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The duality between the gradient and divergence operators on bounded Lipschitz domains

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Abstract

This report gives an exact result on the duality of the divergence and gradient operators, when these are considered as operators between L^2-spaces on a bounded n-dimensional Lipschitz domain. The necessary background is described in detail, with a self-contained exposition.

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Contents

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1 Introduction

It is common knowledge that the formal adjoint of the gradient operator, grad, on an n-dimensional domain is minus the divergence operator, div. However, when one wants to carry out careful analysis, one also needs to determine the precise domains and co-domains of these operators. In spite of our efforts, we could not find a suitable exact statement in the literature.

Hence, in the present work, we compute the adjoint of the gradient operating on a connected, open, and bounded subset $\Omega \subset \mathbb{R}^n$ with a Lipschitz-continuous boundary $\partial\Omega$. As domain of the gradient, we consider an arbitrary vector space G, such that $H_0^1(\Omega) \subset G \subset H^1(\Omega)$. From G we construct a subspace D with $H_0^{\mathrm{div}}(\Omega) \subset D \subset H^{\mathrm{div}}(\Omega)$, for which it holds that $\mathrm{grad}|_G^* = -\mathrm{div}|_D$. See Section 3 for the definitions of the spaces $H^1(\Omega)$, $H_0^1(\Omega)$, $H^{\mathrm{div}}(\Omega)$, and $H_0^{\mathrm{div}}(\Omega)$.

Example 1.1. Let $\Omega \subset \mathbb{R}^n$ and $\partial\Omega$ be as above and consider grad defined on $G = H_0^1(\Omega)$. By the main result below, Theorem 6.2, the adjoint of the gradient with this domain is $-\mathrm{div}$, with domain $D = H^{\mathrm{div}}(\Omega)$. This implies that the operator $A := \left[\begin{smallmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{smallmatrix}\right]$ with domain $\mathrm{dom}\,(A) := \left[\begin{smallmatrix} H_0^1(\Omega) \\ H^{\mathrm{div}}(\Omega)^n \end{smallmatrix}\right]$ is skew-adjoint on $\left[\begin{smallmatrix} L^2(\Omega) \\ L^2(\Omega)^n \end{smallmatrix}\right]$.

We make the exposition self-contained by compiling the necessary background. Our main sources are Tucsnak and Weiss [TW09, Chap. 13], and Girault and Raviart [GR86]. We try to make the text accessible to beginners in the field by filling in details omitted from these two books.

In Section 2, we define test functions and distributions. These are needed in order to define Sobolev spaces, and the divergence and the gradient operators. This is the topic of Section 3. The boundary $\partial\Omega$ of the bounded Lipschitz domain Ω plays a very important role in the duality of the divergence and gradient operators, due to an integration-by-parts formula in Section 5. Therefore we need to include background on Sobolev spaces on Lipschitz manifolds in Section 4. Our contribution is confined to Section 6, which contains the duality results.

This report can be considered as a detailed introduction to [KZ12b], where we develop the results presented here much further. The main motivation for [KZ12b] comes from [ZGM11, ZGMV12, KZ12a], where operators of the type A in Example 1.1 were used extensively for proving the existence of solutions for wave, heat, and Schrödinger equations on n-dimensional spatial domains.

Following [TW09], we work with complex-valued functions. Girault and Raviart work with real-valued functions, but on page 1 they state that their results are equally valid for the complex-valued setting, assuming one makes correct use of the complex conjugate.

2 Test functions and distributions

Throughout this article, we take Ω to be an open subset of $\mathbb{R}^n$, $n = 1, 2, 3, \ldots$, unless anything more is mentioned, and we denote its boundary $\overline{\Omega} \setminus \Omega$ by $\partial\Omega$. For $k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, we denote the space of functions mapping Ω into $\mathbb{C}$ with all partial derivatives up to order k continuous by $C^k(\Omega)$, and moreover $C^\infty(\Omega) = \cap_{k\in\mathbb{Z}_+} C^k(\Omega)$. By $C^k(K)$, for K a closed subset of $\mathbb{R}^n$, we mean the space of restrictions to K of all functions in $C^k(\mathbb{R}^n)$.

A *multi-index* is an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, and we define $|\alpha| := \sum_{k=1}^n \alpha_k$. If $\alpha \in \mathbb{Z}_+^n$ and $f \in C^m(\Omega)$ with $|\alpha| \leq m$, then we may define

$$\partial^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f.$$

In the case where $K \subset \mathbb{R}^n$ is compact, we equip $C^k(K)$ with the norm

$$\|\varphi\|_{C^k(K)} = \sup_{x\in K,\, |\alpha|\leq k} |(\partial^\alpha \varphi)(x)|.$$

For closed $K \subset \mathbb{R}^n$, we denote the intersection of all $C^k(K)$, $k \in \mathbb{Z}_+$, by $C^\infty(K)$.

Writing e.g. $C^k(\Omega)^\ell$, we mean a column vector of $\ell = 1, 2, 3, \ldots$ functions in $C^k(\Omega)$, and we will later also use similar notations for vector-valued distributions. The *support* of a *function* $f \in C(\Omega)$ is the closure of the set $\{\omega \in \Omega \mid f(\omega) \neq 0\}$ in $\mathbb{R}^n$ and it is denoted by $\operatorname{supp} f$.

