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255

Tanja Eisner
Birgit Jacob
André Ran
Hans Zwart
Editors

Operator Theory, Function Spaces, and Applications

International Workshop on Operator Theory and Applications, Amsterdam, July 2014

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Editors
Tanja Eisner
Mathematisches Institut
Universität Leipzig
Leipzig, Germany

Birgit Jacob
Fakultät für Mathematik und Naturwissenschaften
Bergische Universität Wuppertal
Wuppertal, Germany

André Ran
Vrije Universiteit Amsterdam
Amsterdam, The Netherlands

Hans Zwart
Department of Applied Mathematics
University of Twente
Enschede, The Netherlands

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Preface

The IWOTA conference in 2014 was held in Amsterdam from July 14 to 18 at the Vrije Universiteit. This was the second time the IWOTA conference was held there, the first one being in 1985. It was also the fourth time an IWOTA conference was held in The Netherlands. The conference was an intensive week, filled with exciting lectures, a visit to the Rijksmuseum on Wednesday, and a well-attended conference dinner. There were five plenary lectures, twenty semi-plenary ones, and many special sessions. More than 280 participants from all over the world attended the conference.

The book you hold in your hands is the Proceedings of the IWOTA 2014 conference.

The year 2014 marked two special occasions: it was the 80th birthday of Damir Arov, and the 65th birthday of Leiba Rodman. The latter two events were celebrated at the conference on Tuesday and Thursday, respectively, with special session dedicated to their work. Several contributions to these proceedings are the result of these special sessions.

Both Arov and Rodman were born in the Soviet Union at a time when contact with mathematicians from the west was difficult to say the least. Although their lives went on divergent paths, they both worked in the tradition of the Krein school of mathematics.

Arov was a close collaborator of Krein, and stayed and worked in Odessa from his days as a graduate student. His master thesis is concerned with a topic in probability theory, but later on he moved to operator theory with great success. Only after 1989 it was possible for him to get in contact with mathematicians in Western Europe and Israel, and from those days on he worked closely with groups in Amsterdam at the Vrije Universiteit, The Weizmann Institute in Rehovot and in Finland, the Abo Academy in Helsinki. Arov's work focusses on the interplay between operator theory, function theory and systems and control theory, resulting in an ever increasing number of papers: currently MathSciNet gives 117 hits including two books. A description of his mathematical work can be found further on in these proceedings.

Being born 15 years later, Rodman's life took a different turn altogether. His family left for Israel when Leiba was still young, so he finished his studies at Tel Aviv University, graduating also on a topic in the area of probability theory. When Israel Gohberg came to Tel Aviv in the mid seventies, Leiba Rodman was his first PhD student in Israel. After spending a year in Canada, Leiba returned to Israel, but moved in the mid eighties to the USA, first to Arizona, but shortly afterwards to the college of William and Mary in Williamsburg. Leiba's work is very diverse: operator theory, linear algebra and systems and control theory are all well represented in his work. Currently, MathSciNet lists more than 335 hits including 10 books. Leiba was a frequent and welcome visitor at many places, including Vrije Universiteit Amsterdam and Technische Universität Berlin, where he had close collaborators. Despite never having had any PhD student, he influenced many of his collaborators in a profound way. Leiba was also a vice president of the IWOTA Steering Committee; he organized two IWOTA meetings (one in Tempe Arizona, and one in Williamsburg).

When the IWOTA meeting was held in Amsterdam Leiba was full of optimism and plans for future work, hoping his battle with cancer was at least under control. Sadly this turned out not to be the case, and he passed away on March 2, 2015. The IWOTA community has lost one of its leading figures, a person of great personal integrity, boundless energy, and great talent. He will be remembered with fondness by those who were fortunate enough to know him well.

January 2016

Tanja Eisner, Birgit Jacob,
André Ran, Hans Zwart

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My Way in Mathematics: From Ergodic Theory Through Scattering to J-inner Matrix Functions and Passive Linear Systems Theory

Damir Z. Arov

Abstract. Some of the main mathematical themes that I have worked on, and how one theme led to another, are reviewed. Over the years I moved from the subject of my Master's thesis on entropy in ergodic theory to scattering theory and the Nehari problem (in work with V.M. Adamjan and M.G. Krein) and then (in my second thesis) to passive linear stationary systems (including the Darlington method), to generalized bitangential interpolation and extension problems in special classes of matrix-valued functions, and then (in work with H. Dym) to the theory of de Branges reproducing kernel Hilbert spaces and their applications to direct and inverse problems for integral and differential systems of equations and to prediction problems for second-order vector-valued stochastic processes and (in work with O. Staffans) to new developments in the theory of passive linear stationary systems in the direction of state/signal systems theory. The role of my teachers (A.A. Bobrov, V.P. Potapov and M.G. Krein) and my former graduate students will also be discussed.

Mathematics Subject Classification (2010). 30DXX, 35PXX, 37AXX, 37LXX, 42CXX, 45FXX, 46CXX, 47CXX, 47DXX, 93BXX.

Keywords. Entropy, dynamical system, automorphism, scattering theory, scattering matrix, J-inner matrix function, conservative system, passive system, Darlington method, interpolation problem, prediction problem, state/signal system, Nehari problem, de Branges space.

Contents

1. My master's thesis on entropy in the metrical theory of dynamical systems (1956–57). Entropy by Kolmogorov and Sinai. K-systems

My master's research advisor A.A. Bobrov (formerly a graduate student of A.Ya. Hinchin and A.N. Kolmogorov) proposed that I study Shannon entropy in the theory of information, involving two of Hinchin's papers, published in 1953 and 1954. At that time I had been attending lectures by N.I. Gavrilov (formerly a graduate student of I.G. Petrovskii), that included a review of some results in the theory of dynamical systems with invariant measure, the ergodic theorem and the integral spectral representation of a self-adjoint operator in a Hilbert space. In my master's research [11][1] I proposed to use Shannon's entropy in the theory of dynamical systems with invariant measure and I introduced the notion of ε-entropy for a system T^t (flow) on a space Ω with measure μ on some σ-algebra Θ of measurable sets with $\mu(\Omega) = 1$ as follows. Let T be automorphism on Ω, i.e., T is a bijective transform on Ω such that μ is invariant with respect to T:

[1] The entropy chapter of [11] was recently published in [26].

$\mu(TA) = \mu(A)$, $A \in \Theta$. I had introduced the notion of ε-*entropy* $h(T;\varepsilon)$ as a measure of the mixing of T. For the flow T^t I considered $T = T^{t_0}$, where $t_0 > 0$, and I introduced (ε, t_0)-*entropy* $h(\varepsilon; t_0) = h(T;\varepsilon)$. In the definition $h(T;\varepsilon)$ I first of considered a finite partition $\xi = \{A_i\}_1^m$ of Ω on measurable sets and for it I defined

$$H(\xi) = -\sum_1^m \mu(A_i)\log_2 \mu(A_i), \quad h(T;\xi) = \lim_{n\to\infty} \frac{1}{n} H(\vee_0^{n-1} T^k \xi)$$

then,

$$\begin{aligned} h(T;\varepsilon) = \sup\big\{h(T;\xi) : \xi = \{A_i\}_1^m, \\ \mu(A_i) \geq \varepsilon, 1 \leq i \leq m \text{ for some } m\big\}, \varepsilon > 0, \end{aligned} \tag{1}$$

where $T^k\xi = \left\{T^k A_i\right\}_1^m$ and $\zeta = \vee_\alpha \xi_\alpha$ is the intersection (supremum) of the partitions ξ_α.

Since Bobrov was not an expert on this topic, he arranged a journey for me to Moscow University to consult with A.N. Kolmogorov. At that time Kolmogorov was serving as a dean and was very busy with his duties. So, after a brief conversation with me and a quick look at my work, he introduced me to V.M. Alekseev and R.L. Dobrushin. I spoke with them and gave them a draft of my research paper.

Sometime later, at the 1958 Odessa Conference on Functional Analysis, S.V. Fomin presented a preview of Kolmogorov's research that included a notion of entropy for a special class of flows (automorphisms), which after the publication of these results in [56], were called K-flows (K-automorphisms). After Fomin's presentation at the conference, I remarked that in my Master's research I introduced the notion of ε-entropy for a dynamical system with invariant measure, that is connected to Kolmogorov's definition of entropy that was presented by Fomin. Fomin proposed that I show him my work on this subject. As he looked through it, he volunteered to send it to Kolmogorov. I agreed to this. Some time later, Kolmogorov invited me to his home to discuss possible applications of my ε-entropy. Kolmogorov felt that after his work [56] my work did not add anything of scientific interest, but there might be historical interest in how notions of entropy developed. If I wished, he would recommend my work for publication. At that time I gave a negative answer. Then he said that he was preparing a second publication on this topic, and in it he would mention my work. He did so in [57].

Subsequently, Ya. Sinai [61] defined the entropy $h(T)$ of T by the formula

$$h(T) = \sup\{h(T;\xi) : \text{ finite partitions } \xi\}. \tag{2}$$

Thus,

$$h(T) = \lim_{\varepsilon\downarrow 0} h(T;\varepsilon). \tag{3}$$

Kolmogorov introduced the notion of entropy $h_1(T)$ for an automorphism T with an extra property: there exists a partition ζ such that $T^{-1}\zeta \prec \zeta$, the infimum $\wedge_1^\infty T^{-k}\zeta$ is the trivial partition $\{\Omega, \varnothing\}$ and the supremum $\vee_0^\infty T^k\zeta$ is the partition on the points, the maximal partition ζ_{max} of Ω. Such automorphisms are now called K-automorphisms. If ξ is a finitely generated partition, i.e., such that $\vee_{-\infty}^{\infty} T^k\xi = \zeta_{max}$, then T is a K-automorphism and, as was shown by Sinai, Kolmogorov's entropy

$$h_1(T) = h(T) = h(T;\xi).$$

In this case

$$h(T;\varepsilon) = h(T) \text{ for } 0 < \varepsilon \le \varepsilon_0 = \min\{\mu(A_i) : \xi = \{A_i\}_1^m\},$$

where ξ is a generating partition. The notion of entropy $h(T)$ permitted to resolve an old problem on metrical invariants of automorphisms T.

There is a connection between the theory of metrical automorphisms T and the spectral theory of unitary operators: to T corresponds the unitary operator U in the Hilbert space $L_2(d\mu)$ of complex-valued measurable functions f on Ω with $\|f\|^2 = \int_\Omega |f(\mu)|^2 d\mu < \infty$ that is defined by formula

$$(Uf)(p) = f(T^{-1}p), \quad p \in \Omega,\ f \in L_2(d\mu). \tag{4}$$

It is easy to see, that, if two automorphisms T_i on $(\Omega_i, \Theta_i, \mu_i)$, $i = 1, 2$, are metrical isomorphic, i.e., if $T_2 = XT_1X^{-1}$, where X is a bijective measure invariant map from the first space onto the second one, then the unitary operators corresponding to T_i are unitarily equivalent. Thus, the spectral invariants of the unitary operator U are metrical invariants of the corresponding automorphism T. Moreover, it is known that the unitary operators U that correspond to K-automorphisms are unitarily equivalent, since all of them have Lebesgue spectrum with countable multiplicity. This can be shown by consideration of the closed subspace $\mathfrak{D}$ of the functions f from $\mathfrak{H} = L_2(d\mu)$, that are constant on the elements of the Kolmogorov partition ζ. Then

$$U\mathfrak{D} \subset \mathfrak{D}, \quad \cap_0^\infty U^n\mathfrak{D} = \{0\}, \quad \vee_0^\infty U^{-n}\mathfrak{D} = \mathfrak{H}, \tag{5}$$

where the (defect) subspace $\mathfrak{N} = \mathfrak{D} \ominus U\mathfrak{D}$ is an infinite-dimensional subspace of the separable Hilbert space $\mathfrak{H}$, since (Ω, Θ, μ) is assumed to be a Lebesgue space in the Rohlin's sense. From this it follows easily that U has Lebesgue spectrum with countable multiplicity. However, Kolmogorov discovered that there exists K-automorphisms T with different positive entropy $h_1(T)$, i.e., that are not metrically isomorphic, since for nonperiodic K-automorphisms $h(T) = h_1(T)$ is a metrical invariant of T. In particular, as such T are the so-called Bernoulli automorphisms with different entropy. For such an automorphism there exists a finite generating partition $\xi = \{A_i\}_1^m$, such that $\mu\left(\cap_0^n T^k A_{i_k}\right) = \prod_0^n \mu\left(A_{i_k}\right)$ for any $n > 0$. For such T and Bernoulli partition ξ entropy $h(T) = h(T;\xi) = H(\xi)$.

Later Ornstein showed that the entropy of a Bernoulli automorphism defines it up to metrical isomorphism. Thus, for any $h > 0$ and any natural $m > 1$, such that $h \le \log_2 m$, there exists an automorphism T with $h(T) = h$ and with Bernoulli partition that has m elements, and all Bernoulli automorphisms with entropy h are isomorphic to this T. Then it was shown that there exists a K-automorphism, that is not a Bernoulli automorphism, i.e., for it the entropy is not its complete metrical invariant.

As far as I know, the problem of describing a complete set of metrical invariants of K-automorphisms that define a K-automorphism up to metrical isomorphism, is still open. Moreover, in view of above, $h(T;\varepsilon)$ is uniquely defined by $h(T)$ for any Bernoulli automorphism T and any ε, $0<\varepsilon\le\frac{1}{2}$. I do not know if this also holds for K-automorphisms. Similar results were obtained for the K-flows T^t, since $h(T^t)=th(T^1)$. In particular, the group U^t of unitary operators corresponding to a K-flow has a property similar to (5), and all such groups have Lebesgue spectrum with countable multiplicity; hence, they are all unitary equivalent, although the K-flows may have different entropy.

2. My first thesis "Some problems in the metrical theory of dynamical systems" (1964)

In 1959 V.P. Potapov invited me to be his graduate student. In order to overcome the difficulties involved because of my nationality (which in the Soviet slang of that time was referred to as *paragraph* 5), he suggested that I ask Kolmogorov for a letter of recommendation. Kolmogorov wrote such a letter and I was officially accepted as a graduate student at the Odessa Pedagogical Institute from 1959-1962. There I prepared my first dissertation [12]. In this thesis:

1) The entropy $h(T)$ of an endomorphism T of a connected compact commutative group of dimension n (in particular, of n-dimensional torus) was calculated; see [14]. This generalized the results of L.M. Abramov, who dealt with the case $n=1$; my results were later generalized further by S.A. Yuzvinskii (1967).
2) A notion of entropy $m(T)$ for a measurable bijection T of a Lebesgue space that maps a set with zero (positive) measure onto a set with zero (positive) measure was introduced, by consideration of the formula
$$m(T,\xi)=\lim_{n\uparrow\infty}\frac{1}{n}\log_2\mathcal{N}(\vee_{k=0}^{n-1}T^k\xi),$$
where $N(\zeta)$ is the number of sets A_i in the partition ζ and setting
$$m(T,\{\xi_k\})=\lim_{n\to\infty}m(T,\xi_k)$$
for a nondecreasing sequence ξ_k of finite measurable partitions, $m(T)=\inf\{m(T,\{\xi_k\}):\ \{\xi_k\}\}$. It was shown here that $h(T)=m(T)$ for the automorphisms of torus.
3) It was shown that two homeomorphical automorphisms in the connected compact commutative groups X and Y with weight not exceeding the continuum are isomorphic; moreover, if these automorphisms are ergodic, the groups are finite dimensional and G is the homeomorphism under consideration, then G is a product of a shift in X and an isomorphism X onto Y, see [13]; these results were generalized by E.A. Gorin and V.Ya. Lin.

The external review on my first thesis was written by Ya.G. Sinai, the opponents were V.A. Rohlin and I.A. Ibragimov. The thesis was defended in 1964 at Leningrad University.

M.S. Birman invited me to lecture on my joint work with V.M. Adamjan in the V.I. Smirnov seminar a day before my defense in Leningrad. This work developed a connection between the Lax–Phillips scattering scheme and the work of Nagy–Foias on unitary dilations and the characteristic functions of contractions. In particular, we showed that the characteristic function of a simple contraction of the class C_{00} is the scattering matrix of a discrete time Lax–Phillips scattering scheme, which we viewed as the unitary coupling of two simple semi-unitary operators.

We learned about the results of Nagy–Foias from a presentation by Yu.P. Ginzburg in M.G. Krein's seminar and about the Lax–Phillips scattering theme from an unpublished manuscript that M.G. Krein obtained from them at an international conference in Novosibirsk. This manuscript described their recent work on the scattering operator S and scattering matrix $s(\lambda)$ for a continuous group U_t of unitary operators in a Hilbert space H in which there exist subspaces $\mathcal{D}_+$ and $\mathcal{D}_-$ such that

$$\begin{aligned} &\text{(a) } U_{\pm t}\mathcal{D}_\pm \subset \mathcal{D}_\pm,\ t>0, \quad &&\text{(b) } \cap_{t>0} U_{\pm t}\mathcal{D}_\pm = \{0\}, \\ &\text{(c) } \vee_{t<0} U_{\pm t}\mathcal{D}_\pm = H, &&\text{(d) } \mathcal{D}_+ \perp \mathcal{D}_-. \end{aligned} \tag{6}$$

Krein suggested that the work of Lax–Phillips be presented in his seminar and that it would be good to find a connection between the scattering matrix in the Lax–Phillips scheme and the scattering matrix in perturbation theory, where the scattering operator is defined for two groups of unitary operators by consideration of the wave operator under certain conditions. V.M. Adamjan and I found a connection by considering a second group of unitary operators U_t^0 on the space $H_0 = \mathcal{D}_- \oplus \mathcal{D}_+$, such that $V_t^\pm := U_{\pm t} I_{\mathcal{D}_\pm} = U^0_{\pm t} I_{\mathcal{D}_\pm}$, $t \geq 0$. We called the groups U_t and U_t^0 "the couplings of two semigroups of semiunitary operators $V_t^\pm$". Moreover, we discovered that Lax-Phillips scattering matrix $s(\lambda)$ essentially coincides with the Livsic characteristic function of the dissipative operator B, such that iB is the generator of semigroup of contractive operators T_t in the space $X = H \ominus H_0$ of the class C_{00}, i.e., $T_t = e^{iBt}$ has property

$$T_t \mapsto 0 \quad \text{and} \quad T_t^* \to 0 \quad \text{as} \quad t \to +\infty. \tag{7}$$

(Earlier M.S. Livsic in [58] also interpreted the characteristic function of B as a scattering matrix.) More precisely, since at that time the characteristic function of a dissipative operator was defined only for bounded operators B, we considered the Cayley transform $\mathcal{K} = (iI-B)(iI+B)^{-1}$ of B, and showed that $s\left((i-\lambda)/(i+\lambda)\right)$ coincides (up to unitary multipliers) with the Nagy–Foias characteristic function of a contraction $\mathcal{K}$ in the class C_{00}, and it is the scattering matrix of the unitary coupling U of two simple semi-unitary operators $V_\pm$, where U and $V_\pm$ are Cayley transforms of a selfadjoint operator A and a pair of maximal dissipative operators $A_\pm$ that are taken from $U_t = e^{iAt}$ and $V_t^\pm = e^{iA_\pm t}$, respectively; U is the minimal unitary dilation of the contraction $\mathcal{K} \in C_{00}$. This work was published in [2], and then later, in [3], we generalized these results to the case where (c) in (6) was replaced by

$$\text{(c}'\text{)}\ (\vee_{t<0} U_t \mathcal{D}_+) \vee (\vee_{t>0} U_t \mathcal{D}_-) = H.$$

Then the condition (7) is not needed, and $\mathcal{K}$ may be any contraction in X that does not have a unitary part, i.e., it is simple. Moreover, we considered a generalization of the Lax-Phillips scattering scheme, in which the condition (d) in (6) is not assumed. Then instead of a scattering matrix $s(\lambda)$ that is analytic and contractive in the upper half-plane $\mathbb{C}_+$, we considered a scattering suboperator $s(\mu)$ that is contractive on the real axis $\mathbb{R}$. We also showed that $s(\mu)$ is the nontangential boundary value of a scattering matrix $s(\lambda)$ that is analytic and contractive in $\mathbb{C}_+$ if and only if (d) in (6) is satisfied. Our results were presented in detail in [5].

