# Labeled Random Finite Sets in Multi-target Track-Before-Detect

Mélanie Bocquel
Thales Nederland B.V. - Sensors, Hengelo, Nederland.
(e-mail: melanie.bocquel@nl.thalesgroup.com)

**Abstract**

In this paper we address the problem of tracking multiple targets based on raw measurements by means of Particle filtering. Bayesian multitarget tracking, in the Random Finite Set framework, propagates the multitarget posterior density recursively in time. Sequential Monte Carlo (SMC) approximations of the optimal filter are computationally expensive and lead to high-variance estimates as the number of targets increases.

We propose a novel, efficient and reliable Labeled RFS based tracking algorithms suitable for, among others, the TBD surveillance application. This algorithm uses the Interacting Population based MCMC-PF (IP-MCMC-PF), first introduced in [6], as the core engine of a Multiple Cardinality Hypotheses Tracker (MCHT), where each cardinality is treated independently. The proposed multi-target filter is built upon the concept of labeled Random Finite Set (RFS) [40], [41], and formally incorporates the propagation and estimation of track labels within the RFS filtering framework.

Simulation analyses demonstrate that the proposed Multiple Cardinality Hypotheses Particle Filter (MCHPF) yields higher consistency, accuracy and reliability in multitarget tracking.

## I. Introduction

Target tracking is an ever-growing research challenge arising in various disciplines ranging from econometrics, image/signal processing to biomedical engineering [15], [23], [30]. The heart of the matter lies in combining sensed data (uncertain and noisy observations) and a priori knowledge to provide a reliable, accurate and timely estimate of an unknown quantity or outcome (often called state).

Early research in this area focused on the single-target tracking problem. This problem presupposes that a single target is present, and this, throughout the whole process. As a result, the tracking problem reduces to the on-line estimation of the state of this target, based on the observations. The single-target tracking problem can be formulated and solved in a Bayesian setting by representing the target state probabilistically and incorporating statistical models for the sensing action and the target state transition [2], [3]. Research in multi-target tracking (MTT) has shown significant progress in recent years. Compared to single-target tracking, the appealing MTT problem poses both additional challenges and opportunities.

To introduce the MTT problem, let us consider a simplified example: assume an air surveillance system comprising a radar which collects the echo signal received after reflection from aircrafts in the field-of-view (FoV), called targets. Imagine that multiple aircrafts are crossing the FoV of the radar and that weather conditions have deteriorated such that some of the targets are not properly detected during the scan and that false observations are reported. From a Bayesian perspective, the multi-target tracking problem consists in inferring the number of targets, their identities and their individual kinematic properties, from a sequence of noisy and cluttered measurements provided by one or more sensors.

More specifically, a multi-target tracking algorithm should be able to:

- track the behavior of all targets under surveillance,
- recognize new targets as they enter as well as identify when a target disappears,
- distinguish targets as separate entities.

This definition hides much of the complexity of the multi-target tracking and poses significant technical challenges. For example, the algorithm does not know beforehand how many targets need to be tracked, and thus should be ready, at any time instant, to deal with any number of new targets. In addition, targets of interest may be temporally obscured, and the algorithm must be able to keep these targets in track even when they cannot be viewed directly. Targets may also interact, altering each others' behavior. Finally, the observations given to an algorithm are not assigned to unique, identifiable targets beforehand, and thus the way observations correspond to targets must be inferred as well.

The primary focus of this memo is to develop an effective and efficient multi-target tracking algorithm dealing with an unknown and time-varying number of targets.

In many applications involving sensor data, the estimation is often performed on data that have been preprocessed into point measurements. For example, in radar tracking, radar videos are converted to detection plots. Compressing the information on the video into a finite set of points is efficient in terms of memory as well as computational requirements. When combined with point measurement based approaches, multi-target tracking can be very effective for a wide range of applications. However, this approach may be undesirable for applications with low signal-to-noise ratio (SNR), since the information loss incurred during the compression can be significant, and in such cases it can clearly be advantageous to make use of all information contained in the video.

The Track-before-Detect (TBD) approach, which proposes to base the tracking on the raw measurements instead of plots, was first investigated in [4], [13]. Since then a number of TBD techniques and applications have been studied [8], [11], [14], [20], [29], [39]. Solutions using a Particle Filter (PF) based on raw measurements have been proposed in the past ten years [9], [10], [26], [33]. PF approach is a Sequential Monte Carlo (SMC) simulation-based method that approximately solves the full Bayes prediction and update equations recursively using a cloud of stochastic samples [16], [18], [19]. The standard Particle Filter approach resorts to a sequential version of the importance sampling and resampling procedure. The main practical problem with this implementation is the need to perform importance sampling in very high dimensional spaces if many targets are present. First, if too few particles are used, all but a few importance weights will be near zero. Resampling will then lead to a loss of diversity among the particles, the sample impoverishment problem. Second, it can be difficult to find an efficient importance density. A poor selection will lead to a high-variance estimator. In particular, a naive choice of importance density such as the prior density will typically lead to an algorithm whose efficiency decreases exponentially with the number of targets for a fixed number of particles. To deal with multimodal posterior probability density functions in large dimensional state spaces, such as in our application, where many (possibly disappearing and reappearing) targets need to be tracked, Markov Chain Monte Carlo (MCMC) methods can be efficiently used [16], [25], [37].

In this work, a novel, efficient and reliable Labeled RFS based tracking algorithms suitable for, among others, the TBD surveillance application is proposed. This algorithm uses the Interacting Population based MCMC-PF (IP-MCMC-PF), first introduced in [6], as the core engine of a Multiple Cardinality Hypotheses Tracker (MCHT), where each cardinality is treated independently (up to a limiting factor $n_max$). The proposed multi-target filter is built upon the concept of labeled Random Finite Set (RFS) [40], [41], and formally incorporates the propagation and estimation of track labels within the RFS filtering framework.

The paper is organized as follows. Section II recalls some definitions and results on labeled RFS [40], [41], and discusses related work on Sequential MCMC methods. Section III adopts the Random Finite Set (RFS) approach to describe and address the multitarget filtering and estimation problem in the context of TBD tracking. Section IV reports the main novelty of this work, detailed in [5]. Here we give insights for our approach, summarize the main aspects of the proposed algorithm, denoted as Multiple Cardinality Hypotheses Particle Filter (MCHPF) and describe its SMC implementation. Section V collects our simulations results. Finally, we report our conclusions and direction for future research in section VI.

## II. Background

This section is organized as follows. In subsection II-A, we recall the concept of labeled RFS [40], [41] and some properties. Subsection II-B reviews the relevant work on Sequential Markov chain Monte Carlo (MCMC) methods. Specifically we detail the IP-MCMC-PF algorithm [6] and address the proposal density design.

### A. Labeled Random Finite Sets

In applications such as multi-target tracking, apart from the target state, the target identity is also required, so that the target paths are consistently estimated. The targets should be distinctly identified by a label. This can be achieved by appending a label $\ell \in \mathbb{L}$ to each state $x \in \mathbb{X}$ where $\mathbb{L}$ denotes the discrete space of distinct labels, and $\mathbb{X}$ is the state space. To address the uniqueness of labels the concept of labeled RFS is required.

*1) Labeled Random Finite Sets:*

**Definition II.1.** For any $\mathbf{X}_\ell \subset \mathbb{X} \times \mathbb{L}$, let $\mathcal{L} : \mathbb{X} \times \mathbb{L} \to \mathbb{L}$ be the projection $\mathcal{L}(\mathbf{X}_\ell) \triangleq \{\ell : (x, \ell) \in \mathbf{X}_\ell$ for some $x \in \mathbb{X}\}$. A labeled RFS $\mathbf{X}_\ell$ with state space $\mathbb{X}$ and label space $\mathbb{L}$ is an RFS on $\mathbb{X} \times \mathbb{L}$ such that each realization $(x, \ell)$ has distinct labels, i.e.

$$|\mathcal{L}(\mathbf{X}_\ell)| = |\mathbf{X}_\ell|.$$

Note that the set integral of a function $p$ mapping from the class of finite subsets of the space $\mathbb{X} \times \mathbb{L}$, denoted as $\mathcal{F}(\mathbb{X} \times \mathbb{L})$, to the real line, i.e. $p : \mathcal{F}(\mathbb{X} \times \mathbb{L}) \to \mathbb{R}$, is defined by:

$$\int p(\mathbf{X}_\ell)\delta\mathbf{X}_\ell = \sum_{n=0}^{\infty} \frac{1}{n!} \int \sum_{(\ell_1,\ldots,\ell_n)\in\mathbb{L}^n} p(\{(x_1, \ell_1), \ldots , (x_n, \ell_n)\})dx_1 \ldots dx_n.$$

To evaluate set integrals involving labeled RFSs, the following Lemma II.1 is required (the proof is given in [41]):

**Lemma II.1.** *Let $\Delta(\mathbf{X}_\ell)$ denote the distinct label indicator $\delta_{|\mathbf{X}_\ell|}(|\mathcal{L}(\mathbf{X}_\ell)|)$. Then for $h : \mathcal{F}(\mathbb{L}) \to \mathbb{R}$ and $g : \mathbb{X} \times \mathbb{L} \to \mathbb{R}$, integrable on $\mathbb{X}$,*

$$\int \Delta(\mathbf{X}_\ell)h(\mathcal{L}(\mathbf{X}_\ell))g^{\mathbf{X}_\ell}\delta\mathbf{X}_\ell = \sum_{\mathbf{L}\subseteq\mathbb{L}} h(\mathbf{L}) \left[\int g(x, \cdot)dx\right]^{\mathbf{L}}. \tag{1}$$

*where $g^{\mathbf{X}_\ell} = \prod_{(x,\ell)\in\mathbf{X}_\ell} g(x, \ell)$.*

**Example II.1.** A labeled multi-Bernoulli RFS $\mathbf{X}_\ell$ with state space $\mathbb{X}$, label space $\mathbb{L}$ and finite parameter set $\left\{\left(r^{(\theta)}, s^{(\theta)}\right) : \theta \in \Theta\right\}$, is a multi-Bernoulli RFS on $\mathbb{X}$ augmented with distinct labels corresponding to the the successful (non-empty) Bernoulli components, i.e. if the Bernoulli component $\left(r^{(\theta)}, s^{(\theta)}\right)$ yields a non-empty set, then the label of the corresponding state is given by $\alpha(\theta)$, where $\alpha : \Theta \to \mathbb{L}$ define a one-to-one mapping [41]. The set of labeled states generated by the algorithm 1 has probability density given by:

$$\pi(\mathbf{X}_\ell) = \Delta(\mathbf{X}_\ell)1_{\alpha(\Theta)}\left(\mathcal{L}(\mathbf{X}_\ell)\right)\left[\Phi(\mathbf{X}_\ell; \cdot)\right]^{\Theta}, \tag{2}$$

where $\Delta(\cdot)$ is the distinct label indicator, $\mathcal{L}(\mathbf{X}_\ell)$ the set of labels of $\mathbf{X}_\ell$, and

$$\begin{aligned} \Phi(\mathbf{X}_\ell; \theta) &= \sum_{(x,\ell)\in\mathbf{X}_\ell} \delta_{\alpha(\theta)}(\ell)r^{(\theta)}s^{(\theta)}(x) + (1 - 1_{\mathcal{L}(\mathbf{X}_\ell)}(\alpha(\theta)))(1 - r^{(\theta)}), \\ &= \begin{cases} r^{(\theta)}s^{(\theta)}(x), & \text{if } (x, \alpha(\theta)) \in \mathbf{X}_\ell; \\ 1 - r^{(\theta)}, & \text{if } \alpha(\theta) \notin \mathcal{L}(\mathbf{X}_\ell). \end{cases} \end{aligned}$$

The following algorithm 1 illustrates how a sample from such a labeled multi-Bernoulli RFS is generated:

**Algorithm 1**: Sampling a labeled multi-Bernoulli RFS

```
Initialize X = ∅.
for θ ∈ Θ do
    Sample u ~ U[0, 1].
    if u ≤ r^(θ) then
        Sample x ~ s^(θ)(·).
        Set X = X ∪ {(x, α(θ))}.
    end
end
```

For instance, let us consider the probability density that the above procedure (algorithm 1) generates the points $(x_1, \ell_1), \ldots, (x_n, \ell_n)$ in that order:

$$\pi(\{(x_1, \ell_1), \ldots, (x_n, \ell_n)\}) = \delta_n(|\{\ell_1, \ldots, \ell_n\}|) \prod_{\theta \in \Theta} \left(1 - r^{(\theta)}\right) \prod_{j=1}^{n} \frac{\times 1_{\alpha(\Theta)}(\ell_j) r^{(\alpha^{-1}(\ell_j))} s^{(\alpha^{-1}(\ell_j))}(x_j)}{1 - r^{(\alpha^{-1}(\ell_j))}}, \quad (3)$$

where $1_Y(X) = 1$ if $X \subseteq Y$ and $0$ otherwise.