A *test function* on a domain $\Omega \subset \mathbb{R}^n$ is a function $f \in C^\infty(\Omega)$ with $\operatorname{supp} f$ a compact subset of Ω, and we denote the set of test functions on Ω by $\mathcal{D}(\Omega)$. If u is a linear map from $\mathcal{D}(\Omega)$ to $\mathbb{C}$ then the action of u on the test function φ is denoted by (u, φ). We follow the standard convention that (u, φ) is *bilinear, not sesquilinear*, i.e., (u, φ) is linear in both u and φ, unlike an inner product which would normally be conjugate linear in φ.

Definition 2.1 ([TW09, Def. 13.2.1])**.** By a *distribution* on Ω we mean a linear map $u : \mathcal{D}(\Omega) \to \mathbb{C}$ such that for every compact $K \subset \Omega$ there exists an $m \in \mathbb{Z}_+$ and a constant $c \geq 0$, both of which may depend on K, such that

$$|(u, \varphi)| \leq c\|\varphi\|_{C^m(K)} \quad \forall\varphi \in \mathcal{D}(\Omega). \tag{2.1}$$

The vector space of distributions on Ω is denoted by $\mathcal{D}'(\Omega)$. Sometimes we write (u, φ) explicitly as $(u, \varphi)_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}$ for clarity.

The smallest m satisfying (2.1) for all K is called the *order* of u, provided such an m exists. If for a distribution u, there exists an $f \in L^1_{loc}(\Omega)$, i.e., f is Lebesgue integrable over all compact $K \subset \Omega$, such that

$$(u, \varphi) = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x \quad \forall\varphi \in \mathcal{D}(\Omega), \tag{2.2}$$

then u is called *regular*.

Conversely, if $f \in L^1_{loc}(\Omega)$ then $u : \mathcal{D}(\Omega) \to \mathbb{C}$ in (2.2) is a distribution of order zero, which satisfies

$$|(u, \varphi)| \leq \int_K |f(x)| \,\mathrm{d}x \, \|\varphi\|_{C(K)}.$$

Clearly f and u determine each other uniquely if u is a regular distribution, and so we identify a regular distribution u with its representative f. When we write that a distribution lies in e.g. $L^2(\Omega)$, we mean that the distribution is regular and its representative lies in this space. By the inclusion $L^2(\Omega) \subset L^1_{loc}(\Omega)$ and (2.2), we have

$$(f, \varphi)_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = \langle f, \overline{\varphi} \rangle_{L^2(\Omega)} \quad \forall f \in L^2(\Omega),\ \varphi \in \mathcal{D}(\Omega). \tag{2.3}$$

Observe that $\mathcal{D}(\Omega) \subset L^2(\Omega)$, since the elements of $\mathcal{D}(\Omega)$ are continuous with compact support.

Definition 2.2 ([TW09, Def. 13.2.4])**.** The sequence $u_k \in \mathcal{D}'(\Omega)$ of *distributions converges* to $u \in \mathcal{D}'(\Omega)$ if

$$\lim_{k \to \infty} (u_k, \varphi) = (u, \varphi) \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Note that this is a limit in $\mathbb{C}$.

3 Partial derivatives and Sobolev spaces on open subsets Ω of $\mathbb{R}^n$

Definition 3.1 ([TW09, Def. 13.2.6])**.** Let $\Omega \subset \mathbb{R}^n$ be open, $u \in \mathcal{D}'(\Omega)$, and $j \in \{1, \ldots, n\}$. The *partial derivative of the distribution* u with respect to x_j, denoted by $\frac{\partial u}{\partial x_j}$, is the distribution

$$\left(\frac{\partial u}{\partial x_j}, \varphi \right) := - \left(u, \frac{\partial \varphi}{\partial x_j} \right) \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Higher order partial derivatives are defined recursively and we also use multi-index notation with distributions.

If $u \in C^k(\Omega)$, i.e, if u can be identified with a $f \in C^k(\Omega)$ as in (2.2), then the distribution partial derivatives coincide with the classic partial derivatives, again in the sense of (2.2); see [TW09, p. 410].

Lemma 3.2. *The following claims are true:*

1. *The limit in Definition 2.2 is unique and convergence in $L^p(\Omega)$ implies convergence in $\mathcal{D}'(\Omega)$.*

2. *If $u_k \to u$ in $\mathcal{D}'(\Omega)$ then all partial derivatives of all orders of u_k tend to the corresponding partial derivative of u in $\mathcal{D}'(\Omega)$.*

Claim 1 is [TW09, Rem. 13.2.5] and claim 2 is [TW09, Prop. 13.2.9].

Definition 3.3. By $H^k(\Omega)$, $k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, we denote the Sobolev space of distributions on Ω, such that all partial derivatives of order at most k lie in $L^2(\Omega)$. We equip $H^k(\Omega)$ with the inner product

$$\langle f, g\rangle_k = \sum_{|\alpha|\le k} \int_\Omega \partial^\alpha f \, \overline{\partial^\alpha g} \, \mathrm{d}x, \quad f, g \in H^k(\Omega). \tag{3.1}$$

Moreover, $H^{1/2}(\Omega)$ is defined as the space consisting of all $f \in L^2(\Omega)$ such that

$$\|f\|^2_{H^{1/2}(\Omega)} := \|f\|^2_{L^2(\Omega)} + \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^2}{\|x - y\|^{n+1}_{\mathbb{R}^n}} \, \mathrm{d}x \, \mathrm{d}y < \infty. \tag{3.2}$$

The inner product on $H^{1/2}(\Omega)$ is found by polarization of (3.2). The spaces $H^k(\Omega)$ and $H^{1/2}(\Omega)$ are Hilbert spaces; see [TW09, Prop. 13.4.2] and [TW09, p. 416], respectively.