My interest in the Lax–Phillips scattering scheme was partially motivated by the fact that to any K-system with continuous or discrete time (K-flow or K-automorphism) in a space with invariant measure there corresponds an infinite family of Lax–Phillips scattering schemes that satisfy the conditions (a)–(c) in (6) and hence infinitely many scattering suboperators $s(\cdot)$ that are all unitary on the real axis or on the unit circle, respectively. Indeed, as was explained earlier, if T is a K-automorphism, then the operator U defined by formula (4) is unitary in the Hilbert space $H = L_2(d\mu)$ and there exists a closed subspace $\mathcal{D}_+$ of H with property (5) that is defined by a Kolmogorov partition ζ and is invariant under U. Since T^{-1} is a K-automorphism when T is a K-automorphism, a subspace $\mathcal{D}_-$ based on T^{-1} may be obtained similarly so that the discrete group U^n and the subspaces $\mathcal{D}_\pm$ have properties, similar to (a), (b) and (c) in (6). Thus, to different pairs of Kolmogorov partitions ζ_+ and ζ_- of K-automorphisms T and T^{-1} correspond different scattering suboperators $s(.)$, and this family is a metrical invariant for a K-system. I hope that this family $s(\cdot)$, will be useful elsewhere. (Another connection between K-automorphisms and scattering theory may be found in the theory of polymorphisms that is developed by A.M. Vershik, see, e.g., [64] and references inside.)

In [6] V.M.Adamjan and I applied the Lax–Phillips generalized scattering scheme to the problem of predicting the future of one weakly stationary process by past of another weakly stationary process when the cross correlation between these two processes is stationary.

3. From scattering to the Nehari problem. Joint research with V.M. Adamjan and M.G. Krein (1967–71)

Our joint research with V.M. Adamjan led us to consider the problem of describing the set of all the scattering suboperators $s(\mu)$ on $\mathbb{R}$ (or $s\left(e^{i\mu}\right)$ on the unit circle, in the discrete time case) of the set of all unitary couplings U_t (or U, respectively)) into Hilbert spaces $H \supset \mathcal{D} \stackrel{\text{def}}{=} \mathcal{D}_- \vee \mathcal{D}_+$ of two simple semiunitary semigroups $V_t^\pm$ (semiunitary operators $V_\pm$, respectively) on $\mathcal{D}_\pm$, where the angle between $\mathcal{D}_+$ and

$\mathcal{D}_-$ is measured by a Hankel operator with symbol $s(\cdot)$. In the discrete time case the values of $s(e^{i\mu})$ are contractive operators acting between the defect subspaces $\mathfrak{N}_\pm = \mathcal{D}_\pm \ominus V_\pm \mathcal{D}_\pm$ of the operators $V_\pm$ and the Hankel operator $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}(s)$ with symbol $s(e^{i\mu})$ is the operator from $L^2_+(\mathfrak{N}_+)$ into $L^2_-(\mathfrak{N}_-)$ that is defined by the formula

$$(\widehat{\mathcal{T}}\varphi)(e^{i\mu}) = \pi_- M_s \varphi, \quad \varphi \in L^2_+(\mathfrak{N}_+), \tag{8}$$

where

$$L^2_+(\mathfrak{N}) = \left\{\varphi \in L^2(\mathfrak{N}) : \varphi(e^{i\mu}) = \textstyle\sum_0^\infty \varphi_k e^{ik\mu}, \varphi_k \in \mathfrak{N}\right\},$$

$$L^2_-(\mathfrak{N}) = L^2(\mathfrak{N}) \ominus L^2_+(\mathfrak{N}),$$

M_s is operator of “multiplication” by $s(e^{i\mu})$, acting from $L^2(\mathfrak{N}_+)$ into $L^2(\mathfrak{N}_-)$ and π_- is the orthoprojection from $L^2(\mathfrak{N}_-)$ onto $L^2_-(\mathfrak{N}_-)$. This way we came to a problem that we called the “generalized Schur problem.”

In the scalar case the generalized Schur problem problem may be formulated as follows: Given a sequence of complex numbers $\{\gamma_k\}_{k=1}^\infty$ find a function $s \in L^\infty$ with $\|s\|_\infty \leq 1$ such that the coefficient of $e^{-ik\mu}$ in its Fourier series expansion equal γ_k for $k \geq 1$. The classical Schur coefficient problem for functions that are holomorphic and contractive in the unit disk functions is equivalent to the special case of this problem, when $\gamma_k = 0$ for $k > n$.

In our joint work [7] with V.M. Adamjan and M.G. Krein we showed that this problem has a solution if and only if the Hankel operator $\mathcal{T}$ in l^2 defined by the infinite Hankel matrix $(\gamma_{j+k-1})_{j,k=1}^\infty$ is contractive, i.e., if and only if $\|\mathcal{T}\|_\infty \leq 1$. Moreover, in the set $\mathfrak{N}(\mathcal{T})$ of all the solutions to this problem there exists a solution $s(\cdot)$ with $\|s\|_\infty = \|\mathcal{T}\|$. Later, we changed the name of this problem from generalized Schur to Nehari, because we discovered that Nehari had studied this problem before us, and had obtained the same results as in [7] by different methods.

Subsequently in [8] the set $\mathfrak{N}(\mathcal{T})$ was described based on results in the theory of unitary (self-adjoint) extensions U of an isometric (symmetric) operator V. The main tool was a formula of Krein that parametrized the generalized resolvents of a symmetric operator. We obtained a criteria for existence of only one solution, and, in the opposite case, parametrization of the set $\mathfrak{N}(\mathcal{T})$ by the formula

$$s(\varsigma) = [p_-(\zeta)\varepsilon(\zeta) + q_-(\zeta)]\,[q_+(\zeta)\varepsilon(\zeta) + p_+(\zeta)]^{-1}, \tag{9}$$

where ε is an arbitrary scalar function that is holomorphic and contractive in the unit disk, i.e., in terms of the notation $S^{p\times q}$ for the Schur class of $p \times q$ matrix functions that are holomorphic and contractive in the unit disk or upper half-plane, $\varepsilon \in S^{1\times 1}$. The matrix of coefficients in the linear fractional transformation considered in (9) has special properties that will be discussed later.

In the problem under consideration U is the unitary coupling of the simple semi-unitary operators $V_\pm$, defined in the Hilbert space $\mathcal{D} = \mathcal{D}_- \vee \mathcal{D}_+$, U is a unitary extension of the isometric operator V in the Hilbert space $\mathcal{D} = \mathcal{D}_- \vee \mathcal{D}_+$, such that the restriction of V to $\mathcal{D}_+$ is equal to V_+ and restriction of V to $V_-\mathcal{D}_-$ is equal to V_-^*. The problem has unique solution if and only if $U = V$. If not, then

V has defect indices $(1,1)$, and formula (9) was obtained using the Krein formula that was mentioned above. In [9] this formula was generalized to the operator-valued functions in the strictly completely indeterminate case, i.e., when $\|\mathfrak{T}\| < 1$, where the formulas for the coefficients of the linear fractional transformation in (9) in terms of Hankel operator $\mathfrak{T}$ were obtained by a purely algebraical method that is different from the method used in [8]. Then in [10] we established the formula

$$s_k = \min\{\|s - h - r\|_\infty : h + r \in H_{\infty,k}\}, \tag{10}$$

for the singular values $(s_1 \geq s_2 \geq \cdots)$ of a compact Hankel operator $\widehat{\mathfrak{T}}$ with a scalar symbol $s(.)$, where r belongs to the class of rational functions that are bounded on the unit circle with at most k poles in the unit disc (counting multiplicities) and $h \in H_\infty$. Moreover, a formula for the function that minimizes the distance in (10) in terms of the Schmidt pairs of $\widehat{\mathfrak{T}}$ was obtained in [10]. In [1], V.M. Adamjan extended the method that was used in [8] to the operator-valued Nehari problem. In particular, formula (9) was obtained for the matrix-valued Nehari problem in the so-called completely indeterminate case, when $s(\cdot) \in L_\infty^{p\times q}$, $q = \dim \mathfrak{N}_+$, $p = \dim \mathfrak{N}_-$. In this case $\varepsilon \in S^{p\times q}$ in (9). Adamjan also obtained a parametrization formula in the form of the Redheffer transform (see the formula (23) below) that describes the set $\mathfrak{N}(\mathfrak{T})$ of the solutions for the Nehari problem even when it is not in the completely indeterminate case. The matrix coefficients in the linear fractional transform (9) have special properties that were established in [8] for the scalar problem, and in [1] for the matrix-valued problem. These properties will be discussed in the next section.

4. From scattering and Nehari problems to the Darlington method, bitangential interpolation and regular J-inner matrix functions. My second thesis: linear stationary passive systems with losses

V.P. Potapov was my advisor for my first dissertation, and I owe him much for his support in its preparation and even more for sharing his humanistic viewpoint. However, my mathematical interests following the completion of my first dissertation were mostly defined by my participation in Krein's seminar and by my work with him. In this connection I consider both M.G. Krein and V.P. Potapov as my teachers. (See [25].)

I only started to work on problems related to the theory of J-contractive mvf's (matrix-valued functions), which was Potapov's main interest, in the 70s. Although earlier I participated in Potapov's seminar on this theme and in his other seminar, where passive linear electrical finite networks were studied, using the book [60] of S. Seshu and M.B. Reed. In the second seminar, the Darlington method of realizing a real rational scalar function $c(\lambda)$ that is holomorphic with $\Re c(\lambda) > 0$ in the right half-plane (i.e., $c(-i\lambda)$ belongs to the Carathéodory class $\mathcal{C}$), as the impedance of an ideal electrical finite linear two pole with only one resistor was discussed. A generalization by Potapov and E.Ya. Malamud who obtained the representation

$$c(\lambda) = T_A(\tau) \stackrel{\text{def}}{=} [a_{11}(\lambda)\tau + a_{12}(\lambda)]\,[a_{21}(\lambda)\tau + a_{22}(\lambda)]^{-1}, \tag{11}$$

for real rational mvf's $c(\lambda)$ such that $c(-i\lambda)$ belongs to the Carathéodory class $\mathcal{C}^{p\times p}$ of $p \times p$ mvf's, τ is a constant real nonnegative $p \times p$ matrix and the mvf $A(\lambda)$ with four blocks $a_{jk}(\lambda)$ is a real rational mvf such that $A(-i\lambda)$ belongs to the class $\mathcal{U}(J_p)$ of J_p-inner mvf's in the open upper half-plane $\mathbb{C}_+$; see [59] and references therein. Recall that an $m \times m$ matrix J is a signature if it is selfadjoint and unitary. The main examples of signature matrices for this paper are

$$\pm I_m, \quad J_p = \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix}, \quad j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad j_p = j_{pp}. \tag{12}$$

An $m \times m$ mvf $U(\lambda)$ belongs to the Potapov class $\mathcal{P}(J)$ of J-contractive mvf's in the domain Ω (which is equal to either $\mathbb{C}_+$, or $-i\mathbb{C}_+$, or the unit disk D), if it is meromorphic in Ω and

$$U(\lambda)^* J U(\lambda) \le J \quad \text{at holomorphic points in } \Omega. \tag{13}$$

The Potapov–Ginzburg transform

$$S = PG(U) \stackrel{\text{def}}{=} [P_- + P_+U][P_+ + P_-U]^{-1}, \quad \text{where } P_\pm = \left(\frac{1}{2}\right)(I_m \pm J), \tag{14}$$

maps $U \in \mathcal{P}(J)$ into a mvf $S(\lambda)$ in the Schur class $\mathcal{S}^{m\times m}$ in Ω with

$$\det(P_+ + P_-S) \not\equiv 0 \quad \text{in } \Omega.$$

The converse is also true: If $S \in \mathcal{S}^{m\times m}(\Omega)$ and $\det(P_+ + P_-S) \not\equiv 0$ in Ω, then $PG(S) \in \mathcal{P}(J)$. From this it follows, that

$$\mathcal{P}(J) \subseteq \mathcal{N}^{m\times m}, \tag{15}$$

where $\mathcal{N}^{m\times m}$ is the Nevanlinna class of $m \times m$ mvf's that are meromorphic in Ω with bounded Nevanlinna characteristic of growth. Consequently, a mvf $U \in \mathcal{P}(J)$ has nontangential boundary values a.e. on the boundary of Ω. A mvf $U \in \mathcal{P}(J)$ belongs to the class $\mathcal{U}(J)$ of J-inner mvf's, if these boundary values are J-unitary a.e. on the boundary of Ω, i.e.,

$$U(\lambda)^* J U(\lambda) = J \quad \text{a.e. on } \partial\Omega. \tag{16}$$

Moreover U belongs to this class if and only if the corresponding S belongs to the class $\mathcal{S}_{in}^{m\times m}$ of bi-inner $m \times m$ mvf's, i.e., $S \in \mathcal{S}^{m\times m}$ and S has unitary nontangential boundary values a.e. on $\partial\Omega$.

My second dissertation “Linear stationary passive systems with losses” was dedicated to further developments in the theory of passive linear stationary systems with continuous and discrete time. In particular, the unitary operators $U_{\pm t}$ in the passive generalized scattering scheme (a), (b), (c′) and (d) that was considered in (6) were replaced by a pair of contractive semigroups Z_t and Z_t^* for $t \ge 0$. This made it possible to extend the earlier study of simple conservative scattering systems to dissipative (or, in other terminology, passive) systems too. Minimal passive scattering systems with both internal and external losses were studied and passive impedance and transmission systems with losses were analyzed by reduction to the corresponding scattering systems. The Darlington method was generalized as far as possible and was applied to obtain new functional models for simple conservative scattering systems with scattering matrix s and for dissipative scattering systems and minimal dissipative scattering systems.

A number of the results mentioned above were obtained by generalizing the Potapov–Malamud result on Darlington representation (11) to the class $\mathcal{C}^{p\times p}\Pi = \mathcal{C}^{p\times p}\cap\Pi^{p\times p}$, where $\Pi^{p\times p}$ is the class of mvf's f from $\mathcal{N}^{p\times p}$, that have meromorphic pseudocontinuation f_- into exterior Ω_e of Ω, that belong to the Nevanlinna class in Ω_e such that the nontangential boundary value f on $\partial\Omega$ coincides a.e. with the nontangential boundary value of f_-. It is easy to see that this last condition is necessary in order to have the representation (11) with a constant $p \times p$ matrix τ with $\Re\tau \geq 0$ and $A \in \mathcal{U}(J_p)$. The sufficiency of this condition was presented in [15] and with detailed proofs in [16]. This result is intimately connected with an analogous result on the Darlington representation of the Schur class $\mathcal{S}^{p\times q}$ of mvf's s:

$$s(\lambda) = T_W(\varepsilon) \stackrel{\text{def}}{=} [w_{11}(\lambda)\varepsilon + w_{12}(\lambda)]\,[w_{21}(\lambda)\varepsilon + w_{22}(\lambda)]^{-1}\,, \tag{17}$$

where ε is a constant contractive $p \times q$ matrix and the mvf $W(\lambda)$ of the coefficients belongs to $\mathcal{U}(j_{pq})$. In [15] and [16] it was shown that such a representation exists if and only if $s \in \mathcal{S}^{p\times q}\Pi$, where this last class is defined analogously to the class $\mathcal{C}^{p\times p}\Pi$. Moreover, it was shown, that such a representation exists if and only if s may be identified as $s = s_{12}$, where s_{12} is 12-block in the four block decomposition of a bi-inner mvf $S(\lambda)$,

$$S(\lambda) = \begin{pmatrix} s_{11}(\lambda) & s_{12}(\lambda) \\ s_{21}(\lambda) & s_{22}(\lambda) \end{pmatrix}. \tag{18}$$

Furthermore, the set of all such Darlington representations S of minimal size $\tilde{p}\times\tilde{p}$ were described as well as the minimal representations (18) with minimal losses, $\tilde{p} = p+p_l = q+q_l$, where $q_l = \operatorname{rank}(I_p - s(\mu)s(\mu)^*)$, $p_l = \operatorname{rank}(I_q - s(\mu)s(\mu)^*)$ a.e. These mvf's $S(\lambda)$ were used in [15], [17]–[21] to construct functional models of simple conservative scattering systems with scattering matrix $s(\lambda)$ with minimal losses of internal scattering channels and minimal losses of external channels. The operator-valued $s \in S(\mathfrak{N}_+, \mathfrak{N}_-)\Pi$ also was presented as the 12-block of a bi-inner function $S \in S_{in}(\mathfrak{N}_+, \mathfrak{N}_-)$, that is a divisor of a scalar inner function. Independently and at approximately the same time similar results were obtained by R.G. Douglas and J.W. Helton [54]; they obtained them as an operator-valued generalization of the work of P. Dewilde [53], who also independently from author obtained Darlington representation in the form (18) for mvf's. P. Dewilde obtained his result as a generalization to nonrational mvf's of a result of V. Belevich [52], who generalized the Darlington method to ideal finite linear passive electrical multipoles with losses, using the scattering formalism, by representating a rational mvf s that is real contractive in $\mathbb{C}_+$ as a block in a real rational bi-inner mvf S. In [54] the problem of finding criteria for the existence of a bi-inner dilation S (without extra conditions on S) for a given operator function s, was formulated. This problem was solved after more than 30 years by the author with Olof Staffans [48]: a bi-inner dilation S for a Schur class operator function s exists if and only if the two factorization problems

$$I - s(\mu)^* s(\mu) = \varphi(\mu)^* \varphi(\mu) \quad \text{and} \quad I - s(\mu) s(\mu)^* = \psi(\mu) \psi(\mu)^* \quad \text{a.e.} \qquad (19)$$

in the Schur class of operator-valued functions φ and ψ are solvable.

My second dissertation was prepared for defence twice: first in 1977 and then again in 1983, because of anti-semitic problems. In 1977 I planned to defend it at Leningrad University. At that time I had moral support from V.P. Potapov, M.G. Krein, V.A. Yakubovich and A.M. Vershik, but that was not enough.

My contact with V.A. Yakubovich in 1977 led to our joint work [50], which he later built upon to further develop absolutely stability theory.

The defence of the second version of my second dissertation was held at the Institute of Mathematics AN USSR (Kiev, 1986). Again there was opposition because of the prevailing antisemitism, but this time this difficulty was overcome with the combined support of M.G. Krein, Yu.M. Berezanskii and my opponents M.L. Gorbachuk (who, as a gladiator, waged war with a my (so-called) black opponent and with the chief of the joint seminar, where my dissertation was discussed before its presentation for defence), S.V. Hruschev and I.V. Ostrovskii and V.P. Havin, who wrote external report on my dissertation. Moreover, after the defence, I heard that a positive opinion by B.S. Pavlov helped to generate acceptance by "VAK."

This dissertation was dedicated to further developments in the theory of passive linear time invariant systems with discrete and continuous time and with scattering matrices s, that are not bi-inner. In it the Darlington method was generalized so far as possible and was applied to obtain new functional models of conservative simple scattering realizations of scattering matrices s with losses inner scattering channels, as well as to obtain dissipative scattering realizations of s with losses external scattering channels. In particular, minimal dissipative and minimal optimal and minimal $*$-optimal realizations were obtained. Here the results on the generalized Lax–Phillips scattering scheme and the Nehari problem that were mentioned earlier were used and were further developed. Some of the results, that were presented in the dissertation are formulated above and some other will be formulated below.

My work on the Darlington method lead me to deeper investigations of the Nehari problem and to the study of generalized Schur and Carathéodory interpolation problems and their resolvent matrices. I introduced the class of γ-generating matrices

$$\mathfrak{A}(\varsigma) = \begin{pmatrix} p_-(\varsigma) & q_-(\varsigma) \\ q_+(\varsigma) & p_+(\varsigma) \end{pmatrix}, \qquad (20)$$

that describe the set of solutions $\mathcal{N}(\mathcal{T})$ of completely indeterminate Nehari problems by the formulas

$$\mathcal{N}(\mathcal{T}) = T_{\mathfrak{A}}(S^{p\times q}) \stackrel{\text{def}}{=} \{s = T_{\mathfrak{A}}(\varepsilon):\ \varepsilon \in S^{p\times q}\} \tag{21}$$

and (9).