The products $\prod_{\theta \in \Theta} \left(1 - r^{(\theta)}\right) \prod_{j=1}^{n} \frac{\times 1_{\alpha(\Theta)}(\ell_j) r^{(\alpha^{-1}(\ell_j))} s^{(\alpha^{-1}(\ell_j))}(x_j)}{1 - r^{(\alpha^{-1}(\ell_j))}}$ can be written as:

$$\prod_{\theta \in \Theta} \left( \sum_{(x_j, \ell_j) \in \mathbf{X}_\ell} \delta_{\alpha(\theta)}(\ell_j) r^{(\theta)} s^{(\theta)}(x_j) + \left(1 - 1_{\mathcal{L}(\mathbf{X}_\ell)}(\alpha(\theta))\right) \left(1 - r^{(\theta)}\right) \right).$$

2) *Generalized Labeled multi-Bernoulli RFS:*

**Definition II.2.** A generalized labeled multi-Bernoulli RFS is a labeled RFS with state space $\mathbb{X}$ and label space $\mathbb{L}$ distributed according to:

$$\boxed{p(\mathbf{X}_\ell) = \Delta(\mathbf{X}_\ell) \sum_{c \in \mathbb{C}} w^{(c)}(\mathcal{L}(\mathbf{X}_\ell)) \left[p^{(c)}\right]^{\mathbf{X}_\ell},} \quad (4)$$

where,

- $\Delta(\mathbf{X}_\ell)$ is the distinct label indicator $\delta_{|\mathbf{X}_\ell|}(|\mathcal{L}(\mathbf{X}_\ell)|)$,
- $\mathbb{C}$ denotes a discrete index set,
- the weight $w^{(c)}(\mathcal{L}(\mathbf{X}_\ell))$ only depends on the labels and satisfies:

$$\sum_{\mathbf{L} \subseteq \mathbb{L}} \sum_{c \in \mathbb{C}} w^{(c)}(\mathbf{L}) = 1,$$

- the multi-target exponential $\left[p^{(c)}\right]^{\mathbf{X}_\ell}$ depends on the entire multi-target state and satisfies

$$\int p^{(c)}(x, \ell) dx = 1.$$

*Remark.* A generalized labeled multi-Bernoulli can be interpreted as a mixture of multi-target exponentials [41]. Each term in the mixture (4) consists of a weight $w^{(c)}(\mathcal{L}(\mathbf{X}_\ell))$, and a multi-target exponential $\left[p^{(c)}\right]^{\mathbf{X}_\ell}$ that depends on the entire multi-target state.

The cardinality distribution of a generalized labeled multi-Bernoulli RFS is given by:

$$\rho(n) = \sum_{c\in\mathbb{C}} \sum_{\mathbf{L}\in\mathcal{F}(\mathbb{L})} w^{(c)}(\mathbf{L}). \tag{5}$$

*Remark.* It can be easily verified that (5) is indeed a probability distribution, and that the probability density $p$ in (4) integrates to 1. Hence the generalized labeled multi-Bernoulli RFSs cover labeled multi-Bernoulli RFSs discussed above [41]. In particular the labeled multi-Bernoulli RFS described in (3) is a special case of the generalized labeled multi-Bernoulli RFS with:

$$\begin{aligned} p^{(c)}(x,\ell) &= s^{(\ell)}(x), \\ w^{(c)}(\mathbf{L}) &= \prod_{i=1}^{n_{MBer}} \left(1 - r^{(\ell_i)}\right) \prod_{j=1}^{n} \frac{1_{\mathbf{L}}(\ell_j) r^{(\ell_j)}}{1 - r^{(\ell_j)}}. \end{aligned}$$

### *B. Sequential Markov Chain Monte Carlo*

Several approaches combining SMC with MCMC methods have already been proposed in the literature. For instance, MCMC kernels have been used to design efficient regularization steps in order to rejuvenate degenerate particles in the SIR-PF [22]. Recently, sequential MCMC approaches were proposed in [1], [25], [35]. These methods are distinct from the resample-move scheme [22] since an SMC algorithm is used to design efficient high-dimensional proposal distributions for an MCMC sampler. In other words the inefficient importance sampling step of the standard SIR-PF implementation is replaced by a more efficient MCMC sampling step. These methods allow to design effective MCMC algorithms in complex scenarios where standard strategies failed.

#### *1) Sequential Markov Chain Monte Carlo:*

Within the Bayesian estimation framework, we aim at computing the filtering PDF $f_k(\mathbf{X}_k|\mathbf{z}_{1:k})$ recursively by

$$f_k(\mathbf{X}_k|\mathbf{z}_{1:k}) \propto \int \vartheta_k(\mathbf{z}_k|\mathbf{X}_k)\ \Pi_{k|k-1}(\mathbf{X}_k|\mathbf{X}_{k-1})\ f_{k-1}(\mathbf{X}_{k-1}|\mathbf{z}_{1:k-1})\delta\mathbf{X}_{k-1}, \tag{6}$$

where $\Pi_{k|k-1}(\mathbf{X}_k|\mathbf{X}_{k-1})$ is the multi-target transition density and $\vartheta_k(\mathbf{z}_k|\mathbf{X}_k)$ is the multi-target likelihood. Let us represent the density $f_{k-1}(\mathbf{X}_{k-1}|\mathbf{z}_{1:k-1})$ by a set of unweighted particles,

$$f_{k-1}(\mathbf{X}_{k-1}|\mathbf{z}_{1:k-1}) \approx \frac{1}{N}\sum_{i=1}^{N} \delta_{\mathbf{X}_{k-1}^{(i)}}(\mathbf{X}_{k-1}), \tag{7}$$

where $N$ is the number of particles and $(i)$ the particle index. Then, by plugging this particle approximation into eq.(6)

$$f_k(\mathbf{X}_k|\mathbf{z}_{1:k}) \approx \frac{1}{N}\vartheta_k(\mathbf{z}_k|\mathbf{X}_k)\sum_{i=1}^{N} \Pi(\mathbf{X}_k|\mathbf{X}_{k-1}^{(i)}). \tag{8}$$

In [25] a Metropolis Hastings (M-H) algorithm is designed in a sequential setting in order to approximate the filtering distribution eq.(6). This is achieved by using the approximate posterior eq.(8) as the target distribution with a proposal distribution $q(\cdot|\mathbf{X}_k^{n_{M-H}})$. The acceptance ratio is then given by:

$$\begin{aligned} \alpha &= \min\left(1, \frac{f_k(\mathbf{X}_k^*|\mathbf{z}_{1:k})}{f_k(\mathbf{X}_k^{n_{M-H}}|\mathbf{z}_{1:k})}\ \frac{q(\mathbf{X}_k^{n_{M-H}}|\mathbf{X}_k^*)}{q(\mathbf{X}_k^*|\mathbf{X}_k^{n_{M-H}})}\right) \qquad (9) \\ &= \min\left(1, \frac{p(\mathbf{X}_k^*)\ \vartheta_k(\mathbf{z}_k|\mathbf{X}_k^*)}{p(\mathbf{X}_k^{n_{M-H}})\ \vartheta_k(\mathbf{z}_k|\mathbf{X}_k^{n_{M-H}})} \frac{q(\mathbf{X}_k^{n_{M-H}}|\mathbf{X}_k^*)}{q(\mathbf{X}_k^*|\mathbf{X}_k^{n_{M-H}})}\right). \qquad (10) \end{aligned}$$

The desired approximation $\hat{f}_k(\mathbf{X}_k|\mathbf{z}_{1:k})$ is obtained by storing every $N_{thin}^{th}$ accepted sample after the initial burn-in iterations.

To avoid numerical integration of the predictive density at every MCMC iteration, an alternative algorithm has been developed [35]. This algorithm, which considers the joint posterior distribution $f(\mathbf{X}_k, \mathbf{X}_{k-1}|\mathbf{z}_{1:k})$, involves a joint M-H proposal step where both $\mathbf{X}_k$ and $\mathbf{X}_{k-1}$ are updated jointly, as well as individual refinement Metropolis-within-Gibbs steps where $\mathbf{X}_k$ and $\mathbf{X}_{k-1}$ are updated individually. Furthermore, most of the methodologies developed in the SMC setting can be directly reapplied here. This includes Rao-Blackwellisation techniques to reduce the dimensionality of the target distributions [17], [27] or auxiliary particle-type methods to build distributions biased toward "promising" regions of the space [24], [32].

*2) Interacting Population based MCMC-PF:*

In [6], an alternative algorithm well suited to deal with Multitarget tracking problems for a given cardinality has been derived from the Marginalized MCMC-Based Particle [25] method. The proposed algorithm, detailed in [6], denoted by IP-MCMC-PF, is based on parallel usage of multiple population-based M-H samplers and incorporates an interaction procedure for producing improved proposals, thus fully exploits the speed and the parallelism of modern computing architectures and resources. Finally, notions from genetic algorithms and simulated annealing were considered as various sampling improvement strategies.

In the described algorithm, a validation test was performed to check the convergence on the basis of Gelman and Rubin [21] diagnostic. This approach consists, for each state parameter, of three steps: first computing the variance of the samples from each chain (after discarding burn-in); second averaging these within-chain variances; and third comparing this to the variances of all the chains mixed together via the potential scale reduction factor [12], $\hat{\mathbf{R}}$. The parallel chains were considered well-mixed when $\hat{\mathbf{R}}$ is less or equal than $1.1$ for all parameters. Once the set of chains have reached approximate convergence, the M-H sampling chains outputs all mixed together give the new set of particles $\{\mathbf{X}_k^{(i)}\}_{i=1}^N$ which approximates the target distribution $f_k(\mathbf{X}_k|\mathbf{z}_{1:k})$.

## III. Labeled Multi-target Bayes Recursion

Let us consider the multi-target filtering in TBD context, and derive from the RFS formalism an optimal Bayes filter that subsumes a number of popular multi-target Bayesian filtering approaches. For the specific TBD surveillance application, the multi-target estimation problem can be formulated in a Bayesian framework by modeling the hidden set of states $\mathbf{X}_k$ as labeled finite set. In this case computing the posterior distribution of the labeled RFS of states given the raw measurement $\mathbf{z}_k$ requires solving a set integral only for the Chapman Kolmogorov equation. The multi-target likelihood is used directly to update each target.

### A. Labeled Multi-target Bayes Recursion

Each target is identified by a unique label $\ell_k = (k, j)$, with $k$ the time of birth and $j \in \mathbb{N}$ an index to distinguish targets born at the same time. Let us consider the following notations for the label spaces:

- $\mathbb{B}_k$ denotes the label space for targets born at time $k$, i.e.

$$\mathbb{B}_k \triangleq \{k\} \times \mathbb{N}.$$

  Thus a target born at time $k$ has state $\mathbf{x}_k^\ell \in \mathbb{X} \times \mathbb{B}_k$.
- $\mathbb{L}_{0:k}$ denotes the label space for targets born up to and including time $k$ and is constructed recursively by:

$$\mathbb{L}_{0:k} \triangleq \mathbb{L}_{0:k-1} \cup \{k\} \times \mathbb{N}.$$

  Thus the multi-target state $\mathbf{X}_k^\ell$ at time $k$ is a finite subset of $\mathbb{X} \times \mathbb{L}_{0:k}$.

Note that $\mathbb{L}_{0:k-1}$ and $\mathbb{B}_k$ are disjoint.

