{1}{f\left(\left\{s_k^{(1)}, s_k^{(2)}\right\}\middle| Z^{k-1}\right)} \left(p_{BA}^{AB}\left(s_k^{(1)}, s_k^{(2)}\right) + p_{BA}^{BA}\left(s_k^{(1)}, s_k^{(2)}\right)\right) \tag{5}$$

where

$$\begin{aligned}
p_{BA}^{AB}\left(s_k^{(1)}, s_k^{(2)}\right) &= \int\int p\left(s_k^{(1)}\middle| s_{k-1}^{(2)}\right) p\left(s_k^{(2)}\middle| s_{k-1}^{(1)}\right) p_{AB}\left(s_{k-1}^{(1)}, s_{k-1}^{(2)}\right) f\left(\left\{s_{k-1}^{(1)}, s_{k-1}^{(2)}\right\}\middle| Z^{k-1}\right) ds_{k-1}^{(1)} ds_{k-1}^{(2)}\\
p_{BA}^{BA}\left(s_k^{(1)}, s_k^{(2)}\right) &= \int\int p\left(s_k^{(1)}\middle| s_{k-1}^{(1)}\right) p\left(s_k^{(2)}\middle| s_{k-1}^{(2)}\right) p_{BA}\left(s_{k-1}^{(1)}, s_{k-1}^{(2)}\right) f\left(\left\{s_{k-1}^{(1)}, s_{k-1}^{(2)}\right\}\middle| Z^{k-1}\right) ds_{k-1}^{(1)} ds_{k-1}^{(2)}.
\end{aligned}$$

II. Origin of mixed labelling

In multi-target tracking, if $\mathfrak{N}_0^S$ denotes the single-target location state space (for instance, $\mathbb{R}^4$ for position and velocity in x and y coordinates), given enough measurements, the Belief mass associated with $f\left(\left\{s_{k-1}^{(1)}, s_{k-1}^{(2)}\right\}\middle| Z^{k-1}\right)$ will be mostly contained in a small subset of $\mathfrak{N}_0^S$, say $\mathfrak{N}_*^S$. Effectively, the double integrals in (2) and (5) are all taken over $\mathfrak{N}_*^S \times \mathfrak{N}_*^S$, and in good observability conditions, $\mathfrak{N}_*^S$ will be formed by the regions surrounding the true target states.

But if the targets are moving in close proximity with each other, we will have $s_{k-1}^{(1)} \approx s_{k-1}^{(2)}$ for $s_{k-1}^{(1)}, s_{k-1}^{(2)} \in \mathfrak{N}_*^S$, and hence, within $\mathfrak{N}_*^S$, we will have

$$p\left(s_k^{(1)}\middle| s_{k-1}^{(1)}\right) p\left(s_k^{(2)}\middle| s_{k-1}^{(2)}\right) \approx p\left(s_k^{(1)}\middle| s_{k-1}^{(2)}\right) p\left(s_k^{(2)}\middle| s_{k-1}^{(1)}\right) \tag{6}$$

and therefore, $p_{AB}^{AB}\left(s_k^{(1)}, s_k^{(2)}\right) \approx p_{BA}^{AB}\left(s_k^{(1)}, s_k^{(2)}\right)$ and $p_{AB}^{BA}\left(s_k^{(1)}, s_k^{(2)}\right) \approx p_{BA}^{BA}\left(s_k^{(1)}, s_k^{(2)}\right)$, and consequentially

$$p_{AB}\left(s_k^{(1)}, s_k^{(2)}\right) \approx p_{BA}\left(s_k^{(1)}, s_k^{(2)}\right) \tag{7}$$

i.e. there will be “total mixed labelling” as described in [1]. Interestingly, (7) will hold regardless of $s_k^{(1)}, s_k^{(2)}$, implying that total mixed labelling will affect the entire state space of $\mathbf{S}_k = \left\{s_k^{(1)}, s_k^{(2)}\right\}$.

Note that, if the targets are reasonably close to each other, but not that much given observability conditions, the most likely result will be some degree of “partial mixed labelling” (see [1]) instead.

III. Persistence of mixed labelling

If "total mixed labelling" (i.e. the situation given by (7)) affects the entire space of $\mathbf{S}_k$, we can see that the situation will persist indefinitely, even after the targets separate from each other. The reason is that, if $p_{AB}\left(s_{k-1}^{(1)}, s_{k-1}^{(2)}\right) \approx p_{BA}\left(s_{k-1}^{(1)}, s_{k-1}^{(2)}\right)$ for $s_{k-1}^{(1)}, s_{k-1}^{(2)} \in \mathfrak{N}_*^S$, we will have $p_{AB}^{AB}\left(s_k^{(1)}, s_k^{(2)}\right) \approx p_{BA}^{BA}\left(s_k^{(1)}, s_k^{(2)}\right)$ and $p_{AB}^{BA}\left(s_k^{(1)}, s_k^{(2)}\right) \approx p_{BA}^{AB}\left(s_k^{(1)}, s_k^{(2)}\right)$, and (7) will hold again.