Later, in joint work with Harry Dym, the matrix-valued functions in this class were called right regular γ-generating matrices and that class was denoted $\mathfrak{M}_{rR}(j_{pq})$. This class will be described below.

A matrix function $\mathfrak{A}(\zeta)$ with four block decomposition (20) belongs to the class $\mathfrak{M}_r(j_{pq})$ of right γ-generating matrices if it has j_{pq}-unitary values a.e. on the unit circle and its blocks are nontangential limits of mvf's $p_\pm$ and $q_\pm$ such that

$$\begin{gathered} s_{22} \stackrel{\text{def}}{=} p_+^{-1} \in S_{\text{out}}^{q\times q}, \quad s_{11} \stackrel{\text{def}}{=} (p_-^{\#})^{-1} \in S_{\text{out}}^{p\times p}, \\ s_{21} \stackrel{\text{def}}{=} -p_+^{-1}q_+ \ \in S^{q\times p}, \end{gathered} \tag{22}$$

where $S_{\text{out}}^{k\times k}$ is the class of outer matrix functions in the Schur class $S^{k\times k}$, $f^{\#}(z) = f(1/\overline{z})$. Formula (9) may be rewritten as a Redheffer transform:

$$s(\zeta) = R_S(\varepsilon) \stackrel{\text{def}}{=} s_{12}(\zeta) + s_{11}(\zeta)\varepsilon(\zeta)\left(I_q - s_{21}(\zeta)\varepsilon(\zeta)\right)^{-1} s_{22}(\zeta). \tag{23}$$

The matrix function $S(\cdot)$ with four blocks s_{jk} is the Potapov–Ginzburg transform of the matrix function $\mathfrak{A}(\cdot)$. If $\mathfrak{A} \in \mathfrak{M}_r(j_{pq})$ and s_0 is defined by (9) for some $\varepsilon \in S^{p\times q}$ and $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}(s_0)$ is defined by(8), then

$$T_{\mathfrak{A}}(S^{p\times q}) \subseteq \mathcal{N}(\mathcal{T}) \tag{24}$$

with equality if and only if $\mathfrak{A} \in \mathfrak{M}_{rR}(j_{pq})$. This result as well as related results on the description of the set of solutions of a c.i. (completely indeterminate) generalized Schur interpolation problem GSIP$(b_1, b_2; s^0)$ (by a linear fractional transformation based on a regular (later renamed as right regular in joint work with Harry Dym) j_{pq}-inner matrix function $W \in U_{rR}(j_{pq})$ (so-called resolvent matrix of the problem) and analogous results on the c.i. generalized Carathéodory interpolation problem GCIP$(b_3, b_4; c^0)$ and their resolvent matrices were obtained in the second dissertation and presented in [22]–[24].

The classes $\mathfrak{M}_{rR}(j_{pq})$ of right regular γ-generating matrices and $\mathcal{U}_S(J)$ and $\mathcal{U}_{rR}(J)$ of singular and right regular J-inner matrix functions are defined as follows: A J-inner matrix function U belongs to the class $\mathcal{U}_S(J)$ of singular J-inner matrix functions, if it is outer, i.e., if $U \in \mathcal{N}_{\text{out}}^{m\times m}$, where

$$\mathcal{N}_{\text{out}}^{m\times m} = \{f = g^{-1}h:\ h \in \mathcal{S}_{\text{out}}^{m\times m},\ \ g \in \mathcal{S}_{\text{out}}^{1\times 1}\}. \tag{25}$$

If a matrix function in the Nevanlinna class is identified with its nontangential boundary value, then $\mathcal{U}_S(j_{pq}) \subset \mathfrak{M}_r(j_{pq})$. Moreover, the product

$$\mathfrak{A} = \mathfrak{A}_1 W, \quad \text{where} \quad \mathfrak{A}_1 \in \mathfrak{M}_r(j_{pq}) \quad \text{and} \quad W \in \mathcal{U}_S(j_{pq}), \tag{26}$$

belongs to $\mathfrak{M}_r(j_{pq})$; and, by definition, $\mathfrak{A} \in \mathfrak{M}_{rR}(j_{pq})$, if in any of its factorizations (26), the factor W is a constant matrix. Every $\mathfrak{A} \in \mathfrak{M}_r(j_{pq})$ admits an essentially unique factorization (26) with $\mathfrak{A}_1 \in \mathfrak{M}_{rR}(j_{pq})$ and any matrix function $U \in \mathcal{U}(J)$ has an essentially unique factorization

$$U = U_1 U_2, \quad \text{where} \quad U_1 \in \mathcal{U}_{rR}(J) \quad \text{and} \quad U_2 \in \mathcal{U}_S(J). \tag{27}$$

A matrix function $U \in \mathcal{U}_{rR}(J)$, if it does not have nonconstant right divisors in $\mathcal{U}(J)$ that belong to $U_S(J)$. The classes $\mathcal{U}_{rR}(j_{pq})$ and $\mathcal{U}_{rR}(J_p)$ are the classes of resolvent matrices of c.i. GSIP's and GCIP's, respectively.

In a GSIP$(b_1, b_2; s^0)$, the matrix functions $b_1 \in \mathcal{S}_{\text{in}}^{p\times p}$, $b_2 \in \mathcal{S}_{\text{in}}^{q\times q}$ and $s^0 \in S^{p\times q}$ are given and the problem is to describe the set

$$\mathcal{S}(b_1, b_2; s^0) = \{s \in S^{p\times q} : \; b_1^{-1}(s - s^0)b_2^{-1} \in H_\infty^{p\times q}\} \tag{28}$$

This problem is called c.i. (completely indeterminate) if for every nonzero $\xi \in C^p$ there exists an $s \in S(b_1, b_2; s^0)$ such that $s(\lambda)\xi \neq s^0(\lambda)\xi$ for some $\lambda \in C_+$.

In a GCIP$(b_3, b_4; c^0)$, the matrix functions $b_3, b_4 \in \mathcal{S}_{\text{in}}^{p\times p}$ and $c^0 \in \mathcal{C}^{p\times p}$ are given and the problem is to describe the set

$$\mathcal{C}(b_3, b_4; c^0) = \{c \in \mathcal{C}^{p\times p} : \; b_3^{-1}(c - c^0)b_4^{-1} \in N_+^{p\times p}\}, \tag{29}$$

where

$$\mathcal{N}_+^{p\times p} = \{f \in \mathcal{N}^{p\times p} : \; f = g^{-1}h, \;\; g \in \mathcal{S}_{\text{out}} \quad \text{and} \quad h \in S^{p\times p}\}$$

is the Smirnov class of $p \times p$ matrix functions in $\mathbb{C}_+$. The definition of c.i. for such a problem is similar to the definition for a GSIP.

One of my methods for obtaining Darlington representations was based on these generalized interpolation problems. Thus, if $s \in \mathcal{S}^{p\times q}\Pi$ and $\|s\|_\infty < 1$, then it can be shown that there exists a pair $b_1 \in \mathcal{S}_{\text{in}}^{p\times p}$ and $b_2 \in \mathcal{S}_{\text{in}}^{q\times q}$ such that

$$b_2(I_q - s^\# s)^{-1} s^\# b_1 \in \mathcal{N}_+^{q\times p}, \quad \text{where} \;\; s^\#(\lambda) = s(\overline{\lambda})^*.$$

Then, the GSIP$(b_1, b_2; s^0)$ with $s^0 = s$ is s.c.i. (strictly completely indeterminate, i.e., it has a solution s with $\|s\|_\infty < 1$) and there exists a resolvent matrix $W \in \mathcal{U}_{rR}(j_{pq})$ such that $s = T_W(0)$. Thus, a Darlington representation of s is obtained by solving this GSIP. Moreover, if s_{11} and s_{22} are the diagonal blocks of $S = PG(W)$, then

$$b_1^{-1} s_{11} \in S_{\text{out}}^{p\times p} \quad \text{and} \quad s_{22} b_2^{-1} \in S_{\text{out}}^{q\times q}. \tag{30}$$

Later, in work with Harry Dym such a pair of inner mvf's was called an associated pair of W and the set of all associated pairs of W was denoted by ap(W). It was shown that: If $W \in \mathcal{E} \cap \mathcal{U}(j_{pq})$, i.e., if W is entire, and $\{b_1, b_2\} \in$ ap(W) then b_1 and b_2 are entire mvf's too. The converse is true, if W is right regular.

Analogous results were obtained for the Darlington representations of mvf's in the Carathéodory class, by consideration of c.i. GCIP's. In this case, the resolvent matrices $A \in \mathcal{U}(J_p)$ and associated pairs of the first and second kind are defined for such A in terms of the associated pairs of the mvf's

$$W(\lambda) = \mathfrak{V}A(\lambda)\mathfrak{V} \text{ and } B(\lambda) = A(\lambda)\mathfrak{V}, \text{ where } \mathfrak{V} = \frac{1}{\sqrt{2}}\begin{bmatrix} -I_p & I_p \\ I_p & I_p \end{bmatrix}. \tag{31}$$

If $[b_{21}\ b_{22}] = [0\ I_p]B$, then $(b_{21}^{\#})^{-1}$ and b_{22}^{-1} belong to $\mathcal{N}_+^{p\times p}$ and hence they have inner-outer and outer-inner factorizations, respectively. If b_3 and b_4 are inner $p\times p$ mvf's taken from these factorizations, then $\{b_3, b_4\}$ is called an associated pair for B and the set of all associated pairs of B is denoted $\mathrm{ap}(B)$. The set $\mathrm{ap}_I(A)$ and $\mathrm{ap}_{II}(A)$ of associated pairs of first and second kind for A are defined as

$$\mathrm{ap}_I(A) = \mathrm{ap}(W) \quad \text{and} \quad \mathrm{ap}_{II}(A) = \mathrm{ap}(B).$$

Additional details on GSIP's, GCIP's, resolvent matrices and associated pairs of mvf's may be found in the monographs [27], [28] with Harry. Results, related to entire J-inner mvf's are used extensively in [28] in the study of bitangential direct and inverse problems for canonical integral and differential systems.

5. Development of the theory of passive systems by my graduate students

An important contribution to my efforts to develop the theory of passive linear stationary systems, J-inner matrix functions and related problems was made by my graduate students: L. Simakova, M.A. Nudelman (his main advisor was V.A. Yakubovich), L.Z. Grossman, S.M. Saprikin, N.A. Rozhenko, D. Pik (his main advisor was M.A. Kaashoek), see [43]–[46], [37]–[39] and references cited therein. I also helped to advise the works of O. Nitz, D. Kalyuzjnii-Verbovetskii, and M. Bekker (his advisor was M.G. Krein and I was his a nonformal advisor). The main results of Simakova, with complete proofs, may be found in [27]. She studied the mvf's W meromorphic in Ω such that $T_W(\mathcal{S}^{p\times q}) \subset \mathcal{S}^{p\times q}$ and mvf's A such that $T_A(\mathcal{C}^{p\times p}) \subset \mathcal{C}^{p\times p}$. She showed that if $\det W \not\equiv 0$ (resp., $\det A \not\equiv 0$) then the first (resp., second) inclusion holds if and only if $\rho W \varepsilon \mathcal{P}(j_{pq})$ (resp., $\rho A \varepsilon \mathcal{P}(J_p)$) for some scalar function ρ that is meromorphic in Ω. With M. Nudelman we further developed the theory of passive scattering and impedance systems with continuous time. In particular a criterium for all the minimal passive realizations of a given scattering (impedance) matrix to be similar was obtained in [42].

The role of scattering matrices in the theory of unitary extensions of isometric operators was developed with L. Grossman in [36].

The Darlington method was extended with N. Rozhenko in [43] and other papers, cited therein. Darlington representations were extended to mvf's in the generalized Schur class $S_{\chi}^{p\times q}$ with S. Saprikin [45].

A theory of Livsic–Brodskii J-nodes with right strongly regular characteristic mvf's was developed by my daughter Zoya Arova in [51] (her official advisors were I.S. Kac, and M.A. Kaashoek).

6. Joint research with B. Fritzsche and B. Kirstein on J-inner mvf's (1989–97)

After "perestroika" I had the good fortune to work with mathematicians from outside the former Soviet Union. First I worked in Leipzig University with the two Bernds: B.K. Fritzsche and B.E. Kirstein. Mainly we worked on completion problems for (j_p, J_p)-inner matrix functions (see, e.g., [32] and the references inside) and on parametrization formulas for the sets of solutions to c.i. Nehari and GCIP's [34], [33]. We worked together for 10 years, and published 9 papers. In Leipzig University I also collaborated with I. Gavrilyuk on an application the Cayley transform to reduce the solution of a differential equation to the solution of a corresponding discrete time equation (see, e.g., [35]).

7. Joint research on passive scattering theory with M.A. Kaashoek (and D. Pik) with J. Rovnjak (and S. Saprikin)

During the years 1994–2000 I worked in Amsterdam Vrije Universiteit with Rien Kaashoek and our graduate student Derk Pik on further developments in the theory of passive linear scattering systems (see [37], [38] and the references inside). Derk generalized the Darlington method to nonstationary scattering systems. Then in the years 2000 and 2001 I visited University of Virginia for one month each year to work with Jim Rovnyak. Subsequently, Jim invited S.M. Saprikin to visit him for one month in order to help write up our joint work. Our results were published in [21] and [44].

8. Joint research with Olof J. Staffans (and M. Kurula) on passive time-invariant state/signal systems theory (2003–2014)

I first met Olof at the MTNS Conference in 2002, where he presented his view on conservative and passive infinite-dimensional systems [62]. We discovered that we have a common interest in passive linear systems theory. After this meeting he invited me to visit him each year for two or three months to pursue joint work on the further development of passive linear time invariant systems theory. We wrote a number of papers together. In particular, [47] on the Kalman–Yakubovich–Popov inequality for continuous time systems and [48] that was mentioned earlier. However, the main focus of our work was in a new direction that we call "state/signal" (s/s for short) systems theory.

In this new direction instead of input and output data u and y, that are considered in i/s/o (input/state/output) systems theory, only one external signal w in a vector space W with a Hilbert space topology is considered. Thus, in a linear stationary continuous time s/s system a classical trajectory $(x(t), w(t))$ on an interval I is considered, where the state component $x(t)$ is a continuously differentiable function on I with values from a vector space X with a Hilbert space topology ($x \in C^1(X; I)$), signal component $w(t)$ is a continuous function on I with values from W ($w \in C(W, I)$) and they satisfy the conditions

$$dx/dt = F(x(t), w(t)), \tag{32}$$

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \in \mathcal{D}(F), \tag{33}$$

where F is a closed linear operator, acting from $X \times W$ into X with domain $\mathcal{D}(F)$ such that the subset

$$X_0 = \left\{ x \in X : \begin{bmatrix} x \\ w \end{bmatrix} \in \mathcal{D}(F) \text{ for some } w \in W \right\}$$

is dense in X. A generalized trajectory $(x(t), w(t))$ of the system is defined as the limit in $C(X; I) \times L^2_{\mathrm{loc}}(W; I)$ of a sequence of classical trajectories.

Mainly we study the so-called future (or past, or two-sided) solvable systems for which the set of classical trajectories on $\mathbb{R}_+ = [0, \infty)$ (or $\mathbb{R}_- = (-\infty, 0]$, or $\mathbb{R}$) is not empty for any $x(0) \in X_0$.

A discrete time s/s system is defined analogously. The only change is that difference $x(t+1) - x(t)$ is considered instead of the derivative, F is a bounded operator on a closed domain and $X_0 = X$.

If W can be decomposed as (an ordered) direct sum $W = \left[\begin{smallmatrix} U \\ Y \end{smallmatrix}\right]$ of two closed subspaces U and Y such that the system (32), (33) is equivalent to the system

$$\begin{bmatrix} \frac{dx}{dt} \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}(S), \tag{34}$$

where S is a linear closed operator, acting from $X \times U$ into $X \times Y$ with domain $\mathcal{D}(S)$ that has certain properties (in particular, main operator A of the system is defined on a dense domain in X as the projection onto X of the restriction of S to $\mathcal{D}(S) \cap (X \times \{0\})$) and $w(t) = (u(t), y(t))$, then this decomposition is called admissible and the corresponding i/s/o system $\sum_{i/s/o} = (S; X, U, Y)$ (including classical and generalized trajectories $(x(t), u(t), y(t))$ on the intervals I) is called an i/s/o representation of the s/s system $\sum = (V; X, W)$, where V is the graph of the operator F in (32), (33). (In general, we prefer to use the graph V instead of the operator F.) Some results on passive linear stationary continuous time s/s systems and their i/s/o representations we obtained in joint work with Mikael Kurula, a former graduate student of Olof Staffans, see, e.g., [40], [41] and the references inside.

Our results on linear time invariant s/s systems with continuous time are summarized in the monograph [49] that is still in electronic version. In the last chapter of this monograph, passive systems of this kind are considered.

We also plan to write a separate monograph dedicated to passive systems. In a passive s/s system, X is a Hilbert space and W is a Krein space and V is a maximal nonnegative subspace in the Krein (node) space $\mathfrak{K} = X \boxplus X \boxplus W$. Any fundamental decomposition $W = W_+ \boxplus (-W_-)$ of the Krein signal space W is admissible for such a system. The corresponding i/s/o representation of this system is called a scattering representation of the system and is denoted $\sum_{\mathrm{sc}} = (S; X, W_+, W_-)$. The notion of a passive i/s/o scattering system $\sum_{\rm sc} = (S; X, U, Y)$ is introduced and it is shown that any such system is a scattering i/s/o representation of a certain passive s/s system with Krein signal space $W = U \boxplus (-Y)$. Moreover, the transfer function of any scattering passive i/s/o system, scattering matrix, is holomorphic in $\mathbb{C}_+$ and its restriction to $\mathbb{C}_+$ belongs to the Schur class $\mathcal{S}(U, Y)$ of holomorphic contractive functions with values from $B(U, Y)$. If $\dim W_- = \dim W_+$, then the s/s system Σ may have a Lagrangian decomposition $W = U \dotplus Y$, i.e., both closed subspaces U and Y are neutral subspaces in W. The corresponding i/s/o representation of Σ is called an impedance representation and it is denoted by $\sum_{\rm imp} = (S; X, U, Y)$. A third significant class of i/s/o representations of a passive s/s system is the class of transmission representations $\sum_{\rm tr} = (S; X, U, Y)$ in which U and Y are orthogonal in the Krein signal space W. Thus a passive s/s system with a Krein signal space W with indefinite metric has infinitely many scattering representations and may also have impedance and transmission representations. Correspondingly, it has infinitely many scattering matrices and may have impedance and transmission matrices, transfer functions of these representations.

If V is a Lagrangian subspace in a Krein node space, then the system is called conservative. To each such system there correspond conservative scattering (impedance and transmission) representations. The notions of dilation and compression may be introduced for an s/s system Σ and an i/s/o system. A conservative s/s system is called simple, if it is not the dilation of another conservative s/s system. A passive s/s system that is not a dilation of an other s/s system is called minimal. It is shown that every conservative s/s system is the dilation of a simple conservative system and every passive s/s system is the dilation of a minimal passive s/s system. The notions of incoming and outgoing scattering channels are introduced for a passive s/s system in a natural way. The scattering matrices of a passive i/s/o system and its compression coincide in $\mathbb{C}_+$.