Suppose that at time $k$ there are $n_k$ target states $\{\mathbf{x}^\ell_{k,1}, \ldots, \mathbf{x}^\ell_{k,n_k}\}$, each taking values in the (labeled) state space $\mathbb{X} \times \mathbb{L}$. In the RFS formulation [28] the finite set of targets at time $k$ is referred as the multi-target state:

$$\mathbf{X}^\ell_k = \{\mathbf{x}^\ell_{k,1}, \ldots, \mathbf{x}^\ell_{k,n_k}\} \in \mathcal{F}(\mathbb{X} \times \mathbb{L}),$$

where $\mathcal{F}(\mathbb{X} \times \mathbb{L})$ is the multi-target state space, i.e. the finite subsets of $\mathbb{X} \times \mathbb{L}$. In TBD the multi-target observation at time $k$ is an array $\mathbf{z}_k = [\mathbf{z}^1_k, \ldots, \mathbf{z}^m_k] \in \mathbb{Z}$, where each $\mathbf{z}^c_k \in \mathbb{C}$ is the complex (I/Q) signal in the radar cell indexed by $c$.

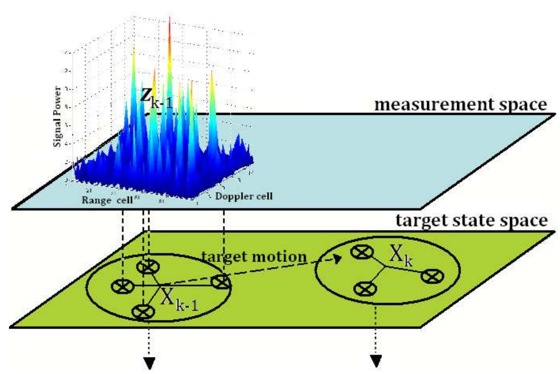
Let $\mathbf{z}_{1:k} = \{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ denote the set of measurements collected up to and including time $k$. Then, the labeled multi-target Bayes recursion propagates the multi-target posterior density $f_k(.|\mathbf{z}_{1:k})$ in time according to the following update and prediction:

$$\begin{aligned} &\text{PREDICTION:} \\ f_{k|k-1}(\mathbf{X}^\ell_k|\mathbf{z}_{1:k-1}) &= \int \Pi_{k|k-1}(\mathbf{X}^\ell_k|\mathbf{X}^\ell_{k-1})\ f_{k-1}(\mathbf{X}^\ell_{k-1}|\mathbf{z}_{1:k-1})\delta\mathbf{X}^\ell_{k-1}, \quad (11) \\ &\text{UPDATE:} \\ f_k(\mathbf{X}^\ell_k|\mathbf{z}_{1:k}) &= \frac{\vartheta_k(\mathbf{z}_k|\mathbf{X}^\ell_k)\ f_{k|k-1}(\mathbf{X}^\ell_k|\mathbf{z}_{1:k-1})}{\int \vartheta_k(\mathbf{z}_k|\mathbf{X}^\ell)\ f_{k|k-1}(\mathbf{X}^\ell|\mathbf{z}_{1:k-1})\delta\mathbf{X}^\ell}, \quad (12) \end{aligned}$$

where the integrals above are set integrals, $\Pi_{k|k-1}(.|.)$ is the labeled RFS multi-target transition kernel, from time $k-1$ to $k$, and $\vartheta_k(.|.)$ denote the labeled RFS multi-target likelihood function at time $k$.

### *B. Multi-target Dynamic Model: Prediction*

The following is a specification of the standard RFS multi-target dynamical model considered in this memo. More general dynamical models are available within the RFS framework. However, these are not falling into the scope of this memo.




$$f_{k-1}(\mathbf{X}^\ell_{k-1}|\mathbf{z}_{1:k-1}) \xrightarrow{prediction} \underbrace{f_{k|k-1}(\mathbf{X}^\ell_k|\mathbf{z}_{1:k-1})}_{\int \Pi_{k|k-1}(\mathbf{X}^\ell_k|\mathbf{X}^\ell_{k-1})\ f_{k-1}(\mathbf{X}^\ell_{k-1}|\mathbf{z}_{1:k-1})\delta\mathbf{X}^\ell_{k-1}}$$

Let $\mathbb{L}_{k-1}$ denote the label space at time $k-1$, $\mathbb{B}_k$ the label space for targets born at time $k$, $\mathbb{S}_k \subseteq \mathbb{L}_{k-1}$ the label space for targets surviving from time $k-1$ to $k$ and $\mathbb{L}_k$ the label space at time $k$ such that $\mathbb{L}_{k-1} \cap \mathbb{B}_k = \emptyset$ and $\mathbb{L}_k = \mathbb{S}_k \cup \mathbb{B}_k$. The multi-target dynamical model is described by the Markov multi-target transition kernel $\Pi_{k|k-1}(\mathbf{X}^\ell_k|\mathbf{X}^\ell_{k-1})$ involving thinning of disappearing targets, Markov shifts of surviving targets and superposition of new targets.

Given the multi-target state $\mathbf{X}^\ell_{k-1}$ at time $k-1$, each $(\mathbf{x}_{k-1}, \ell_{k-1}) \in \mathbf{X}^\ell_{k-1}$ either continues to exist at time $k$ with probability $p_{S,k}(\mathbf{x}_{k-1}, \ell_{k-1})$ and moves to a new state $(\mathbf{x}_k, \ell_k)$ with probability density $\pi_{k|k-1}(\mathbf{x}_k|\mathbf{x}_{k-1}, \ell_{k-1})\ \delta_{\ell_{k-1}}(\ell_k)$, or dies with probability $q_{S,k}(\mathbf{x}_{k-1}, \ell_{k-1}) = 1 - p_{S,k}(\mathbf{x}_{k-1}, \ell_{k-1})$.

Thus, assuming that, conditional on $\mathbf{X}_{k-1}^{\ell}$, the transition of the target kinematic states are mutually independent, then the set $\mathbf{X}_{S}^{\ell} \subset \mathbf{X}_{k}^{\ell}$ of surviving targets at time $k$ is modeled by:

- a LABELED MULTI-BERNOULLI RFS $S_{k|k-1}(\mathbf{x}_{k-1}, \ell_{k-1})$ with parameter set:

$$\{r^{(\mathbf{x}_{k-1},\ell_{k-1})}, s^{(\mathbf{x}_{k-1},\ell_{k-1})} : (\mathbf{x}_{k-1}, \ell_{k-1}) \in \mathbf{X}_{k-1}^{\ell}\}$$

  with $r^{(\mathbf{x}_{k-1},\ell_{k-1})} = p_{S,k}(\mathbf{x}_{k-1}, \ell_{k-1})$ and $s^{(\mathbf{x}_{k-1},\ell_{k-1})}(\cdot) = \pi_{k|k-1}(\cdot|\mathbf{x}_{k-1}, \ell_{k-1})$.
  The labeling function $\alpha : \mathbf{X}_{k-1}^{\ell} \to \mathbb{L}_{0:k-1}$ is defined by $\alpha(\mathbf{x}_{k-1}, \ell_{k-1}) = \ell_{k-1}$.

*Remark.* Note that the label of the target is preserved in the transition, only the kinematic part of state changes.

Hence $\mathbf{X}_{S}^{\ell}$, the set of surviving targets at time $k$, is distributed according to:

$$\boxed{\Pi_S(\mathbf{X}_S^{\ell}|\mathbf{X}_{k-1}^{\ell}) = \Delta(\mathbf{X}_S^{\ell})\ 1_{\mathcal{L}(\mathbf{X}_{k-1}^{\ell})}(\mathcal{L}(\mathbf{X}_S^{\ell}))\ \left[\Phi_{k|k-1}(\mathbf{X}_S^{\ell};\cdot)\right]^{\mathbf{X}_{k-1}^{\ell}},} \tag{13}$$

where,

$$\begin{aligned}
\Phi_{k|k-1}(\mathbf{X}_S^{\ell};\mathbf{x}_{k-1},\ell_{k-1}) &= \sum_{(\mathbf{x}_k,\ell_k)\in\mathbf{X}_S^{\ell}} \delta_{\ell_{k-1}}(\ell_k) p_{S,k}(\mathbf{x}_{k-1},\ell_{k-1})\pi_{k|k-1}(\mathbf{x}_k|\mathbf{x}_{k-1},\ell_{k-1}) \\
&\quad + \left(1 - 1_{\mathcal{L}(\mathbf{X}_S^{\ell})}(\ell_{k-1})\right) q_{S,k}(\mathbf{x}_{k-1},\ell_{k-1}) \\
&= \begin{cases} p_{S,k}(\mathbf{x}_{k-1},\ell_{k-1})\pi_{k|k-1}(\mathbf{x}_k|\mathbf{x}_{k-1},\ell_{k-1}), & \text{if } (\mathbf{x}_{k-1},\ell_{k-1}) \in \mathbf{X}_S^{\ell}; \\ q_{S,k}(\mathbf{x}_{k-1},\ell_{k-1}), & \text{if } \ell_{k-1} \notin \mathcal{L}(\mathbf{X}_S^{\ell}). \end{cases}
\end{aligned}$$

In addition, the set $\mathbf{X}_B^{\ell} \subset \mathbf{X}_k^{\ell}$ of new born targets, assumed independent of the surviving targets, can be modeled by a labeled RFS with label space $\mathbb{B}_k$. We consider the birth density to be of the following form:

$$\boxed{\Pi_B(\mathbf{X}_B^{\ell}) = \Delta(\mathbf{X}_B^{\ell})\ w_B(\mathcal{L}(\mathbf{X}_B^{\ell}))\ \left[\Phi_{B,k}(\cdot)\right]^{\mathbf{X}_B^{\ell}}.} \tag{14}$$

This birth model (14) covers both:

1) LABELED POISSON RFS with intensity function $\psi_k(.) = \mu_b\ b_{k|k-1}(.)$:

$$\begin{aligned}
w_B(\mathcal{L}(\mathbf{X}_B^{\ell})) &= \frac{e^{-\lambda}\lambda^n}{n!} \text{ with } \lambda \triangleq \int \psi_k(x)dx \text{ and } n \triangleq |\mathbf{X}_B^{\ell}|, \\
\Phi_{B,k}(x,\ell) &= \frac{\psi_k(x)}{\int \psi_k(x)dx}.
\end{aligned}$$

   Hence, the birth density is given by:

$$\Pi_B(\mathbf{X}_B^{\ell}) = \delta_{\mathbb{B}_k(|\mathbf{X}_B^{\ell}|)}(L(\mathbf{X}_B^{\ell}))\ \frac{e^{-\int \psi_k(x)dx}}{|\mathbf{X}_B^{\ell}|!} \prod_{(x,l)\in\mathbf{X}_B^{\ell}} \psi_k(x). \tag{15}$$

2) LABELED MULTI-BERNOULLI RFS with the multi-Bernoulli parameters $\{p_{B,k}(l), b_{k|k-1}^{(l)}(.)\}_{l\in\mathbb{B}_k}$

$$\begin{aligned}
w_B(\mathcal{L}(\mathbf{X}_B^{\ell})) &= \prod_{l\in\mathbb{B}_k} (1 - p_{B,k}(l)) \prod_{\ell\in\mathcal{L}(\mathbf{X}_B^{\ell})} \frac{1_{\mathbb{L}_k}(\ell)p_{B,k}(\ell)}{1-p_{B,k}(\ell)}, \\
\Phi_{B,k}(x,\ell) &= b_{k|k-1}^{(\ell)}(x).
\end{aligned}$$

Hence, the birth density is given by:

$$\Pi_B(\mathbf{X}_B^\ell) = \delta_{\mathbb{B}_k(|\mathbf{X}_B^\ell|)}(L(\mathbf{X}_B^\ell)) \prod_{l\in\mathbb{B}_k}(1-p_{B,k}(l)) \prod_{\ell\in\mathcal{L}(\mathbf{X}_B^\ell)} \frac{p_{B,k}(\ell)b_{k|k-1}^{(\ell)}(x)}{1-p_{B,k}(\ell)}. \quad (16)$$

Thus, given a realization $\mathbf{X}_{k-1}^\ell$ of the multi-target state at time $k-1$, the multi-target state $\mathbf{X}_k^\ell$ at time $k$ is given by the superposition of the sets of surviving and new born targets $\mathbf{X}_k^\ell = \mathbf{X}_S^\ell \cup \mathbf{X}_B^\ell$.