Example 3.4. The Heaviside step function is not a member of $H^{1/2}(-1, 1)$. Indeed, $|f(x) - f(y)| = 1$ for all x and y on opposite sides of 0, and

$$\begin{aligned}
\|f\|^2_{H^{1/2}(-1,1)} &= 1 + 2 \int_{-1}^0 \int_0^1 \frac{\mathrm{d}x \, \mathrm{d}y}{(x - y)^2} \\
&\ge 1 + 2 \int_{-1}^{-\varepsilon} \int_\varepsilon^1 \frac{\mathrm{d}x \, \mathrm{d}y}{(x - y)^2} = 1 - 2\ln 2 + 4\ln(1 + \varepsilon) - 2\ln(2\varepsilon)
\end{aligned}$$

for all $\varepsilon \in (0, 1)$. Letting $\varepsilon \to 0$, we obtain $\|f\|^2_{1/2} = \infty$, and so $f \notin H^{1/2}(-1, 1)$.

Definition 3.5. The *divergence operator* is the operator $\operatorname{div} : \mathcal{D}'(\Omega)^n \to \mathcal{D}'(\Omega)$ given by

$$\operatorname{div} v = \frac{\partial v_1}{\partial x_1} + \ldots + \frac{\partial v_n}{\partial x_n},$$

and the *gradient operator* is the operator $\operatorname{grad} : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)^n$ defined by

$$\operatorname{grad} w = \left(\frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_n} \right).$$

We will consider grad as an unbounded operator from $L^2(\Omega)$ into $L^2(\Omega)^n$ with domain contained in $H^1(\Omega)$. We will show that the adjoint of this operator is $-\operatorname{div}$ considered as an unbounded operator from $L^2(\Omega)^n$ into $L^2(\Omega)$ with domain contained in the space

$$H^{\operatorname{div}}(\Omega) := \left\{ v \in L^2(\Omega)^n \mid \operatorname{div} v \in L^2(\Omega) \right\},$$

equipped with the graph norm of div.

Lemma 3.6. *The space $H^{\mathrm{div}}(\Omega)$ is a Hilbert space with the inner product*

$$\langle x,z\rangle_{H^{\mathrm{div}}(\Omega)} := \langle x,z\rangle_{L^2(\Omega)^n} + \langle \operatorname{div} x, \operatorname{div} z\rangle_{L^2(\Omega)}, \quad x,z\in H^{\mathrm{div}}(\Omega). \tag{3.3}$$

Proof. We prove only completeness, leaving it to the reader to verify that (3.3) satisfies the axioms of an inner product. Let x_k be a Cauchy sequence in $H^{\mathrm{div}}(\Omega)$, so that x_k and $\operatorname{div} x_k$ are Cauchy sequences in $L^2(\Omega)^n$ and $L^2(\Omega)$, respectively. By the completeness of $L^2(\Omega)$, there exist $x_0 \in L^2(\Omega)^n$ and $y_0 \in L^2(\Omega)$, such that $x_k \to x_0$ and $\operatorname{div} x_k \to y_0$ as $k\to\infty$. From Definition 2.2 and Lemma 3.2 it now follows that

$$(\operatorname{div} x_0, \varphi) = \lim_{k\to\infty} (\operatorname{div} x_k, \varphi) = (y_0, \varphi) \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Hence, $\operatorname{div} x_0 = y_0$ as distributions, and by construction $y_0 \in L^2(\Omega)$. This implies that $x_0 \in H^{\mathrm{div}}(\Omega)$, and thus $H^{\mathrm{div}}(\Omega)$ is complete. □

The following subspaces of $H^1(\Omega)$ and $H^{\mathrm{div}}(\Omega)$ will turn out to be very useful for us:

Definition 3.7. We denote the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$ by $H^1_0(\Omega)$. Similarly, the closure of $\mathcal{D}(\Omega)^n$ in $H^{\mathrm{div}}(\Omega)$ is denoted by $H^{\mathrm{div}}_0(\Omega)$.

4 Sobolev spaces on Lipschitz manifolds $\partial\Omega$

We will make use of Sobolev spaces on the boundary $\partial\Omega$ of Ω, and in order to do this we need to introduce some notions on topological manifolds; see [TW09, Sect. 13.5] or [Spi65] for more background. We begin by introducing the concept of Lipschitz-continuous boundary.

We call a $\mathbb{C}^m$-valued *function* ϕ defined on a subset of $\mathbb{R}^n$ *(globally) Lipschitz continuous* if there exists a *Lipschitz constant* $L \geq 0$, such that $\|\phi(x)-\phi(y)\|_{\mathbb{C}^m} \leq L\|x-y\|_{\mathbb{R}^n}$ for all $x,y\in \operatorname{dom}(\phi)$. Here $\operatorname{dom}(\phi)$ denotes the domain of the function ϕ.

