

A Comparison of Random Walks in Dependent Random Environments

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Abstract

Although the theoretical behavior of one-dimensional random walks in random environments is well understood, the actual evaluation of various characteristics of such processes has received relatively little attention. This paper develops new methodology for the exact computation of the drift in such models. Focusing on random walks in *dependent* random environments, including k -dependent and moving average environments, we show how the drift can be characterized and found using Perron-Frobenius theory. We compare random walks in various dependent environments and show that their drift behavior can differ significantly.

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1 Introduction

Random walks in random environments (RWREs) are well-known mathematical models for motion through disorganized (random) media. They generalize ordinary random walks, usually on the d -dimensional lattice \mathbb{Z}^d , via a two-stage random procedure. First, the environment is generated according to some probability distribution (e.g., on a set $\mathcal{U}^{\mathbb{Z}}$, where \mathcal{U} is the set of all possible environment states at any position). Second, the walker performs an ordinary random walk $\{X_n, n = 0, 1, \dots\}$ in which the transition probabilities at any state are determined by the environment at that state. RWREs exhibit interesting and unusual behavior that is not seen in ordinary random walks. For example, the walk can tend to infinity almost surely, while the speed (also called drift) is 0; that is, $\mathbb{P}(\lim_{n \rightarrow \infty} X_n = \infty) = 1$, while $\lim_{n \rightarrow \infty} X_n/n = 0$. The reason for such surprising behavior is that RWREs can spend a long time in (rare) regions from which it is difficult to escape — in effect, the walker becomes “trapped” for a long time.

Since the late 1960s a vast body of knowledge has been built up on the behavior of RWREs. Early applications can be found in Chernov [4] and Temkin [16]; see also Kozlov [9] and references therein. Recent applications to charge transport in designed materials are given in Brereton et al. [3] and Stenzel et al. [14]. The mathematical framework for RWREs was laid by Solomon [13], who proved conditions for recurrence/transience for one-dimensional RWREs and also derived law of large number properties for such processes. Kesten et al. [8] were the first to establish central limit-type scaling laws for transient RWREs, and Sinai [12] proved

such results for the recurrent case, showing remarkable “sub-diffusive” behavior. Large deviations for these processes were obtained in Greven and Den Hollander [6]. The main focus in these papers was on one-dimensional random walks in independent environments. Markovian environments were investigated in Dolgopyat [5] and Mayer-Wolf et al. [10]. Alili [1] showed that in the one-dimensional case much of the theory for independent environments could be generalized to the case where the environment process is stationary and ergodic. Overviews of the current state of the art, with a focus on higher-dimensional RWREs, can be found, for example, in Hughes [7], Sznitman [15], Zeitouni [17, 18], and Révész [11].

Although from a theoretical perspective the behavior of one-dimensional RWREs is well understood, from an applied and computational point of view significant gaps in our understanding remain. For example, exact drift computations and comparisons (as opposed to comparisons using simulation) between dependent random environments seem to be entirely missing from the literature. The reason is that such exact computations are not trivial and require additional insights.

The contribution of this paper is twofold. First, we provide new methodology and explicit expressions for the computation of the drift of one-dimensional random walks in various dependent environments, focusing on so-called ‘swap models’. In particular, our approach is based on Perron–Frobenius theory, which allows easy computation of the drift as well as various cutoff points for transient/recurrent behavior. Second, we compare the drift behaviour between various dependent environments, including moving average and k -dependent environments. We show that the behavior can deviate considerably from that in the (known) independent case.

The rest of the paper is organized as follows. In Section 2 we formulate the model for a one-dimensional RWRE in a stationary and ergodic environment and review some of the key results from [1]. We then formulate special cases for the environment: the iid, the Markovian, the k -dependent, and the moving average environment. In Section 3 we derive explicit (computable) expressions for the drift for each of these models, using a novel construction involving an auxiliary Markov chain. Conclusions and directions for future research are given in Section 4.

2 Model and preliminaries

In this section we review some key results on one-dimensional RWREs and introduce the class of “swap-models” that we will study in more detail.

2.1 General theory

Consider a stochastic process $\{X_n, n = 0, 1, 2, \dots\}$ with state space \mathbb{Z} , and a stochastic “Underlying” environment \mathbf{U} taking values in some set $\mathcal{U}^{\mathbb{Z}}$, where \mathcal{U} is the set of possible environment states for each site in \mathbb{Z} . We assume that \mathbf{U} is stationary (under \mathbb{P}) as well as ergodic (under the natural shift operator on \mathbb{Z}). The evolution of $\{X_n\}$ depends on the realization of \mathbf{U} , which is random but fixed over time. For any realization \mathbf{u} of \mathbf{U} the process $\{X_n\}$ behaves as a simple random walk with transition probabilities

$$\begin{aligned} \mathbb{P}(X_{n+1} = i + 1 \mid X_n = i, \mathbf{U} = \mathbf{u}) &= \alpha_i(\mathbf{u}) \\ \mathbb{P}(X_{n+1} = i - 1 \mid X_n = i, \mathbf{U} = \mathbf{u}) &= \beta_i(\mathbf{u}) = 1 - \alpha_i(\mathbf{u}). \end{aligned} \tag{2.1}$$

The theoretical behavior of $\{X_n\}$ is well understood, as set out in the seminal work of Solomon [13]. In particular, Theorems 2.1 and 2.2 below completely describe the transience/recurrence behavior and the Law of Large Numbers behavior of $\{X_n\}$. We follow the notation of Alili [1] and first give the key quantities that

appear in these theorems. Define

$$\sigma_i = \sigma_i(\mathbf{u}) = \frac{\beta_i(\mathbf{u})}{\alpha_i(\mathbf{u})}, \quad i \in \mathbb{Z}, \quad (2.2)$$

and let

$$S = 1 + \sigma_1 + \sigma_1 \sigma_2 + \sigma_1 \sigma_2 \sigma_3 + \cdots \quad (2.3)$$

and

$$F = 1 + \frac{1}{\sigma_{-1}} + \frac{1}{\sigma_{-1} \sigma_{-2}} + \frac{1}{\sigma_{-1} \sigma_{-2} \sigma_{-3}} + \cdots \quad (2.4)$$

Theorem 2.1. (Theorem 2.1 in [1])

1. If $\mathbb{E}[\ln \sigma_0] < 0$, then almost surely $\lim_{n \rightarrow \infty} X_n = \infty$.
2. If $\mathbb{E}[\ln \sigma_0] > 0$, then almost surely $\lim_{n \rightarrow \infty} X_n = -\infty$.
3. If $\mathbb{E}[\ln \sigma_0] = 0$, then almost surely $\liminf_{n \rightarrow \infty} X_n = -\infty$ and $\limsup_{n \rightarrow \infty} X_n = \infty$.

Theorem 2.2. (Theorem 4.1 in [1])

1. If $\mathbb{E}[S] < \infty$, then almost surely $\lim_{n \rightarrow \infty} \frac{X_n}{n} = \frac{1}{\mathbb{E}[(1 + \sigma_0)S]} = \frac{1}{2\mathbb{E}[S] - 1}$.
2. If $\mathbb{E}[F] < \infty$, then almost surely $\lim_{n \rightarrow \infty} \frac{X_n}{n} = \frac{-1}{\mathbb{E}[(1 + \sigma_0^{-1})F]} = \frac{-1}{2\mathbb{E}[F] - 1}$.
3. If $\mathbb{E}[S] = \infty$ and $\mathbb{E}[F] = \infty$, then almost surely $\lim_{n \rightarrow \infty} \frac{X_n}{n} = 0$.

Note that we have added the second equalities in statements 1. and 2. of Theorem 2.2. These follow directly from the stationarity of \mathbf{U} . In particular, if θ denotes the shift operator on \mathbb{Z} , then

$$\begin{aligned} \mathbb{E}[\sigma_0 \sigma_1 \cdots \sigma_{n-1}] &= \mathbb{E} \left[\frac{\beta_0(\mathbf{U}) \beta_1(\mathbf{U}) \cdots \beta_{n-1}(\mathbf{U})}{\alpha_0(\mathbf{U}) \alpha_1(\mathbf{U}) \cdots \alpha_{n-1}(\mathbf{U})} \right] \\ &= \mathbb{E} \left[\frac{\beta_1(\theta \mathbf{U}) \beta_2(\theta \mathbf{U}) \cdots \beta_n(\theta \mathbf{U})}{\alpha_1(\theta \mathbf{U}) \alpha_2(\theta \mathbf{U}) \cdots \alpha_n(\theta \mathbf{U})} \right] \\ (\text{apply } \theta \mathbf{U} \stackrel{d}{=} \mathbf{U}) &= \mathbb{E} \left[\frac{\beta_1(\mathbf{U}) \beta_2(\mathbf{U}) \cdots \beta_n(\mathbf{U})}{\alpha_1(\mathbf{U}) \alpha_2(\mathbf{U}) \cdots \alpha_n(\mathbf{U})} \right] \\ &= \mathbb{E}[\sigma_1 \sigma_2 \cdots \sigma_n], \end{aligned}$$

from which it follows that $\mathbb{E}[(1 + \sigma_0)S] = 2\mathbb{E}[S] - 1$.

We will call $\lim_{n \rightarrow \infty} X_n/n$ the *drift* of the process $\{X_n\}$, and denote it by V . Note that, as mentioned in the introduction, it is possible for the chain to be transient with drift 0 (namely when $\mathbb{E}[\ln \sigma_0] \neq 0$, $\mathbb{E}[S] = \infty$ and $\mathbb{E}[F] = \infty$).