The LABELED RFS MULTI-TARGET TRANSITION KERNEL is the convolution:

$$\boxed{\Pi_{k|k-1}(\mathbf{X}_k^\ell|\mathbf{X}_{k-1}^\ell) = \sum_{\mathbf{X}_S^\ell\subseteq\mathbf{X}_k^\ell} \Pi_S(\mathbf{X}_S^\ell|\mathbf{X}_{k-1}^\ell)\ \Pi_B(\mathbf{X}_k^\ell \setminus \mathbf{X}_S^\ell).} \quad (17)$$

The labeled RFS multi-target transition equation (17) incorporates target motion, birth and death.

*Remark.* Note that since the label space $\mathbb{B}_k$ of the birth targets and the label space $\mathbb{L}_{0:k-1}$ of the surviving targets are mutually exclusive, the superposition of both sets of targets is indeed a labeled RFS.

**Proposition III.1.** *If the prior distribution is a generalized labeled multi-Bernoulli RFS then under the labeled RFS multi-target transition kernel (17), the predicted distribution is also a generalized labeled multi-Bernoulli RFS:*

$$\boxed{\Pi_{k|k-1}(\mathbf{X}_k^\ell|\mathbf{X}_{k-1}^\ell) = \Delta(\mathbf{X}_k^\ell)\sum_{c\in\mathbb{C}} w_k^{(c)}(\mathcal{L}(\mathbf{X}_k^\ell))\left[p_k^{(c)}(\cdot)\right]^{\mathbf{X}_k^\ell},} \quad (18)$$

where,

- the weight $w_k^{(c)}(\mathcal{L}(\mathbf{X}_k^\ell))$ is the product of the weight $w_B(\mathcal{L}(\mathbf{X}_B^\ell))$ of the birth labels and the weight $w_S^{(c)}(\mathcal{L}(\mathbf{X}_S^\ell))$ of the surviving labels, i.e.

  $$w_k^{(c)}(\mathcal{L}(\mathbf{X}_k^\ell)) = w_B(\mathcal{L}(\mathbf{X}_B^\ell))w_S^{(c)}(\mathcal{L}(\mathbf{X}_S^\ell)),$$

  where,

  $$w_S^{(c)}(L_S) = \left[\eta_S^{(c)}(\cdot)\right]^{L_S} \sum_{I\subseteq\mathcal{L}(\mathbf{X}_{k-1}^\ell)} 1_I(L_S)w_{k-1}^{(c)}(I)\left[q_S^{(c)}(\cdot)\right]^{I\setminus L_S},$$

  with,

  $$q_s^{(c)}(\ell) = \int q_{S,k}(\mathbf{x}_{k-1},\ell)\ p_{k-1}^{(c)}(x,\ell)dx,$$

  and

  $$\eta_S^{(c)}(\ell) = \int\left(\int p_{S,k}(\mathbf{x}_{k-1},\ell)\ \pi_{k|k-1}(x|\mathbf{x}_{k-1},\ell)\ p_{k-1}^{(c)}(\mathbf{x}_{k-1},\ell)d\mathbf{x}_{k-1}\right)dx.$$

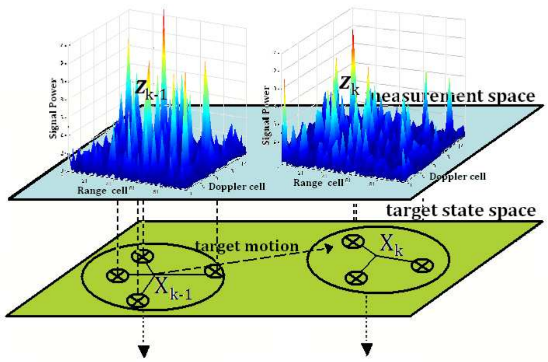
  The survival set weight $w_S^{(c)}(L_S)$ involves a weighted sum of the prior weights $w_{k-1}^{(c)}(\cdot)$ over all label sets that contains the surviving set $L_S$.
- the predicted single-target density for a given label $p_k^{(c)}(\cdot,\ell)$ is either the density $\Phi_{B,k}(\cdot,\ell)$ of a newly born target or the density of a surviving target computed from the prior density $p_{k-1}^{(c)}(\cdot,\ell)$ via the single-target prediction with transition $\pi_{k|k-1}(\cdot|\cdot,\ell)$ weighted by the probability of survival $p_{S,k}(\cdot,\ell)$, i.e.

  $$\begin{aligned} p_k^{(c)}(x,\ell) &= 1_{\mathcal{L}(\mathbf{X}_S^\ell)}(\ell)\Phi_{S,k}^{(c)}(x,\ell) + 1_{\mathcal{L}(\mathbf{X}_B^\ell)}(\ell)\Phi_{B,k}(\cdot,\ell), \\ \Phi_{S,k}^{(c)}(x,\ell) &= \frac{\int p_{S,k}(y,\ell)\ \pi_{k|k-1}(x|y,\ell)\ p_{k-1}^{(c)}(y,\ell)dy}{\eta_S^{(c)}(\ell)}, \end{aligned}$$

  with the normalization constant $\eta_s^{(c)}(\ell)$.

### C. Multi-target Observation Model: Update

The following is a specification of the RFS multi-target observation model in the TBD context. The TBD approach defines a model for the raw measurement in terms of a multi-target state hypothesis. In this study, the target return signals measured by the radar are assumed to fluctuate according to the Swerling return amplitude fluctuation models [38]. These models are widely used as a means of estimating radar detection performance [29], [36]. Here the Swerling fluctuation models are incorporated into the likelihood function of the filter, to account for the target return fluctuations.




$$f_{k|k-1}(\mathbf{X}_k^\ell|\mathbf{z}_{1:k-1}) \xrightarrow{\;update\;} \underbrace{f_k(\mathbf{X}_k^\ell|\mathbf{z}_{1:k})}_{\frac{\vartheta_k(\mathbf{z}_k|\mathbf{X}_k^\ell) f_{k|k-1}(\mathbf{X}_k^\ell|\mathbf{z}_{1:k-1})}{\int \vartheta_k(\mathbf{z}_k|\mathbf{X}_k^\ell) f_{k|k-1}(\mathbf{X}_k^\ell|\mathbf{z}_{1:k-1}) \delta \mathbf{X}_k^\ell}} \tag{19}$$

For the specific TBD surveillance application, the labeled multi-target likelihood, i.e. the probability density of the observation $\mathbf{z}_k$ conditioned on the labeled multi-target state $\mathbf{X}_k^\ell$, is used directly to update each target. Note that:

$$\boxed{\vartheta_k(\mathbf{z}_k|\mathbf{X}_k^\ell) = \vartheta_k(\mathbf{z}_k|\mathbf{X}_k).} \tag{20}$$

A given target $\mathbf{x}_{k,j}^\ell \in \mathbf{X}_k^\ell$, at time $k$, illuminates a set of cells denoted by $C(\mathbf{x}_{k,j}^\ell)$. For example, $C(\mathbf{x}_{k,j}^\ell)$ could be the set of cells whose centers fall within a certain distance from the position of the target. The complex (I/Q) signal in each cell of the observation image at time $k$, $\mathbf{z}_k^c$, is assumed to be:

$$\mathbf{z}_k^c = \begin{cases} \sum_{j:c\in C(\mathbf{x}_{k,j}^\ell)} h^c(\mathbf{x}_{k,j}^\ell) + \mathbf{w}_k^c, & \mathcal{H}_1\text{: if at least one target illuminates the cell } c; \\ \mathbf{w}_k^c, & \mathcal{H}_0\text{: if there is no target;} \end{cases} \tag{21}$$

where $h^c(\mathbf{x}_{k,j}^\ell)$, the contribution of a target with state $\mathbf{x}_{k,j}^\ell$ to the signal in cell $c$, depends on the so-called sensor point spread function (in this case pulse form), the target location and the complex target echo. $\mathbf{w}_k^c$ is the measurement noise in cell $c$ with known statistics. The noise is assumed to be independent from cell to cell and identically distributed according to the PDF $\varphi^c$.

Let the measurement likelihood in cell $c$ in the presence of at least one target be denoted by $\phi^c(\mathbf{z}_k^c|\mathbf{X}_k^\ell)$, and and the likelihood under the hypothesis of no targets be $\phi^c(\mathbf{z}_k^c)$. Since conditioned on the multi-target state $\mathbf{X}_k^\ell$, the values of the cells are independently distributed, the multi-target likelihood of the whole observation $\mathbf{z}_k$ is simply the product over the cells given by:

$$\vartheta_k(\mathbf{z}_k|\mathbf{X}_k^\ell) = \left( \prod_{c\in C(\mathbf{x}):\mathbf{x}\in\mathbf{X}_k^\ell} \phi^c(\mathbf{z}_k^c|\mathbf{X}_k^\ell) \right) \left( \prod_{c\notin \bigcup\limits_{\mathbf{x}\in\mathbf{X}_k^\ell} C(x)} \varphi^c(\mathbf{z}_k^c) \right). \tag{22}$$

Hence,

$$\vartheta_k(\mathbf{z}_k|\mathbf{X}_k^\ell) = f(\mathbf{z}_k) \prod_{\mathbf{x}\in\mathbf{X}_k^\ell} g_{\mathbf{z}_k}(x), \quad (23)$$

where,

$$f(\mathbf{z}_k) = \prod_{c=1}^{m} \varphi^c(\mathbf{z}_k^c), \quad (24)$$

$$g_{\mathbf{z}_k}(\mathbf{x}) = \prod_{c\in C(\mathbf{x}):\mathbf{x}\in\mathbf{X}_k^\ell} \mathfrak{L}(\mathbf{z}_k^c|\mathbf{X}_k^\ell), \quad (25)$$

with $\mathfrak{L}(\mathbf{z}_k^c|\mathbf{X}_k^\ell)$, the likelihood ratio for cell $c$ given by:

$$\mathfrak{L}(\mathbf{z}_k^c|\mathbf{X}_k^\ell) = \frac{\phi^c(\mathbf{z}_k^c|\mathbf{X}_k^\ell)}{\varphi^c(\mathbf{z}_k^c)}. \quad (26)$$

In many cases it is more convenient to deal with the likelihood ratio of the data, rather than the measurement PDF.

**Example III.1.** Consider a radar system, located at the Cartesian origin, that collects, at discrete instants $k$, a measurement $\mathbf{z}_k$. Assuming that the measurement $\mathbf{z}_k = [\mathbf{z}_k^1 \ \dots \ \mathbf{z}_k^m]^T$ consists of the power signal for all $m$ range-Doppler-bearing cells. The power measurements per (range-Doppler-bearing) cell is given by:

$$\mathbf{z}_k^c = |\mathbf{z}_{A,k}^c|^2 \qquad k \in \mathbb{N},$$

where $\mathbf{z}_{A,k}^c$ represents the complex (I/Q) signal in cell $c$, i.e.

$$\mathbf{z}_{A,k}^c = \sum_{\mathbf{x}\in\mathbf{X}_k:c\in C(\mathbf{x})} A(\mathbf{x}) \ h_A(\mathbf{x}) + \mathbf{w}_k,$$

where $A(\mathbf{x})$ is the complex echo of the target of state $\mathbf{x}$, $h_A(\mathbf{x})$ represents the point spread function for a target of state $\mathbf{x}$ and $\mathbf{w}_k$ is the measurement noise.

In particular, assume there is $n_k$ closely spaced targets, then the complex (I/Q) signal is given by:

$$\mathbf{z}_{A,k} = \sum_{j=1}^{n_k} A(\mathbf{x}_{k,j}) \ h_A(\mathbf{x}_{k,j}) + \mathbf{w}_k.$$

Here,

- $A(\mathbf{x}_{k,j})$, the complex echo of the target $j$, is modeled by:

  $$A(\mathbf{x}_{k,j}) = \bar{A}(\mathbf{x}_{k,j}) \ e^{\iota\varphi_k} + \mathbf{a}_k^{(j)},$$

  with $\bar{A}(\mathbf{x}_{k,j})$ a known amplitude, $\varphi_k$ an unknown phase, uniformly distributed on $[0, 2\pi)$, and $\mathbf{a}_k^{(j)}$ a zero-mean complex Gaussian variable with variance $\sigma^2_{\mathbf{a}_k^{(j)}}$.
- $h_A(\mathbf{x}_{k,j}) = [h_A(\mathbf{x}_{k,j})^1 \dots h_A(\mathbf{x}_{k,j})^m]^T$ represents the point spread function for the target of state vector $\mathbf{x}_{k,j}$,
- $\mathbf{w}_k$ is a $m$-dimensional vector representing the measurement noise assumed to be a zero mean, white complex Gaussian noise with variance $\sigma^2_{\mathbf{w}_k}$.