By focusing on the Laplacian transformations of the components of the trajectories, we came to the notion of the resolvent set $\rho(\Sigma)$ of an s/s system $\Sigma = (V; X; W)$. The systems for which $\rho(\Sigma) \neq \varnothing$ (i.e., the class of regular systems) are studied and the i/s/o resolvent functions $\widehat{\mathfrak{G}}(\lambda)$ for Σ and in its four block decomposition its four blocks $\widehat{\mathfrak{A}}(\lambda)$ (s/s resolvent function), $\widehat{\mathfrak{B}}(\lambda)$ (i/s resolvent function), $\widehat{\mathfrak{C}}(\lambda)$ (s/o resolvent function) and $\widehat{\mathfrak{D}}(\lambda)$ (i/o resolvent function) are defined by a frequency domain admissible ordered sum decomposition $W = U \dotplus Y = \left[\begin{smallmatrix} U \\ Y \end{smallmatrix}\right]$ of W as follows. A point $\lambda \in \rho(\Sigma)$ if there exists a (frequency domain admissible) decomposition $W = \left[\begin{smallmatrix} U \\ Y \end{smallmatrix}\right]$ such that for any $x_0 \in X$ and $\widehat{u}(\lambda) \in U$ the condition

$$\begin{bmatrix} \lambda \widehat{x}(\lambda) - x_0 \\ \widehat{x}(\lambda) \\ \widehat{w}(\lambda) \end{bmatrix} \in V \quad \text{with} \quad \widehat{w}(\lambda) = \begin{bmatrix} \widehat{u}(\lambda) \\ \widehat{y}(\lambda) \end{bmatrix}$$

is equivalent to the equation

$$\begin{bmatrix} \widehat{x}(\lambda) \\ \widehat{y}(\lambda) \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \begin{bmatrix} x_0 \\ \widehat{u}(\lambda) \end{bmatrix},$$

where four block operator on the right-hand side is bounded and acts between vector spaces with Hilbert space topologies. In a natural way the notions of Ω-dilation, Ω-compression, Ω-restriction, Ω-projection are introduced for two regular s/s systems $\sum_i = (V_i; X_i, W)$ and an open set $\Omega \subseteq \rho(\Sigma_1) \cap \rho(\Sigma_2)$. The notions of dilation, compression, restriction and projection we introduced and study in the time domain for s/s and i/s/o systems too and even for so-called s/s pre-systems, in which the generating subspace V may be the graph of a multi-valued closed operator F, and for i/s/o pseudo-systems, in which the operator S may be multi-valued. In the time domain these notions are mainly reasonable for the so-called well-posed i/s/o systems and the well-posed s/s systems. A chapter in our monograph [49] is devoted to well-posed i/s/o systems that is adapted from the monograph [63] by Olof. Another chapter is devoted to well-posed s/s systems, i.e., to systems that have at least one well-posed i/s/o representation. In particular, any passive s/s system is well posed.

9. Joint research with Harry Dym on the theories of J-inner mvf's and de Branges spaces and their applications to interpolation, extrapolation and inverse problems and prediction (1992–2014)

I started to work with Harry Dym on the development of the theory of J-contractive matrix functions and related problems in 1992. Every year since then I have visited the Weizmann Institute of Science (for 3 or more months). The results of the more than 20 years of our joint research were published in a series of papers that are mostly summarized in our monographs [27], [28] (where can be founded references to our other publications). The history of the start of our joint work may be found in the introduction to [27]. At the outset I was familiar with Harry's monograph [55], with his papers with I. Gohberg on the Nehari problem, with P. Dewilde on Darlington representation and the entropy functional, with D. Alpay on J-inner matrix functions, de Branges RKHS's (Reproducing Kernel Hilbert Spaces) and some of their applications to inverse problems and to the Krein resolvent matrices for symmetric operators. I found that Harry was familiar with much of the work that was done by M.G. Krein and his school. He also had experience in the development of L. de Branges theory of RKHS's and their applications to the interpolation problems and inverse problems. Before I began to work with Harry, I had no experience with de Branges RKHS's and their applications.

As I noted earlier, the results of our joint work up to 2012 are mainly summarized in our monographs [27], [28]. In particular, these volumes include applications of our results on right regular and strongly right regular mvf's to interpolation and extension problems in special classes of mvf's (Schur, Carathéodory, positive definite, helical) and inverse problems for canonical integral and differential systems of equations and for Dirac–Krein system. Functional models for nonselfadjoint operators (Livsic–Brodskii operator nodes and their characteristic functions) are also presented; other models may be found in [51].

After this we worked on the application of these results to prediction problems for second-order multi-dimensional stochastic processes: ws (weakly stationary) processes and processes with ws-increments. In the course of this work the theory of de Branges RKHS's, J-inner matrix functions, extension problems and inverse problems for canonical integral and differential systems were developed further. Some of these more recent results are summarized in the papers [29], [30] and in a monograph [31], which is currently being prepared for publication. Below I will mention only some highlights of our results on the classes $\mathcal{U}_{rR}(J)$ and $\mathcal{U}_{rsR}(J)$ of right regular and right strongly regular J-inner mvf's, and two classes of de Branges spaces that are connected with them: $\mathcal{H}(U)$ and $\mathcal{B}(\mathcal{E})$. Both of these spaces are RKHS's (Reproducing kernel Hilbert Spaces).

Recall that for every $U \in \mathcal{U}(J)$, there corresponds a RKHS $\mathcal{H}(U)$ with the RK (Reproducing Kernel)

$$K_\omega^U = \frac{J - U(\lambda)JU(\omega)^*}{-2\pi i(\lambda - \overline{\omega})},$$

$\lambda, \omega \in h_U$ (extended to $\lambda = \overline{\omega}$ by continuity), where $\mathfrak{h}_U$ denotes the domain of holomorphy of the mvf U in the complex plane. Then $\mathcal{H}(U)$ is the Hilbert space of (holomorphic) $m \times 1$ vector functions on $\mathfrak{h}_U$ such that:

1) $K_\omega^U \xi \in \mathcal{H}(U)$ for every $\omega \in \mathfrak{h}_U$ and $\xi \in \mathbb{C}^m$.
2) $\xi^* f(\lambda) = (f, K_\lambda^U \xi)_{\mathcal{H}(U)}$ for every $\xi \in \mathbb{C}^m$, $f \in \mathcal{H}(U)$ and $\lambda \in \mathfrak{h}_U$.

It was shown that $\mathcal{H}(U) \subset \Pi^m$ and that $\mathcal{H}(U) \subset \mathcal{E} \cap \Pi^m$ (the entire vector functions in Π^m) if and only if U is an entire J-inner mvf (i.e., if and only if $U \in \mathcal{E} \cap U(J)$)

There exist a number of different ways to characterize the classes $U_S(J)$, $U_{rR}(J)$ and $U_{rsR}(J)$. In particular (upon identifying vvf's in Π^m with their non-tangential boundary values):

1) $U \in \mathcal{U}_S(J)$ if and only if $\mathcal{H}(U) \cap L_2^m = \{0\}$;
2) $U \in \mathcal{U}_{rR}(J)$ if and only if $\mathcal{H}(U) \cap L_2^m$ is dense in $\mathcal{H}(U)$;
3) $U \in \mathcal{U}_{rsR}(J)$ if and only if $\mathcal{H}(U) \subset L_2^m$.

The last condition led us to a criteria for right strongly regularity in terms of the matricial Treil–Volberg version of the Muckenhoupt condition for a matricial weight, defined by the mvf U.

The class $\mathcal{E} \cap \mathcal{U}_{rR}(J_p)$ coincides with the class of resolvent matrices of c.i. generalized Krein helical extension problems and we extensively exploited results on this class in the study of direct and inverse problems for canonical systems. Moreover, the classes $U_{rsR}(j_{pq})$ and $U_{rsR}(J_p)$ coincide with the classes of resolvent matrices for strictly completely indeterminate generalized Schur and Carathéodory interpolation problems. We presented algebraic formulas for resolvent matrices in this last setting in terms of the given data of the problems.

Another kind of de Branges RKHS that we studied and exploited for spectral analysis and prediction problems is the space $\mathcal{B}(\mathcal{E})$, that is defined by a $p \times 2p$ mvf $\mathcal{E} = [E_- \; E_+]$ that is meromorphic in $\mathbb{C}_+$ with two $p \times p$ blocks $E_\pm$ such that

$$\det E_+ \not\equiv 0 \quad \text{and} \quad E_+^{-1} E_- \in S_{\text{in}}^{p \times p}.$$

For such a mvf, the RK

$$K_\omega^{\mathcal{E}} = \frac{\mathcal{E}(\lambda)j_p\mathcal{E}(\omega)^*}{2\pi i(\lambda - \overline{\omega})},$$

(extended to $\lambda = \overline{\omega}$ by continuity) is positive on $\mathfrak{h}_{\mathcal{E}} \times \mathfrak{h}_{\mathcal{E}}$. Our main interest in the class of de Branges matrices is in the subclass $\mathcal{I}(j_p)$ of de Branges matrices $\mathcal{E}$ for which $B(\mathcal{E})$ is invariant under the generalized backwards shift operator

$$(R_\alpha f)(\lambda) = \begin{cases} \dfrac{f(\lambda) - f(\alpha)}{\lambda - \alpha} & \text{for } \lambda \neq \alpha \quad \text{and} \\ f'(\alpha) & \text{for } \lambda = \alpha, \end{cases}$$

for $f \in \mathcal{B}(\mathcal{E})$ and $\alpha \in \mathfrak{h}_{\mathcal{E}}$. The formula

$$\mathcal{E}^U = \begin{bmatrix} E_-^U & E_+^U \end{bmatrix} = \begin{bmatrix} UP_+ + P_- & UP_- + P_+ \end{bmatrix}, \text{ where } P_\pm = \frac{1}{2}(I_m \pm J),$$

associates a de Branges matrix $\mathcal{E}^U \in \mathcal{I}(J_m)$ with every $U \in \mathcal{U}(J)$. Moreover, U is an entire mvf if and only if $\mathcal{E}^U$ is an entire mvf, and $U \in \mathcal{U}^0(J)$ (i.e., U is holomorphic at 0 with $U(0) = I_m$) if and only if $\mathcal{E}^U \in \mathcal{I}^0(j_m)$ (i.e., $\mathcal{E}^U$ is holomorphic at 0 and $\mathcal{E}^U(0) = [I_m \ I_m]$). Furthermore, it is easy to check that $K_\omega^{\mathcal{E}^U} = K_\omega^U$ and hence that $\mathcal{B}(\mathcal{E}^U) = \mathcal{H}(U)$. This connection between the two kinds de Branges RKHS's was exploited in [29], [30].

Another correspondence between the classes $\mathcal{U}(J_p)$ and $\mathcal{I}(j_p)$ is established by the formula

$$\mathcal{E}_A = [E_- \ E_+] = [a_{22} - a_{21} \ \ a_{22} + a_{21}] = \sqrt{2}[0 \ I_p]B, \quad \text{for } A \in \mathcal{U}(J_p),$$

where B is defined in (31). Moreover, A is an entire mvf if and only if $\mathcal{E}_A$ is an entire mvf and, if $A \in \mathcal{U}^0(J_p)$, then $\mathcal{E}_A \in \mathcal{I}^0(j_p)$.

A mvf $A \in \mathcal{U}(J_p)$ is said to be perfect if the mvf $c = T_A(I_p)$ satisfies the condition

$$\lim_{\nu \to \infty} \nu^{-1} \Re c(i\nu) = 0.$$

For each $\mathcal{E} \in \mathcal{I}^0(j_p)$, there exists exactly one perfect mvf $A \in \mathcal{U}^0(J_p)$ such that $\mathcal{E} = \mathcal{E}_A$. This two-sided connection between the classes $U^0(J_p)$ and $\mathcal{I}^0(j_p)$ was extensively exploited in our study of direct and inverse spectral problems, as well as in a number of extension and prediction problems and their bitangential generalizations.

References

[1] V.M. Adamjan, Nondegenerate unitary couplings of semiunitary operators, Funkc. Anal. and its Applic., 7, no. 4 (1973), 1–16.

[2] V.M. Adamjan and D.Z. Arov, On a class of scattering operators and characteristic functions of contractions, DAN SSSR, 160, no. (1965), 9–12.

[3] V.M. Adamjan and D.Z. Arov, On scattering operators and contraction semigroups in Hilbert space, DAN SSSR, 165, no. (1965), 9–12.

[4] V.M. Adamjan and D.Z. Arov, Unitary couplings of semiunitary operators, Akad. Nauk Armjan. SSR Dokl., 43, no. 5 (1966), 257–263.

[5] V.M. Adamjan, D.Z. Arov, On the unitary coupling of semi-unitary operators, Math. Issl., Kishinev, v. 1, no. 2 (1966), 3–64.

[6] V.M. Adamjan and D.Z. Arov, A general solution of a certain problem in the linear prediction of stationary processes, Teor. Verojatn. i Primenen., 13 (1968), 419–431.

[7] V.M. Adamjan, D.Z. Arov and M.G. Krein, Infinite Hankel matrices and generalized problems of Carathéodory–Fejér and F. Riesz, Functional. Anal. I Prilojen. 2, no. 1 (1968), 1–19.

[8] V.M. Adamjan, D.Z. Arov and M.G. Krein, Infinite Hankel matrices and generalized Carathéodory–Fejér and I. Schur problems, Functional. Anal. i Prilojen. 2, no. 4 (1968), 1–17.

[9] V.M. Adamjan, D.Z. Arov and M.G. Krein, Infinite Hankel block matrices and related problems of extension, Izv. Akad. Nauk Armjan. SSR, Ser. Mat., 6, no. 2-3 (1971), 87–112.

[10] V.M. Adamjan, D.Z. Arov and M.G.Krein, Analytic properties of the Schmidt pairs of a Hankel operator and the generalized Schur–Takagy problem, Math. Sborn. (N.S.) 86 (1971), 34–75.

[11] D.Z. Arov, The theory of information and its transfer through communication channels, unpublished student work, Odessa State University, 1957.

[12] D.Z. Arov, Some problems of metrical theory of dynamical systems, unpublished manuscript (the dissertation, Odessa State Pedagogical Institute, 1964).

[13] D.Z. Arov, Topological similarity of automorphisms and translations of compact commutative groups, Uspekhi Mat. Nauk 18, no. 5 (1963), 133–138.

[14] D.Z. Arov, The calculation of the entropy for a class of the groups endomorphisms, Zap. Meh. Mat. Fak. Har'kov Gos. Univ. I Har'kov Mat. Obsc., 4, 30(1964), 48–69.

[15] D.Z. Arov, Darlington's method in the study of dissipative systems (Russian), DAN SSSR, v. 201, no. 3 (1971), 559–562; translation in Soviet Physics Dokl. 16 (1971), 954–956.

[16] D.Z. Arov, Realization of matrix valued functions according to Darlington (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 37 (1973), 1299–1331.

[17] D.Z. Arov, Unitary couplings with losses (a theory of scattering with losses), (Russian) Functional Anal. i Prilozhen. 8 (1974), no. 4, 5–22.

[18] D.Z. Arov, Scattering theory with dissipative of energy. (Russian) Dokl. Akad. Nauk SSSR, 216(1974), 713–716.

[19] D.Z. Arov, Realization of a canonical system with a dissipative boundary condition at one of the segment in terms of the coefficient of dynamical compliance (Russian) Sibirsk. Mat. Zh. 16 (1975), no. 3, 440–463, 643.

[20] D.Z. Arov, Passive linear steady-state dynamical systems (Russian), Sibirsk. Mat. Zh. 20 (1979), no. 2, 211–228, 457.

[21] D.Z. Arov, Stable dissipative linear stationary dynamical scattering systems (Russian) J. Operator Theory 2 (1979), no. 1, 95–126; translation with appendices by the author and J. Rovnjak in Oper. Theory: Adv. and Appl., 134, (1999), 99–136.

[22] D.Z. Arov, Regular J-inner matrix-functions and related continuation problems. Linear operators in function spaces (Timisoara, 1988), 63–87, Operator Theory Adv. Appl., 43, 1990.

[23] D.Z. Arov, Gamma-generating matrices, J-inner matrix functions and related extrapolation problems, I, II, III, IV. Teor. Functii, Funct. Anal. i Priloz. no.51 (1989), 61–67; no. 52 (1989), 103–109; no. 53 (1990), 57–65; Mat. Fiz. Anal. Geom. 2 (1995), no. 1, 3–14.

[24] D.Z. Arov, The generalized bitangential Carathéodory–Nevanlinna–Pick problem and (j, J_0)-inner matrix functions. Izv. Ross. Akad. Nauk, Ser. Mat. 57 (1993), no. 1, 3–32.

[25] D.Z. Arov, The influence of V.P. Potapov and M.G. Krein on my scientific work, Operator theory: Advances and Applications, v. 72 (1994), 1–16.

[26] D.Z. Arov, On the origin history of the notion of the ε-entropy of a Lebesgue space automorphism and the notion of the (ε, T)-entropy of a dynamical system with continuous time (with a comment by A.M. Vershik). Zapiski Nauchnyh Seminarov POMI 436 (2015), "Representation theory, dynamical systems, combinatorial and algorithmic methods" XXV, 76–100. (Russian)

[27] D.Z. Arov and H. Dym, J-contractive matrix valued functions and related topics. In Encyclopedia of Mathematics and its Applications, v. 116, Cambridge University Press, 2008.

[28] D.Z. Arov and H. Dym, Bitangential direct and inverse problems for systems of integral and differential equations. In Encyclopedia of Mathematics and its Applications, v. 145, Cambridge University Press, 2012.

[29] D.Z. Arov and H. Dym, de Branges spaces of vector valued functions, electronic version in Operator Theory, Springer.

[30] D.Z. Arov and H. Dym, Applications of de Branges spaces of vector valued functions, electronic version in Operator Theory, Springer.

[31] D.Z. Arov and H. Dym, Multivariate prediction, de Branges spaces and related extension and inverse problems, (unpublished monograph, 280 pp.).

[32] D.Z. Arov, B. Fritzsche and B. Kirstein, On block completion problems for (j_p, J_p)-inner functions. II. The case of a given $q \times q$ block, Int. Eq. Op. Th., v. 18 (1994), no. 3, 245–260.

[33] D.Z. Arov, B. Fritzsche and B. Kirstein, A function-theoretic approach to a parametrization of the set of solutions of a completely indeterminate matricial Nehari problem, Int. Eq. OT, v. 30 (1998), no. 1, 1–66.

[34] D.Z. Arov, B. Fritzsche and B. Kirstein, On a Parametrization Formula for the Solution Set of Completely Indeterminate Generalized Matricial Carathéodory–Fejér Problem, Math. Nachr., v. 219 (2000), no. 1, 5–43.

[35] D.Z. Arov and I.P. Gavrilyuk, A method for solving initial value problems for linear differential equations in Hilbert space based on Cayley transform. Numer. Funct. Anal. Optim. 14 (1993) no. 5-6, 459–473.

[36] D.Z. Arov and L.Z. Grossman, Scattering matrices in the theory of unitary extension of isometric operators, Math. Nachr. 157 (1992), 105–123.

[37] D.Z. Arov, M.A. Kaashoek and D.R. Pik, Minimal and optimal linear discrete time-invariant dissipative scattering systems, Integral Equations Operator Theory 29 (1997), no. 2, 127–154.

[38] D.Z. Arov, M.A. Kaashoek and D.R. Pik, Optimal time-variant systems and factorization of operators. II. Factorization, J. Operator Theory 43 (2000), no. 2, 263–294.

[39] D.Z. Arov, M.A. Kaashoek and D.R. Pik, The Kalman–Yakubovich–Popov inequality for discrete time systems of infinite dimension, J. Operator Theory 55 (2006), no. 2, 393–438.

[40] D.Z. Arov, M. Kurula and O.J. Staffans, Boundary control state/signal systems and boundary triplets, Chapter 3 in Operator Methods for Boundary Value Problem, Cambridge University Press, 2012, 35–72.

[41] D.Z. Arov, M. Kurula and O.J. Staffans, Passive state/signal systems and conservative boundary relations, Chapter 4 in Operator Methods for Boundary Value Problem, Cambridge University Press, 2012, 73–119.

[42] D.Z. Arov and M.A. Nudelman, Conditions for the similarity of all minimal passive realizations of a given transfer function (scattering and resistence matrices). Mat. Sb., 193 (2002), no. 5-6, 791–810.

[43] D.Z. Arov and N.A. Rozhenko, Realization of stationary stochastic processes: applications of passive systems theory. Methods Funct. Anal. Topology, 18, (2012), no. 4, 305–312.

[44] D.Z. Arov, J. Rovnjak and S.M. Saprikin, Linear passive stationary scattering systems with Pontryagin state spaces. Math Nachr. 279 (2006), no. 13-14, 1396–1424.