2.2 Swap model

We next focus on what we will call *swap* models, as studied by Sinai [12]. Here, $\mathcal{U} = \{-1, 1\}$; that is, we assume that all elements U_i of the process \mathbf{U} take value either -1 or $+1$. We assume that the transition probabilities in state i only depends on U_i , and not on other elements of \mathbf{U} , as follows. When $U_i = -1$, the transition probabilities of $\{X_n\}$ from state i to states $i+1$ and $i-1$ are swapped with respect to the values they have when $U_i = +1$. Thus, for some fixed value p in $(0, 1)$ we

let $\alpha_i(\mathbf{u}) = p$ (and $\beta_i(\mathbf{u}) = 1 - p$) if $u_i = 1$, and $\alpha_i(\mathbf{u}) = 1 - p$ (and $\beta_i(\mathbf{u}) = p$) if $u_i = -1$. Thus, (2.1) becomes

$$\mathbb{P}(X_{n+1} = i + 1 \mid X_n = i, \mathbf{U} = \mathbf{u}) = \begin{cases} p & \text{if } u_i = 1 \\ 1 - p & \text{if } u_i = -1 \end{cases}$$

and

$$\mathbb{P}(X_{n+1} = i - 1 \mid X_n = i, \mathbf{U} = \mathbf{u}) = \begin{cases} 1 - p & \text{if } u_i = 1 \\ p & \text{if } u_i = -1. \end{cases}$$

Next, we need to choose a dependence structure for \mathbf{U} . The standard case [12] simply assumes that the $\{U_i\}$ are iid (independent and identically distributed):

Iid environment. Let the $\{U_i\}$ be iid with

$$\mathbb{P}(U_i = 1) = \alpha, \quad \mathbb{P}(U_i = -1) = 1 - \alpha$$

for some $0 < \alpha < 1$. In this case the model has two parameters: α and p .

This can be extended to a more general model using the following dependence assumption.

Assumption 2.1. Let \mathbf{U} be generated by a stationary and ergodic Markov chain $\{Y_i, i \in \mathbb{Z}\}$ taking values in a finite set $\{1, \dots, m\}$. In particular, we let $U_i = g(Y_i)$, where $g: \{0, \dots, m\} \rightarrow \{-1, 1\}$ is a given function.

Despite its simplicity, this formalism covers a number of interesting dependence structures on \mathbf{U} , discussed next.

Markov environment. Let $U_i = Y_i$, where $\{Y_i\}$ is a stationary discrete-time Markov chain on $\{-1, 1\}$, with one-step transition matrix P given by

$$P = \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix},$$

for some $a, b \in (0, 1)$. The $\{U_i\}$ form a dependent Markovian environment depending on a and b .

k -dependent environment. Let $k \geq 1$ be a fixed integer. Our goal is to obtain a generalization of the Markovian environment in which the conditional distribution of U_i given all other variables is the same as the conditional distribution of U_i given only U_{i-k}, \dots, U_{i-1} (or, equivalently, given U_{i+1}, \dots, U_{i+k}). To this end we define a k -dimensional Markov chain $\{Y_i, i \in \mathbb{Z}\}$ on $\{-1, 1\}^k$ as follows. From any state $(u_{i-k}, \dots, u_{i-1})$ in $\{-1, 1\}^k$, $\{Y_i\}$ has two possible one-step transitions, given by

$$(u_{i-k}, \dots, u_{i-1}) \rightarrow (u_{i-k+1}, \dots, u_{i-1}, u_i), \quad u_j \in \{-1, 1\},$$

with corresponding probabilities $1 - a_{(u_{i-k}, \dots, u_{i-2})}$, $a_{(u_{i-k}, \dots, u_{i-2})}$, $b_{(u_{i-k}, \dots, u_{i-2})}$, and $1 - b_{(u_{i-k}, \dots, u_{i-2})}$, for (u_{i-1}, u_i) equal to $(-1, -1)$, $(-1, 1)$, $(1, -1)$, and $(1, 1)$, respectively. Now let U_i denote the last component of Y_i . Then $\{U_i, i \in \mathbb{Z}\}$ is a k -dependent environment, and $Y_i = (U_{i-k+1}, \dots, U_i)$.

Note the correspondence in notation with the (1-dependent) Markov environment: a indicates transition probabilities from $U_{i-1} = -1$ to $U_i = +1$, and b from $U_{i-1} = +1$ to $U_i = -1$, where in both cases the subindex denotes the dependence on U_{i-k}, \dots, U_{i-2} .

Moving average environment. Consider a “moving average” environment, which is built up in two phases as follows. First, start with an iid environment $\{\widehat{U}_i\}$ as in the iid case, with $\mathbb{P}(\widehat{U}_i = 1) = \alpha$. Let $Y_i = (\widehat{U}_i, \widehat{U}_{i+1}, \widehat{U}_{i+2})$. Hence, $\{Y_i\}$ is

a Markov process with states $1 = (-1, -1, -1), 2 = (-1, -1, 1), \dots, 8 = (1, 1, 1)$ (lexicographical order). The corresponding transition matrix clearly is given by

$$P = \begin{bmatrix} 1-\alpha & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\alpha & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\alpha & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-\alpha & \alpha \\ 1-\alpha & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\alpha & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\alpha & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-\alpha & \alpha \end{bmatrix}. \quad (2.5)$$

Now define $U_i = g(Y_i)$, where $g(Y_i) = 1$ if at least two of the three random variables $\widehat{U}_i, \widehat{U}_{i+1}$ and \widehat{U}_{i+2} are 1, and $g(Y_i) = -1$ otherwise. Thus,

$$(g(1), \dots, g(8)) = (-1, -1, -1, 1, -1, 1, 1, 1), \quad (2.6)$$

and we see that each U_i is obtained by taking the moving average of $\widehat{U}_i, \widehat{U}_{i+1}$ and \widehat{U}_{i+2} , as illustrated in Figure 2.2.

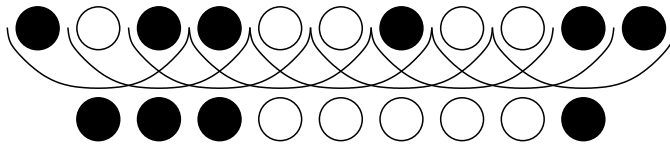


Figure 1: Moving average environment.

3 Evaluating the drift

As a starting point for the analysis, we begin in Section 3.1 with the solution for the iid environment, based on first principles. As mentioned earlier, this model was studied by Sinai [12]. Then, in Section 3.2 we give the general solution approach for the Markov-based swap model, followed by sections with results on the transience/recurrence and on the drift for the random environments mentioned in Section 2.2: the Markov environment, the 2-dependent environment, and the moving average environment (all based on Section 3.2).

3.1 Iid environment

As a warm-up we consider the iid case first, with $\mathbb{P}(U_i = 1) = \alpha = 1 - \mathbb{P}(U_i = -1)$. Here,

$$\mathbb{E}[\ln \sigma_0] = \mathbb{E}[U_0] \ln \sigma = (1 - 2\alpha) \ln \frac{1-p}{p}.$$

Hence, by Theorem 2.1, we have the following findings, consistent with intuition. $X_n \rightarrow +\infty$ a.s. if and only if either $\alpha > 1/2$ and $p > 1/2$, or $\alpha < 1/2$ and $p < 1/2$; $X_n \rightarrow -\infty$ a.s. if and only if either $\alpha > 1/2$ and $p < 1/2$, or $\alpha < 1/2$ and $p > 1/2$; and $\{X_n\}$ is recurrent a.s. if and only if either $\alpha = 1/2$, or $p = 1/2$, or both.

Moving on to Theorem 2.2, we define $\sigma = (1-p)/p$, so that we have

$$\mathbb{E}[S] = \sum_{n=0}^{\infty} \mathbb{E} \left[\sigma^{\sum_{i=1}^n U_i} \right] = \sum_{n=0}^{\infty} (\mathbb{E}[\sigma^{U_1}])^n = \sum_{n=0}^{\infty} (\sigma^{-1}(1-\alpha) + \sigma\alpha)^n, \quad (3.1)$$

which is finite if and only if $\sigma^{-1}(1-\alpha) + \sigma\alpha < 1$; that is, $\mathbb{E}[S] < \infty$ if and only if either $\alpha > 1/2$ and $p \in (1/2, \alpha)$, or $\alpha < 1/2$ and $p \in (\alpha, 1/2)$. Similarly (replace σ

by σ^{-1} and p by $1-p$), $\mathbb{E}[F] = \sum_{n=0}^{\infty} (\sigma(1-\alpha) + \sigma^{-1}\alpha)^n < \infty$ if and only if either $\alpha > 1/2$ and $p \in (1-\alpha, 1/2)$, or $\alpha < 1/2$ and $p \in (1/2, 1-\alpha)$.

Clearly the cases with respect to $\mathbb{E}[S]$ and $\mathbb{E}[F]$ do not entirely cover the cases we concluded to be transient above. E.g., when $\alpha > 1/2$ and $p \in [\alpha, 1]$, the process tends to $+\infty$, but the drift is zero. We summarize these findings in the following proposition.

Proposition 3.1. *We distinguish between transient cases with and without drift, and the recurrent case as follows.*

- 1a. *If either $\alpha > 1/2$ and $p \in (1/2, \alpha)$ or $\alpha < 1/2$ and $p \in (\alpha, 1/2)$, then almost surely $\lim_{n \rightarrow \infty} X_n = \infty$ and*

$$V = (2p-1) \frac{\alpha-p}{\alpha(1-p) + (1-\alpha)p} > 0. \quad (3.2)$$

- 1b. *If either $\alpha > 1/2$ and $p \in (1-\alpha, 1/2)$ or $\alpha < 1/2$ and $p \in (1/2, 1-\alpha)$, then almost surely $\lim_{n \rightarrow \infty} X_n = -\infty$ and*

$$V = -(1-2p) \frac{\alpha-(1-p)}{\alpha p + (1-\alpha)(1-p)} < 0. \quad (3.3)$$

- 2a. *If either $\alpha > 1/2$ and $p \in [\alpha, 1]$ or $\alpha < 1/2$ and $p \in [0, \alpha]$, then almost surely $\lim_{n \rightarrow \infty} X_n = \infty$, but $V = 0$.*

- 2b. *If either $\alpha > 1/2$ and $p \in [0, 1-\alpha]$ or $\alpha < 1/2$ and $p \in [1-\alpha, 1]$, then almost surely $\lim_{n \rightarrow \infty} X_n = -\infty$, but $V = 0$.*

3. *Otherwise (when $\alpha = 1/2$ or $p = 1/2$ or both), $\{X_n\}$ is recurrent and $V = 0$.*

Proof. Immediate from the above; (3.2) follows from (3.1) by using $\sigma = (1-p)/p$; and similar for (3.3). \square

We illustrate the drift as a function of α and p in Figure 2.