Under the assumption of a non-fluctuating target model (Swerling 0), the multi-target observation likelihood $\phi^c(z^c|\mathbf{X})$ is a non-central chi-squared distribution and reduces to an exponential distribution if

$z^c$ only contains clutter noise and thermal noise. In fact, in this case the complex echo of the target $j$ is modeled by:

$$A(\mathbf{x}) = \bar{A}(\mathbf{x})e^{i\theta}, \quad \theta \in [0, 2\pi) \tag{27}$$

Let $\Sigma_{h^c} := \sum_{\mathbf{x}\in\mathbf{X}:c\in C(\mathbf{x})} A(\mathbf{x})h_A(\mathbf{x})$ for notational convenience, then the power signal for cell $c$ is given by:

$$\begin{aligned} z^c &= |\Sigma_{h^c}e^{i\theta} + w^c|^2 \\ &= (\Sigma_{h^c}\cos(\theta) + \Re(w^c))^2 + (\Sigma_{h^c}\sin(\theta) + \Im(w^c))^2 \\ &= U_R^2 + U_I^2 \end{aligned}$$

where $U_R \sim \mathcal{N}(\Sigma_{h^c}\cos(\theta), \sigma^2_{w^c}/2)$ and $U_I \sim \mathcal{N}(\Sigma_{h^c}\sin(\theta), \sigma^2_{w^c}/2)$ are statistically independent normal random variables. Then $\sqrt{U_R^2 + U_I^2}$ has a Ricean distribution, and reduces to a Rayleigh distribution when $\Sigma_{h^c} = 0$. Thus, the measurement likelihood is given by:

$$\phi^c(z^c|\mathbf{X}) = \frac{1}{\sigma^2_{w^c}} \exp\left(-\frac{z^c + \Sigma_{h^{(i)}}}{\sigma^2_{w^c}}\right) I_0\left(\sqrt{\frac{4z^c\Sigma_{h^c}}{\sigma^2_{w^c}}}\right) \tag{28}$$

$$\varphi^c(z^c) = \frac{1}{\sigma^2_{w^c}} \exp\left(-\frac{z^c}{\sigma^2_{w^c}}\right) \tag{29}$$

where $I_0(\cdot)$ is the modified Bessel function of zero order defined by:

$$I_0(y) := \sum_{j=0}^{\infty} \frac{\left(\frac{y^2}{4}\right)^j}{j!\ \Gamma(j+1)}.$$

## IV. Multiple Cardinality Hypotheses Particle Filter (MCHPF)

This section is organized as follows. In subsection IV-A, we give insights for our approach. Subsection IV-B introduces the proposed Multiple Cardinality Hypotheses Tracker (MCHT) and describes its SMC implementation.

### *A. Justifications behind the MCHT*

The basic strategy is to propagate a time-varying number of hypothesized tracks, which are characterized by an identity or a label. Note that care must be taken in the application of the multi-target Bayes recursion to ensure distinctness of labels. To address the uniqueness of labels the proposed algorithm is built upon the concept of labeled RFS [41], and formally incorporates the propagation and estimation of track labels within the RFS filtering framework.

The basic idea is as follows. From a given configuration of targets at time $k-1$, possible configurations at time $k$ are proposed based on the dynamical and birth models. Then, the likelihood associated to each of these scenarios given the measurement is computed. Finally, only the hypotheses with the most significant probability are retained for the next step.

A three-level tree-based structure, that allows efficient and systematic management of all the possible configurations of targets, is used. The first level, referred as "Cardinality", lists hypotheses on the number of targets. Each of these cardinality hypotheses in turn gives rise to a series of combination of targets. The second level, denoted as "Label component", expands explicitly the discrete label component of the targets. Finally, the third level, called "Track", represents as particles sets the kinematic components of the targets state. The proposed structure also leads to simple and efficient schemes for pruning of unlikely components, and helps reduce the computational cost of the recursion.

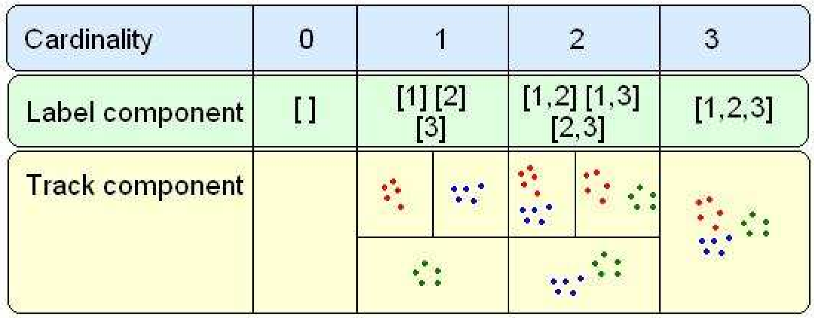



**Fig. 1:** Three-level tree-based structure.

The multi-target prediction and update steps are recursively applied to propagate the multi-target posterior forward in time on component by component basis. The multi-target dynamical model involves thinning, Markov shifts and superposition of new targets. The prediction for each existence probability is essentially the previous posterior density integrated with the survival probability, and subsequently scaled down by the previous existence probability. The prediction for each hypothesized track density is very similar to the single-target Bayes prediction, except for the incorporation of the survival probability and subsequent normalization.

**Example IV.1.** To better understand the MCHT algorithm, let us consider the following example:

1) First, suppose, at time $k-1$, there are three target configurations:
   - $\mathbf{H}_1$ A single target is present, with label $\ell_{k-1,1} = (k-2, 1)$.
   - $\mathbf{H}_2$ A single target is present, with label $\ell_{k-1,1} = (k-1, 1)$.
   - $\mathbf{H}_3$ Two targets are present, with labels $\ell_{k-1,1} = (k-2, 1)$ and $\ell_{k-1,2} = (k-1, 1)$ respectively.

   A particle set represents the kinematic components of each target configuration.
2) Let us assume the birth process is a multi-Bernoulli density $\pi_\Gamma = \{(p_{B,k}(j), b_k^{(j)})\}_{j=1}^{|\mathcal{I}_{d_k}|}$ where $b_k^{(j)}(x) = \mathcal{N}(x; m_\gamma^{(j)}, P_\gamma)$ with $m_\gamma^{(j)}$ the estimated state of the $j^{th}$ detection plot $\mathbf{x}_{k,d_j}$. In addition, consider a single detection plot. Then, at time $k$, the possible target configurations can be inferred from the dynamical and birth models.

   Under $\mathbf{H}_1$ the new target configuration can be:
   - $\mathbf{H}_0$ No target present.
   - $\mathbf{H}_1$ A single target is present, with label $\ell_{k,1} = (k-2, 1)$.
   - $\mathbf{H}_2$ A single target is present, with label $\ell_{k,1} = (k, 1)$.
   - $\mathbf{H}_3$ Two targets are present, with labels $\ell_{k,1} = (k-2, 1)$ and $\ell_{k,2} = (k, 1)$ respectively.

   Under $\mathbf{H}_2$ the new target configuration can be:
   - $\mathbf{H}_0$ No target present.
   - $\mathbf{H}_1$ A single target is present, with label $\ell_{k,1} = (k-1, 1)$.
   - $\mathbf{H}_2$ A single target is present, with label $\ell_{k,1} = (k, 1)$.
   - $\mathbf{H}_3$ Two targets are present, with labels $\ell_{k,1} = (k-1, 1)$ and $\ell_{k,2} = (k, 1)$ respectively.

   Under $\mathbf{H}_3$ the new target configuration can be:
   - $\mathbf{H}_0$ No target present.
   - $\mathbf{H}_1$ A single target is present, with label $\ell_{k,1} = (k-2, 1)$.
   - $\mathbf{H}_2$ A single target is present, with label $\ell_{k,1} = (k-1, 1)$.
   - $\mathbf{H}_3$ A single target is present, with label $\ell_{k,1} = (k, 1)$.
   - $\mathbf{H}_4$ Two targets are present, with labels $\ell_{k,1} = (k-2, 1)$ and $\ell_{k,2} = (k-1, 1)$ respectively.
   - $\mathbf{H}_5$ Two targets are present, with labels $\ell_{k,1} = (k-2, 1)$ and $\ell_{k,2} = (k, 1)$ respectively.
   - $\mathbf{H}_6$ Two targets are present, with labels $\ell_{k,1} = (k-1, 1)$ and $\ell_{k,2} = (k, 1)$ respectively.

- $\mathbf{H}_7$ Three targets are present, with labels $\ell_{k,1} = (k-2, 1)$, $\ell_{k,2} = (k-1, 1)$ and $\ell_{k,3} = (k, 1)$ respectively.

Going ahead in this way, for further steps, the tree of hypotheses grows exponentially. To ensure tractability, the predicted and updated multi-target densities should be truncated by keeping only components with the most significant probability.

*B. Particle MCHT Implementation*

The MCHT is implemented via the previously derived closed form solution. The (continuous) kinematic component of the target state, and hence the individual target tracks, are represented as sets of particles which are predicted and updated using the IP-MCMC-PF filter. The labels of the target state uniquely identify target tracks. The multi-target prediction and update steps are then calculated on component by component basis, using Murty's algorithm [31] to calculate newly predicted and updated components in descending order of probability of occurrence. For each existing component, the number of new components calculated and stored in each forward propagation is set to be proportional to the weight of the original component, subject to each cardinality retaining a minimum of $C_{min}$ terms. The resultant posterior at each time step is then further truncated to a maximum of $C_{tot,max}$ terms. Components with weights below a predefined threshold are discarded. Let $\mathfrak{T}_{k-1}$ denote the three-level tree-based structure containing $M_{k-1}$ hypothesized tracks with their probability of occurrence. Thus, the MCHPF can be summarized as Algorithm 2.

---

**Algorithm 2**: MCHPF algorithm

**input** : $\mathfrak{T}_{k-1}$ and a new measurement, $\mathbf{z}_k$
**output**: $\mathfrak{T}_k$

1 - *Generate $M$-best ranked hypotheses using Murty's algorithm:*
Set up a cost matrix defining the probability of occurrence at time $k$ of new target configurations (cardinality, and label component) given the previous hypothesized configurations, based on the previous posterior density, the survival probability, the previous existence probability. Apply Murty's algorithm to find out the $M$-best ranked hypotheses $H_m$.

2 - *Prediction Step:*
```
for m ← 1 to M do
    for i ← 1 to N do
        Sample X^ℓ_{k,m} from X^ℓ_{k-1,m} according to the model evolution described by the hypothesis H_m
        see eq.(17).
    end
end
```

3 - *Update Step:*
Run $M$ IP-MCMC-PF algorithms (as given in Algorithm 3 below) in parallel; one for each of the $M$-best ranked hypothesized tracks

$\{\mathbf{X}^{\ell}_{k,m}\}_{i=1}^{N}$ = IP-MCMC-PF$\Big(\{\mathbf{X}^{\ell}_{k,m}\}_{i=1}^{N}, \mathbf{z}_k\Big)$.
Update the existence probabilities associated to each hypothesized tracks.

4 - *Pruning Step:*;
Discard the hypothesized tracks with existence probabilities below a threshold $\tau_P$, that is, $M_k$ hypothesized tracks are kept at the next time step.

---

**Algorithm 3**: IP-MCMC-PF algorithm

**input** : $\{\mathbf{X}_k^{(i)}\}_{i=1}^N$ and a new measurement, $\mathbf{z}_k$.
**output**: $\{\mathbf{X}_k^{(i)}\}_{i=1}^N$.

1 - *Initialize the* $N_{MCMC}$ *M-H samplers:*
Select a random subset $\{\mathbf{X}_s^{(0)}\}_{s=1}^{N_{\text{MCMC}}} \subset \{\mathbf{X}_k^{(i)}\}_{i=1}^N$.

Compute the associated joint log-likelihood $\ell_s^{(0)} = \log \vartheta_k(\mathbf{z}_k|\mathbf{X}_s^{(0)})$. Store the seeds $\{\mathbf{X}_s^{(0)}; \ell_s^{(0)}\}_{s=1}^{N_{\text{MCMC}}}$ in a cache.