[45] D.Z. Arov and S.M. Saprikin, Maximal solutions for embedding problem for a generalized Schur function and optimal dissipative scattering systems with Pontryagin state spaces, Methods Funct. Anal. Topology 7(2001), no. 4, 69–80.

[46] D.Z. Arov and L.A. Simakova, The boundary values of a convergenct sequence of J-contractive matrix valued functions, Mat. Zametki 19 (1976), no. 4, 491–500.

[47] D.Z. Arov and O.J. Staffans, The infinite-dimensional continuous time Kalman–Yakubovich–Popov inequality. The extended field of operator theory (M. Dritschel, ed.) Oper. Theory Adv. Appl., 171, 2007, 37–72.

[48] D.Z. Arov and O.J. Staffans, Bi-inner dilations and bi-stable passive scattering realizations of Schur class operator-valued functions, Integral Equations Operator Theory 62 (2008), no. 1, 29–42.

[49] D.Z. Arov and O.J. Staffans, Linear stationary systems in continuous time, electronic version, 2014.

[50] D.Z. Arov and V.A. Yakubovich, Conditions of semiboundness of quadratic functionals on Hardy spaces. Vestnik Leningrad Univ., Math., Mekh. Astron.(1982), no. 1, 11–18, Engl. Transl. in Vestnik Leningrad. Univ. Math. 15 (1982).

[51] Z. Arova, Operator nodes with strongly regular characteristic functions, Printed by Huisdrukkerij Vrije Universiteit, Amsterdam, The Netherlands, 2003.

[52] V. Belevich, Factorizations of scattering matrices with applications to passive network synthesis, Phillips Research Reports, 18 (1963), 275–317.

[53] P. Dewilde, Roomy scattering matrix synthesis, Technical Report, Berkeley, 1971.

[54] R.G. Douglas and J.W. Helton, Inner dilation of analytic matrix function and Darlington synthesis, Acta Sci. Math. (Szeged), 34 (1973), 61–67.

[55] H. Dym, J-contractive matrix functions, reproducing kernel Hilbert spaces and interpolation, CBMS Regional Conference series, 71, AMS, Providence, RI, 1989.

[56] A.N. Kolmogorov, A new metrical invariant of transitive dynamical systems and automorphismes of Lebesquespace, DAN SSSR, 119, no. 5 (1958), 861–864.

[57] A.N. Kolmogorov, On the entropy on the unit of time as a metrical invariant of the automorphismes, DAN SSSR, 124, no. 4 (1959), 754–759.

[58] M.S. Livsic, On the application nonselfadjoint operators in the scattering theory, Journ. experiment. and theoret. Physics, 31, no. 1 (1956), 121–131.

[59] E.Ya. Malamud, On a generalization of a Darlington theorem, Izvest. AN Arm. SSR, v.7, no. 3 (1972), 183–195.

[60] S. Seshu and M.B. Reed, Linear graphs and electrical networks, Addison-Wesley publishing company, London, England, 1961.

[61] Ya.G. Sinai, On the notion of entropy of a dynamical system, DAN SSSR, 124, no. 4 (1959), 768–771.

[62] O.J. Staffans, Passive and conservative infinite dimensional impedance and scattering systems (from a personal point of view). In Mathematical Systems Theory in Biology, Communication and Finance, v. 134 of IMA Volumes in Mathematics and Applications, pp. 375–414.

[63] O.J. Staffans, *Well-posed linear systems*. Encyclopedia of Mathematics and its Applications, 103. Cambridge University Press, Cambridge, 2005.

[64] A.M. Vershik, Polymorphisms, Markov processes, and quasi-similarity, Discrete Contin. Dyn. Syst., 13, no. 5, 1305–1324 (2005).

Damir Z. Arov
South Ukrainian National Pedagogical University
Institute of Physics and Mathematics
Division of Informatics and Applied Mathematics
Staroportofrankovskaya st. 26
65020 Odessa, Ukraine
e-mail: arov_damir@mail.ru

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Generic rank-k Perturbations of Structured Matrices

Leonhard Batzke, Christian Mehl, André C.M. Ran
and Leiba Rodman

Abstract. This paper deals with the effect of generic but structured low rank perturbations on the Jordan structure and sign characteristic of matrices that have structure in an indefinite inner product space. The paper is a follow-up of earlier papers in which the effect of rank one perturbations was considered. Several results that are in contrast to the case of unstructured low rank perturbations of general matrices are presented here.

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Keywords. H-symmetric matrices, H-selfadjoint matrices, indefinite inner product, sign characteristic, perturbation analysis, generic structured low rank perturbation.

1. Introduction

In the past two decades, the effects of generic low rank perturbations on the Jordan structure of matrices and matrix pencils with multiple eigenvalues have been extensively studied, see [5, 9, 20, 21, 23, 24]. Recently, starting with [15] the same question has been investigated for generic structure-preserving low rank perturbations of matrices that are structured with respect to some indefinite inner product. While the references [5, 9, 20, 21, 23, 24] on unstructured perturbations have dealt with the general case of rank k, [15] and the follow-up papers [16]–[19] on structure-preserving perturbations focussed on the special case $k = 1$. The reason for this restriction was the use of a particular proof technique that was based on the so-called Brunovsky form which is handy for the case $k = 1$ and may be for the case $k = 2$, but becomes rather complicated for the case $k > 2$. Nevertheless,

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A large part of this work was done while Leiba Rodman visited at TU Berlin and VU Amsterdam. We are very sad that shortly after finalizing this paper, Leiba passed away on March 2, 2015. We will remember him as a dear friend and we will miss discussing with him as well as his stimulating interest in matters concerning matrices and operators in indefinite inner product spaces.

the papers [15]–[19] (see also [6, 10]) showed that in some situations there are surprising differences in the changes of Jordan structure with respect to general and structure-preserving rank-one perturbations. This mainly has to do with the fact that the possible Jordan canonical forms for matrices that are structured with respect to indefinite inner products are restricted. This work has later been generalized to the case of structured matrix pencils in [1]–[3], see also [4]. Although a few questions remained open, the effect of generic structure-preserving rank-one perturbations on the Jordan structure and the sign characteristic of matrices and matrix pencils that are structured with respect to an indefinite inner product seems now to be well understood.

In this paper, we will consider the more general case of generic structure-preserving rank-k perturbations, where $k \geq 1$. Numerical experiments with random perturbations support the following meta-conjecture.

Meta-Conjecture 1.1. *Let $A \in \mathbb{F}^{n,n}$ be a matrix that is structured with respect to some indefinite inner product and let $B \in \mathbb{F}^{n,n}$ be a matrix of rank k so that $A+B$ is from the same structure class as A. Then generically the Jordan structure and sign characteristic of $A+B$ are the same that one would obtain by performing a sequence of k generic structure-preserving rank-one perturbations on A.*

Here and throughout the paper, $\mathbb{F}$ denotes one of the fields $\mathbb{R}$ or $\mathbb{C}$. Moreover, the term generic is understood in the following way. A set $\mathcal{A} \subseteq \mathbb{F}^n$ is called *algebraic* if there exist finitely many polynomials p_j in n variables, $j = 1, \ldots, k$ such that $a \in \mathcal{A}$ if and only if

$$p_j(a) = 0 \quad \text{for} \quad j = 1, \ldots, k.$$

An algebraic set $\mathcal{A} \subseteq \mathbb{F}^n$ is called *proper* if $\mathcal{A} \neq \mathbb{F}^n$. Then, a set $\Omega \subseteq \mathbb{F}^n$ is called *generic* if $\mathbb{F}^n \setminus \Omega$ is contained in a proper algebraic set.

A proof of Conjecture 1.1 on the meta level seems to be hard to achieve. We illustrate the difficulties for the special case of H-symmetric matrices $A \in \mathbb{C}^{n \times n}$, i.e., matrices satisfying $A^T H = HA$, where $H \in \mathbb{C}^{n \times n}$ is symmetric and invertible. An H-symmetric rank-one perturbation of A has the form $A + uu^T H$ while an H-symmetric rank-two perturbation has the form $A + [u, v][u, v]^T H = A + uu^T H + vv^T H$, where $u, v \in \mathbb{C}^n$. Here, one can immediately see that the rank-two perturbation of A can be interpreted as a sequence of two independent rank-one perturbations, so the only remaining question concerns genericity. Now the statements on generic structure-preserving rank-one perturbations of H-symmetric matrices from [15] typically have the form that they assert the existence of a generic set $\Omega(A) \subseteq \mathbb{C}^n$ such that for all $u \in \Omega(A)$ the spectrum of $A + uu^T H$ shows the generic behavior stated in the corresponding theorem. Clearly, this set $\Omega(A)$ depends on A and thus, the set of all vectors $v \in \mathbb{C}^n$ such that the spectrum of the rank-one perturbation $A + uu^T H + vv^T H$ of $A + uu^T H$ shows the generic behavior is given by $\Omega(A + uu^T H)$. On the other hand, the precise meaning of a *generic H-symmetric rank-two perturbation* $A + uu^T H + vv^T H$ of A is the existence of a generic set $\Omega \subseteq \mathbb{C}^n \times \mathbb{C}^n$ such that $(u, v) \in \Omega$. Thus, the statement of Conjecture 1.1

can be translated by asserting that the set

$$\Omega = \bigcup_{u \in \Omega(A)} \Big(\{u\} \times \Omega(A + uu^T H) \Big)$$

is generic. Unfortunately, this fact cannot be proved without more detailed knowledge on the structure of the generic sets $\Omega(A)$ as the following example shows. Consider the set

$$\mathbb{C}^2 \setminus \big\{ (x, e^x) \,\big|\, x \in \mathbb{C} \big\} = \bigcup_{x \in \mathbb{C}} \Big(\{x\} \times \big(\mathbb{C} \setminus \{e^x\} \big) \Big).$$

Clearly, the sets $\mathbb{C}$ and $\mathbb{C} \setminus \{e^x\}$ are generic for all $x \in \mathbb{C}$. However, the set $\mathbb{C}^2 \setminus \big\{ (x, e^x) \,\big|\, x \in \mathbb{C} \big\}$ is not generic as $\Gamma := \big\{ (x, e^x) \,\big|\, x \in \mathbb{C} \big\}$, the graph of the natural exponential function, is not contained in a proper algebraic set.

Still, the set Γ from the previous paragraph is a *thin* set in the sense that it is a set of measure zero, so one might have the idea to weaken the term *generic* to sets whose complement is contained in a set of measure zero. However, this approach would have a significant drawback when passing to the real case. In [17, Lemma 2.2] it was shown that if $W \subseteq \mathbb{C}^n$ is a proper algebraic set in $\mathbb{C}^n$, then $W \cap \mathbb{R}^n$ is a proper algebraic set in $\mathbb{R}^n$ – a feature that allows to easily transfer results on generic rank-one perturbations from the complex to the real case. Clearly, a generalization of [17, Lemma 2.2] to sets of measure zero would be wrong as the set $\mathbb{R}^n$ itself is a set of measure zero in $\mathbb{C}^n$. Thus, using the term *generic* as defined here does not only lead to stronger statements, but also eases the discussion of the case that the matrices and perturbations under consideration are real.

The classes of structured matrices we consider in this paper are the following. Throughout the paper let $A^\star$ denote either the transpose A^T or the conjugate transpose A^* of a matrix A. Furthermore, let $H^\star = H \in \mathbb{F}^{n \times n}$ and $-J^T = J \in \mathbb{F}^{n \times n}$ be invertible. Then $A \in \mathbb{F}^{n \times n}$ is called

1. H-selfadjoint, if $\mathbb{F} = \mathbb{C}$, $\star = *$, and $A^* H = HA$;
2. H-symmetric, if $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $\star = T$, and $A^T H = HA$;
3. J-Hamiltonian, if $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $\star = T$, and $A^T J = -JA$.

There is no need to consider H-skew-adjoint matrices $A \in \mathbb{C}^{n,n}$ satisfying $A^* H = -HA$, because this case can be reduced to the case of H-selfadjoint matrices by considering iA instead. Similarly, it is not necessary to discuss inner products induced by a skew-Hermitian matrix $S \in \mathbb{C}^{n,n}$ as one can consider iS instead. On the other hand, we do not consider H-skew-symmetric matrices $A \in \mathbb{F}^{n,n}$ satisfying $A^T H = -HA$ or J-skew-Hamiltonian matrices $A \in \mathbb{F}^{n,n}$ satisfying $A^T J = JA$ for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, because in those cases rank-one perturbations do not exist and thus Conjecture 1.1 cannot be applied.

The remainder of the paper is organized as follows. In Section 2 we provide preliminary results. In Sections 3 and 4 we consider structure-preserving rank-k perturbations of H-symmetric, H-selfadjoint, and J-Hamiltonian matrices with the focus on the change of Jordan structures in Section 3 and on the change of the sign characteristic in Section 4.

2. Preliminaries

We start with a series of lemmas that will be key tools in this paper. First, we recap [2, Lemma 2.2] and also give a proof for completeness.

Lemma 2.1 ([2]). *Let $\mathcal{B} \subseteq \mathbb{F}^\ell$ not be contained in any proper algebraic subset of $\mathbb{F}^\ell$. Then, $\mathcal{B} \times \mathbb{F}^k$ is not contained in any proper algebraic subset of $\mathbb{F}^\ell \times \mathbb{F}^k$.*

Proof. First, we observe that the hypothesis that $\mathcal{B}$ is not contained in any proper algebraic subset of $\mathbb{F}^\ell$ is equivalent to the fact that for any nonzero polynomial p in ℓ variables there exists an $x \in \mathcal{B}$ (possibly depending on p) such that $p(x) \neq 0$. Letting now q be any nonzero polynomial in $\ell + k$ variables, then the assertion is equivalent to showing that there exists an $(x, y) \in \mathcal{B} \times \mathbb{F}^k$ such that $q(x, y) \neq 0$. Thus, for any such q consider the set

$$\Gamma_q := \big\{ y \in \mathbb{F}^k \mid q(\,\cdot\,, y) \text{ is a nonzero polynomial in } \ell \text{ variables} \big\}$$

which is not empty (otherwise q would be constantly zero). Now, for any $y \in \Gamma_q$, by hypothesis there exists an $x \in \mathcal{B}$ such that $q(x, y) \neq 0$ but then $(x, y) \in \mathcal{B} \times \mathbb{F}^k$. □

Lemma 2.2 ([15]). *Let $Y(x_1, \dots, x_r) \in \mathbb{F}^{m\times n}[x_1, \dots, x_r]$ be a matrix whose entries are polynomials in $x_1, \dots, x_r$. If $\operatorname{rank} Y(a_1, \dots, a_r) = k$ for some $[a_1, \dots, a_r]^T \in \mathbb{F}^r$, then the set*

$$\big\{ [b_1, \dots, b_r]^T \in \mathbb{F}^r \,\big|\, \operatorname{rank} Y(b_1, \dots, b_r) \geq k \big\} \tag{2.1}$$

is generic.

Lemma 2.3. *Let $H^\star = H \in \mathbb{F}^{n\times n}$ be invertible and let $A \in \mathbb{F}^{n\times n}$ have rank k. If n is even, let also $-J^T = J \in \mathbb{F}^{n\times n}$ be invertible.*

(1) *Let $\mathbb{F} = \mathbb{C}$ and $\star = *$, or let $\mathbb{F} = \mathbb{R}$ and $\star = T$. If $A^\star H = HA$, then there exists a matrix $U \in \mathbb{F}^{n\times k}$ of rank k and a signature matrix $\Sigma = \operatorname{diag}(s_1, \dots, s_k) \in \mathbb{R}^{k\times k}$, where $s_j \in \{+1, -1\}$, $j = 1, \dots, n$ such that $A = U\Sigma U^\star H$.*

(2) *If $\mathbb{F} = \mathbb{C}$, $\star = T$, and A is H-symmetric, then there exists a matrix $U \in \mathbb{C}^{n\times k}$ of rank k such that $A = UU^T H$.*

(3) *If $\mathbb{F} = \mathbb{R}$ and A is J-Hamiltonian, then there exists a matrix $U \in \mathbb{R}^{n\times k}$ of rank k and a signature matrix $\Sigma = \operatorname{diag}(s_1, \dots, s_k) \in \mathbb{R}^{k\times k}$, where $s_j \in \{+1, -1\}$, $j = 1, \dots, n$, such that $A = U\Sigma U^T J$.*

(4) *If $\mathbb{F} = \mathbb{C}$ and A is J-Hamiltonian, then there exists a matrix $U \in \mathbb{C}^{n\times k}$ of rank k such that $A = UU^T J$.*

Proof. If $\star = *$ and A is H-selfadjoint, then AH^{-1} is Hermitian. By Sylvester's Law of Inertia, there exists a nonsingular matrix $\widetilde{U} \in \mathbb{C}^{n\times n}$ and a matrix $\widetilde{\Sigma} = \operatorname{diag}(s_1, \dots, s_n) \in \mathbb{C}^{n\times n}$ such that $AH^{-1} = \widetilde{U}\widetilde{\Sigma}\widetilde{U}^*$, where we have $s_1, \dots, s_k \in \{+1, -1\}$ and $s_{k+1} = \cdots = s_n = 0$ as A has rank k. Letting $U \in \mathbb{C}^{n\times k}$ contain the first k columns of $\widetilde{U}$ and $\Sigma := \operatorname{diag}(s_1, \dots, s_k) \in \mathbb{C}^{k\times k}$, we obtain that $A = U\Sigma U^* H$ which proves (1). The other parts of the lemma are proved analogously using adequate factorizations like a nonunitary version of the Takagi factorization. □

Lemma 2.4. *Let $A, G \in \mathbb{C}^{n\times n}$, $R \in \mathbb{C}^{k\times k}$, let G, R be invertible, and let A have the pairwise distinct eigenvalues $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ with algebraic multiplicities $a_1, \dots, a_m$. Suppose that the matrix $A+URU^\star G$ generically (with respect to the entries of $U \in \mathbb{C}^{n\times k}$ if $\star = T$ and with respect to the real and imaginary parts of the entries of $U \in \mathbb{C}^{n\times k}$ if $\star = *$) has the eigenvalues $\lambda_1, \dots, \lambda_m$ with algebraic multiplicities $\widetilde{a}_1, \dots, \widetilde{a}_m$, where $\widetilde{a}_j \le a_j$ for $j = 1, \dots, m$.*

Furthermore, let $\varepsilon > 0$ be such that the discs

$$D_j := \{\mu \in \mathbb{C} \,|\, |\lambda_j - \mu| < \varepsilon^{2/n}\}, \quad j = 1, \dots, m$$

are pairwise disjoint. If for each $j = 1, \dots, m$ there exists a matrix $U_j \in \mathbb{C}^{n\times k}$ with $\|U_j\| < \varepsilon$ such that the matrix $A + U_jRU_j^\star G$ has exactly $(a_j - \widetilde{a}_j)$ simple eigenvalues in D_j different from λ_j, then generically (with respect to the entries of U if $\star = T$ and with respect to the real and imaginary parts of the entries of U if $\star = *$) the eigenvalues of $A+URU^\star G$ that are different from the eigenvalues of A are simple.

Lemma 2.4 was proved in [18, Lemma 8.1] for the case $k = 1$, $\star = T$, and $R = I_k$, but the proof remains valid (with obvious adaptions) for the more general statement in Lemma 2.4.

Definition 2.5. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two finite nonincreasing sequences of positive integers given by $n_1 \ge \cdots \ge n_m$ and $\eta_1 \ge \cdots \ge \eta_\ell$, respectively. We say that $\mathcal{L}_2$ *dominates* $\mathcal{L}_1$ if $\ell \ge m$ and $\eta_j \ge n_j$ for $j = 1, \dots, m$.

Part (3) of the following theorem will be a key tool used in the proofs of our main results in this paper.