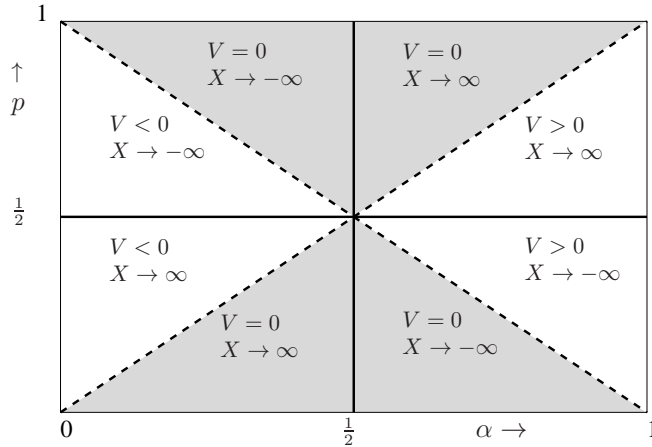


Figure 2: Graphical representation of Proposition 3.1. Solid lines, where the process is recurrent, divide the remaining parameter space in four quadrants. In quadrants I and III, $\{X_n\}$ moves to the right; in quadrants II and IV, $\{X_n\}$ moves to the left. In gray areas (including dashed boundaries and boundaries at $p = 0, 1$), the drift is zero. In white areas (including boundaries at $\alpha = 0, 1$), the drift is nonzero.

3.2 General solution for swap models

Due to the choice of notation for the states in $\mathcal{U} = \{-1, 1\}$, we can, for any swap model, write σ_i (defined in (2.2)) as:

$$\sigma_i = \frac{p}{1-p} \mathbb{I}(U_i = -1) + \frac{1-p}{p} \mathbb{I}(U_i = 1) = \sigma^{U_i},$$

where $\sigma = (1-p)/p$. Consequently, for the key quantity in Theorem 2.1 we find:

$$\mathbb{E}[\ln \sigma_0] = \mathbb{E}[U_0 \ln \sigma] = \ln \sigma \mathbb{E}[U_0], \quad (3.4)$$

the sign of which (and hence the a.s. limit of X_n) only depends on whether p is less than or greater than $1/2$, and on whether $\mathbb{E}[U_0]$ is positive or negative. This holds regardless of the dependence structure between the $\{U_i\}$ (even when Assumption 2.1 is not satisfied).

Furthermore, for the key quantities in Theorem 2.2 we have:

$$\mathbb{E}[S] = \sum_{n=0}^{\infty} \mathbb{E} \left[\sigma^{\sum_{i=1}^n U_i} \right] \quad \text{and} \quad \mathbb{E}[F] = \sum_{n=0}^{\infty} \mathbb{E} \left[\sigma^{-\sum_{i=1}^n U_i} \right]. \quad (3.5)$$

In what follows we will focus on $\mathbb{E}[S]$, since analogous results for $\mathbb{E}[F]$ follow by replacing σ with σ^{-1} and p with $1-p$. This follows from the stationarity of \mathbf{U} , which implies that for any n the product $\sigma_{-1}\sigma_{-2}\cdots\sigma_{-n}$ has the same distribution as $\sigma_1\sigma_2\cdots\sigma_n$ (apply a shift over $n+1$ positions).

Consider now the RWRE swap model with a random environment generated by a Markov chain $\{Y_i, i \in \mathbb{Z}\}$, as specified in Assumption 2.1. Thus,

$$\mathbb{E}[S] = \sum_{n=0}^{\infty} \mathbb{E} \left[\sigma^{\sum_{i=1}^n U_i} \right] = \sum_{n=0}^{\infty} \mathbb{E} \left[\sigma^{\sum_{i=1}^n g(Y_i)} \right].$$

Define

$$G_y^{(n)}(\sigma) = \mathbb{E} \left[\sigma^{\sum_{i=1}^n g(Y_i)} \mid Y_0 = y \right], \quad y = 1, \dots, m$$

and let $P = (P_{y,y'})$ be the one-step transition matrix of $\{Y_i\}$. Then, by conditioning on Y_1 ,

$$\begin{aligned} G_y^{(n+1)}(\sigma) &= \mathbb{E} \left[\sigma^{\sum_{i=1}^{n+1} g(Y_i)} \mid Y_0 = y \right] = \mathbb{E} \left[\sigma^{\sum_{i=2}^{n+1} g(Y_i)} \sigma^{g(Y_1)} \mid Y_0 = y \right] \\ &= \sum_{y'=1}^m P_{y,y'} \sigma^{g(y')} G_{y'}^{(n)}(\sigma). \end{aligned}$$

In matrix notation, with $\mathbf{G}^{(n)}(\sigma) = (G_1^{(n)}(\sigma), \dots, G_m^{(n)}(\sigma))^\top$, we can write this as

$$\mathbf{G}^{(n+1)}(\sigma) = PD\mathbf{G}^{(n)}(\sigma),$$

where

$$D = \text{diag}(\sigma^{g(1)}, \dots, \sigma^{g(m)}).$$

It follows, also using $G_y^{(0)}(\sigma) = 1$, that

$$\mathbf{G}^{(n)}(\sigma) = (PD)^n \mathbf{G}^{(0)}(\sigma) = (PD)^n \mathbf{1},$$

where $\mathbf{1} = (1, 1)^\top$, and hence

$$\mathbb{E}[S] = \sum_{n=0}^{\infty} \boldsymbol{\pi} \mathbf{G}^{(n)}(\sigma) = \boldsymbol{\pi} \sum_{n=0}^{\infty} (PD)^n \mathbf{1},$$

where $\boldsymbol{\pi}$ denotes the stationary distribution vector for $\{Y_i\}$. The matrix series $\sum_{n=0}^{\infty} (PD)^n$ converges if and only if $\text{Sp}(PD) < 1$, where $\text{Sp}(\cdot)$ denotes the spectral radius, and in that case the limit is $(I - PD)^{-1}$, which leads to an explicit expression for $\mathbb{E}[S]$. We summarize these findings in the following theorem.

Theorem 3.1. *For $\mathbb{E}[\ln \sigma_0]$ of Theorem 2.1 we have*

$$\mathbb{E}[\ln \sigma_0] = \ln \sigma \mathbb{E}[U_0]. \quad (3.6)$$

Thus, the a.s. limit of X_n only depends on whether p is less than or larger than $1/2$, and on whether $\mathbb{E}[U_0]$ is positive or negative, regardless of the dependence structure between the $\{U_i\}$.

For $\mathbb{E}[S]$ of Theorem 2.2 we have, with matrices P and D as defined above

$$\mathbb{E}[S] = \begin{cases} \boldsymbol{\pi}(I - PD)^{-1}\mathbf{1} & \text{if } \text{Sp}(PD) < 1 \\ \infty & \text{else.} \end{cases} \quad (3.7)$$

Based on the above, the following subsections will give results on the transience/recurrence and on the drift for the random environments mentioned in Section 2.2. As will turn out, it is not trivial to determine when $\text{Sp}(PD) < 1$.

3.3 Markov environment

The quantity $\mathbb{E}[\ln \sigma_0]$ in Theorem 2.1, which determines whether X_n will diverge to $+\infty$ or $-\infty$, or is recurrent, is given by

$$\mathbb{E}[\ln \sigma_0] = \frac{b}{a+b} \ln \sigma^{-1} + \frac{a}{a+b} \ln \sigma = \frac{a-b}{a+b} \ln \frac{1-p}{p}.$$

Hence, $X_n \rightarrow +\infty$ a.s. if and only if either $a > b$ and $p > 1/2$, or $a < b$ and $p < 1/2$; $X_n \rightarrow -\infty$ a.s. if and only if either $a > b$ and $p < 1/2$, or $a < b$ and $p > 1/2$; and $\{X_n\}$ is recurrent a.s. if and only if either $a = b$, or $p = 1/2$, or both.

Next we study $\mathbb{E}[S]$ to find the drift. In the context of Section 3.2 the processes $\{U_i\}$ and $\{Y_i\}$ are identical and the function g is the identity on the state space $\mathcal{U} = \{-1, 1\}$. Thus, the matrix D is given by $D = \text{diag}(\sigma^{-1}, \sigma)$, and since P is as in Section 2.2, the matrix PD is given by

$$PD = \begin{bmatrix} (1-a)\sigma^{-1} & a\sigma \\ b\sigma^{-1} & (1-b)\sigma \end{bmatrix},$$

for which we have the following.

Lemma 3.1. *The matrix series $\sum_{n=0}^{\infty} (PD)^n$ converges to*

$$(I - PD)^{-1} = \frac{1}{\det(I - PD)} \begin{bmatrix} 1 - (1-b)\sigma & a\sigma \\ b\sigma^{-1} & 1 - (1-a)\sigma^{-1} \end{bmatrix}, \quad (3.8)$$

with $\det(I - PD) = 2 - a - b - (\frac{1-a}{\sigma} + (1-b)\sigma)$, iff σ lies between 1 and $\frac{1-a}{1-b}$.

Note that the condition that σ lies between 1 and $\frac{1-a}{1-b}$ can either mean $1 < \sigma < \frac{1-a}{1-b}$ (when $a < b$), or $\frac{1-a}{1-b} < \sigma < 1$ (when $a > b$).

Proof. The series $\sum_{n=0}^{\infty} (PD)^n$ converges if and only if $\text{Sp}(PD) < 1$, where $\text{Sp}(\cdot)$ denotes the spectral radius $\max_i |\lambda_i|$. The eigenvalues λ_1, λ_2 follow from

$$|\lambda I - PD| = \lambda^2 - A\lambda + (1-a-b) = 0, \quad \text{where} \quad A = (1-a)\sigma^{-1} + (1-b)\sigma.$$

The discriminant of this quadratic equation is

$$A^2 - 4(1-a)(1-b) + 4ab = \left(\frac{1-a}{\sigma} - (1-b)\sigma \right)^2 + 4ab > 0,$$

so the spectral radius is given by the largest eigenvalue,

$$\text{Sp}(PD) = \frac{A + \sqrt{A^2 - 4(1-a-b)}}{2}.$$

Clearly $\text{Sp}(PD) < 1$ if and only if $\sqrt{A^2 - 4(1-a-b)} < 2 - A$, or equivalently $A < 2 - a - b$. Substituting the definition of A and multiplying by σ this leads to

$$(1-b)\sigma^2 - (2-a-b)\sigma + (1-a) < 0,$$

or equivalently,

$$(\sigma - 1)((1-b)\sigma - (1-a)) < 0.$$

Since the coefficient of σ^2 in the above is $1-b > 0$, the statement of the lemma now follows immediately. \square

This leads to the following proposition.