2 - *Apply random-walk M-H sampling procedure:*
Run $s = 1, \dots, N_{\text{MCMC}}$ random-walk M-H samplers (Algorithm 4) in parallel

$$\{\mathbf{X}_s^{(l)}; \ell_s^{(l)}\}_{l=1}^{N_{\text{M-H}}} = \text{RW-MCMC}\Big(\{\mathbf{X}_s^{(0)}; \ell_s^{(0)}\}, \{\mathbf{X}_k^{(i)}\}_{i=1}^N\Big).$$

3 - *Check convergence:*
Compute $\hat{\text{R}}$ [6].
**if** $\hat{\text{R}} \geq 1.1$ **then**
  Run the chains out longer to improve convergence to the stationary distribution and increase the precision of inferences.
**else**
  Mix all the $N_{\text{MCMC}}$ sets of simulations together to obtain $\{\mathbf{X}_k^{(i)}; \ell_k^{(k)}\}_{i=1}^N$.
**end**

**Algorithm 4**: Random Walk M-H sampling procedure

**input** : $\{\mathbf{X}_s^{(0)}; \ell_s^{(0)}\}$ initial configuration and $\{\mathbf{X}_k^{(i)}\}_{i=1}^N$.
**output**: $\{\mathbf{X}_s^{(l)}; \ell_s^{(l)}\}_{l=1}^{N_{\text{M-H}}}$ with $N_{\text{M-H}} := N/N_{\text{MCMC}}$.

**for** $l \leftarrow 1$ **to** $B + N_{M\text{-}H}$ **do**
  1 - *Propose a new configuration* $\mathbf{X}_s^*$*:*
  Randomly select a partition $\mathbf{x}_{k-1,j}^{(i)}$ out of $\{\mathbf{X}_{k-1}^{(i)}\}_{i=1}^N$.
  Given $\mathbf{x}_{k-1,j}^{(i)}$, draw $\mathbf{x}_{s,j}^* = \mathbf{x}_{k-1,j}^{(i)} + \epsilon,\ \epsilon \sim \mathcal{N}(0, \sigma_{RW}^2)$.
  Propose new configuration $\mathbf{X}_s^*$, such that
  $\mathbf{X}_s^* = \{\mathbf{x}_{s,1}^{(l-1)}, \dots, \mathbf{x}_{s,j-1}^{(l-1)}, \mathbf{x}_{s,j}^*, \mathbf{x}_{s,j+1}^{(l-1)}, \dots, \mathbf{x}_{s,n_k}^{(l-1)}\}$.
  Compute the joint log-likelihood $\ell_s^* = \ell_s^{(l-1)} - \mathbf{p}\ell_{s,j}^{(l-1)} + \mathbf{p}\ell_{s,j}^*$.

  2 - *Accept/Reject:*
  Sample $u \sim \mathcal{U}_{[0,1]}$, $\mathcal{U}_{[0,1]}$ a uniform distribution in $[0, 1]$.
  Calculate the M-H acceptance ratio: $\alpha = \frac{\ell_s^*}{\ell_s^{(l-1)}}$.

  **if** $u \leq \min(1, \alpha)$ **then**
    Accept move: $\mathbf{X}_s^{(l)} = \mathbf{X}_s^*, \quad \ell_s^{(l)} = \ell_s^*$,
  **else**
    Reject move: $\mathbf{X}_s^{(l)} = \mathbf{X}_s^{(l-1)}, \quad \ell_s^{(l)} = \ell_s^{(l-1)}$.
  **end**
**end**
Discard $B$ initial burn-in samples to allow the M-H sampler to converge to the stationary distribution.
Store the remaining samples $\{\mathbf{X}_s^{(l)}; \ell_s^{(l)}\}_{l=B+1}^{B+N_{\text{M-H}}} \to \{\mathbf{X}_s^{(l)}; \ell_s^{(l)}\}_{l=1}^{N_{\text{M-H}}}$.

## V. Simulation setup and results

Demonstrations and numerical studies of the proposed implementations are now presented.

This scenario considers a nonlinear TBD problem in which a maximum of $10$ targets are present at any time. There are various target births and deaths throughout the scenario duration of $T_{max} = 50$ seconds. Targets are initialized at separate locations. The single target state $\mathbf{x}_{k,j}$ comprises only the position and velocity of the $j^{th}$ target, $\mathbf{s}_{k,j} = [x_{k,j}, \dot{x}_{k,j}, y_{k,j}, \dot{y}_{k,j}]$, and a measure of the average target return amplitude, $I_{k,j}$:

$$\mathbf{x}_{k,j} = [\mathbf{s}_{k,j}, I_{k,j}]^T \,. \tag{30}$$

A nearly constant velocity (NCV) model is adopted to describe target position and velocity $\mathbf{s}_{k,j}$. This model presumes that the target moves along a straight line with (nearly) constant speed. In addition the fluctuation of the target return amplitude $I_{k,j}$ is modeled as a random walk.
The model uncertainty is handled by the process noise $\mathbf{v}_{k,j}$, which is assumed to be standard white Gaussian noise with covariance $G$. Under these assumptions, the dynamical model is linear Gaussian given by:

$$\mathbf{x}_{k,j} = F\ \mathbf{x}_{k-1,j} + \mathbf{v}_{k-1,j}, \tag{31}$$

with the transition matrix $F$:

$$F = \begin{bmatrix} F_s & 0 & 0 \\ 0 & F_s & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ F_s = \begin{bmatrix} 1 & \Delta_T \\ 0 & 1 \end{bmatrix}, \tag{32}$$

and the process noise covariance $G$:

$$G = \begin{bmatrix} \sigma_{v_x}^2 G_s & 0 & 0 \\ 0 & \sigma_{v_y}^2 G_s & 0 \\ 0 & 0 & \sigma_i^2 \Delta_T \end{bmatrix}, \ G_s = \begin{bmatrix} \frac{\Delta_T^3}{3} & \frac{\Delta_T^2}{2} \\ \frac{\Delta_T^2}{2} & \Delta_T \end{bmatrix}, \tag{33}$$

with a fixed sampling period $\Delta_T = 1$ s and where the parameters $\sigma_{v_x} = \sigma_{v_y} = 2$ m.s$^{-2}$ and $\sigma_i = (2/3)$ dB.m$^{-2}$ denote the standard deviations of the process noise in object motion and amplitude, respectively. Furthermore, target velocities are assumed bounded. This prior information is target dependent and will be used for state initialization.

*Remark.* In general, the return amplitude will be a function of the target's distance from the sensor ($\propto \frac{1}{r_{k,j}^4}$, where $r_{k,j}$ is the range to the target). For the purpose of this discussion, the return amplitude is assumed approximately independent of the position of the target, for trajectories which are short in comparison to the distance from the sensor to the target.

The true tracks on a 2-D plane are depicted in Figure 2.

**TABLE I:** Targets initial parameters

| Target | $k_b$ | $k_d$ | SNR | $\lvert v\rvert$ | Target | $k_b$ | $k_d$ | SNR | $\lvert v\rvert$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 11 | 8 | 10 | 11 | 14 | 28 | 13 | 8 |
| 2 | 1 | 13 | 12 | 10 | 12 | 18 | 30 | 9 | 30 |
| 3 | 1 | 14 | 10 | 19 | 13 | 18 | 33 | 8 | 15 |
| 4 | 1 | 13 | 13 | 17 | 14 | 18 | 32 | 9 | 10 |
| 5 | 1 | 13 | 11 | 20 | 15 | 19 | 31 | 8 | 17 |
| 6 | 1 | 20 | 11 | 24 | 16 | 20 | 35 | 11 | 7 |
| 7 | 2 | 16 | 8 | 15 | 17 | 21 | 36 | 13 | 20 |
| 8 | 4 | 17 | 12 | 9 | 18 | 21 | 31 | 11 | 11 |
| 9 | 4 | 19 | 9 | 20 | 19 | 22 | 39 | 14 | 9 |
| 10 | 11 | 29 | 11 | 27 | 20 | 33 | 51 | 13 | 26 |

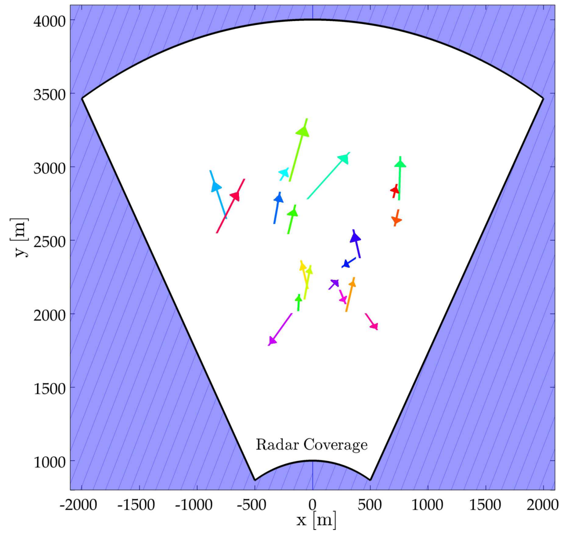



**Fig. 2:** True tracks on a 2-D plane for Scenario 5.

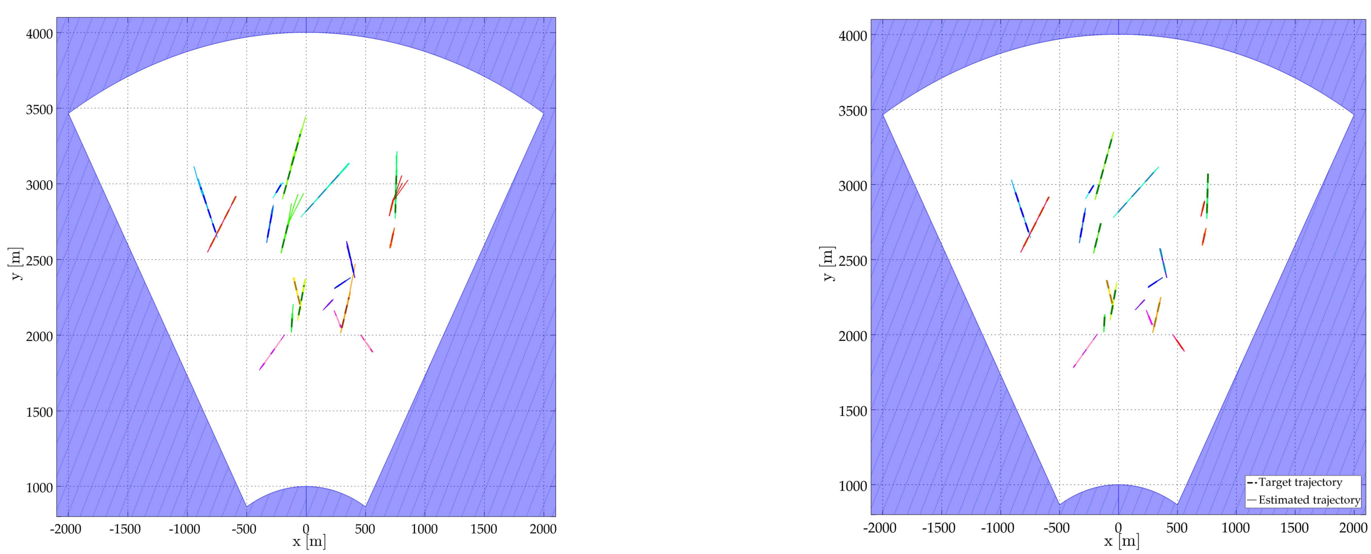
We compared the tracking accuracy and the track initiation/termination performances of the conventional (a) IP-RJMCMC-PF ( [7]) and the (b) MCHPF (Algorithm 2) over 100 Monte-Carlo simulations. For these comparisons, both algorithms are given the same initialization, so as to ensure a fair overall comparison.

The Radar parameters and the MCHPF settings used in simulation are reported in Table II.