Theorem 2.6. *Let $A, G, R \in \mathbb{C}^{n\times n}$, let G, R be invertible, and let $k \in \mathbb{N} \setminus \{0\}$. Furthermore, let $\lambda \in \mathbb{C}$ be an eigenvalue of A with geometric multiplicity $m > k$ and suppose that $n_1 \ge n_2 \ge \cdots \ge n_m$ are the sizes of the Jordan blocks associated with λ in the Jordan canonical form of A, i.e., the Jordan canonical form of A takes the form*

$$\mathcal{J}_{n_1}(\lambda) \oplus \mathcal{J}_{n_2}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widetilde{\mathcal{J}},$$

where $\lambda \notin \sigma(\widetilde{\mathcal{J}})$. Then, the following statements hold:

(1) *If $U_0 \in \mathbb{C}^{n\times k}$, then the Jordan canonical form of $A + U_0RU_0^\star G$ is given by*

$$\mathcal{J}_{\eta_1}(\lambda) \oplus \mathcal{J}_{\eta_2}(\lambda) \oplus \cdots \oplus \mathcal{J}_{\eta_\ell}(\lambda) \oplus \widehat{\mathcal{J}}; \qquad \eta_1 \ge \cdots \ge \eta_\ell,$$

where $\lambda \notin \sigma(\widehat{\mathcal{J}})$ and where $(\eta_1, \dots, \eta_\ell)$ dominates $(n_{k+1}, \dots, n_m)$, that is, we have $\ell \ge m - k$, and $\eta_j \ge n_{j+k}$ for $j = 1, \dots, m - k$.

(2) *Assume that for all $U \in \mathbb{C}^{n\times k}$ the algebraic multiplicity a_U of λ as an eigenvalue of $A + URU^\star G$ satisfies $a_U \ge a_0$ for some $a_0 \in \mathbb{N}$. If there exists one matrix $U_0 \in \mathbb{C}^{n\times k}$ such that $a_{U_0} = a_0$, then the set*

$$\Omega := \{U \in \mathbb{C}^{n\times k} \mid a_U = a_0\}$$

is generic (with respect to the entries of U if $\star = T$ and with respect to the real and imaginary parts of the entries of U if $\star = *$).

(3) *Assume that there exists one particular matrix* $U_0 \in \mathbb{C}^{n\times k}$ *such that the Jordan canonical form of* $A + U_0 R U_0^\star G$ *is described as in the statements* (a) *and* (b) *below:*
   (a) *The Jordan structure at* λ *is given by*

$$\mathcal{J}_{n_{k+1}}(\lambda) \oplus \mathcal{J}_{n_{k+2}}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widehat{\mathcal{J}},$$

   where $\lambda \notin \sigma(\widehat{\mathcal{J}})$.
   (b) *All eigenvalues that are not eigenvalues of* A *are simple.*

Then, there exists a generic set $\Omega \subseteq \mathbb{C}^{n\times k}$ (*with respect to the entries of* $U \in \mathbb{C}^{n\times k}$ *if* $\star = T$ *and with respect to the real and imaginary parts of the entries of* $U \in \mathbb{C}^{n\times k}$ *if* $\star = *$) *such that the Jordan canonical form of* $A + URU^\star G$ *is as described in* (a) *and* (b) *for all* $U \in \Omega$.

Proof. (1) is a particular case of [5, Lemma 2.1]. (Note that no assumption on the rank of U_0 is needed.)

In the remainder of this proof, the term *generic* is always meant in the sense 'generic with respect to the entries of $U \in \mathbb{C}^{n\times k}$' if $\star = T$ and 'generic with respect to the real and imaginary parts of the entries of $U \in \mathbb{C}^{n\times k}$' if $\star = *$.

(2) In this argument, let $Y(U) := (A + URU^\star G - \lambda I_n)^n$. By hypothesis, we have that $\operatorname{rank}\big(Y(U_0)\big) = n - a_0$ for some matrix $U_0 \in \mathbb{C}^{n,k}$. Thus, we can apply Lemma 2.2 to $Y(U)$, which yields that the set

$$\Omega := \big\{U \in \mathbb{C}^{n\times k} \mid \operatorname{rank}\big(Y(U)\big) \geq n - a_0\big\}$$

is generic. Observe that the condition $\operatorname{rank}\big(Y(U)\big) \geq n - a_0$ is equivalent to $a_U \leq a_0$, and since $a_U \geq a_0$ by hypothesis, it is even equivalent to $a_U = a_0$. Hence, Ω is the desired generic set from the assertion.

(3) By (1), the list of partial multiplicities in $A + URU^\star G$ at λ dominates the list $(n_{k+1}, \ldots, n_m)$, and hence, the algebraic multiplicity a_U of $A + URU^\star G$ at λ must be greater than or equal to $a_0 := n_{k+1} + \cdots + n_m$. However, by hypothesis there exists one U_0 so that $A + U_0 R U_0^\star G$ has exactly the partial multiplicities $(n_{k+1}, \ldots, n_m)$, so in particular it has the algebraic multiplicity $a_{U_0} = a_0$. Therefore, by (2) the set Ω_1 of all $U \in \mathbb{C}^{n\times k}$ satisfying $a_U = a_0$ is generic and for all $U \in \Omega_1$. Since $(n_{k+1}, \ldots, n_m)$ is the only possible list of partial multiplicities that dominates $(n_{k+1}, \ldots, n_m)$ and leads to the algebraic multiplicity a_0, we find that the perturbed matrix $A + URU^\star G$ satisfies condition (a). Moreover, since $A + U_0 R U_0^\star G$ already satisfies condition (b), by Lemma 2.4 the set Ω_2 of all $U \in \mathbb{C}^{n\times k}$ satisfying (b) is also generic. Thus, $\Omega = \Omega_1 \cap \Omega_2$ is the desired set. $\square$

We end this section by collecting important facts about the canonical forms of matrices that are structured with respect to some indefinite inner products. These forms are available in many sources, see, e.g., [8, 11, 14] or [12, 13, 26] in terms of pairs of Hermitian or symmetric and/or skew-symmetric matrices. We do not need the explicit structures of the canonical forms for the purpose of this paper, but only information on paring of certain Jordan blocks and on the *sign characteristic*. The sign characteristic is an important invariant of matrices that are structured with respect to indefinite inner products, we refer the reader to [7, 8] for details. To give a brief impression, consider the following example.

Example 2.7. Let $\lambda \in \mathbb{R}$ and consider the matrices

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_1 = \mathcal{J}_2(\lambda) := \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad A_2 = \begin{bmatrix} \lambda & -1 \\ 0 & \lambda \end{bmatrix}.$$

Then A_1 and A_2 are both H-selfadjoint and they are similar. However, they are not "equivalent as H-selfadjoint matrices" in the sense that there does not exist a nonsingular matrix $S \in \mathbb{C}^{2\times 2}$ so that $S^{-1}A_1S = A_2$ and $S^*HS = H$. (Note that this transformation corresponds to a change of basis in $\mathbb{C}^2$ with transformation matrix S). Indeed, any transformation matrix S that changes A_1 into A_2 would transform H into $-H$. In fact, $(\mathcal{J}_2(\lambda), H)$ and $(\mathcal{J}_2(\lambda), -H)$ are the canonical forms of the pairs (A_1, H) and (A_2, H), respectively, and they differ by a sign $\sigma \in \{+1, -1\}$ as a scalar factor of the matrix inducing the indefinite inner product. This sign is an additional invariant that can be thought of as being attached to the partial multiplicity 2 of the eigenvalue λ of A_1 (or A_2).

In general, if $H \in \mathbb{C}^{n\times n}$ is invertible and $\lambda \in \mathbb{R}$ is an eigenvalue of the H-selfadjoint matrix $A \in \mathbb{C}^{n\times n}$, then in the canonical form of (A, H) there is a sign for any partial multiplicity n_i of λ as an eigenvalue of A. The collection of all these signs then forms the *sign characteristic* of the eigenvalue λ. As in the example, we interpret the sign to be attached to the particular partial multiplicity. The following theorem states which eigenvalues of matrices that are structured with respect to indefinite inner products have a sign characteristic and it also lists possible restrictions in the Jordan structure of particular eigenvalues if there are any.

Theorem 2.8 (Sign characteristic and restriction of Jordan structures). *Let $H^\star = H \in \mathbb{F}^{n\times n}$ be invertible and let $A \in \mathbb{F}^{n\times n}$. If n is even, let also $-J^T = J \in \mathbb{F}^{n\times n}$ be invertible. Moreover, let $\lambda \in \mathbb{C}$ be an eigenvalue of A.*

(1) *Let either $\mathbb{F} = \mathbb{C}$ and $\star = *$, or $\mathbb{F} = \mathbb{R}$ and $\star = T$, and let $A^\star H = HA$. If λ is real, then each partial multiplicity of λ has a sign in the sign characteristic of λ.*
(2) *Let $\mathbb{F} = \mathbb{C}$ and $\star = T$, and let A be H-symmetric. Then λ does not have a sign characteristic.*
(3) *Let $\mathbb{F} = \mathbb{C}$ and let A be J-Hamiltonian. Then λ does not have a sign characteristic. If $\lambda = 0$, then the partial multiplicities of λ as an eigenvalue of A of each fixed odd size n_0 occur an even number of times.*
(4) *Let $\mathbb{F} = \mathbb{R}$ and let A be J-Hamiltonian. If $\lambda \neq 0$ is purely imaginary, then each partial multiplicity of λ has a sign in the sign characteristic of λ. If $\lambda = 0$, then the partial multiplicities of λ as an eigenvalue of A of each fixed odd size n_0 occur an even number of times. Furthermore, each even partial multiplicity of the eigenvalue $\lambda = 0$ has a sign in the sign characteristic of λ.*

3. Jordan structure under rank-k perturbations

In this section, we aim to investigate the effect of structure-preserving rank-k perturbations on the Jordan structure of H-selfadjoint, H-symmetric, and J-Hamiltonian matrices.

In our first result, we will consider the class of (complex) H-selfadjoint matrices. Recall that any H-selfadjoint rank-k perturbation has the form that is described in Lemma 2.3(1).

Theorem 3.1. *Let $H \in \mathbb{C}^{n,n}$ be invertible and Hermitian and let $A \in \mathbb{C}^{n,n}$ be H-selfadjoint. Furthermore let $\Sigma = \operatorname{diag}(s_1, \ldots, s_k)$ with $s_j \in \{-1, +1\}$ for $j = 1, \ldots, k$. Then, there exists a generic set $\Omega_k \subseteq \mathbb{C}^{n\times k}$ (with respect to the real and imaginary parts of the entries of $U \in \mathbb{C}^{n\times k}$) such that for all $U \in \Omega_k$ and $B := U\Sigma U^*H$ the following statements hold:*

(1) *Let $\lambda \in \mathbb{C}$ be any eigenvalue of A and let m denote its geometric multiplicity. If $k \geq m$, then λ is not an eigenvalue of $A + B$. Otherwise, suppose that $n_1 \geq n_2 \geq \cdots \geq n_m$ are the sizes of the Jordan blocks associated with λ in the Jordan canonical form of A, i.e., the Jordan canonical form of A takes the form*

$$\mathcal{J}_{n_1}(\lambda) \oplus \mathcal{J}_{n_2}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widetilde{\mathcal{J}},$$

where $\lambda \notin \sigma(\widetilde{\mathcal{J}})$. Then, the Jordan canonical form of $A + B$ is given by

$$\mathcal{J}_{n_{k+1}}(\lambda) \oplus \mathcal{J}_{n_{k+2}}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widehat{\mathcal{J}},$$

where $\lambda \notin \sigma(\widehat{\mathcal{J}})$.

(2) *If $\mu \in \mathbb{C}$ is an eigenvalue of $A+B$, but not of A, then μ is a simple eigenvalue of $A + B$.*

Proof. In this proof, the term *generic* is meant it the sense 'generic with respect to the real and imaginary parts of the entries of $U \in \mathbb{C}^{n\times k}$'. We show that there exist two generic subsets $\Omega_{k,1}$ and $\Omega_{k,2}$ of $\mathbb{C}^{n,k}$ so that property (1) is satisfied on $\Omega_{k,1}$ and property (2) on $\Omega_{k,2}$. Then, $\Omega_k := \Omega_{k,1} \cap \Omega_{k,2}$ is the desired generic set.

Concerning (1): By part (3) of Theorem 2.6 it is sufficient to construct one particular H-selfadjoint rank-k perturbation, such that the Jordan structure is as claimed. We do this by constructing a sequence of k rank-one perturbations with the desired properties.

Now, by [16, Theorem 3.3], for a generic rank-1 perturbation of the form s_1uu^*H the perturbed matrix $A + s_1uu^*H$ will have the partial multiplicities $n_2, \ldots, n_m$ at each eigenvalue λ. ([16, Theorem 3.3] was formulated and proved for the case $s_1 = 1$ only, but if $s_1 = -1$, one can still apply this result, by considering rank-1 perturbations of the form $uu^*(-H)$ of the $(-H)$-selfadjoint matrix A.) We consider now a fixed u_1 so that $A_1 := A + s_1u_1u_1^*H$ has this property. Then, [16, Theorem 3.3], can be applied anew to the matrix A_1 showing that there exists a vector u_2 such that $A_2 = A + s_1u_1u_1^*H + s_2u_2u_2^*H$ has the partial multiplicities $n_3, \ldots, n_m$ at each eigenvalue λ. Repeating this step $k - 2$ more times results in an H-selfadjoint matrix $A_k = A + s_1u_1u_1^*H + \cdots + s_ku_ku_k^*H$ that has the partial multiplicities $n_{k+1}, \ldots, n_m$ at each eigenvalue λ.

Concerning (2): We assert that the particular rank-k perturbation of the form $A + u_1u_1^*H + \cdots + u_ku_k^*H$ constructed above has the property that all eigenvalues different from those of A are simple. In fact, since in each step $j = 2, \ldots, k$ we generate $A_j := A_{j-1} + s_ju_ju_j^*H$, only the eigenvalues of A_j that have been eigenvalues of A_{j-1} can be multiple, so that these have been also eigenvalues of A. Thus, the existence of the desired generic set $\Omega_{k,2}$ follows from Lemma 2.4. □

Next, we turn to H-symmetric matrices, where we will treat both cases, $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$, at once. Note that by Lemma 2.3, any H-symmetric rank k perturbation of A has the form $U\Sigma U^TH$ where $U \in \mathbb{F}^{n,k}$ and where $\Sigma \in \mathbb{R}^{k,k}$ is a diagonal matrix with ± 1's on the diagonal (in case $\mathbb{F} = \mathbb{R}$) or $+1$'s on the diagonal (in case $\mathbb{F} = \mathbb{C}$). Still, even in the case $\mathbb{F} = \mathbb{C}$, we will allow -1's on the diagonal of Σ, which does not lead to a more general statement but allows a unified treatment of both cases.

Theorem 3.2. *Let $H \in \mathbb{F}^{n,n}$ be invertible with $H^T = H$ and let $A \in \mathbb{F}^{n,n}$ be H-symmetric. Furthermore let $\Sigma = \operatorname{diag}(s_1, \ldots, s_k)$ with $s_j \in \{-1, +1\}$ for $j = 1, \ldots, k$. Then, there exists a generic set $\Omega_k \subseteq \mathbb{F}^{n\times k}$ such that for all $U \in \Omega_k$ and $B := U\Sigma U^TH$ the following statements hold:*

(1) *Let $\lambda \in \mathbb{C}$ be any eigenvalue of A and let m denote its geometric multiplicity. If $k \geq m$, then λ is not an eigenvalue of $A + B$. Otherwise, suppose that $n_1 \geq n_2 \geq \cdots \geq n_m$ are the sizes of the Jordan blocks associated with λ in the Jordan canonical form of A, i.e., the Jordan canonical form of A takes the form*

$$\mathcal{J}_{n_1}(\lambda) \oplus \mathcal{J}_{n_2}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widetilde{\mathcal{J}},$$

where $\lambda \notin \sigma(\widetilde{\mathcal{J}})$. Then, the Jordan canonical form of $A + B$ is given by

$$\mathcal{J}_{n_{k+1}}(\lambda) \oplus \mathcal{J}_{n_{k+2}}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widehat{\mathcal{J}},$$

where $\lambda \notin \sigma(\widehat{\mathcal{J}})$.

(2) *If $\mu \in \mathbb{C}$ is an eigenvalue of $A+B$, but not of A, then μ is a simple eigenvalue of $A + B$.*

Proof. We sketch the proof of this theorem in the complex case only, since the real case is then obtained by the fact that for a generic set $\Omega_k \subseteq \mathbb{C}^{n,k}$, the set $\Omega_k \cap \mathbb{R}^{n,k}$ is generic as well, see [17, Lemma 2.2]. Then, the proof of the complex case proceeds as the proof of Theorem 3.1, by showing that there exist two generic subsets $\Omega_{k,1}$ and $\Omega_{k,2}$ of $\mathbb{C}^{n,k}$ so that the intersection of them is the desired generic set (note that these sets are actually generic and not just generic with respect to the real and imaginary parts of their entries). Thus, for the sake of brevity, we refrain from giving a complete proof but point out that the only difference (beside replacing $*$ by T) is that for rank-1 perturbations of the form suu^TH (with $s = \pm 1$) we refer to the result [15, Theorem 5.1] instead of [16, Theorem 3.3]. □

Now, we turn to J-Hamiltonian matrices. As we saw in Theorem 2.8, the Jordan blocks of Hamiltonian matrices at 0 have to be paired in a certain way. This restriction produced surprising results in the case of Hamiltonian rank-one perturbations of Hamiltonian matrices, see [15, Theorem 4.2]. We will in the following see that also in the case of rank-k perturbations, taking care of this pairing of certain blocks will be the most challenging task.

As in the previous theorem, we will treat both cases, $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$ at the same time. Thus, also as before, we will in the case $\mathbb{F} = \mathbb{C}$ consider perturbations of the form $U\Sigma U^T J$, with Σ possibly having some -1's on the diagonal, so that we can treat complex and real perturbations at once.

Theorem 3.3. *Let $J \in \mathbb{F}^{n,n}$ be skew-symmetric and invertible, let $A \in \mathbb{F}^{n,n}$ be J-Hamiltonian. Furthermore, let $\Sigma = \operatorname{diag}(s_1, \ldots, s_k)$ with $s_j \in \{-1, +1\}$ for $j = 1, \ldots, k$. Then, there exists a generic set $\Omega_k \subseteq \mathbb{F}^{n\times k}$ such that for all $U \in \Omega_k$ and $B := U\Sigma U^T J$ the following statements hold:*

(1) *Let $\lambda \in \mathbb{C}$ be any eigenvalue of A and let m denote its geometric multiplicity. If $k \geq m$, then λ is not an eigenvalue of $A + B$. Otherwise, suppose that $n_1 \geq n_2 \geq \cdots \geq n_m$ are the sizes of the Jordan blocks associated with λ in the Jordan canonical form of A, i.e., the Jordan canonical form of A takes the form*

$$\mathcal{J}_{n_1}(\lambda) \oplus \mathcal{J}_{n_2}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widetilde{\mathcal{J}},$$

where $\lambda \notin \sigma(\widetilde{\mathcal{J}})$. Then:

(1a) *If either $\lambda \neq 0$ or $\lambda = 0$ and $n_1 + \cdots + n_k$ is even, then the Jordan canonical form of $A + B$ is given by*

$$\mathcal{J}_{n_{k+1}}(\lambda) \oplus \mathcal{J}_{n_{k+2}}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widehat{\mathcal{J}},$$

where $\lambda \notin \sigma(\widehat{\mathcal{J}})$.

(1b) *If $\lambda = 0$ and $n_1 + \cdots + n_k$ is odd, then the Jordan canonical form of $A + B$ is given by*

$$\mathcal{J}_{n_{k+1}+1}(\lambda) \oplus \mathcal{J}_{n_{k+2}}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widehat{\mathcal{J}},$$

where $\lambda \notin \sigma(\widehat{\mathcal{J}})$.

(2) *If $\mu \in \mathbb{C}$ is an eigenvalue of $A+B$, but not of A, then μ is a simple eigenvalue of $A + B$.*

Proof. We provide the proof of this theorem in the complex case only, since the real case is then obtained by the fact that for a generic set $\Omega_k \subseteq \mathbb{C}^{n,k}$, the set $\Omega_k \cap \mathbb{R}^{n,k}$ is generic as well, see [17, Lemma 2.2]. We show that there exist two generic sets $\Omega_{k,1}$ and $\Omega_{k,2}$ so that property (1) is satisfied on $\Omega_{k,1}$ and property (2) on $\Omega_{k,2}$, so that $\Omega_k := \Omega_{k,1} \cap \Omega_{k,2}$ is the desired generic set.