Proposition 3.2. *We distinguish between transient cases with and without drift, and the recurrent case as follows.*

1a. *If either $a > b$ and $p \in (\frac{1}{2}, \frac{1-b}{(1-a)+(1-b)})$ or $a < b$ and $p \in (\frac{1-b}{(1-a)+(1-b)}, \frac{1}{2})$, then almost surely $\lim_{n \rightarrow \infty} X_n = \infty$ and*

$$V = (2p-1) \frac{(1-b)(1-p) - (1-a)p}{\left(b + \frac{a-b}{a+b}\right)(1-p) + \left(a - \frac{a-b}{a+b}\right)p} > 0. \quad (3.9)$$

1b. *If either $a > b$ and $p \in (\frac{1-a}{(1-a)+(1-b)}, \frac{1}{2})$ or $a < b$ and $p \in (\frac{1}{2}, \frac{1-a}{(1-a)+(1-b)})$, then almost surely $\lim_{n \rightarrow \infty} X_n = -\infty$ and*

$$V = -(1-2p) \frac{(1-b)p - (1-a)(1-p)}{\left(b + \frac{a-b}{a+b}\right)p + \left(a - \frac{a-b}{a+b}\right)(1-p)} < 0. \quad (3.10)$$

2a. *If either $a > b$ and $p \in [\frac{1-b}{(1-a)+(1-b)}, 1]$ or $a < b$ and $p \in [0, \frac{1-b}{(1-a)+(1-b)}]$, then almost surely $\lim_{n \rightarrow \infty} X_n = \infty$, but $V = 0$.*

2b. *If either $a > b$ and $p \in [0, \frac{1-a}{(1-a)+(1-b)}]$ or $a < b$ and $p \in [\frac{1-a}{(1-a)+(1-b)}, 1]$, then almost surely $\lim_{n \rightarrow \infty} X_n = -\infty$, but $V = 0$.*

3. *Otherwise (when $a = b$ or $p = 1/2$ or both), $\{X_n\}$ is recurrent and $V = 0$.*

Proof. Substitution of (3.8) and $\boldsymbol{\pi} = \frac{1}{a+b}(b, a)$ in (3.7) leads to

$$\begin{aligned} V^{-1} &= 2\mathbb{E}[S] - 1 \\ &= \frac{2}{(a+b)\det(I-PD)} (b, a) \begin{bmatrix} 1 - (1-b)\sigma & a\sigma \\ b\sigma^{-1} & 1 - (1-a)\sigma \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1 \\ &= \frac{2}{\det(I-PD)} \frac{(a+b) - (1-a-b)(b\sigma + a\sigma^{-1})}{a+b} - 1 \\ &= \frac{1 + \sigma \left(b + \frac{a-b}{a+b}\right) \sigma + \left(a - \frac{a-b}{a+b}\right)}{1 - \sigma \left((1-b)\sigma - (1-a)\right)} \\ &= \frac{1}{2p-1} \frac{\left(b + \frac{a-b}{a+b}\right)(1-p) + \left(a - \frac{a-b}{a+b}\right)p}{(1-b)(1-p) - (1-a)p}. \end{aligned}$$

When σ lies between 1 and $\frac{1-a}{1-b}$, i.e. when $p = (1 + \sigma)^{-1}$ lies between $1/2$ and $(1 - b)/((1 - a) + (1 - b))$, it follows by Lemma 3.1 that the process has positive drift, given by the reciprocal of the above. This proves (3.9). The proof of (3.10) follows from replacing σ by σ^{-1} and p by $1 - p$, and adding a minus sign. The other statements follow immediately. \square

When we take $a + b = 1$ we obtain the iid case of the previous section, with $\alpha = a/(a + b)$. Indeed, the proposition then becomes identical to Proposition 3.1. In the following subsection we make a comparison between the Markov case and the iid case.

3.3.1 Comparison with the iid environment

To study the impact of the (Markovian) dependence, we reformulate the expression for the drift in Proposition 3.2. Note that the role of α in the iid case is played by $P(U_0 = 1) = a/(a + b)$ in the Markov case. Furthermore, we can show that the correlation coefficient between two consecutive U_i 's satisfies

$$\varrho \equiv \varrho(U_0, U_1) = \frac{\text{Cov}(U_0, U_1)}{\text{Var}(U_0)} = \frac{\frac{a+b-4ab}{a+b} - \left(\frac{a-b}{a+b}\right)^2}{1 - \left(\frac{a-b}{a+b}\right)^2} = 1 - a - b.$$

So ϱ depends on a and b only through their sum $a + b$, with extreme values 1 (for $a = b = 0$; i.e., $U_i \equiv U_0$) and -1 (for $a = b = 1$; that is, $U_{2i} \equiv U_0$ and $U_{2i+1} \equiv -U_0$). The intermediate case $a + b = 1$ leads to $\varrho = 0$ and corresponds to the iid case, as we have seen before. To express V in terms of α and ϱ we solve the system of equations $\frac{a}{a+b} = \alpha$ and $1 - a - b = \varrho$, leading to the solution

$$\begin{aligned} a &= (1 - \varrho)\alpha \\ b &= (1 - \varrho)(1 - \alpha). \end{aligned}$$

Substitution in the expression for V (here in case of positive drift only, see (3.9)) and rewriting yields

$$V = (2p - 1) \frac{\alpha - p + \varrho(1 - \alpha - p)}{(\alpha(1 - p) + (1 - \alpha)p)(1 + \varrho) - \varrho}.$$

This enables us not only to immediately recognize the result (3.2) for the iid case (take $\varrho = 0$), but also to study the dependence of the drift V on ϱ . Note that due to the restriction that a and b are probabilities, it must hold that $\varrho > \max\{1 - 1/\alpha, 1 - 1/(1 - \alpha)\}$.

Figures 3 and 4 illustrate various aspects of the difference between iid and Markov cases. Clearly, compared to the iid case (for the same value of α), the Markov case with positive correlation coefficient has lower drift, but also a lower ‘cutoff value’ of p at which the drift becomes zero. For negative correlation coefficients we see a higher cutoff value, but not all values of α are possible (since we should have $a < 1$). Furthermore, for weak correlations the drift (if it exists) tends to be larger than for strong correlations (both positive and negative), depending on p and α . Note that Figure 4 seems to suggest there are two cutoff values in terms of the correlation coefficient. However, it should be realized that drift curves corresponding to some α are no longer drawn for negative correlations since the particular value of α cannot be attained. E.g., when ϱ is close to -1 , then a and b are both close to 1, hence α can only be close to $1/2$.

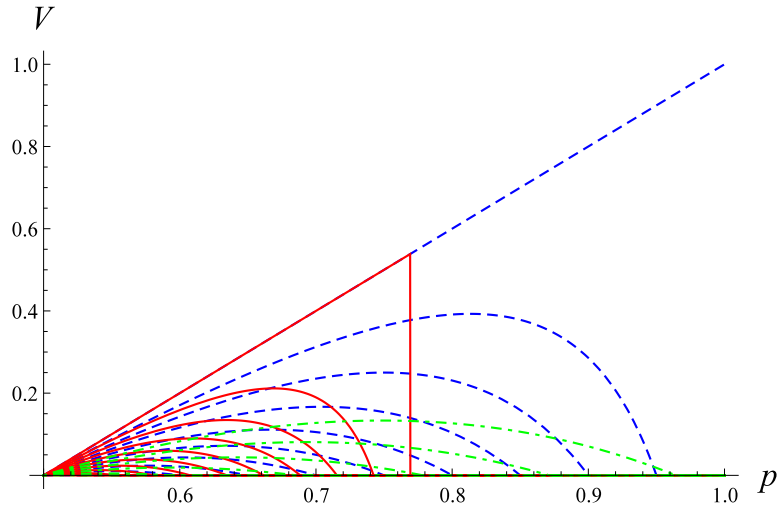


Figure 3: Drift for $\varrho = 0$ (blue, dashed), $\varrho = 0.3$ (red, solid), and $\varrho = -0.3$ (green, dotdashed) as a function of p . From highest to lowest curves for $\alpha = 1, 0.95, \dots, 0.55$ (for $\varrho = 0$ and $\varrho = 0.3$), and for $\alpha = 0.75, 0.70, \dots, 0.55$ (for $\varrho = -0.3$).

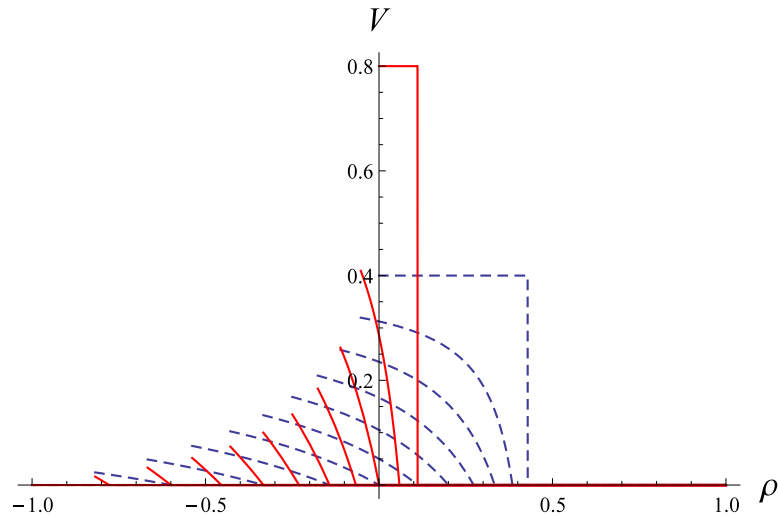


Figure 4: Drift for $p = 0.7$ (blue, dashed) and $p = 0.9$ (red, solid) as a function of the correlation coefficient ϱ , for $\alpha = 1, 0.95, \dots, 0.55$ (from highest to lowest curves). The values at $\varrho = 0$ give the drift for the independent case. Note that ϱ must be greater than or equal to $1/\alpha$.