If we have instead "partial mixed labelling", it is possible that mixed labelling disappears with time. An interesting question, however, is whether partial mixed labelling may disappear after the targets become well-separated again. To illustrate this situation, let us assume that at time $k-1$, we have $\mathfrak{N}_*^S = \Omega^{(1)} \cup \Omega^{(2)}$, with $\Omega^{(1)} \cap \Omega^{(2)} = \emptyset$, which would be the case if the targets are well-separated and $\Omega^{(1)}, \Omega^{(2)}$ are the regions surrounding each of the true target states.

We also assume that $\Omega^{(1)}$ and $\Omega^{(2)}$ are small enough such that the probability (conditioned on Z^{k-1}) that a certain element of $\Omega^{(1)}$ corresponds to A and that a certain element of $\Omega^{(2)}$ corresponds to B is more-or-less constant and equal to P_{k-1}. Conversely, P_{k-1}^* would be the probability that an element of $\Omega^{(2)}$ corresponds to A and an element of $\Omega^{(1)}$ corresponds to B.

Since the targets are well-separated, clearly (6) does not hold. We then assume, without loss of generality, that, for given $s_k^{(1)}, s_k^{(2)}$, we have

$$\begin{aligned}
p\left(s_k^{(1)} \middle| s_{k-1}^{(1)}\right) p\left(s_k^{(2)} \middle| s_{k-1}^{(2)}\right) \gg p\left(s_k^{(1)} \middle| s_{k-1}^{(2)}\right) p\left(s_k^{(2)} \middle| s_{k-1}^{(1)}\right) \text{ if } s_{k-1}^{(1)} \in \Omega^{(1)} \text{ and } s_{k-1}^{(2)} \in \Omega^{(2)} \\
p\left(s_k^{(1)} \middle| s_{k-1}^{(1)}\right) p\left(s_k^{(2)} \middle| s_{k-1}^{(2)}\right) \ll p\left(s_k^{(1)} \middle| s_{k-1}^{(2)}\right) p\left(s_k^{(2)} \middle| s_{k-1}^{(1)}\right) \text{ if } s_{k-1}^{(1)} \in \Omega^{(2)} \text{ and } s_{k-1}^{(2)} \in \Omega^{(1)}
\end{aligned}$$

such that effectively, we have

$$\begin{aligned}
p_{AB}^{AB}\left(s_k^{(1)}, s_k^{(2)}\right) &= \int_{\Omega^{(2)}} \int_{\Omega^{(1)}} p\left(s_k^{(1)} \middle| s_{k-1}^{(1)}\right) p\left(s_k^{(2)} \middle| s_{k-1}^{(2)}\right) p_{AB}\left(s_{k-1}^{(1)}, s_{k-1}^{(2)}\right) f\left(\left\{s_{k-1}^{(1)}, s_{k-1}^{(2)}\right\} \middle| Z^{k-1}\right) ds_{k-1}^{(1)} ds_{k-1}^{(2)} \\
p_{AB}^{BA}\left(s_k^{(1)}, s_k^{(2)}\right) &= \int_{\Omega^{(1)}} \int_{\Omega^{(2)}} p\left(s_k^{(1)} \middle| s_{k-1}^{(2)}\right) p\left(s_k^{(2)} \middle| s_{k-1}^{(1)}\right) p_{BA}\left(s_{k-1}^{(1)}, s_{k-1}^{(2)}\right) f\left(\left\{s_{k-1}^{(1)}, s_{k-1}^{(2)}\right\} \middle| Z^{k-1}\right) ds_{k-1}^{(1)} ds_{k-1}^{(2)} \\
p_{BA}^{AB}\left(s_k^{(1)}, s_k^{(2)}\right) &= \int_{\Omega^{(1)}} \int_{\Omega^{(2)}} p\left(s_k^{(1)} \middle| s_{k-1}^{(2)}\right) p\left(s_k^{(2)} \middle| s_{k-1}^{(1)}\right) p_{AB}\left(s_{k-1}^{(1)}, s_{k-1}^{(2)}\right) f\left(\left\{s_{k-1}^{(1)}, s_{k-1}^{(2)}\right\} \middle| Z^{k-1}\right) ds_{k-1}^{(1)} ds_{k-1}^{(2)} \\
p_{BA}^{BA}\left(s_k^{(1)}, s_k^{(2)}\right) &= \int_{\Omega^{(2)}} \int_{\Omega^{(1)}} p\left(s_k^{(1)} \middle| s_{k-1}^{(1)}\right) p\left(s_k^{(2)} \middle| s_{k-1}^{(2)}\right) p_{BA}\left(s_{k-1}^{(1)}, s_{k-1}^{(2)}\right) f\left(\left\{s_{k-1}^{(1)}, s_{k-1}^{(2)}\right\} \middle| Z^{k-1}\right) ds_{k-1}^{(1)} ds_{k-1}^{(2)}.
\end{aligned}$$