**TABLE II:** Parameters used in simulation

| | | |
|---|---|---|
| Radar parameters | Beamwidth in bearing<br>Range-quant size<br>Doppler-bin size | $\Delta_b = 1\ [degree]$<br>$\Delta_r = 10\ [m]$<br>$\Delta_d = 2\ [m.s^{-1}]$ |
| MCHPF settings | Number of particles per track<br>Max. number of targets<br>Min. number of components per cardinality<br>Max. number of components<br>Number of best-ranked hypotheses | 500<br>$n_{max} = 15$<br>$C_{min} = 40$<br>$C_{tot,max} = 1000$<br>$M = 500$ |

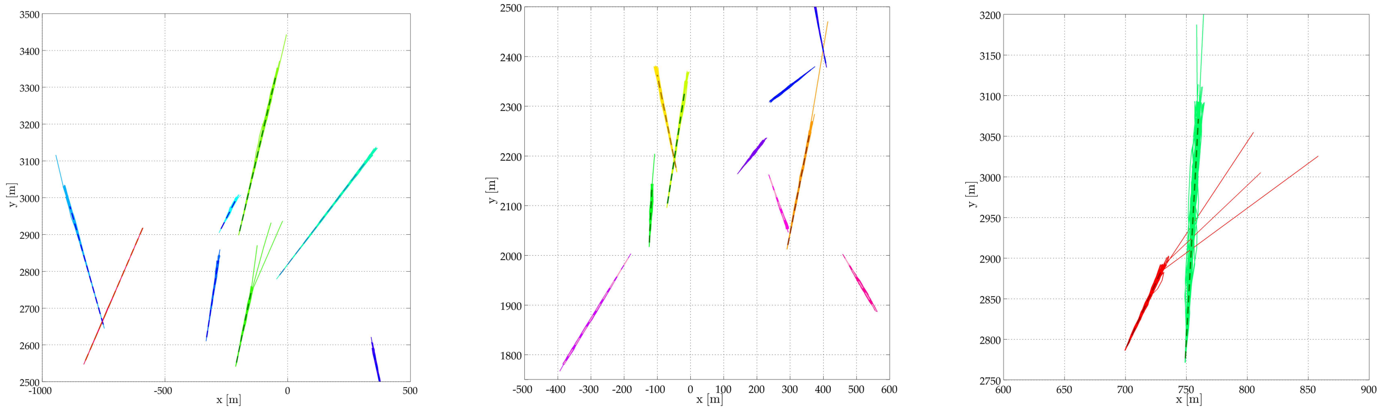
The tracks produced by each algorithm over 100 Monte Carlo trials are shown in Figure 3.




**(a)** IP-RJMCMC-PF

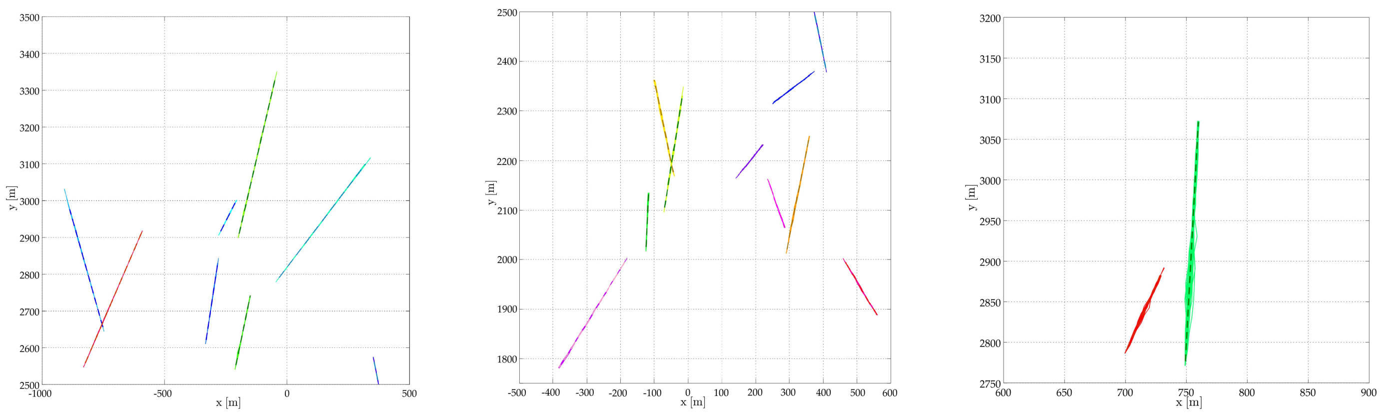
**(b)** MCHPF

**Fig. 3:** Estimated trajectories (i.e., particles-based conditional mean) over 100 Monte Carlo runs. Each filter uses $N = 1000$ particles. The proposed MCHPF provides better tracking performance than the IP-RJMCMC-PF filter.




**(a)** **(b)** **(c)**

**Fig. 4:** Zoom on tracks produced by the IP-RJMCMC-PF filter.




**(a)** **(b)** **(c)**

**Fig. 5:** Zoom on tracks produced by the MCHPF filter.

Figure 6 compares the OSPA penalty [34] $(p = 2,\ c = 100\ \text{m})$ for the two filters, further confirming that the MCHPF filter generally outperforms the IP-RJMCMC-PF filter. Figure 7 shows the OSPA localization and cardinality components.

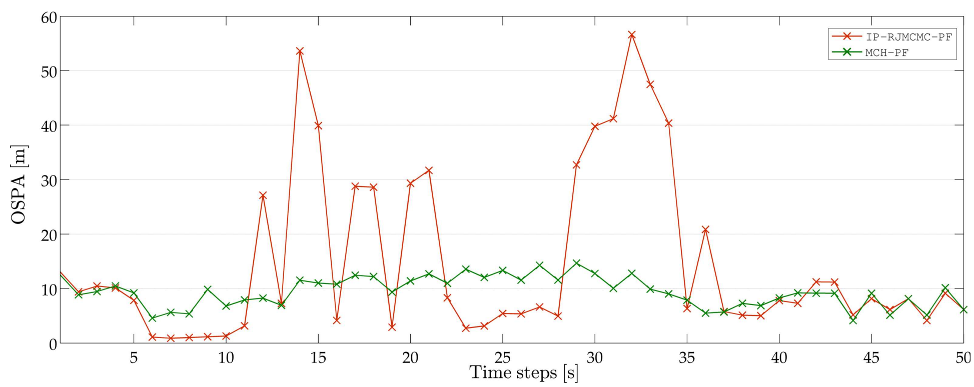



**Fig. 6:** OSPA penalty $(p = 2,\ c = 100\ \text{m})$

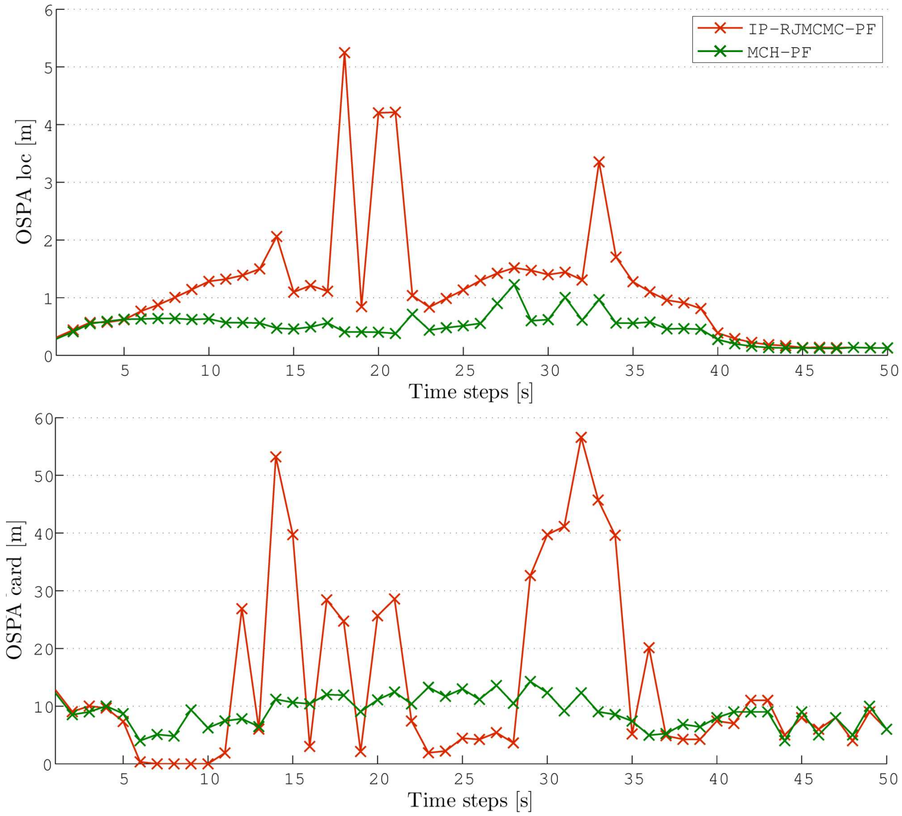
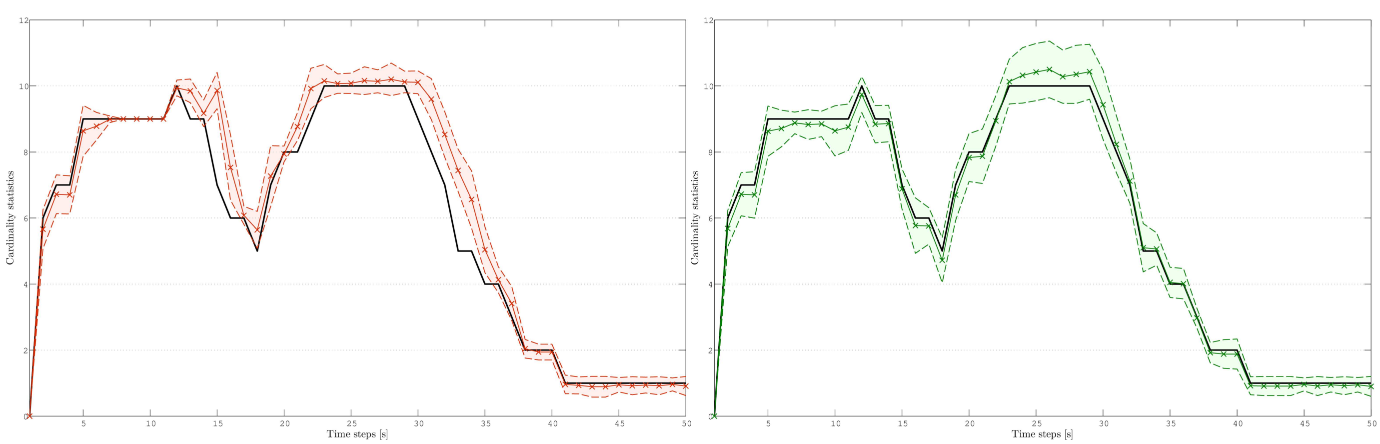



**Fig. 7:** OSPA localization and cardinality components

Preliminary results indicate that the proposed MCHPF filter performs remarkably well, and significantly outperforms the IP-RJMCMC-PF filter in both localization and cardinality estimation. State estimation is performed with the following procedure. A target is declared present if the estimated probability of existence is greater than $0.5$, otherwise no target is declared. If a target is declared, the state estimate is given by the mean of the posterior state distribution, otherwise if no target is declared, there is no state estimate.

Figure 8 reports the mean and standard deviation of the estimated cardinality distribution versus time for both filters, further confirming that the MCHPF filter outperforms the IP-RJMCMC-PF filter.




**(a)** IP-RJMCMC-PF with $N = 1000$ particles.

**(b)** MCHPF with $N = 1000$ particles.

**Fig. 8:** Cardinality statistics (mean and standard deviation) versus time. The MCHPF filter initiates and terminates tracks with a very short delay.

Finally Figure 9 shows the existence probability associated to each target versus time for the two filters.