3.4 2-dependent environment

In this section we treat the k -dependent environment for $k = 2$. For this case we have the transition probabilities

$$P_{u_{i-2}u_{i-1},u_i} = \mathbb{P}(U_i = u_i \mid U_{i-2} = u_{i-2}, U_{i-1} = u_{i-1}), \quad u_j \in \{-1, 1\},$$

so that the one-step transition matrix of the Markov chain $\{Y_i, i \in \mathbb{Z}\}$ with $Y_i = (U_{i-1}, U_i)$ is given by

$$P = \begin{bmatrix} P_{-1-1,-1} & P_{-1-1,+1} & 0 & 0 \\ 0 & 0 & P_{-1+1,-1} & P_{-1+1,+1} \\ P_{+1-1,-1} & P_{+1-1,+1} & 0 & 0 \\ 0 & 0 & P_{+1+1,-1} & P_{+1+1,+1} \end{bmatrix} = \begin{bmatrix} 1-a_- & a_- & 0 & 0 \\ 0 & 0 & b_- & 1-b_- \\ 1-a_+ & a_+ & 0 & 0 \\ 0 & 0 & b_+ & 1-b_+ \end{bmatrix}.$$

Thus, the model has five parameters, a_-, a_+, b_-, b_+ , and p . Also note that the special case $a_- = a_+ (= a)$ and $b_- = b_+ (= b)$ corresponds to the (1-dependent) Markovian case in Section 3.3.

We first note that the stationary distribution (row) vector $\boldsymbol{\pi}$ is given by

$$\boldsymbol{\pi} = \left(2 + \frac{1-a_+}{a_-} + \frac{1-b_-}{b_+} \right)^{-1} \left(\frac{1-a_+}{a_-}, 1, 1, \frac{1-b_-}{b_+} \right), \quad (3.11)$$

so assuming stationarity we have $\mathbb{P}(U_0 = 1) = \pi_{-1,1} + \pi_{1,1}$ and $\mathbb{P}(U_0 = -1) = \pi_{-1,-1} + \pi_{1,-1}$. It follows that $\mathbb{P}(U_0 = 1) > \mathbb{P}(U_0 = -1)$ if and only if $\frac{a_-}{1-a_+} > \frac{b_+}{1-b_-}$. This is important to determine the sign of $\mathbb{E}[\ln \sigma_0]$, which satisfies (with $\sigma = \frac{1-p}{p}$ as before),

$$\mathbb{E}[\ln \sigma_0] = (2\mathbb{P}(U_0 = 1) - 1) \ln \sigma.$$

Hence, $X_n \rightarrow +\infty$ a.s. if and only if either $\frac{a_-}{1-a_+} > \frac{b_+}{1-b_-}$ and $p > 1/2$, or $\frac{a_-}{1-a_+} < \frac{b_+}{1-b_-}$ and $p < 1/2$; $X_n \rightarrow -\infty$ a.s. if and only if either $\frac{a_-}{1-a_+} > \frac{b_+}{1-b_-}$ and $p < 1/2$, or $\frac{a_-}{1-a_+} < \frac{b_+}{1-b_-}$ and $p > 1/2$; and $\{X_n\}$ is recurrent a.s. if and only if either $\frac{a_-}{1-a_+} = \frac{b_+}{1-b_-}$, or $p = 1/2$, or both.

Next we consider the drift. As before we have when $\mathbb{E}[S] < \infty$ that $V^{-1} = 2\mathbb{E}[S] - 1$. So in view of (3.7) we need to consider the matrix PD where $D = \text{diag}(\sigma^{-1}, \sigma, \sigma^{-1}, \sigma)$, so

$$PD = \begin{bmatrix} (1-a_-)\sigma^{-1} & a_- \sigma & 0 & 0 \\ 0 & 0 & b_- \sigma^{-1} & (1-b_-)\sigma \\ (1-a_+)\sigma^{-1} & a_+ \sigma & 0 & 0 \\ 0 & 0 & b_+ \sigma^{-1} & (1-b_+)\sigma \end{bmatrix}$$

and hence

$$\begin{aligned} V^{-1} &= 2\boldsymbol{\pi} \left(\sum_{n=0}^{\infty} (PD)^n \right) \mathbf{1} - \mathbf{1} \\ &= 2\boldsymbol{\pi} (I - PD)^{-1} \mathbf{1} - \mathbf{1} \end{aligned}$$

if $\text{Sp}(PD) < 1$. Unfortunately, the eigenvalues of PD are now the roots of a 4-degree polynomial, which are hard to find explicitly. However, using Perron–Frobenius theory and the implicit function theorem it is possible to prove the following lemma, which has the same structure as in the Markovian case.

Lemma 3.2. *The matrix series $\sum_{n=0}^{\infty} (PD)^n$ converges to $(I - PD)^{-1}$, which is*

$$\begin{bmatrix} 1 - a_+ b_- - \sigma + \sigma B & a_- \sigma ((b_+ - 1)\sigma + 1) & a_- (-\sigma b_- + b_- + b_+ \sigma) & -a_- (b_- - 1)\sigma^2 \\ \frac{(a_+ - 1)(b_- (\sigma - 1) - b_+ \sigma)}{\sigma^2} & \frac{(a_- + \sigma - 1)((b_+ - 1)\sigma + 1)}{\sigma} & -\frac{(a_- + \sigma - 1)(b_- (\sigma - 1) - b_+ \sigma)}{\sigma^2} & -(b_- - 1)(a_- + \sigma - 1) \\ -\frac{(a_+ - 1)((b_+ - 1)\sigma + 1)}{\sigma} & (a_- + a_+ (\sigma - 1))((b_+ - 1)\sigma + 1) & \frac{(a_- + \sigma - 1)((b_+ - 1)\sigma + 1)}{\sigma} & -(b_- - 1)(a_- + a_+ (\sigma - 1))\sigma \\ \frac{b_+ - a_+ b_+}{\sigma^2} & \frac{b_+ (a_- + a_+ (\sigma - 1))}{\sigma} & \frac{b_+ (a_- + \sigma - 1)}{\sigma^2} & \frac{1 - A + \sigma - a_+ b_- \sigma}{\sigma} \end{bmatrix}$$

divided by $\det(I - PD) = -\sigma^{-1}(\sigma - 1)((1 - B)\sigma - (1 - A))$, iff σ lies between 1 and $\frac{1-A}{1-B}$. Here, $A = a_- + a_+ b_- - a_- b_-$ and $B = b_+ + a_+ b_- - a_+ b_+$.

Proof. To find out for which values of σ we have $\text{Sp}(PD) < 1$, first we denote the (possibly complex) eigenvalues of PD by $\lambda_i(\sigma)$, $i = 0, 1, 2, 3$, as continuous functions of σ . Since PD is a nonnegative irreducible matrix for any $\sigma > 0$, we can apply Perron–Frobenius to claim that there is always a unique eigenvalue with largest absolute value (the other $|\lambda_i|$ being strictly smaller), and that this eigenvalue is real and positive (so in fact it always equals $\text{Sp}(PD)$). When $\sigma = 1$ the matrix is stochastic and we know this eigenvalue to be 1, and denote it by $\lambda_0(1)$.

Now, moving σ from 1 to any other positive value, $\lambda_0(\sigma)$ *must* continue to play the role of the Perron–Frobenius eigenvalue; i.e., none of the other $\lambda_i(\sigma)$ can at some point take over this role. If this were not true, then the continuity of the $\lambda_i(\sigma)$ would imply that one value $\hat{\sigma}$ exists where (say) λ_1 ‘overtakes’ λ_0 , meaning that $|\lambda_1(\hat{\sigma})| = |\lambda_0(\hat{\sigma})|$, which is in contradiction with the earlier Perron–Frobenius statement.

Thus, it remains to find out when $\lambda_0(\sigma) < 1$, which can be established using the implicit function theorem, since λ_0 is implicitly defined as a function of σ by $f(\sigma, \lambda_0) = 0$, with $f(\sigma, \lambda) = \det(\lambda I - PD)$ together with $\lambda_0(1) = 1$. Using $\det(D) = 1$, we find that

$$\begin{aligned} f(\sigma, \lambda) &= \det((\lambda D^{-1} - P)D) = \det(\lambda D^{-1} - P) = \\ &= \sigma[\lambda(a_+b_- - a_-b_+) + \lambda^3(b_+ - 1)] \\ &\quad + \sigma^{-1}[\lambda(a_+b_- - a_-b_-) + \lambda^3(a_- - 1)] \\ &\quad + \lambda^4 + (1 - a_- - b_+ + a_-b_+ - a_+b_-)\lambda^2 + a_-b_- - a_-b_+ - a_+b_- + a_+b_+. \end{aligned}$$

Setting $\lambda = 1$ in this expression gives $\det(I - PD)$ as given in the lemma, with two roots for σ . Thus, there is only an eigenvalue 1 when $\sigma = 1$, which we already called $\lambda_0(1)$, or when $\sigma = \frac{1-A}{1-B}$. In the latter case this must be $\lambda_0(\frac{1-A}{1-B})$, i.e., it cannot be $\lambda_i(\frac{1-A}{1-B})$ for some $i \neq 0$, again due to continuity. As a result we have either $\lambda_0(\sigma) > 1$ or $\lambda_0(\sigma) < 1$ when σ lies between 1 and $\frac{1-A}{1-B}$. Whether $\frac{1-A}{1-B} < 1$ or $\frac{1-A}{1-B} > 1$ depends on the parameters:

$$\frac{1-A}{1-B} > 1 \quad \Leftrightarrow \quad \frac{a_-}{1-a_+} < \frac{b_+}{1-b_-}, \quad (3.12)$$

where we used that $1 - B = 1 - b_+ - a_+b_- + a_+b_+ > (1 - b_+)(1 - a_+) > 0$. Now we apply the implicit function theorem:

$$\left. \frac{d\lambda_0(\sigma)}{d\sigma} \right|_{\sigma=1} = - \left. \frac{\frac{\partial f(\sigma, \lambda_0)}{\partial \sigma}}{\frac{\partial f(\sigma, \lambda_0)}{\partial \lambda_0}} \right|_{\sigma=1, \lambda_0=1} \quad (3.13)$$

$$= - \frac{b_+(1-a_+) - a_-(1-b_-)}{a_-(1-b_- + b_+) + b_+(1-a_+ + a_-)} \quad (3.14)$$

$$= \frac{\frac{a_-}{1-a_+} - \frac{b_+}{1-b_-}}{\frac{a_-}{1-a_+} \left(1 + \frac{b_+}{1-b_-}\right) + \frac{b_+}{1-b_-} \left(1 + \frac{a_-}{1-a_+}\right)}, \quad (3.15)$$

which due to (3.12) is < 0 iff $\frac{1-A}{1-B} > 1$ and is > 0 iff $\frac{1-A}{1-B} < 1$, so that indeed $\text{Sp}(PD) = \lambda_0(\sigma) < 1$ if and only if σ lies between 1 and $\frac{1-A}{1-B}$. \square

Note that for the case $\frac{a_-}{1-a_+} = \frac{b_+}{1-b_-}$ the series never converges, as there is no drift, $\mathbb{P}(U_0 = 1) = \mathbb{P}(U_0 = -1)$. This corresponds to $a = b$ in the Markovian case and $\alpha = 1/2$ in the iid case.