Proof of (1): We first mention that in the case $\lambda = 0$, all odd-sized multiplicities have to occur an even number of times by Theorem 2.8. This implies in particular that $n_1 + \cdots + n_m$ is even. Therefore, if the number $n_1 + \cdots + n_k$ is even, then odd entries in both subsequences $n_1, \ldots, n_k$ and $n_{k+1}, \ldots, n_m$ occur an even number of times so that, in particular, there is no fundamental obstruction to the sequence $n_{k+1}, \ldots, n_m$ of partial multiplicities occurring in some Hamiltonian matrix at 0.

On the other hand, if $n_1+\cdots+n_k$ is odd, then there must occur an odd number of blocks of size $n_k = n_{k+1}$ in both subsequences $n_1, \ldots, n_k$ and $n_{k+1}, \ldots, n_m$. In particular, it is thus not possible for the partial multiplicities $n_{k+1}, \ldots, n_m$ to be realized in some Hamiltonian matrix at 0.

Case (1a): By part (3) of Theorem 2.6 it is sufficient to construct a sequence of k Hamiltonian rank-one perturbations such that the Jordan structure is as claimed.

If $\lambda \neq 0$, then by [15, Theorem 4.2] for a generic rank-1 perturbation of the form $s_1 uu^T J$, the perturbed matrix $A + s_1 uu^T J$ will have the partial multiplicities $n_2, \ldots, n_m$ at λ (if $s_1 = -1$, this also holds by applying [15, Theorem 4.2] to $-A$). We now consider a fixed u_1 so that $A_1 := A + s_1 u_1 u_1^T J$ has this property. Then [15, Theorem 4.2] can be applied anew to the matrix A_1. Repeating this step $k-1$ more times finally results in a Hamiltonian matrix $A_k = A + s_1 u_1 u_1^T J + \cdots + s_k u_k u_k^T J$ that has the partial multiplicities $n_{k+1}, \ldots, n_m$ at λ.

Next, let us consider the case that $\lambda = 0$ but $n_1 + \cdots + n_k$ is even. We aim to proceed as for $\lambda \neq 0$ applying [15, Theorem 4.2]. In this case, a generic rank-1 perturbation of the form $s_1 uu^T J$ will in the perturbed matrix $A + s_1 uu^T J$ at 0 create the partial multiplicities $n_2, \ldots, n_m$ if n_1 is even and $n_2 + 1, n_3, \ldots, n_m$ if n_1 is odd. We now fix u_1 so that

$$A_1 := A + s_1 u_1 u_1^T J$$

has this property. Again, by [15, Theorem 4.2] for a generic vector v the matrix $A_1 + s_2 vv^T J$ will at 0 have the partial multiplicities $n_3, \ldots, n_m$ if $n_1 + n_2$ is even (this includes the case that $n_1 = n_2$ are odd as in this case the block of size $n_2 + 1$ will simply disappear) and $n_3 + 1, n_4, \ldots, n_m$ if $n_1 + n_2$ is odd. We fix u_2 with this property setting

$$A_2 := A + s_1 u_1 u_1^T J + s_2 u_2 u_2^T J.$$

After $k-2$ more steps of this procedure, we obtain a Hamiltonian matrix $A_k = A + s_1 u_1 u_1^T J + \cdots + s_k u_k u_k^T J$ with the partial multiplicities $n_{k+1}, \ldots, n_m$ at 0 as $n_1 + \cdots + n_k$ is even.

Case (1b): Let us assume that $\lambda = 0$ and $n_1 + \cdots + n_k$ is odd, which immediately implies $k + 1 \leq m$. As mentioned above, the partial multiplicity sequence $n_{k+1}, \ldots, n_m$ contains the odd entry n_{k+1} an odd number of times, and thus cannot be realized in a Hamiltonian matrix at 0. Hence, the minimum algebraic multiplicity of $A + B$ at zero is $n_{k+1} + \cdots + n_m + 1$. By Theorem 2.6(2), this is the generic algebraic multiplicity of $A + B$ at 0 if we can find a particular perturbation that creates this algebraic multiplicity. However, such a perturbation is easily constructed as in Case (1a), $\lambda = 0$, using [15, Theorem 4.2].

In order to determine the precise partial multiplicities of $A + B$ in this case, we employ an argument that was initially used to prove [2, Theorem 3.4], see also the following Example 3.4 for an illustration of this argument: Let us assume that

$B = U\Sigma U^T J$, where U is an element of a generic set $\widetilde{\Omega}_k \subseteq \mathbb{C}^{n\times k}$ such that (1a) holds for all nonzero eigenvalues of A and the algebraic multiplicity of $A+B$ at zero is $n_{k+1}+\cdots+n_m+1$. It remains to determine the generic partial multiplicities of $A+B$ at 0. Let us group together Jordan blocks of the same size, i.e., let

$$(n_1, n_2, n_3, \ldots, n_m) = (\underbrace{p_1, \ldots, p_1}_{t_1 \text{ times}}, \underbrace{p_2, \ldots, p_2}_{t_2 \text{ times}}, \ldots, \underbrace{p_\nu, \ldots, p_\nu}_{t_\nu \text{ times}}),$$

and let ℓ be such that $p_\ell = n_k = n_{k+1}$. Then p_ℓ is odd, t_ℓ is even, and

$$(n_{k+1}, \ldots, n_m) = (\underbrace{p_\ell, \ldots, p_\ell}_{d \text{ times}}, \underbrace{p_{\ell+1}, \ldots, p_{\ell+1}}_{t_{\ell+1} \text{ times}}, \ldots, \underbrace{p_\nu, \ldots, p_\nu}_{t_\nu \text{ times}}),$$

where d is odd. Now, $A+B$ has the algebraic multiplicity $n_{k+1}+\cdots+n_m+1$ at zero and by Theorem 2.6, the list of descending partial multiplicities of $A+B$ at zero dominates $(n_{k+1}, \ldots, n_m)$. Therefore, either one of the blocks corresponding to the partial multiplicities $n_k, \ldots, n_m$ has grown in size by exactly one, or a new block of size one has been created. Moreover, the Hamiltonian matrix $A+B$ must have an even number of Jordan blocks of size p_ℓ at 0. If $\nu > \ell$ and $p_{\ell+1} < p_\ell - 1$, then these restrictions can only be realized by the list of partial multiplicities given by

$$(p_\ell + 1, \underbrace{p_\ell, \ldots, p_\ell}_{(d-1) \text{ times}}, \underbrace{p_{\ell+1}, \ldots, p_{\ell+1}}_{t_{\ell+1} \text{ times}}, \ldots, \underbrace{p_\nu, \ldots, p_\nu}_{t_\nu \text{ times}}). \tag{3.2}$$

Only when $\nu > \ell$ and $p_{\ell+1} = p_\ell - 1$, or when $\nu = \ell$ and $p_\ell = 1$ then also a list different from (3.2) can be realized, namely

$$(\underbrace{p_\ell, \ldots, p_\ell}_{(d+1) \text{ times}}, \underbrace{p_{\ell+1}, \ldots, p_{\ell+1}}_{(t_{\ell+1}-1) \text{ times}}, \ldots, \underbrace{p_\nu, \ldots, p_\nu}_{t_\nu \text{ times}}). \tag{3.3}$$

Hereby, in the latter case of $\nu = \ell$ and $p_\ell = 1$, the above list is given by $(p_\ell, \ldots, p_\ell)$ (repeated $(d+1)$ times), and this interpretation shall be applied to the following lists as well. Then, aiming to prove that the partial multiplicities in (3.2) are generically realized in $A+B$ at 0, let us assume the opposite: assume for some Hamiltonian matrix A that $A+B$ has the partial multiplicities from (3.3) at 0 for all $U \in \mathcal{B}$, where $\mathcal{B}$ is not contained in any proper algebraic subset of $\mathbb{C}^{n,k}$. Then, we apply a further Hamiltonian rank-1 perturbation $suu^T J$ to $A+B$ (again, $s \in \{-1, +1\}$). By Theorem 2.6(1), for all $[U, u] \in \mathcal{B} \times \mathbb{C}^n$, the sequence of partial multiplicities at 0 of the Hamiltonian matrix $A + B + suu^T J$ dominates

$$(\underbrace{p_\ell, \ldots, p_\ell}_{d \text{ times}}, \underbrace{p_{\ell+1}, \ldots, p_{\ell+1}}_{(t_{\ell+1}-1) \text{ times}}, \ldots, \underbrace{p_\nu, \ldots, p_\nu}_{t_\nu \text{ times}}). \tag{3.4}$$

On the other hand, applying the already proved part (1a) to the case $k+1$, we find that there exists a generic set $\Gamma \subseteq \mathbb{C}^{n\times(k+1)}$ such that the partial multiplicities of $A + [U, u](\Sigma \oplus [s])[U, u]^T J$ at 0 are given by

$$(\underbrace{p_\ell, \ldots, p_\ell}_{(d-1) \text{ times}}, \underbrace{p_{\ell+1}, \ldots, p_{\ell+1}}_{t_{\ell+1} \text{ times}}, \ldots, \underbrace{p_\nu, \ldots, p_\nu}_{t_\nu \text{ times}}),$$

for all $[U, u] \in \Gamma$. Observe that the latter sequence does not dominate the one in (3.4). Thus, a contradiction is obtained as by Lemma 2.1 the set $\mathcal{B} \times \mathbb{C}^n$ is not contained in any proper algebraic subset of $\mathbb{C}^{n,k+1}$ and thus, clearly, $(\mathcal{B} \times \mathbb{C}^n) \cap \Gamma$ is not empty.

Proof of (2): Analogous to (2) of Theorem 3.1. □

Example 3.4. Let $A \in \mathbb{F}^{n,n}$ be a J-Hamiltonian matrix for some invertible skew-symmetric matrix $J \in \mathbb{F}^{n,n}$. Assume that A has the partial multiplicities (6, 5, 5, 4, 3, 3, 2) at 0 and apply a J-Hamiltonian rank-three perturbation C to A. Then the J-Hamiltonian matrix $A+C$ has a list of partial multiplicities at 0 that dominates $(4, 3, 3, 2)$. Since there are examples for J-Hamiltonian rank-three perturbations that lead to these partial multiplicities at 0, it follows from part (3) of Theorem 2.6 that this is the generic case.

Now assume that a J-Hamiltonian rank-two perturbation B is applied to A. Then the J-Hamiltonian matrix $A + B$ has a list of partial multiplicities at 0 that dominates $(5, 4, 3, 3, 2)$. This list, however, cannot be realized in a Hamiltonian matrix at 0, because the partial multiplicity 5 only occurs once, but not an even number of times. Thus, the minimal algebraic multiplicity $a_0 = 17 = 5+4+3+3+2$ of the eigenvalue zero cannot be realized in $A + B$, but there are examples for the algebraic multiplicity $a_0 + 1 = 18$. The only possible lists of partial multiplicities at 0 that lead to the algebraic multiplicity 18 and that can be realized in a Hamiltonian matrix are

$$(6, 4, 3, 3, 2) \quad \text{and} \quad (5, 5, 3, 3, 2).$$

The proof of Theorem 3.3 shows that the list $(6, 4, 3, 3, 2)$ is the one that generically occurs: Suppose that there exists a set $\mathcal{B} \subseteq \mathbb{F}^n \times \mathbb{F}^n$ that is not contained in a proper algebraic set, so that for all rank-two perturbations parametrized by elements of $\mathcal{B}$, the second list $(5, 5, 3, 3, 2)$ is realized. Then, any further rank-one perturbation would create partial multiplicities dominating $(5, 3, 3, 2)$ at 0, so for all rank-three perturbations parametrized by elements of $\mathcal{B} \times \mathbb{F}^n$, which is not contained in a proper algebraic set, the partial multiplicities would dominate $(5, 3, 3, 2)$. This is in contradiction to the fact that the subset of $\mathbb{F}^{n,3}$ parametrizing *all* rank-three perturbations that do not lead to the partial multiplicities $(4, 3, 3, 2)$ at 0 is contained in a proper algebraic set.

We provide a second, more simple example that looks rather surprising at first sight.

Example 3.5. Consider the (real or complex) matrices

$$A = \mathcal{J}_6(0) \oplus \begin{bmatrix} \mathcal{J}_5(0) & 0 \\ 0 & \mathcal{J}_5(0) \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 0 & I_3 \\ -I_3 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & I_5 \\ -I_5 & 0 \end{bmatrix}.$$

Then A is J-Hamiltonian. Applying a generic J-Hamiltonian rank-2 perturbation of the form $B = uu^T J + vv^T J$ for some $(u, v) \in \Omega_2$, where Ω_2 is the generic set from Theorem 3.3, results in a J-Hamiltonian matrix $A+B$ having the Jordan canonical form

$$\mathcal{J}_6(0) \oplus \widetilde{A}, \tag{3.5}$$

where $\widetilde{A}$ has ten simple nonzero eigenvalues. At first, this example looks as if the two smaller Jordan blocks of A at 0 have disappeared and the largest Jordan block of size $J_6(0)$ has remained, which is in complete contrast to the well-known results from [20] that state that under a generic unstructured rank-two perturbation, the two largest Jordan blocks associated with each eigenvalue disappear. However, interpreting B as a sequence of two rank-one perturbations, we see that in the first step $A + uu^T J$ generically has the Jordan canonical form

$$\mathcal{J}_5(0) \oplus \mathcal{J}_5(0) \oplus \widehat{A},$$

where $\widehat{A}$ has six simple nonzero eigenvalues if $u \in \Omega_1$, where Ω_1 is as in Theorem 3.3. Applying now the rank-one perturbation $vv^T J$ to $A + uu^T J$, we obtain that $A + uu^T J + vv^T J$ has the Jordan canonical form as in (3.5), because the single Jordan block $\mathcal{J}_5(0)$ cannot be realized in any Hamiltonian matrix as Jordan blocks of odd sizes associated with the eigenvalue zero have to occur an even number of times. Therefore, the "remaining" block has to grow in size by one. From this point of view, the Jordan block of size six did not remain, but was destroyed by the first rank-one perturbation and then recreated by the second one. This interpretation is in line with the fact that by Theorem 3.3 a generic Hamiltonian rank-two perturbation of a Hamiltonian matrix with the Jordan canonical form $\mathcal{J}_{2m}(0) \oplus \mathcal{J}_5(0) \oplus \mathcal{J}_5(0)$ will for any $m \geq 3$ result in a Hamiltonian matrix having the Jordan canonical form $\mathcal{J}_6(0) \oplus \widetilde{A}$ with $\widetilde{A}$ having only simple nonzero eigenvalues.

Remark 3.6. We conclude this section by mentioning that in each of the cases in Theorems 3.1–3.3, the generic set $\Omega_k = \Omega_k(A, s_1, \ldots, s_k)$ does not only depend on the matrix A, but also on the choices of the parameters $s_1, \ldots, s_k \in \{-1, +1\}$. However, since there are only finitely many combinations of the parameters (namely 2^k different choices) the intersection $\widetilde{\Omega}_k(A)$ of these 2^k generic sets $\Omega_k(A, s_1, \ldots, s_k)$ is still generic. Thus, the statement of each of the Theorems 3.1–3.3 can be strengthened in such a way that for all $U \in \widetilde{\Omega}_k(A)$, and all $\Sigma = \mathrm{diag}(s_1, \ldots, s_k)$ with $s_1, \ldots, s_k \in \{-1, +1\}$, the matrix $A + U\Sigma U^* H$ or $A + U\Sigma U^T H$ or $A + U\Sigma U^T J$, respectively, has the properties as stated in Theorems 3.1, 3.2, or 3.3, respectively.

4. Sign characteristic under rank-k perturbations

Since the behavior of the Jordan structure of matrices under rank-k perturbations was already established in the previous section, we now turn to the question of the change of the sign characteristic of (complex) H-selfadjoint, real H-symmetric, and real J-Hamiltonian matrices. (We recall that the other types of matrices considered in this paper, i.e., complex H-symmetric and complex J-Hamiltonian ones, do not have a sign characteristic by Theorem 2.8.)

First, we turn to H-selfadjoint and real H-symmetric matrices. Recall from Theorem 2.8 that each partial multiplicity n_{ij} of a real eigenvalue λ_i of a matrix A that is H-selfadjoint or real H-symmetric has a sign $\sigma_{ij} \in \{+1, -1\}$ in the *sign characteristic* of λ_i. We go on to prove a theorem on the sign characteristic of H-selfadjoint matrices under H-selfadjoint rank-k perturbations. However, since this theorem will come without an explicit genericity hypothesis, we will later be able to also apply it to real H-symmetric matrices.

Theorem 4.1. *Let $H \in \mathbb{C}^{n\times n}$ be invertible and Hermitian let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint. Let $\Sigma = \operatorname{diag}(s_1, \dots, s_k)$ with $s_j \in \{-1, +1\}$ and let $\lambda_1, \dots, \lambda_p$ be the pairwise distinct real eigenvalues of A and $\lambda_{p+1}, \dots, \lambda_q$ be the pairwise distinct nonreal eigenvalues of A. Furthermore, (in difference to before) let $n_{1,j} > \cdots > n_{m_j,j}$ be the distinct block sizes of A at some eigenvalue λ_j such that there exist $\ell_{i,j}$ blocks of size $n_{i,j}$ at λ_j and, whenever $j \in \{1, \dots, p\}$, let A have the signs $\{\sigma_{1,i,j}, \dots, \sigma_{\ell_{i,j},i,j}\}$ attached to its blocks of size $n_{i,j}$ at λ_j. Then, whenever $U \in \mathbb{C}^{n,k}$ is such that for $B := U\Sigma U^*H$ the statement (1) below is satisfied, also (2) holds.*

1. *The perturbed matrix $A + B$ has the Jordan structure as described in (1) of Theorem 3.1. More precisely, for each $j = 1, \dots, q$, the matrix $A+B$ has the distinct block sizes $n_{\kappa_j,j} > n_{\kappa_j+1,j} > \cdots > n_{m_j,j}$ occurring $\ell'_{\kappa_j,j}$, $\ell_{\kappa_j+1,j}$, ..., $\ell_{m_j,j}$ times, respectively, at λ_j, where $\ell'_{\kappa_j,j} = \ell_{1,j} + \cdots + \ell_{\kappa_j,j} - k$ and κ_j is the smallest integer with $\ell'_{\kappa_j,j} \geq 1$.*
2. *For each $j = 1, \dots, p$, let $\{\sigma'_{1,\kappa_j,j}, \dots, \sigma'_{\ell'_{\kappa_j,j},\kappa_j,j}\}$ be the signs of $A + B$ at blocks of size $n_{\kappa_j,j}$ at λ_j and let $\{\sigma'_{1,i,j}, \dots, \sigma'_{\ell_{i,j},i,j}\}$ be the signs at blocks of size $n_{i,j}$ at λ_j for $i = \kappa_j + 1, \dots, m_j$. Then,*

$$\sum_{s=1}^{\ell_{i,j}} \sigma_{s,i,j} = \sum_{s=1}^{\ell_{i,j}} \sigma'_{s,i,j}, \qquad i = \kappa_j + 1, \dots, m_j, \quad j = 1, \dots, p \tag{4.6}$$

and

$$\left| \sum_{s=1}^{\ell_{\kappa_j,j}} \sigma_{s,\kappa_j,j} - \sum_{s=1}^{\ell'_{\kappa_j,j}} \sigma'_{s,\kappa_j,j} \right| \leq \ell_{\kappa_j,j} - \ell'_{\kappa_j,j}, \quad j = 1, \dots, p. \tag{4.7}$$

Proof. In the first step of the proof, we show that there exists some set $\Omega'_k \subseteq \mathbb{C}^{n,k}$, that is generic with respect to the real and imaginary parts of the entries of its elements (and the term "generic" is understood in this way in the remainder of this proof), so that for all $U \in \Omega'_k$, the statements from (1) and (2) above hold.