Definition 4.1 ([GR86, Def. I.1.1])**.** Let Ω be an open subset of $\mathbb{R}^n$ with boundary $\partial\Omega := \overline{\Omega}\backslash\Omega$. We say that the *boundary* $\partial\Omega$ is *Lipschitz continuous* if for every $x\in\partial\Omega$ there exists a neighbourhood $\mathcal{O}_x$ of x in $\mathbb{R}^n$ and new orthogonal coordinates $y=(y_1,\ldots,y_n)$, with the following properties:

1. $\mathcal{O}_x$ is an open hypercube in the new coordinates, i.e., there exist $a_1,\ldots,a_n > 0$, such that $\mathcal{O}_x = \{y \mid -a_j < y_j < a_j \ \forall j=1,\ldots,n\}$.

2. Denoting $y' := (y_1,\ldots,y_{n-1})$, there exists a Lipschitz-continuous function ϕ_x defined on $\mathcal{O}'_x = \{y' \mid -a_j < y_j < a_j \ \forall j = 1,\ldots,n-1\}$, mapping into $\mathbb{R}$, which locally describes Ω and its boundary near x in the following sense:

(a) $|\phi_x(y')| \le a_n/2$ for all $y' \in \mathcal{O}'_x$,

(b) $\Omega \cap \mathcal{O}_x = \{y \mid y_n < \phi_x(y')\}$, and

(c) $\partial\Omega \cap \mathcal{O}_x = \{y \mid y_n = \phi_x(y')\}$.

By a *Lipschitz domain in* $\mathbb{R}^n$, we mean an *open connected subset* of $\mathbb{R}^n$ whose boundary is Lipschitz continuous. We usually work with bounded Lipschitz domains. It is often possible to treat an open subset of $\mathbb{R}^n$, with Lipschitz continuous boundary and finitely many, say N, disconnected components, as N separate Lipschitz domains. One then solves the problem at hand on one domain at a time and combines the partial solutions.

Example 4.2. Every open bounded convex subset of $\mathbb{R}^n$ is a bounded Lipschitz domain, [Gri85, Cor. 1.2.2.3].

We will now see that the boundary of a Lipschitz domain is a so-called topological manifold, or more precisely, it is a *Lipschitz manifold.*

Definition 4.3. A pair (X, τ), where X is a set and τ is a set of subsets of X, is a *topological space* if τ is a *topology* on X, i.e.,

1. $\emptyset, X \in \tau$ and

2. τ is closed under arbitrary union and finite intersection.

The elements of τ are called the *open subsets* of X.

We have the following lemma:

Lemma 4.4. *If $Y \subset X$, where (X, τ) is a topological space, then (Y, Σ) is also a topological space with the topology*

$$\Sigma := \{Y \cap T \mid T \in \tau\}. \tag{4.1}$$

Proof. Trivially $\emptyset = Y \cap \emptyset \in \Sigma$, and since $Y \subset X$, we also have $Y = Y \cap X \in \Sigma$. Moreover, for any collection of $S_\alpha \in \Sigma$, there by definition exists a collection of $T_\alpha \in \tau$, such that $S_\alpha = Y \cap T_\alpha$. As (X, τ) is a topological space, $\cup_\alpha T_\alpha \in \tau$, and therefore also

$$\cup_\alpha S_\alpha = \cup_\alpha (Y \cap \tau_\alpha) = Y \cap (\cup_\alpha T_\alpha) \in \Sigma.$$

If the collection is finite, then $\cap_\alpha T_\alpha \in \tau$ and

$$\cap_\alpha S_\alpha = \cap_\alpha (Y \cap \tau_\alpha) = Y \cap (\cap_\alpha T_\alpha) \in \Sigma.$$

□

If (X, τ) is a topological space and $Y \subset X$, then we call (Y, Σ) a *topological subspace* of (X, τ), where Σ is the *subspace topology* (4.1).

A *neighbourhood* of a point $x \in X$ is an open set $T \in \tau$ such that $x \in T$. A topological space is a *Hausdorff space* if distinct points have disjoint neighbourhoods:

$$x, y \in X,\ x \neq y \quad \Longrightarrow \quad \exists T_x, T_y \in \tau:\ x \in T_x,\ y \in T_y,\ T_x \cap T_y = \emptyset.$$

Let (X, τ) and (Y, ρ) be two topological spaces. A function ψ from (X, τ) into (Y, ρ) is *continuous at the point* $x \in X$, if for every neighbourhood R of $\psi(x)$, there exists a neighbourhood T of x such that $\psi(T) \subset R$. The function is *continuous* if it is continuous at every $x \in X$, and every continuous function has the property

$$R \in \rho \quad \Longrightarrow \quad \{x \in X \mid f(x) \in R\} \in \tau,$$

which is often also taken as the definition of continuity.

A continuous function ψ that maps X onto Y and has a continuous inverse is called a *homeomorphism.* If there exists a homeomorphism ψ from (X, τ) to (Y, ρ), then these topological spaces are said to be *homeomorphic*, meaning that they have precisely the same topological structure: $T \in \tau$ if and only if $\psi(T) \in \rho$.