We conclude that if σ lies between 1 and $\frac{1-A}{1-B}$, or equivalently, if p lies between $1/2$ and $\frac{1-B}{1-A+1-B}$, the drift is given by $V = (2\pi(I - PD)^{-1} \mathbf{1} - 1)^{-1}$, where π is

given in (3.11) and $(I - PD)^{-1}$ follows from Lemma 3.2. Using computer algebra, this can be shown to equal

$$V = (2p - 1) \frac{dp(1-p)((1-B)(1-p) - (1-A)p)}{\sum_{i=0}^3 c_i p^i} \quad (3.16)$$

where

$$\begin{aligned} d &= a_-(b_- - b_+ - 1) + b_+(a_+ - a_- - 1) \\ c_0 &= 2a_-b_+(b_- - b_+) \\ c_1 &= -c_0(2 + a_+ + a_-) + (B - A)(1 - B) \\ c_2 &= -c_0 - c_1 - c_3 \\ c_3 &= (B - A)(2 - A - B). \end{aligned}$$

Including the transience/recurrence result from the first part of this section, and including the cases with negative drift, we obtain the following analog to Propositions 3.1 and 3.2.

Proposition 3.3. *We distinguish between transient cases with and without drift, and the recurrent case in the same way as for the Markov environment in Proposition 3.2. In particular, all statements (1a.), ..., (3) in Proposition 3.2 also hold for the 2-dependent environment if we replace a and b by A and B respectively, (3.9) by (3.16), and (3.10) by minus the same expression (3.16) but with p replaced by $1 - p$.*

3.4.1 Comparison with the Markov environment

To facilitate a comparison between the drifts for the two-dependent and Markov environments it is convenient to write the probability distribution vector of (U_0, U_1, U_2) as πR , where π is the distribution vector of (U_0, U_1) , see (3.11), and

$$R = \begin{bmatrix} 1 - a_- & a_- & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_- & 1 - b_- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 - a_+ & a_+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_+ & 1 - b_+ \end{bmatrix}.$$

Thus, $\pi R = c \left(\frac{(1-a_+)(1-a_-)}{a_-}, 1 - a_+, b_-, 1 - b_-, 1 - a_+, a_+, 1 - b_-, \frac{(1-b_-)(1-b_+)}{b_+} \right)$, where $c = \left(2 + \frac{1-a_+}{a_-} + \frac{1-b_-}{b_+} \right)^{-1}$. If we also define

$$M_0 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, M_{01} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, M_{02} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

then the probability distribution vector of U_0 , (U_0, U_1) , and (U_0, U_2) are respectively given by

$$\begin{aligned} \pi R M_0 &= c \left(\frac{1 - a_+}{a_-} + 1, \frac{1 - b_-}{b_+} + 1 \right), \\ \pi R M_{01} &= \pi = c \left(\frac{1 - a_+}{a_-}, 1, 1, \frac{1 - b_-}{b_+} \right), \\ \pi R M_{02} &= c \left(\frac{(1 - a_+)(1 - a_-)}{a_-} + b_-, 2 - a_+ - b_-, 2 - a_+ - b_-, a_+ + \frac{(1 - b_-)(1 - b_+)}{b_+} \right). \end{aligned}$$

Various characteristics of the distribution of (U_0, U_1, U_2) are now easily found. In particular, the probability $\mathbb{P}(U_0 = 1)$ is

$$\alpha = \frac{a_-(1 - b_- + b_+)}{a_-(1 - b_- + b_+) + b_+(1 - a_+ + a_-)},$$

the correlation coefficient between U_0 and U_1 is

$$\varrho_{01} = 1 - \frac{a_-}{a_- + 1 - a_+} - \frac{b_+}{b_+ + 1 - b_-},$$

the correlation coefficient between U_0 and U_2 is

$$\begin{aligned} \varrho_{02} &= 1 - (2 - a_+ - b_-) \left(\frac{a_-}{a_- + 1 - a_+} + \frac{b_+}{b_+ + 1 - b_-} \right) \\ &= 1 - (2 - a_+ - b_-)(1 - \varrho_{01}), \end{aligned}$$

and $\mathbb{E}[U_0 U_1 U_2]$ is

$$e_{012} = \frac{4a_-b_+(b_- - a_+) + a_-(1 - b_- + b_+) - b_+(1 - a_+ + a_-)}{a_-(1 - b_- + b_+) + b_+(1 - a_+ + a_-)}.$$

The original parameters can be expressed in terms of α , ϱ_{01} , ϱ_{02} , and e_{012} as follows:

$$\begin{aligned} a_- &= -\frac{2\alpha(2\alpha(\varrho_{02} - 1) - 2\varrho_{02} + 1) + e_{012} + 1}{8(\alpha - 1)(\alpha(\varrho_{01} - 1) + 1)} \\ b_- &= \frac{2\alpha(\alpha(4\varrho_{01} - 2(\varrho_{02} + 1)) - 4\varrho_{01} + 2\varrho_{02} + 1) + e_{012} + 1}{8(\alpha - 1)\alpha(\varrho_{01} - 1)} \\ a_+ &= -\frac{2\alpha(2\alpha(-2\varrho_{01} + \varrho_{02} + 1) + 4\varrho_{01} - 2\varrho_{02} - 3) + e_{012} + 1}{8(\alpha - 1)\alpha(\varrho_{01} - 1)} \\ b_+ &= \frac{2\alpha(-2\alpha(\varrho_{02} - 1) + 2\varrho_{02} - 3) + e_{012} + 1}{8\alpha(\alpha(\varrho_{01} - 1) - \varrho_{01})}. \end{aligned}$$

Note that due to the restriction that a_- , a_+ , b_- , and b_+ are probabilities, $(\alpha, \varrho_{01}, \varrho_{02}, e_{012})$ can only take values in a strict subset of $[0, 1] \times [-1, 1]^3$.

An illustration of the different behavior that can be achieved for two-dependent environments (as opposed to Markovian environments) is given in Figure 5. Here, $\alpha = 0.95$ and $\varrho_1 = 0.3$. The drift for the corresponding Markovian case is indicated in the figure. The cutoff value is here approximately 0.75. By varying ϱ_2 and e_{012} one can achieve a considerable increase in the drift. It is not difficult to verify that the smallest possible value for ϱ_2 is here $(\alpha - 1)/\alpha = -1/19$, in which case e_{012} can only take the value $3 + 2\alpha(-5 - 4\alpha(-1 + \varrho_1) + 4\varrho_1) = 417/500$. This gives a maximal cutoff value of 1. The corresponding drift curve is indicated by the ‘‘maximal’’ label in Figure 5. For $\varrho_2 = 0$, the parameter e_{012} can at most vary from $-1 + 2\alpha(-1 + \alpha(2 - 4\varrho_1) + 4\varrho_1) = 103/123 = 0.824$ to $7 + 2\alpha(-9 + \alpha(6 - 4\varrho_1) + 4\varrho_1) = 211/250 = 0.844$. The solid red curves show the evolution of the drift between these extremes. The dashed blue curve corresponds to the drift for the independent case with $\alpha = 0.95$.

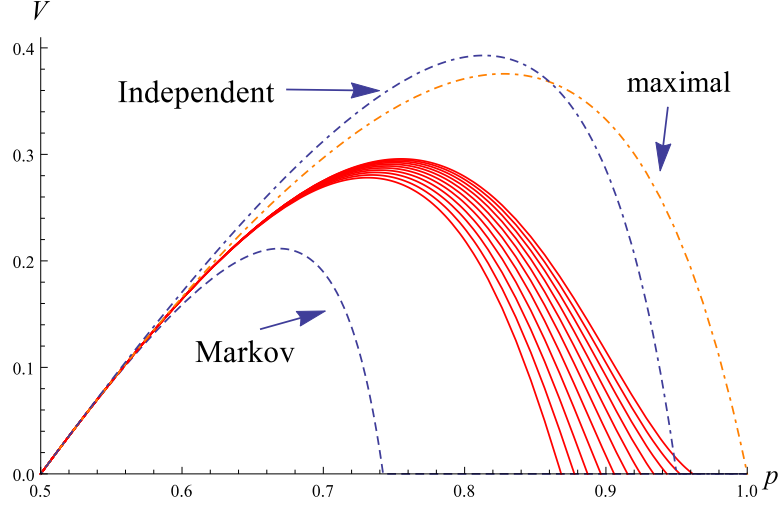


Figure 5: Drift for $\alpha = 0.95$ and $\varrho_1 = 0.3$ for various ϱ_2 and e_{012} . The solid red curves show the drift for $\varrho_2 = 0$ and e_{012} varying from 0.824 to 0.844. The smallest dashed blue curve corresponds to the Markov case. The “maximal” dotdashed orange curve corresponds to the case $\varrho_2 = -1/19$ and $e_{012} = 417/500$. The middle dashed blue line gives the independent case.