Using the probabilities P_{k-1} and P_{k-1}^* that we have defined, we obtain

$$\begin{aligned}
p_{AB}^{AB}\left(s_k^{(1)}, s_k^{(2)}\right) &\approx P_{k-1} \int_{\Omega^{(2)}} \int_{\Omega^{(1)}} p\left(s_k^{(1)} \middle| s_{k-1}^{(1)}\right) p\left(s_k^{(2)} \middle| s_{k-1}^{(2)}\right) f\left(\left\{s_{k-1}^{(1)}, s_{k-1}^{(2)}\right\} \middle| Z^{k-1}\right) ds_{k-1}^{(1)} ds_{k-1}^{(2)} \\
p_{AB}^{BA}\left(s_k^{(1)}, s_k^{(2)}\right) &\approx P_{k-1} \int_{\Omega^{(1)}} \int_{\Omega^{(2)}} p\left(s_k^{(1)} \middle| s_{k-1}^{(2)}\right) p\left(s_k^{(2)} \middle| s_{k-1}^{(1)}\right) f\left(\left\{s_{k-1}^{(1)}, s_{k-1}^{(2)}\right\} \middle| Z^{k-1}\right) ds_{k-1}^{(1)} ds_{k-1}^{(2)} \\
p_{BA}^{AB}\left(s_k^{(1)}, s_k^{(2)}\right) &\approx P_{k-1}^* \int_{\Omega^{(1)}} \int_{\Omega^{(2)}} p\left(s_k^{(1)} \middle| s_{k-1}^{(2)}\right) p\left(s_k^{(2)} \middle| s_{k-1}^{(1)}\right) f\left(\left\{s_{k-1}^{(1)}, s_{k-1}^{(2)}\right\} \middle| Z^{k-1}\right) ds_{k-1}^{(1)} ds_{k-1}^{(2)} \\
p_{BA}^{BA}\left(s_k^{(1)}, s_k^{(2)}\right) &\approx P_{k-1}^* \int_{\Omega^{(2)}} \int_{\Omega^{(1)}} p\left(s_k^{(1)} \middle| s_{k-1}^{(1)}\right) p\left(s_k^{(2)} \middle| s_{k-1}^{(2)}\right) f\left(\left\{s_{k-1}^{(1)}, s_{k-1}^{(2)}\right\} \middle| Z^{k-1}\right) ds_{k-1}^{(1)} ds_{k-1}^{(2)}
\end{aligned}$$

and finally

$$p_{AB}\left(s_k^{(1)}, s_k^{(2)}\right) \propto P_{k-1}, \tag{8}$$

$$p_{BA}\left(s_k^{(1)}, s_k^{(2)}\right) \propto P_{k-1}^* \tag{9}$$

also regardless of $s_k^{(1)}$ and $s_k^{(2)}$. Therefore, although partial mixed labelling may disappear with time, this will generally not happen after the targets are separated enough.

IV. Mixed labelling and non-kinematic states

It is possible that we have a situation of closely space targets, but mixed labelling does not arise. This may happen when the unlabelled state vector $S_k^{(i)}$ contains entries corresponding to non-kinematic quantities, such as the target's classification, or the target's Identification Friend-or-Foe (IFF) code, or the callsign attributed by the Air Traffic Control. Let $N_k^{(i)}$ be this non-kinematic quantity. Typically, $N_k^{(i)}$ has a very small (or zero) probability of changing between two subsequent time steps.

As consequence, the condition for the appearance of mixed labelling (6) may only hold if $s_k^{(1)}$ and $s_k^{(2)}$ contain the same value of $N_k^{(i)}$. If two targets do not share the same value for $N_k^{(i)}$, and we can effectively estimate this quantity (for instance, when the targets' callsigns are provided in the observations), then a pair of locations $\left\{\hat{s}_k^{(1)}, \hat{s}_k^{(2)}\right\}$ in a high probability area should also contain different values of $N_k^{(i)}$. As consequence, there will be no mixed labelling associated with $\left\{\hat{s}_k^{(1)}, \hat{s}_k^{(2)}\right\}$.

REFERENCES

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