A neighbourhood T of a point $x \in X$ is called *Euclidean* if it is homeomorphic to a subset $\mathcal{O}$ of $\mathbb{R}^n$ for some $n \in \mathbb{Z}_+$, say with homeomorphism $\psi : T \to \mathcal{O}$. In this case we call the pair (T, ψ) a *chart* on X. In particular, $\mathcal{O} = \psi(T)$ is an open subset of $\mathbb{R}^n$, since $T \in \tau$ is open. A topological space X is *locally Euclidean* if there exists an $n \in \mathbb{Z}_+$ such that all $x \in X$ have a neighbourhood that is homeomorphic to an open hypercube in $\mathbb{R}^n$. This is equivalent to saying that every $x \in X$ has a neighbourhood homeomorphic to all of the Euclidean space $\mathbb{R}^n$. Noting that n may not depend on x, we make the following definition:

Definition 4.5. A locally Euclidean Hausdorff space, where every neighbourhood is homeomorphic to a hypercube in $\mathbb{R}^n$, is called an *n-dimensional topological manifold.*

Let $S \subset X$ with X a topological space. A collection $(T_j)_{j \in J} \subset \tau$ is an *(open) covering* of S if $S \subset \bigcup_{j \in J} T_j$. A locally Euclidean space has a covering $(T_j)_{j \in J}$ of Euclidean neighbourhoods, and we call the corresponding family $(T_j, \psi_j)_{j \in J}$ of charts on X an *atlas* for X. On the overlap $T_j \cap T_k$ of two charts we define the *transition map* $\psi_{j,k}$ by $\psi_j \circ \psi_k^{-1}$, a homeomorphism on $T_j \cap T_k$ which allows us to change charts from (T_k, ψ_k) to (T_j, ψ_j).

In the notation of Definition 4.1, if Ω is a Lipschitz domain in $\mathbb{R}^n$ then we can view $\partial\Omega$ locally as an $n-1$-dimensional topological sub-manifold of $\mathbb{R}^n$ using the mapping $\Phi_x(y') := (y', \phi_x(y'))$, which maps $\mathcal{O}'_x$ one-to-one onto $\partial\Omega \cap \mathcal{O}_x$. This mapping satisfies for all $z', y' \in \mathcal{O}'_x$:

$$\begin{aligned}\|z'-y'\|^2_{\mathbb{R}^{n-1}} \leq \|z'-y'\|^2_{\mathbb{R}^{n-1}} + |\phi_x(z')-\phi_x(y')|^2 = \|\Phi_x(z')-\Phi_x(y')\|^2_{\mathbb{R}^n} \\ \leq (1+L_x^2)\|z'-y'\|^2_{\mathbb{R}^{n-1}},\end{aligned}$$

where L_x is the Lipschitz constant of ϕ_x. Hence $(\partial\Omega \cap \mathcal{O}_x, \Phi_x^{-1})$ is a chart onto $\mathcal{O}'_x$, which is in addition *bi-Lipschitz*, i.e., both Φ_x and Φ_x^{-1} are Lipschitz continuous.[1] This clearly implies that all transition maps $\Phi_x^{-1} \circ \Phi_y$ are bi-Lipschitz too, and in this way every open covering $(T_j)_{j\in J}$ of $\partial\Omega$ gives rise to a bi-Lipschitz atlas (T_j, Φ_j) for $\partial\Omega$.

Definition 4.6. An $n-1$-dimensional topological manifold is an $n-1$-*dimensional Lipschitz manifold* if it has an atlas whose charts are all bi-Lipschitz.

Above we showed that the boundary of a bounded Lipschitz domain in $\mathbb{R}^n$ is an $n-1$-dimensional Lipschitz manifold. Next we define the Sobolev spaces $H^{\pm 1/2}(\Gamma)$, where Γ is an open subset of $(\partial\Omega, \Sigma)$, with Σ the subspace topology.

Definition 4.7 ([TW09, Defs 13.5.7])**.** Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $\Gamma \subset \partial\Omega$ be an open set in the subspace topology of $\partial\Omega$.

The space $H^{1/2}(\Gamma)$ consists of those $f \in L^2(\Gamma)$ for which

$$f \circ \Phi_x \in H^{1/2}(\Phi_x^{-1}(\Gamma \cap \mathcal{O}_x)) \tag{4.2}$$

for all $x \in \Gamma$, and $\mathcal{O}_x$, where $\Phi_x(y') := (y', \phi_x(y'))$ with ϕ_x as in Definition 4.1 as before, and $H^{1/2}(\Phi_x^{-1}(\Gamma \cap \Omega_x))$ is defined in Definition 3.3.

We thus use an atlas $(\partial\Omega \cap \mathcal{O}_x, \Phi_x^{-1})_{x\in\partial\Omega}$ for $\partial\Omega$ to define $H^{1/2}(\Gamma)$. By [TW09, p. 422], condition (4.2) holds for every atlas of $\partial\Omega$ if and only if it holds for one atlas. For a bounded Lipschitz domain Ω, the boundary $\partial\Omega$ is closed and bounded in $\mathbb{R}^n$. Hence the boundary is compact, and so we can find a finite atlas to check the condition (4.2) on.

For a fixed finite atlas $(\partial\Omega \cap \mathcal{O}_j, \Phi_j^{-1})_{j=1}^N$ of $\partial\Omega$, we equip $H^{1/2}(\Gamma)$ with the norm given by

$$\|f\|^2_{H^{1/2}(\Gamma)} := \sum_{j=1}^N \|f \circ \Phi_j\|^2_{H^{1/2}(\Phi_j^{-1}(\Gamma\cap\mathcal{O}_j))}. \tag{4.3}$$

[1]This is consistent with the notation in [TW09], but [GR86] use the charts Φ_x instead of Φ_x^{-1}.