3.5 Moving average environment

Recall that the environment is given by $U_i = g(Y_i)$ where the Markov process $\{Y_i\}$ is given by $Y_i = (\widehat{U}_i, \widehat{U}_{i+1}, \widehat{U}_{i+2})$. The sequence $\{\widehat{U}_i\}$ is iid with $\mathbb{P}(\widehat{U}_i = 1) = \alpha = 1 - \mathbb{P}(\widehat{U}_i = -1)$. Thus, $\{Y_i\}$ has states $1 = (-1, -1, -1), 2 = (-1, -1, 1), \dots, 8 = (1, 1, 1)$ (in lexicographical order) and transition matrix P given by (2.5). The deterministic function g is given by (2.6); see also Figure 2.2.

The almost sure behavior of $\{X_n\}$ again depends only on $\mathbb{E}[U_0]$ which equals $-4\alpha^3 + 6\alpha^2 - 1 = (2\alpha - 1)(-2\alpha^2 + 2\alpha + 1)$. Since $-2\alpha^2 + 2\alpha + 1 > 0$ for $0 \leq \alpha \leq 1$, the sign of $\mathbb{E}[U_0]$ is the same as the sign of $\mathbb{E}[\widehat{U}_0] = 2\alpha - 1$, so the almost sure behavior is precisely the same as in the iid case; we will not repeat it here (but see Proposition 3.4).

To study the drift, we need the stationary vector of $\{Y_i\}$, which is given by

$$\pi = \left\{ (1 - \alpha)^3, (1 - \alpha)^2\alpha, (1 - \alpha)^2\alpha, (1 - \alpha)\alpha^2, \right. \\ \left. (1 - \alpha)^2\alpha, (1 - \alpha)\alpha^2, (1 - \alpha)\alpha^2, \alpha^3 \right\}, \quad (3.17)$$

and the convergence behavior of $\sum (PD)^n$, with $D = \text{diag}(\sigma^{-1}, \sigma^{-1}, \sigma^{-1}, \sigma, \sigma^{-1}, \sigma, \sigma, \sigma)$. This is given in the following lemma.

Lemma 3.3. *The matrix series $\sum_{n=0}^{\infty} (PD)^n$ converges to $(I - PD)^{-1}$ iff σ lies between 1 and σ_{cutoff} , which is the unique root $\neq 1$ of*

$$\det(I - PD) = -\frac{\alpha(1 - \alpha)^2}{\sigma^3} + \frac{\alpha^2(1 - \alpha)^2}{\sigma^2} - \frac{(1 - \alpha)(1 - \alpha + \alpha^2)}{\sigma} + 1 \\ - 2\alpha^2(1 - \alpha)^2 - \alpha^2(1 - \alpha)\sigma^3 + \alpha^2(1 - \alpha)^2\sigma^2 - \alpha(1 - \alpha + \alpha^2)\sigma. \quad (3.18)$$

Proof. The proof is similar to that of Lemma 3.2; we only give an outline, leaving details for the reader to verify. Again, denote the possibly complex eigenvalues of PD by $\lambda_i(\sigma), i = 0, \dots, 7$ and use Perron-Frobenius theory to conclude that for any $\sigma > 0$ we have $\text{Sp}(PD) = \lambda_0(\sigma)$, say, with $\lambda_0(1) = 1$.

To find out when $\lambda_0(\sigma) < 1$ we again use the implicit function theorem on $f(\sigma, \lambda_0) = 0$, with $f(\sigma, \lambda) = \det(\lambda I - PD)$. Setting $\lambda = 1$ gives (3.18). It can be shown that $f(\sigma, 1)$ is zero at $\sigma = 1$, that $f(\sigma, 1) \rightarrow \infty$ for $\sigma \downarrow 0$, and that $(\partial^2/\partial\sigma^2)f(\sigma, 1) < 0$ for all $\sigma > 0$ (for the latter, consider $0 < \sigma < 1$ and $\sigma \geq 1$ separately). Thus we can conclude that $f(\sigma, 1)$ has precisely two roots for $\sigma > 0$, at $\sigma = 1$ and at $\sigma = \sigma_{\text{cutoff}}$.

As a result we have either $\lambda_0(\sigma) > 1$ or $\lambda_0(\sigma) < 1$ when σ lies between 1 and σ_{cutoff} . For the location of σ_{cutoff} it is helpful to know that $(\partial/\partial\sigma)f(\sigma, 1)|_{\sigma=1} = (2\alpha - 1)(2\alpha^2 - 2\alpha - 1)$, which is positive for $0 < \alpha < 1/2$ and negative for $1/2 < \alpha < 1$. Thus we have $\sigma_{\text{cutoff}} > 1$ iff $\alpha < 1/2$. Also $(\partial/\partial\lambda)f(1, 1) = 1$ so that the implicit function theorem gives $(d/d\sigma)\lambda_0(\sigma)|_{\sigma=1} = -(2\alpha - 1)(2\alpha^2 - 2\alpha - 1)$, so that indeed $\lambda_0(\sigma) < 1$ iff σ lies between 1 and σ_{cutoff} . \square

The cutoff value for p is now easily found as $(1 + \sigma_{\text{cutoff}})^{-1}$, which can be numerically evaluated. The values are plotted in Figure 6.

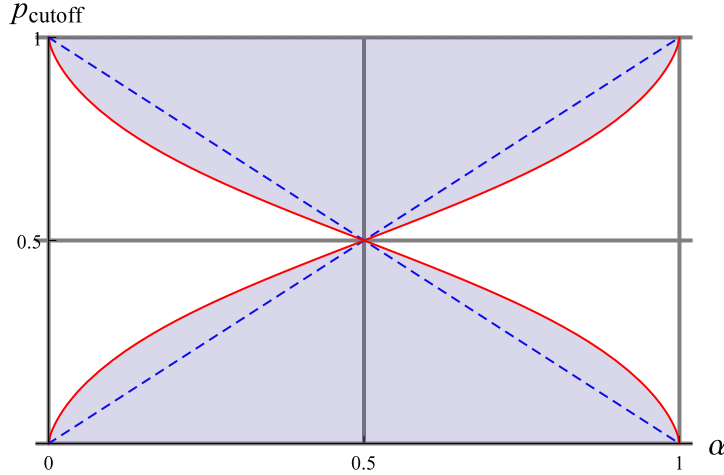


Figure 6: Relation between cutoff value for p , and α . The solid red curve is for the moving average process. For comparison, the dashed blue line is the iid case (see also Figure 2).

When p lies between $1/2$ and p_{cutoff} , the drift is given by $V = (2\pi(I - PD)^{-1} \mathbf{1} - 1)^{-1}$, where π is given in (3.17) and $(I - PD)^{-1}$ follows from Lemma 3.3. Using computer algebra we can find a rather unattractive, but explicit expression for the value of the drift; it is given by the quotient of

$$\begin{aligned} & \alpha^4 \left(-(1 - 2p)^2 (p - 1)p + \alpha^3 (1 - 2p((p - 2)p(2p - 5) + 6) + 4) \right) \\ & + \alpha^2 (2p - 1)(p(3p((p - 2)p + 3) - 5) + 1) - \alpha(1 - 2p)^2 p^2 - (p - 1)^2 p^3 (2p - 1) \end{aligned}$$

and

$$\begin{aligned} & - 2\alpha^5 (2p - 1)^3 - \alpha^4 (1 - 2p)^2 ((p - 11)p + 6) + \alpha^3 (2p - 1)(2p(p^3 - 9p + 10) \\ & - 5) - \alpha^2 (p + 1)(2p - 1)(p(p(3p - 7) + 6) - 1) + \alpha p^2 (2p - 1) + (p - 1)^2 p^3. \end{aligned}$$

Proposition 3.4. *Let $p_{\text{cutoff}} = (1 + \sigma_{\text{cutoff}})^{-1}$, where σ_{cutoff} follows from Lemma 3.3. Then $p_{\text{cutoff}} > 1/2$ iff $\alpha > 1/2$. We distinguish between transient cases with and without drift, and the recurrent case as follows.*

- 1a. *If either $\alpha > 1/2$ and $p \in (1/2, p_{\text{cutoff}})$ or $\alpha < 1/2$ and $p \in (p_{\text{cutoff}}, 1/2)$, then almost surely $\lim_{n \rightarrow \infty} X_n = \infty$ and the drift $V > 0$ is given as above.*

- 1b. If either $\alpha > 1/2$ and $p \in (1-p_{\text{cutoff}}, 1/2)$ or $\alpha < 1/2$ and $p \in (1/2, 1-p_{\text{cutoff}})$, then almost surely $\lim_{n \rightarrow \infty} X_n = -\infty$ and the drift $V < 0$ is given as minus the same expression as above but with p replaced by $1-p$.
- 2a. If either $\alpha > 1/2$ and $p \in [p_{\text{cutoff}}, 1]$ or $\alpha < 1/2$ and $p \in [0, p_{\text{cutoff}}]$, then almost surely $\lim_{n \rightarrow \infty} X_n = \infty$, but $V = 0$.
- 2b. If either $\alpha > 1/2$ and $p \in [0, 1-p_{\text{cutoff}}]$ or $\alpha < 1/2$ and $p \in [1-p_{\text{cutoff}}, 1]$, then almost surely $\lim_{n \rightarrow \infty} X_n = -\infty$, but $V = 0$.
3. Otherwise (when $\alpha = 1/2$ or $p = 1/2$ or both), $\{X_n\}$ is recurrent and $V = 0$.

Figure 7 compares the drifts for the moving average and independent environments.

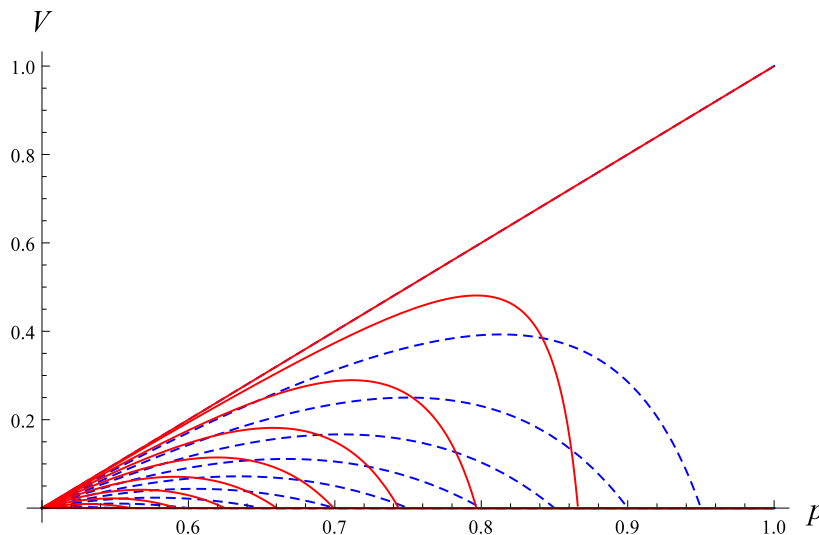


Figure 7: Red: Drift for the moving average environment as a function of p for $\alpha = 1, 0.95, \dots, 0.55$ (from highest to lowest curves). Blue: comparison with the independent case.