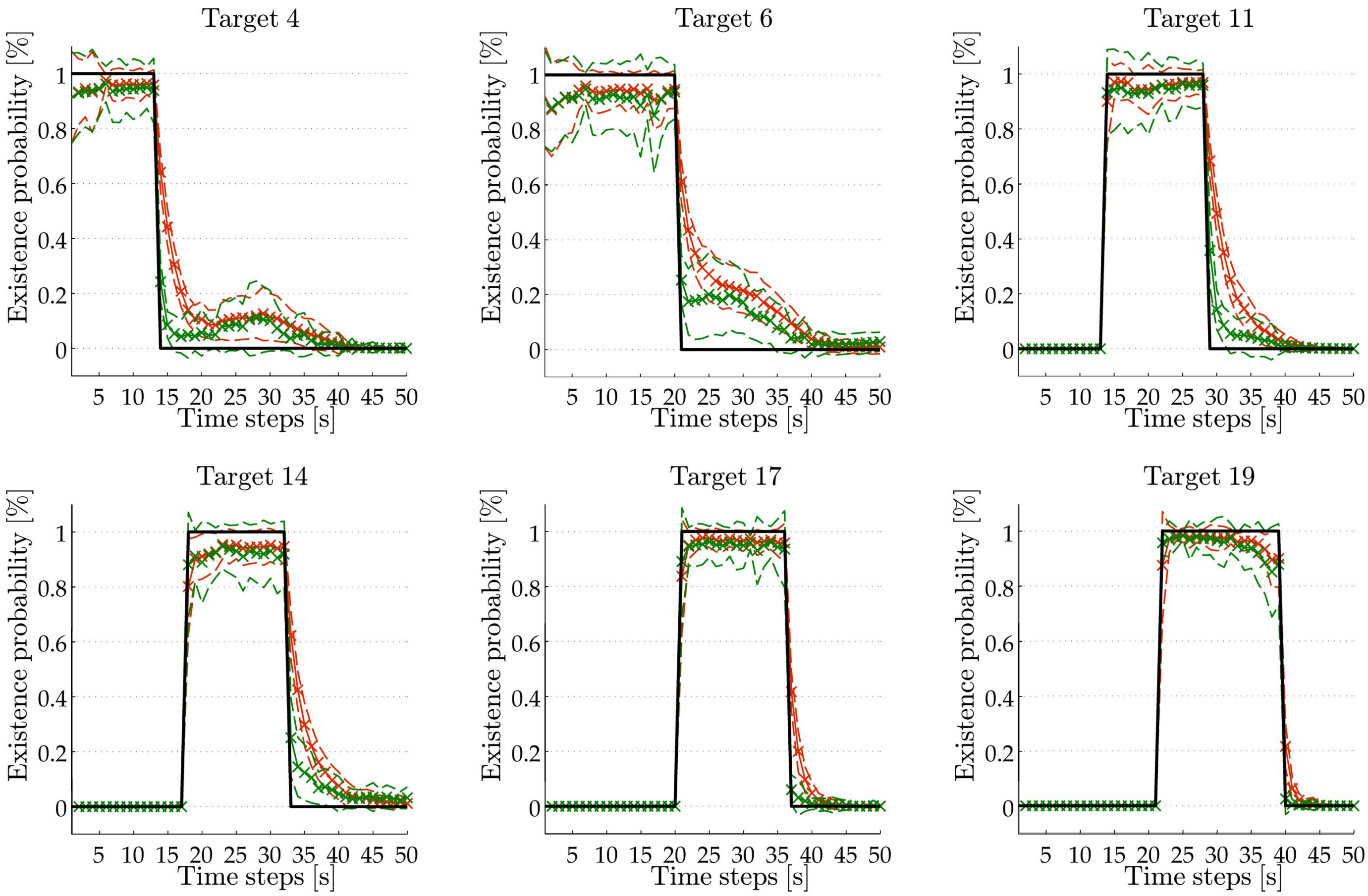



**Fig. 9:** Existence probability associated to each target versus time.

## VI. Conclusion

This work has addressed the Bayesian multi-target tracking (MTT) problem dealing with an unknown and possibly time varying number of targets. MTT sets a major challenge for researchers in the broad fields of estimation and information fusion. In this memo, we have designed a novel, efficient and reliable tracking algorithm suitable for a Track-Before-Detect (TBD) radar surveillance application. The so called Multiple Cardinality Hypotheses Particle Filter (MCHPF) algorithm is built upon the concepts of generalized labeled multi-Bernoulli RFS and multi-target likelihood. In simulations, the proposed MCHPF filter performs remarkably well, and outperforms the IP-RJMCMC-PF filter as well as the conventional multi-target SIR-PF in terms of track accuracy and consistency. Given the wide range of applicability of multi-target tracking and the great number of engineering problems related to the recursive estimation of the state of multiple targets, many questions remain open. There are some important possible extensions to the work discussed in this memo. They include, on one hand, generalizations of our algorithms to handle more complex scenarios and, on the other hand, the development of solutions for problems dealing with specific aspects of a tracking system.


## VII. Acknowledgement

The research leading to these results has received funding from the EU's Seventh Framework Programme under grant agreement n$^o$ 238710. The research has been carried out in the MC IMPULSE project: https://mcimpulse.isy.liu.se.

# REFERENCES


[1] C. Andrieu and G. O. Roberts. The Pseudo-marginal Approach for Efficient Monte Carlo Computations. *Annals of Statistics*, 37(2):697–725, 2009.

[2] S. Arulampalam, N. Gordon, and R. Ristic. *Beyond the Kalman Filter - Particle Filters for Tracking Applications*. Boston - London: Artech House, 2004.

[3] Y. Bar-Shalom and T.E. Fortmann. *Tracking and Data Association*, volume 179 of *Mathematics in Science and Engineering*. Academic Press Professional, Inc., 1987.

[4] Y. Barniv. *Dynamic Programming Algorithm for Detecting Dim Moving Targets*, chapter 4. Artech House, multitarget-multisensor tracking: advanced applications edition, 1990.

[5] M. Bocquel. *Random Finite Sets in Multi-target Tracking: Efficient Sequential MCMC implementation*. PhD thesis, Faculty of Electrical Engineering, Mathematics and Computer Science. Department of Applied Mathematics. University of Twente, 2013.

[6] M. Bocquel, H. Driessen, and A. Bagchi. Multitarget Tracking with Interacting Population-based MCMC-PF. *Proc. of the 15th Int. Conf. on Information Fusion*, pages 74–81, Singapore, July 2012.

[7] M. Bocquel, H. Driessen, and A. Bagchi. Multitarget Tracking with IP Reversible Jump MCMC-PF. *Proc. of the 16th Int. Conf. on Information Fusion*, Istanbul, July 2013.

[8] Y. Boers and H. Driessen. Particle Filter Track-Before-Detect Application Using Hard Inequality Constraints. *IEEE Trans. on Aerospace and Electronic Systems*, 41(4):1483Ű1489, 2005.

[9] Y. Boers and J. N. Driessen. Multitarget Particle Filter Track-Before-Detect Application. *IEE Proc. on Radar, Sonar and Navigation*, 151:351–357, 2004.

[10] Y. Boers and J.N. Driessen. Particle Filter based Detection for Tracking. *Proc. of the 2001 American Control Conference*, pages 4393–4397, 2001.

[11] Y. Boers, F. Ehlers, W. Koch, T. Luginbuhl, L.D. Stone, and R.L. Streit. Track Before Detect Algorithms. *Eurasip Journal on applied Signal Processing*, 2008.

[12] S. P. Brooks and A. Gelman. General Methods for Monitoring Convergence of Iterative Simulations. *Journal of Computational and Graphical Statistics*, 7(4):434–455, 1998.

[13] B.D. Carlson, E.D. Evans, and S.L. Wilson. Search Radar Detection and Track with the Hough Transform. Part I: System Concept. *IEEE Trans. Aerospace & Electronic Systems*, 30:102–108, 1994.

[14] S. Davies, M. Rutten, and B. Cheung. A Comparison of Detection Performance for Several Track-before-Detect Algorithms. *EURASIP Journal on Advances in Signal Processing*, 2008:Article ID 428036, 2008.

[15] T.M. Deserno, editor. *Biomedical Image Processing*. Springer-Verlag, New York, NY, USA, 2011.

[16] A. Doucet, N. de Freitas, and N. Gordon. *Sequential Monte Carlo Methods in Practice*. Springer-Verlag, Berlin, 2001.

[17] A. Doucet, N. de Freitas, and K. Murphy. Rao-blackwellised particle filtering for dynamic bayesian networks. Technical report, Stuart Russel, 2000.

[18] A. Doucet, S. Godsill, and C. Andrieu. On Sequential Monte Carlo Sampling Methods for Bayesian Filtering. *Statistics and Computing*, 10:197–208, 2000.

[19] A. Doucet and A. M. Johansen. A Tutorial on Particle Filtering and Smoothing: Fifteen Years Later. *In Handbook of Nonlinear Filtering, University Press*, 2009.

[20] M. Fallon and S. Godsill. Acoustic Source Localization and Tracking Using Track Before Detect. *IEEE Transactions on Audio, Speech and Language Processing*, 18(6):1228–1242, 2010.

[21] A. Gelman and D. B. Rubin. Inference from Iterative Simulation Using Multiple Sequences (with discussion). *Statistical Science*, 7:457–511, 1992.

[22] W. R. Gilks and C. Berzuini. Following a moving target-Monte Carlo inference for dynamic Bayesian models. *Journal of the Royal Statistical Society: Series B*, 63:127–146, 2001.

[23] A.C. Harvey. *Applications of the Kalman Filter in Econometrics*. Cambridge University Press, 1987.

[24] D.M. Higdon. Auxiliary Variable Methods for Markov Chain Monte Carlo with Applications. *Journal of the American Statistical Association*, 93(442):585–595, June 1998.

[25] Z. Khan, T. Balch, and F. Dellaert. MCMC-Based Particle Filtering for Tracking a Variable Number of Interacting Targets. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 27(11):1805–1819, 2005.

[26] C. Kreucher, K. Kastella, and A.O. Hero. Tracking Multiple Targets using a Particle Filter Representation of the Joint Multitarget Probability Density. *Proc. of SPIE - Signal and Data Processing of Small Targets*, 5204:258–269, 2003.

[27] J. Liu and R. Chen. Sequential Monte Carlo Methods for Dynamic Systems. *Journal of the Royal Statistical Society: Series B*, 93(443):1032–1044, 1998.

[28] R.P.S. Mahler. *Statistical Multisource-Multitarget Information Fusion*. Artech House, 2007.

[29] M. McDonald and B. Balaji. Track-before-Detect using Swerling 0,1, and 3 Target Models for Small Manoeuvring Maritime Targets. *EURASIP Journal on Advances in Signal Processing*, 2008:Article ID 326259, 2008.

[30] E. Meijering, I. Smal, and Danuser G. Tracking in Molecular Bioimaging. *IEEE Signal Processing Magazine*, 23(3):46–53, June 2006.

[31] K.G. Murty. An Algorithm for Ranking all the Assignements in order of Increasing Cost. *Operations Research*, 16(3):682–687, 1968.

[32] M.K. Pitt and N. Shephard. Filtering via Simulation: Auxiliary Particle Filters. *Journal of the American Statistical Association*, 94(446):590–599, June 1999.

[33] D. J. Salmond and H. Birch. A Particle Filter for Track-Before-Detect. *Proc. of the 2001 American Control Conference,*, pages 3755–3760, 2001.

[34] D. Schuhmacher, B.T. Vo, and B.N. Vo. A Consistent Metric for Performance Evaluation of Multi-Object Filters. *IEEE Transactions on Signal Processing*, 56(8):3447–3457, 1998.

[35] F. Septier, S.K. Pang, A. Carmi, and S.J. Godsill. On MCMC-Based Particle Methods for Bayesian filtering: Application to Multitarget Tracking. *3rd IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP)*, pages 360–363, 2009.

[36] M.I. Skolnik. *Introduction to Radar Systems*. McGraw-Hill, Inc, Singapore, 2nd ed., 1981.

[37] K. Smith, S. Ba, D. Gatica-Perez, and J.M. Odobez. Tracking the Visual Focus of Attention for a Varying Number of Wandering People. *IEEE Transaction Pattern Anal. Mach. Intell.*, 30:1212–1229, 2008.

[38] P. Swerling. Probability of Detection for Fluctuating Targets. *Information Theory, IRE Transactions on*, 6(2):269–308, 1960.

[39] S. Tonissen and Y. Bar-Shalom. Maximum likelihood track-before-detect with fluctuating target amplitude. *IEEE Trans. on Aerospace and Electronic Systems*, 34(3):796–809, 1998.

[40] B.T. Vo and B.N. Vo. A Random Finite Set Conjugate Prior and Application to Multi-Target Tracking. *Proc. 7th Int. Conf. Intelligent Sensors, Sensor Networks, and Information Processing (ISSNIP 2011)*, Adelaide, Australia, December 2011.

[41] B.T. Vo and B.N. Vo. Labeled Random Finite Sets and Multi-Object Conjugate Priors. *IEEE Transactions on Signal Processing*, 61(13):3460–3475, July 2013.