According to [TW09, p. 423], $H^{1/2}(\Gamma)$ is a Hilbert space with this norm. Moreover, any norm of this type is by [TW09, p. 423] equivalent to the norm

$$\|f\|^2_{H^{1/2}(\Gamma)} := \|f\|^2_{L^2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|f(x) - f(y)|^2}{\|x - y\|^n_{\mathbb{R}^n}} \,\mathrm{d}\sigma_x \,\mathrm{d}\sigma_y, \tag{4.4}$$

where $\mathrm{d}\sigma_x$ is a surface element in $\partial\Omega$ at $x \in \Gamma$. In the sequel we always consider $H^{1/2}(\Gamma)$ with the norm (4.4).

5 Boundary traces and integration by parts

We need duality with respect to a pivot space; see e.g. [TW09, Sect. 2.9] for more details on this.

Definition 5.1. Let V be a Hilbert space densely and continuously embedded in the Hilbert space W. The *dual V' of V with pivot space W* is the completion of W with respect to the *duality norm*

$$\|w\|_{V'} := \sup_{v \in V,\ v \neq 0} \frac{|\langle w, v\rangle_W|}{\|v\|_V}.$$

Every element $\widetilde{w}$ in this completion is a sequence of $w_k \in W$, which is Cauchy in the duality norm, and this $\widetilde{w}$ is identified with the linear functional

$$v \mapsto (\widetilde{w}, v)_{V',V} := v \mapsto \lim_{k\to\infty} \langle w_k, v\rangle_W \tag{5.1}$$

on V. The space W is embedded into V' by identifying $w \in W$ with the Cauchy sequence $(w, w, w, \ldots)$.

Lemma 5.2. *For a bounded Lipschitz domain with boundary $\partial\Omega$, the space $H^{1/2}(\partial\Omega)$ is dense in $L^2(\partial\Omega)$.*

This result follows from [TW09, Cor. 13.6.11].

Definition 5.3. For a bounded Lipschitz domain with boundary $\partial\Omega$, the space $H^{-1/2}(\partial\Omega)$ is defined as the dual of $H^{1/2}(\partial\Omega)$ with pivot space $L^2(\partial\Omega)$.[2]

If Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, then the outward unit normal vector field is defined for almost all $x \in \partial\Omega$ using local coordinates, and we can define a vector field ν in a neighbourhood of $\overline{\Omega}$ that coincides with the outward unit normal vector field for almost every $x \in \partial\Omega$; see [TW09, Def. 13.6.3] and the remarks following. According, to [TW09, p. 424–425], $\nu \in L^\infty(\partial\Omega)^n$.

Theorem 5.4 ([GR86, Thms I.1.5, I.2.5, and I.2.6, Cor. I.2.8])**.** *For a bounded Lipschitz domain Ω the following hold:*

[2]This definition is consistent with [TW09]; see p. 432. In [GR86, Def. I.1.4], $H^{-1/2}(\partial\Omega)$ defined differently, using $H^{-1/2}(\Omega)$ and an atlas.

1. *The boundary trace mapping* $g \mapsto g|_{\partial\Omega} : C^1(\overline{\Omega}) \to C(\partial\Omega)$ *has a unique continuous extension* γ_0 *that maps* $H^1(\Omega)$ *onto* $H^{1/2}(\partial\Omega)$. *The space* $H_0^1(\Omega)$ *in Definition 3.7 equals* $\left\{g \in H^1(\Omega) \mid \gamma_0 g = 0\right\}$.

2. *The normal trace mapping* $u \mapsto \nu \cdot \gamma_0 u : H^1(\Omega)^n \to L^2(\partial\Omega)$ *has a unique continuous extension* $\gamma_\perp$ *that maps* $H^{\mathrm{div}}(\Omega)$ *onto* $H^{-1/2}(\partial\Omega)$. *Here the dot* $\cdot$ *denotes the inner product in* $\mathbb{R}^n$, $p \cdot q = q^\top p$. *Furthermore,*
$$H_0^{\mathrm{div}}(\Omega) = \left\{f \in H^{\mathrm{div}}(\Omega) \mid \gamma_\perp f = 0\right\}.$$

We call γ_0 the *Dirichlet trace map* and $\gamma_\perp$ the *normal trace map*. The following "integration by parts" formula is the foundation for our main duality result; see e.g. [Gri85, Thm 1.5.3.1] or [Neč12, Thm 3.1.1]:

Theorem 5.5. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$. *For all* $f \in H^{\mathrm{div}}(\Omega)$ *and* $g \in H^1(\Omega)$ *it holds that*
$$\langle \operatorname{div} f, g\rangle_{L^2(\Omega)} + \langle f, \operatorname{grad} g\rangle_{L^2(\Omega)^n} = (\gamma_\perp f, \gamma_0 g)_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}. \tag{5.2}$$

6 Duality of the divergence and gradient operators

We will make use of the following general result:

Lemma 6.1. *Let* T *be a closed linear operator from* $\operatorname{dom}(T) \subset X$ *into* Y, *where* X *and* Y *are Hilbert spaces. Equip* $\operatorname{dom}(T)$ *with the graph norm of* T, *in order to make it a Hilbert space. Let* R *be a restriction of the operator* T. *The following claims are true:*

1. *The closure of the operator* R *is* $\overline{R} = T|_{\overline{\operatorname{dom}(R)}}$, *where* $\overline{\operatorname{dom}(R)}$ *is the closure of* $\operatorname{dom}(R)$ *in the graph norm of* T. *In particular,* R *is a closed operator if and only if* $\operatorname{dom}(R)$ *is a closed subspace of* $\operatorname{dom}(T)$.