It is interesting to note that the cutoff points (where V becomes 0) are significantly lower in the moving average case than the iid case, using the same α , while at the same time the maximal drift that can be achieved is *higher* for the moving average case than for the iid case. This is different behavior from the Markovian case; see also Figure 3.

4 Conclusions

Random walks in random environments can exhibit interesting and unusual behavior due to the trapping phenomenon. The dependency structure of the random environment can significantly affect the drift of the process. We showed how to conveniently construct dependent environment processes, including k -dependent and moving average environments, by using an auxiliary Markov chain. For the well-known swap RWRE model, this approach allows for easy computation of drift, as well as explicit conditions under which the drift is positive, negative, or zero. The cutoff values where the drift becomes zero, are determined via Perron–Frobenius theory. Various generalizations of the above environments can be considered in

the same (swap model) framework, and can be analyzed along the same lines, e.g., replacing iid by Markovian $\{\widehat{U}_i\}$ in the moving average model, or taking moving averages of more than 3 neighboring states.

Other possible directions for future research are (a) extending the two-state dependent random environment to a k -state dependent random environment; (b) replacing the transition probabilities for swap model with the more general rules in Eq.(2.1); and (c) generalizing the single-state random walk process to a multi-state discrete-time *quasi birth and death process* (see, e.g., [2]). By using an infinite “phase space” for such processes, it might be possible to bridge the gap between the theory for one- and multi-dimensional RWREs.

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Single-class: Each queue has a dedicated (separate) server. This is a generalized Jackson network.

Multi-class: Queues 1 and 2 are served by the same server under a non-pre-emptive priority policy giving priority to queue 1. All other queues have their own server. Note that in this case the load on the server of queues 1 and 2 is $\nu_1/\mu_1 + \nu_2/\mu_2 \approx 0.97 < 1$. I.e. it is quite heavily loaded but is still stable. Note in general having a load of less than unity does not immediately imply that the system is stable yet for this simple case it can be shown that stability holds under such a priority policy (c.f. [2]).

Besides exemplifying the correctness of our theoretical results, the goal in this simulation set-up is to illustrate that while the asymptotic variability parameters do not depend on service times and scheduling policies, the shape of the variance curve is in general influenced by such factors.

We ran 2×10^5 simulation runs of each case (single-class and multi-class) each for 1,000 time units, starting at time $t = 0$ with the system empty¹. We then estimated $\text{Var}(D_{5 \rightarrow 4}(t))$ for each run over a grid of time points $t = 20, 40, 60, \dots, 1000$, by taking the sample variance at each time point over 2×10^5 observations. Note that we purposely observe the flow $5 \rightarrow 4$ which is not directly adjacent to the multi-class server serving 1 and 2.

Our main theorem applied to this example implies that in both the single-class and multi-class case, for non-small t ,

$$\text{Var}(D_{5 \rightarrow 4}(t)) \approx \sigma_{5 \rightarrow 4}^2 t = \frac{199}{18} t = 11.05\bar{5} t.$$

This is illustrated in Figure 2 (top) where we plot the variance curves versus the approximation $\sigma_{5 \rightarrow 4}^2 t$. To take a closer look at the effect of single-class vs. multi-class we then plot the bias, $\sigma_{5 \rightarrow 4}^2 t - \text{Var}(D_{5 \rightarrow 4}(t))$ on Figure 2 (bottom). It is indeed evident that different system characteristics yield different variance curves.

It is somewhat expected that the multi-class case will have a higher bias, since in this case the server of 1 and 2 is under a heavier load (0.97). Further, in that case one can expect more “bursts” on the flow $2 \rightarrow 4$ since queue 2 is served with low-priority. These bursts perhaps “propagate” to flow $4 \rightarrow 5$ and ultimately to the flow which we measure: $5 \rightarrow 4$. Nevertheless, such phenomena are not captured by the asymptotic quantities found in the current paper. It should be noted that in [17] second order properties of this sort are explored for elementary queueing systems such as the stable M/G/1 queue. It is not clear how to extend such an investigation to networks.

7 Conclusion

While stable queueing networks have been analysed for decades, up to now, exact expressions for the asymptotic variability of flows have not been known. In this paper we put forward easy computable expressions together with a simple diffusion limit theorem for the flows. It is interesting to see if and how the manufacturing queueing modeling (c.f. [5]) and queueing network decomposition community will adopt our results and incorporate them in heuristic decomposition schemes.

¹The simulation was carried out using a simulation package written in C++: PRONETSIM. See [25], Appendix A, for details about this software.

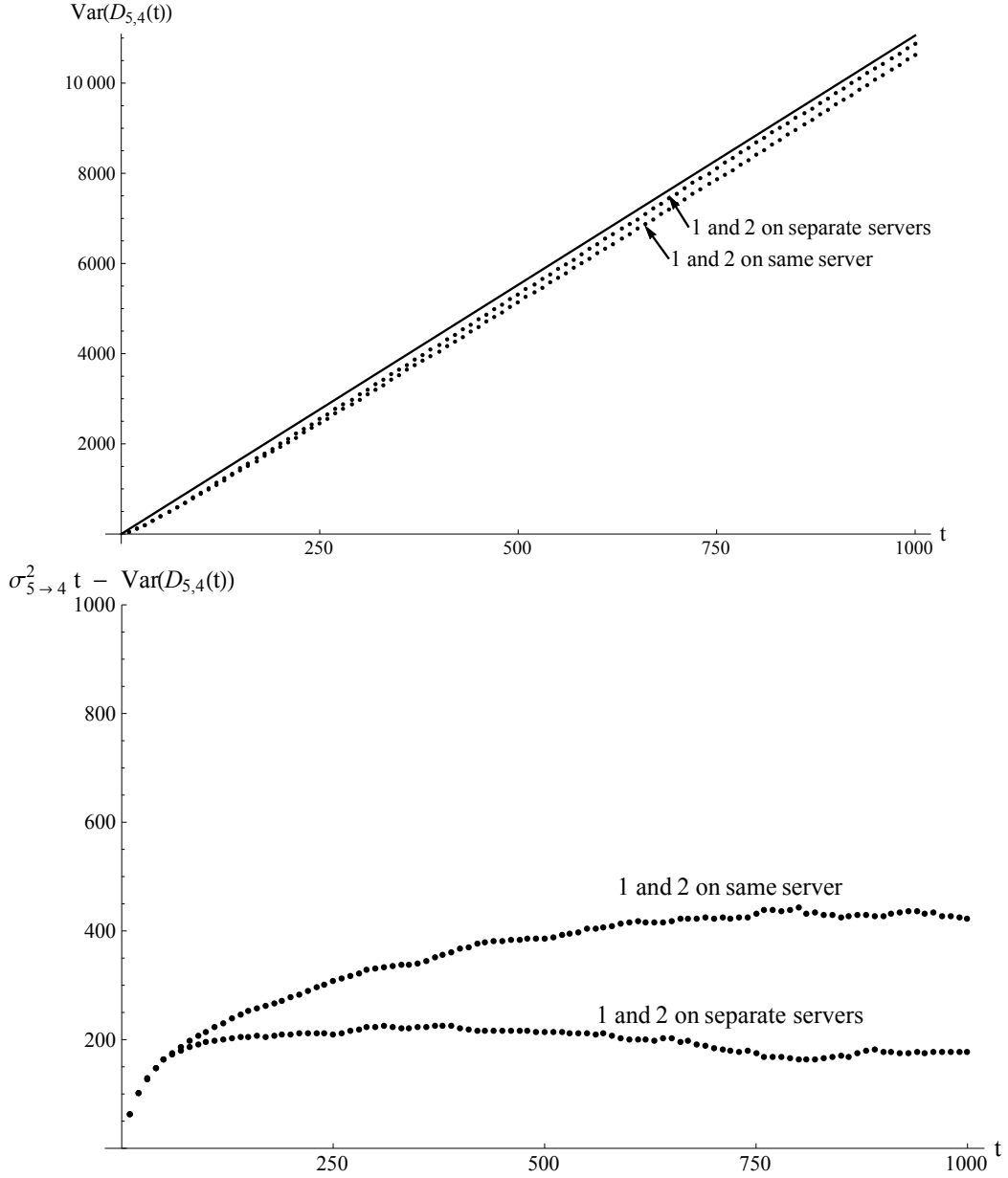


Figure 2: Simulation estimates of $\text{Var}(D_{5 \rightarrow 4}(t))$ for two cases: single-class (1 and 2 on separate servers) and multi-class (1 and 2 on same server with a priority policy). The top graph illustrates the variance curve estimates (dotted) vs. the solid line $\sigma_{5 \rightarrow 4}^2 t$. The bottom graph shows the bias: $\sigma_{5 \rightarrow 4}^2 t - \text{Var}(D_{5 \rightarrow 4}(t))$. As is illustrated, both systems have the same asymptotic variance for $D_{5 \rightarrow 4}(t)$, yet their variance curves differ for finite t .

The queueing networks we considered in this paper are assumed to be open and stable. This stands in contrast with the more general case handled in [6] (where nodes are allowed to be either under-loaded, over-loaded or critical). It should be mentioned that our results easily carry over to the case where some nodes are over-loaded. In this case, the service times of over-loaded nodes contributes to the exogenous arrivals in a straightforward manner (see

for example [13] for an early treatment of this idea). On the contrary, the case in which some nodes are critical is more challenging. In that case, the single-server queue was only recently handled with some difficulty in [1]. There the authors observed a BRAVO effect (Balancing Reduces Asymptotic Variance of Outputs). We do not handle this in the network context. Thus the challenge of finding the asymptotic variability of flows in critical queueing networks remains.

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