2. *Let* γ *be a linear operator from* $\operatorname{dom}(T)$ *into a Hilbert space* Z. *If*
$$\gamma \operatorname{dom}(R) = \gamma \operatorname{dom}(T) \quad \text{and} \quad \ker(\gamma) \subset \operatorname{dom}(R), \tag{6.1}$$
then necessarily also $\operatorname{dom}(R) = \operatorname{dom}(T)$.

Proof. The following chain of equivalences, where $G(R) = \left[\begin{smallmatrix} I \\ R \end{smallmatrix}\right] \operatorname{dom}(R)$ is the graph of R, proves that $\overline{R} = T|_{\overline{\operatorname{dom}(R)}}$:

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} \in G(\overline{R}) \quad &\overset{(i)}{\Longleftrightarrow} \quad \exists x_k \in \operatorname{dom}(R): \ x_k \overset{X}{\to} x, \ Rx_k \overset{Y}{\to} y \\
&\overset{(ii)}{\Longleftrightarrow} \quad \exists x_k \in \operatorname{dom}(R): \ x_k \overset{X}{\to} x, \ Tx_k \overset{Y}{\to} y \\
&\overset{(iii)}{\Longleftrightarrow} \quad \exists x_k \in \operatorname{dom}(R): \ x_k \overset{\operatorname{dom}(T)}{\to} x, \ Tx = y \\
&\overset{(iv)}{\Longleftrightarrow} \quad x \in \overline{\operatorname{dom}(R)}, \ Tx = y,
\end{aligned}$$

where we have used that (i): $G(\overline{R}) = \overline{G(R)}$ by the definition of operator closure, (ii): $G(R) \subset G(T)$, (iii): $\mathrm{dom}\,(T)$ has the graph norm of T closed, and (iv): $\mathrm{dom}\,(R)$ has the same norm as $\mathrm{dom}\,(T)$.

Now it follows easily that R is closed if and only if $\mathrm{dom}\,(R)$ is closed in $\mathrm{dom}\,(T)$:

$$R = \overline{R} \quad \Longrightarrow \quad T\big|_{\mathrm{dom}(R)} = T\big|_{\mathrm{dom}(\overline{R})} \quad \Longrightarrow \quad \mathrm{dom}\,(R) = \overline{\mathrm{dom}\,(R)},$$

and moreover, assuming $\mathrm{dom}\,(R) = \overline{\mathrm{dom}\,(R)}$, we obtain that

$$R = T\big|_{\mathrm{dom}(R)} = T\big|_{\overline{\mathrm{dom}(R)}} = \overline{R}.$$

Regarding assertion 2, it follows from $R \subset T$ that $\mathrm{dom}\,(R) \subset \mathrm{dom}\,(T)$. For the converse inclusion, choose $x \in \mathrm{dom}\,(T)$ arbitrarily. By the first assumption in (6.1), we can find an $\xi \in \mathrm{dom}\,(R)$ such that $\gamma x = \gamma \xi$. Then $x - \xi \in \ker(\gamma) \subset \mathrm{dom}\,(R)$, by the second assumption in (6.1), so that $x = x - \xi + \xi \in \mathrm{dom}\,(R)$. □

We have the following general result:

Theorem 6.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $H_0^1(\Omega) \subset G \subset H^1(\Omega)$. Setting*

$$D := \left\{ f \in H^{\mathrm{div}}(\Omega) \mid (\gamma_\perp f, \gamma_0 g)_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} = 0 \ \forall g \in G \right\}, \qquad (6.2)$$

we obtain the following:

1. *The set D is a closed subspace of $H^{\mathrm{div}}(\Omega)$ that contains $H_0^{\mathrm{div}}(\Omega)$, i.e., $H_0^{\mathrm{div}}(\Omega) \subset D \subset H^{\mathrm{div}}(\Omega)$.*
2. *When we identify $L^2(\Omega)$ and $L^2(\Omega)^n$ with their own duals, and we consider $\mathrm{grad}\big|_G$ as an unbounded operator mapping the dense subspace G of $L^2(\Omega)$ into $L^2(\Omega)^n$, we have $\mathrm{grad}\big|_G^* = -\mathrm{div}\big|_D$.*
3. *Let G be closed in $H^1(\Omega)$. Then $D = H^{\mathrm{div}}(\Omega)$ if and only if $G = H_0^1(\Omega)$, and $D = H_0^{\mathrm{div}}(\Omega)$ if and only if $G = H^1(\Omega)$.*

Proof. We prove assertion 2 first. Since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, necessarily also G which contains $H_0^1(\Omega)$ is dense. From (5.2) it immediately follows that for all $f \in D$:

$$\langle \mathrm{div}\, f, g \rangle_{L^2(\Omega)} + \langle f, \mathrm{grad}\, g \rangle_{L^2(\Omega)^n} = 0 \quad \forall g \in G, \qquad (6.3)$$

and hence $-\mathrm{div}\big|_D \subset \mathrm{grad}\big|_G^*$. We now prove the converse inclusion.

Assume therefore that $f \in \mathrm{dom}\left(\mathrm{grad}\big|_G^*\right) \subset L^2(\Omega)^n$, i.e., that there exists an $h \in L^2(\Omega)$, such that

$$\langle h, g \rangle_{L^2(\Omega)} + \langle f, \mathrm{grad}\, g \rangle_{L^2(\Omega)^n} = 0 \quad \forall g \in G. \qquad (6.4)$$

Since $\mathcal{D}(\Omega) \subset H_0^1(\Omega) \subset G$, (6.4) holds in particular for all $g \in \mathcal{D}(\Omega)$, and thus by (2.3) it holds for all $\overline{g} \in \mathcal{D}(\Omega)$ that:

$$\begin{aligned} 0 &= (h, \overline{g})_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + \big(f, \overline{\operatorname{grad} g}\big)_{\mathcal{D}'(\Omega)^n,\mathcal{D}(\Omega)^n} \\ &= (h, \overline{g})_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + \sum_{k=1}^{n} \left(f_k, \overline{\frac{\partial g}{\partial x_k}} \right)_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \\ &= (h, \overline{g})_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} - \sum_{k=1}^{n} \left(\frac{\partial f_k}{\partial x_k}, \overline{g} \right)_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \\ &= (h - \operatorname{div} f, \overline{g})_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}. \end{aligned}$$

Hence, in the sense of distributions, $\operatorname{div} f = h \in L^2(\Omega)$, which implies that $f \in H^{\operatorname{div}}(\Omega)$.

We have now proved that the existence of an $h \in L^2(\Omega)$, such that (6.4) holds, implies (6.3), and combining this with the integration by parts formula (5.2), we obtain that

$$(\gamma_\perp f, \gamma_0 g)_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)} = 0 \quad \forall g \in G,$$

i.e., that $f \in D$. From (6.4), we moreover have $h = -\operatorname{grad}\big|_G^* f$. Summarizing, we have shown that (6.4) implies that $-\operatorname{grad}\big|_G^* f = h = \operatorname{div} f$, hence $\operatorname{grad}\big|_G^* \subset -\operatorname{div}\big|_D$. We are finished proving assertion 2.

Next we show how assertion 3 is a consequence of Theorem 5.4, assertion 2 of Lemma 6.1, and (6.2). It follows from (6.2) and Theorem 5.4 that $H_0^{\operatorname{div}}(\Omega) \subset D \subset H^{\operatorname{div}}(\Omega)$. Indeed, trivially $D \subset H^{\operatorname{div}}(\Omega)$ by (6.2), and moreover by Theorem 5.4 and (6.2):

$$f \in H_0^{\operatorname{div}}(\Omega) \quad \Longrightarrow \quad f \in H^{\operatorname{div}}(\Omega),\ \gamma_\perp f = 0 \quad \Longrightarrow \quad f \in D.$$

We prove the equivalence $G = H_0^1(\Omega) \iff D = H^{\operatorname{div}}(\Omega)$. In Lemma 6.1, take $T := \operatorname{grad}\big|_G$ and $R := \operatorname{grad}\big|_{H_0^1(\Omega)}$. Moreover, set $\gamma := \gamma_0$, which by Theorem 5.4 has kernel $H_0^1(\Omega) \subset G$, where the inclusion is by assumption. Now Lemma 6.1 and Theorem 5.4 give

$$G = H_0^1(\Omega) \quad \iff \quad \gamma_0 G = \gamma_0 H_0^1(\Omega) = \left\{ 0 \in H^{1/2}(\partial\Omega) \right\}.$$

Next one uses Lemma 6.1 to obtain that

$$D = H^{\operatorname{div}}(\Omega) \quad \iff \quad \gamma_\perp D = \gamma_\perp H^{\operatorname{div}}(\Omega) = H^{-1/2}(\partial\Omega) \qquad (6.5)$$

by taking $T := \operatorname{div}$, defined on $H^{\operatorname{div}}(\Omega)$, and $R := \operatorname{div}\big|_D$, with $\gamma := \gamma_\perp$, $\ker(\gamma_\perp) = H_0^{\operatorname{div}}(\Omega) \subset D$. (The last equality on the right-hand side of (6.5) holds by Theorem 5.4.) The argument is completed by showing that

$$\gamma_0 G = \{0\} \quad \iff \quad \gamma_\perp D = H^{-1/2}(\partial\Omega).$$

Assume first that $\gamma_0 G = \{0\}$. Then $D = H^{\mathrm{div}}(\Omega)$ by (6.2), and Theorem 5.4 gives that $\gamma_\perp D = H^{-1/2}(\partial\Omega)$. Conversely, assume that $\gamma_\perp D = H^{-1/2}(\partial\Omega)$. Then (6.2) yields that $\gamma_0 G = \{0\}$, which by Theorem 5.4 implies that $G \subset \ker(\gamma_0) = H_0^1(\Omega)$. Combining this with the assumption that $H_0^1(\Omega) \subset G$, we obtain $G = H_0^1(\Omega)$, which by Theorem 5.4 implies that $\gamma_0 G = \{0\}$.

The equivalence $D = H_0^{\mathrm{div}}(\Omega) \Longleftrightarrow G = H^1(\Omega)$ is proved similarly.

The proof of assertion 1 is now short: We established in the proof of assertion 3 that $H_0^{\mathrm{div}}(\Omega) \subset D \subset H^{\mathrm{div}}(\Omega)$. Moreover, since $\mathrm{grad}\big|_G^* = -\mathrm{div}\big|_D$ is a closed operator, we obtain from assertion 1 of Lemma 6.1 that D is closed in the norm of $H^{\mathrm{div}}(\Omega)$. □

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