Letting $\Omega_1, \dots, \Omega_k$ be the generic sets constructed in Theorem 3.1, we define

$$\Omega'_k := (\Omega_1 \times \mathbb{C}^{n,k-1}) \cap (\Omega_2 \times \mathbb{C}^{n,k-2}) \cap \cdots \cap \Omega_k,$$

which is (as the intersection of finitely many generic sets) clearly a generic subset of $\mathbb{C}^{n,k}$. Now, let $U := [u_1, \ldots, u_k] \in \Omega'_k$, then clearly the Jordan structure of $A_1 := A + s_1 u_1 u_1^* H$ is as described in (1) and (2) of Theorem 3.1 for $k = 1$. Therefore, by [19, Theorem 4.6] for all $j = 1, \ldots, p$ all signs of A attached to blocks at λ_j of size $n_{2,j}, \ldots, n_{m_j,j}$ are preserved, i.e., they are the same in A and A_1. Further, of the $\ell_{1,j}$ signs attached to blocks of size $n_{1,j}$ in A at λ_j, exactly $\ell_{1,j} - 1$ are attached to blocks of size $n_{1,j}$ in A_1, i.e., if η is the sum of the $\ell_{1,j}$ signs attached to blocks of size $n_{1,j}$ in A at λ_j and if $\widetilde{\eta}$ is the sum of the $\ell_{1,j} - 1$ signs attached to blocks of size $n_{1,j}$ in A_1, then $|\eta - \widetilde{\eta}| = 1$. (If there are both signs $+1$ and -1 among the list of $\ell_{1,j}$ signs attached to the blocks of size $n_{1,j}$, then it depends on the particular perturbations whether the sign that has been dropped to obtain the list of $\ell_{1,j} - 1$ signs is positive or negative.)

Now, we consider the perturbed matrix $A_2 := A_1 + s_2 u_2 u_2^* H$. Since $[u_1, u_2] \in \Omega_2$, clearly A_2 has the Jordan structure as described in (1) and (2) of Theorem 3.1 for $k = 1$, whereby we consider A_1 instead of A as the unperturbed matrix in that theorem. Hence, again applying [19, Theorem 4.6] for all $j = 1, \ldots, p$, all signs of A_1 attached to blocks of size $n_{3,j}, \ldots, n_{m_j,j}$ are preserved, i.e., they are the same in A_2 and A_1. Further, if $\ell_{1,j} \geq 2$, then also all signs of A_1 at blocks of size $n_{2,j}$ at λ_j are preserved and of the $\ell_{1,j} - 1$ signs of A_1 at blocks of size $n_{1,j}$, exactly $\ell_{1,j} - 2$ are preserved, i.e., attached to blocks of size $n_{1,j}$ in A_2 (the remaining sign does not occur anymore since the corresponding block was destroyed under perturbation). In the remaining case $\ell_{1,j} = 1$, the matrix A_1 does not have a Jordan block of size $n_{1,j}$ at λ_j, thus of its $\ell_{2,j}$ signs attached to blocks of size $n_{2,j}$, exactly $\ell_{2,j} - 1$ are attached to blocks of size $n_{2,j}$ in A_2.

Now, repeating this argument $k - 2$ more times, we arrive at $A_k = A + B$ letting the largest Jordan block of A_k at λ_j have size $n_{\kappa_j,j}$ with exactly $\ell'_{\kappa_j,j} = \ell_{1,j} + \cdots + \ell_{\kappa_j,j} - k$ copies. Then, the signs at blocks of size $n_{\kappa_j+1,j}, \ldots, n_{m_j,j}$ are preserved, i.e., they are the same in A and in A_k (4.6), and of the signs attached to blocks of size $n_{\kappa_j,j}$ in A, exactly $\ell_{\kappa'_j,j}$ are attached to blocks of size $n_{\kappa_j,j}$ in A_k which is equivalent to (4.7).

At last, we turn to the second step of the proof by following the lines of the proof of [19, Theorem 4.6]. Thus, let us assume for some $U \in \mathbb{C}^{n,k}$ that the property (1) from above holds but $U \notin \Omega'_k$. Then, by [22, Theorem 3.4], there exists $\delta > 0$ such that for every $U_0 \in \mathbb{C}^{n,k}$ with $\|U - U_0\| < \delta$ and with $(A + U_0 \Sigma U_0 H, H)$ satisfying property (1) (where B is replaced by $U_0 \Sigma U_0 H$), the sign characteristic of $(A + U_0 \Sigma U_0 H, H)$ coincides with that of $(A + U \Sigma U H, H)$. It remains to choose $U_0 \in \Omega'_k$, which is possible in view of the genericity of Ω'_k. $\square$

Now, if A is H-selfadjoint, it is immediately clear that for the generic (with respect to the real and imaginary parts of the entries) set $\Omega_k \subseteq \mathbb{C}^{n,k}$ from Theorem 3.1, both (1) and (2) from the above Theorem hold, i.e., the behavior described in (1) and (2) above is generic (with respect to the real and imaginary parts of U).

On the other hand, if H is real and A is real H-symmetric, one can interpret A as being (complex) H-selfadjoint and still apply the above theorem. Since for the real generic set $\Omega_k \subseteq \mathbb{R}^{n,k}$ from Theorem 3.2 (in the case $\mathbb{F} = \mathbb{R}$) condition (1) from the above theorem is satisfied, also (2) holds whenever $U \in \Omega_k$. Note that since there was no explicit genericity hypothesis in Theorem 4.1, it could be applied to both the H-selfadjoint and the H-symmetric case, despite the two different notions of genericity in Theorems 3.1 and 3.2.

Next, let us turn to rank-k perturbations of real J-Hamiltonian matrices, whereby Theorem 4.1 will be a key ingredient. Again, the J-Hamiltonian case will be more difficult since the partial multiplicities of a J-Hamiltonian matrix behave differently under structured low-rank perturbations.

By Theorem 2.8, each partial multiplicity n_{ij} of a purely imaginary but nonzero eigenvalue λ_i of a matrix A that is real J-Hamiltonian has a sign $\sigma_{ij} \in \{+1, -1\}$ in the *sign characteristic* of λ_i. Moreover, if $\lambda = 0$ is an eigenvalue of a real J-Hamiltonian matrix, then only even partial multiplicities will have a sign in the sign characteristic. In order to allow a unified treatment of purely imaginary eigenvalues including the eigenvalue $\lambda = 0$, we will extend the notion of sign characteristic and define each odd partial multiplicity at the eigenvalue zero to have the "sign" zero in the sign characteristic.

Theorem 4.2. *Let $J \in \mathbb{R}^{n\times n}$ be invertible and skew-symmetric (thus n is even), and let $A \in \mathbb{R}^{n\times n}$ be J-Hamiltonian. Let $\Sigma = \operatorname{diag}(s_1, \dots, s_k)$ with $s_j \in \{-1, +1\}$ and let $\lambda_1, \dots, \lambda_p$ be the purely imaginary eigenvalues of A and $\lambda_{p+1}, \dots, \lambda_q$ be the non purely imaginary eigenvalues of A. Further, let $n_{1,j} > \dots > n_{m_j,j}$ be the distinct block sizes of A at some eigenvalue λ_j such that there exist $\ell_{i,j}$ blocks of size $n_{i,j}$ at λ_j and, whenever $j \in \{1, \dots, p\}$, let A have the signs $\{\sigma_{1,i,j}, \dots, \sigma_{\ell_{i,j},i,j}\}$ attached to its blocks of size $n_{i,j}$ at λ_j. Then, whenever $U \in \mathbb{R}^{n,k}$ is such that for $B := U\Sigma U^T J$ the statement (1) below is satisfied, also (2) holds.*

(1) *The perturbed matrix $A + B$ has the Jordan structure as described in (1) of Theorem 3.3. To be more precise, for each $j = 1, \dots, q$, letting $\ell'_{\kappa_j,j} = \ell_{1,j} + \dots + \ell_{\kappa_j,j} - k$ and letting κ_j be the smallest integer with $\ell'_{\kappa_j,j} \geq 1$, then:*

   (a) *If $\lambda_j \neq 0$, or if $\lambda_j = 0$ and $\ell_{1,j}n_{1,j} + \dots + \ell_{\kappa_j-1,j}n_{\kappa_j-1,j} + (\ell_{\kappa_j,j} - \ell'_{\kappa_j,j})n_{\kappa_j,j}$ is even, $A + B$ has the distinct block sizes*

$$n_{\kappa_j,j} > n_{\kappa_j+1,j} > \dots > n_{m_j,j} \tag{4.8}$$

   occurring $\ell'_{\kappa_j,j}, \ell_{\kappa_j+1,j}, \dots, \ell_{m_j,j}$ times, respectively, at λ_j.

   (b) *If $\lambda_j = 0$ and $\ell_{1,j}n_{1,j} + \dots + \ell_{\kappa_j-1,j}n_{\kappa_j-1,j} + (\ell_{\kappa_j,j} - \ell'_{\kappa_j,j})n_{\kappa_j,j}$ is odd, $A + B$ has the distinct block sizes*

$$n_{\kappa_j,j} + 1 > n_{\kappa_j,j} > n_{\kappa_j+1,j} > \dots > n_{m_j,j} \tag{4.9}$$

   occurring $1, (\ell'_{\kappa_j,j} - 1), \ell_{\kappa_j+1,j}, \dots, \ell_{m_j,j}$ times, respectively, at 0.

(2) *For each $j = 1, \dots, p$, let $\{\sigma'_{1,\kappa_j,j}, \dots, \sigma'_{\ell'_{\kappa_j,j},\kappa_j,j}\}$ be the signs of $A + B$ at blocks of size $n_{\kappa_j,j}$ at λ_j and let $\{\sigma'_{1,i,j}, \dots, \sigma'_{\ell_{i,j},i,j}\}$ be the signs at blocks of size $n_{i,j}$ at λ_j for $i = \kappa_j + 1, \dots, m_j$. Then the following statements hold for $j = 1, \dots, p$:*

(a1) *If $\lambda_j \neq 0$, then the signs of $A+B$ satisfy*

$$\sum_{s=1}^{\ell_{i,j}} \sigma_{s,i,j} = \sum_{s=1}^{\ell_{i,j}} \sigma'_{s,i,j}, \qquad i = \kappa_j + 1, \ldots, m_j,$$

and

$$\left| \sum_{s=1}^{\ell_{\kappa_j,j}} \sigma_{s,\kappa_j,j} - \sum_{s=1}^{\ell'_{\kappa_j,j}} \sigma'_{s,\kappa_j,j} \right| \leq \ell_{\kappa_j,j} - \ell'_{\kappa_j,j}.$$

(a2) *If $\lambda_j = 0$ and $\ell_{1,j}n_{1,j} + \cdots + \ell_{\kappa_j-1,j}n_{\kappa_j-1,j} + (\ell_{\kappa_j,j} - \ell'_{\kappa_j,j})n_{\kappa_j,j}$ is even, the signs of $A+B$ satisfy*

$$\sum_{s=1}^{\ell_{i,j}} \sigma_{s,i,j} = \sum_{s=1}^{\ell_{i,j}} \sigma'_{s,i,j}$$

for $i = \kappa_j + 1, \ldots, m_j$, where both sums are zero whenever $n_{i,j}$ is odd. Furthermore, if $n_{\kappa_j,j}$ is odd, then the above also holds for $i = n_{\kappa_j,j}$ (as in that case both sums are zero), and if $n_{\kappa_j,j}$ is even, then

$$\left| \sum_{s=1}^{\ell_{\kappa_j,j}} \sigma_{s,\kappa_j,j} - \sum_{s=1}^{\ell'_{\kappa_j,j}} \sigma'_{s,\kappa_j,j} \right| \leq \ell_{\kappa_j,j} - \ell'_{\kappa_j,j}.$$

(b) *If $\lambda_j = 0$ and $\ell_{1,j}n_{1,j} + \cdots + \ell_{\kappa_j-1,j}n_{\kappa_j-1,j} + (\ell_{\kappa_j,j} - \ell'_{\kappa_j,j})n_{\kappa_j,j}$ is odd, the signs of $A+B$ satisfy*

$$\sum_{s=1}^{\ell_{i,j}} \sigma_{s,i,j} = \sum_{s=1}^{\ell_{i,j}} \sigma'_{s,i,j}$$

for $i = \kappa_j + 1, \ldots, m_j$, where both sums are zero whenever $n_{i,j}$ is odd. (In particular, $n_{\kappa_j,j}$ is odd, so all corresponding signs are zero.)

Proof. We proceed using [19, Theorem 4.1] in order to identify the signs attached to blocks in (A, J) with ones attached to blocks in (iA, iJ), where iA is an iJ-selfadjoint (complex) matrix.

We first consider the case (a1), i.e., $\lambda_j = i\alpha$ is different from zero. Now, for any $U \in \mathbb{R}^{n,k}$ such that the perturbed matrix $A + U\Sigma U^T J$ has the partial multiplicities in (4.8) at λ_j, also $iA + iU\Sigma U^T J$, which is iJ-selfadjoint, has these multiplicities at $-\alpha$. Hence, by Theorem 4.1, the signs of $iA + iU\Sigma U^T J$ at $-\alpha$ are obtained as follows: All signs at blocks of sizes $n_{\kappa_j+1,j}, \ldots, n_{m_j,j}$ are preserved, and of the signs at blocks of size $n_{\kappa_j,j}$, exactly $\ell'_{\kappa_j,j}$ ones are preserved. Now, the same procedure applies to the signs of $A + U\Sigma U^T J$ by [19, Theorem 4.1], i.e., the signs satisfy the assertion in (a1).

The next case is (a2), i.e., we have $\lambda_j = 0$ and the number

$$\ell_{1,j}n_{1,j} + \cdots + \ell_{\kappa_j-1,j}n_{\kappa_j-1,j} + (\ell_{\kappa_j,j} - \ell'_{\kappa_j,j})n_{\kappa_j,j}$$

is even. This number is the sum of the sizes of all blocks at λ_j that are destroyed under perturbation in this case. Since $\ell_{1,j}n_{1,j}, \dots, \ell_{\kappa_j-1,j}n_{\kappa_j-1,j}$ are all even, this implies that either $\ell_{\kappa_j,j} - \ell'_{\kappa_j,j}$ or $n_{\kappa_j,j}$ is even (or both), i.e., an even number of odd-sized blocks is destroyed under perturbation.

Again, let $U \in \mathbb{R}^{n,k}$ be such that the perturbed matrix $A + U\Sigma U^T J$ has the partial multiplicities from (4.8) at 0. Then the same is true for the iJ-selfadjoint matrix $iA + iU\Sigma U^T J$ at 0. Hence, by Theorem 4.1, the signs of $iA + iU\Sigma U^T J$ are obtained as follows: All signs at blocks of sizes $n_{\kappa_j+1,j}, \dots, n_{m_j,j}$ are preserved, and of the signs at blocks of size $n_{\kappa_j,j}$, exactly $\ell'_{\kappa_j,j}$ ones are preserved. By [19, Theorem 4.1] this translates to the signs of $A+U\Sigma U^T J$ at 0: All signs at blocks of even sizes smaller than $n_{\kappa_j,j}$ are preserved. Further, if $n_{\kappa_j,j}$ is even, then exactly $\ell'_{\kappa_j,j}$ signs at this block size are preserved, i.e., the signs satisfy the assertion in (a2).

Finally, let $\lambda_j = 0$ and let $\ell_{1,j}n_{1,j} + \dots + \ell_{\kappa_j-1,j}n_{\kappa_j-1,j} + (\ell_{\kappa_j,j} - \ell'_{\kappa_j,j})n_{\kappa_j,j}$ be odd. From this immediately follows that $(\ell_{\kappa_j,j} - \ell'_{\kappa_j,j})n_{\kappa_j,j}$ must be odd, i.e., $n_{\kappa_j,j}$ and $(\ell_{\kappa_j,j} - \ell'_{\kappa_j,j})$ are both odd, and since $\ell_{\kappa_j,j}$ is even, $\ell'_{\kappa_j,j}$ is odd. In particular, as $n_{\kappa_j,j}$ is odd, there are no signs attached to blocks of this size in neither A nor $A+U\Sigma U^T J$. Also, we note that $\ell'_{\kappa_j,j} - 1$ may be 0 so that in the perturbed matrix, there do not occur blocks of this size.

Concerning the Jordan structure of the perturbed matrix, again we assume that $U \in \mathbb{R}^{n\times k}$ is such that the perturbed matrix $A + U\Sigma U^T J$ has the partial multiplicities in (4.9) at 0. Concerning the sign characteristic, we cannot apply Theorem 4.1 in this case (note that the partial multiplicities in (4.9) differ from the ones required in Theorem 4.1) so that we continue with a strategy similar to the one from the proof of Theorem 3.3:

Let $s_{k+1} \in \{-1, +1\}$ and let $u \in \Omega_1$ be a vector from the generic set Ω_1 in Theorem 3.3 applied for the case $k = 1$ to the matrix $A+B$. Then at the eigenvalue $\lambda_j = 0$, the matrix $A + B + s_{k+1}uu^T J$ has the partial multiplicities

$$n_{\kappa_j,j} > n_{\kappa_j+1,j} > \dots > n_{m_j,j}$$

occurring $(\ell'_{\kappa_j,j} - 1), \ell_{\kappa_j+1,j}, \dots, \ell_{m_j,j}$ times, respectively, i.e., only the newly generated block of size $n_{\kappa_j,j} + 1$ at $\lambda_j = 0$ in $A + B$ has vanished.

Let $\{\sigma''_{1,i,j}, \dots, \sigma''_{\ell_{i,j},i,j}\}$ be the signs of $A + B + s_{k+1}uu^T J$ at blocks of size $n_{i,j}$ at $\lambda_j = 0$ for $i = \kappa_j + 1, \dots, m_j$. (The signs on the blocks of size $n_{\kappa_j,j}$ are zero by definition as $n_{\kappa_j,j}$ is odd, so there is no need for considering these signs in the following.) Observe that $A + B + s_{k+1}uu^T J$ is a rank-one perturbation of $A + B$ that satisfies the hypotheses of (1) and (a2), so applying the already proved part (a2) for the rank-one case to the matrix $A + B$, we obtain that

$$\sum_{s=1}^{\ell_{i,j}} \sigma'_{s,i,j} = \sum_{s=1}^{\ell_{i,j}} \sigma''_{s,i,j}, \quad i = \kappa_j + 1, \dots, m_j. \tag{4.10}$$

On the other hand, the matrix $A+B+s_{k+1}uu^T J$ is a rank-$(k+1)$ perturbation of A that also satisfies the hypotheses of (1) and (a2), so applying the already proved part (a2) for the rank-$(k+1)$ case to the matrix A, we obtain that

$$\sum_{s=1}^{\ell_{i,j}} \sigma_{s,i,j} = \sum_{s=1}^{\ell_{i,j}} \sigma''_{s,i,j}, \quad i = \kappa_j + 1, \ldots, m_j. \tag{4.11}$$

Combining (4.10) and (4.11), we see that the assertion in (b) is satisfied. □

In particular, since the case $k = 1$ is included in the above theorem, we have hereby proved [19, Conjecture 4.8]. Then again, in the above theorem, there is no statement on the sign at the newly generated block of (even) size $n_{\kappa_j,j} + 1$ in the case (2b). Examples show that this sign can either be $+1$ or -1 depending on the particular perturbation; see also [19, Conjecture 4.4], [4].

5. Conclusion

We have completely described the canonical form, i.e., Jordan structure and sign characteristic (whenever applicable) of structured matrices under generic, structure-preserving rank-k perturbations. In particular, the Meta-Conjecture 1.1 from the Introduction of this paper was proved for the cases of H-selfadjoint matrices over the field $\mathbb{C}$, and for H-symmetric and J-Hamiltonian matrices over both the fields $\mathbb{R}$ and $\mathbb{C}$.

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References

[1] L. Batzke. Generic rank-one perturbations of structured regular matrix pencils. *Linear Algebra Appl.*, 458:638–670, 2014.

[2] L. Batzke. Generic rank-two perturbations of structured regular matrix pencils. *Oper. Matrices*, 10:83–112, 2016.

[3] L. Batzke. Sign characteristics of regular Hermitian matrix pencils under generic rank-1 and rank-2 perturbations. *Electron. J. Linear Algebra*, 30:760–794, 2015.

[4] L. Batzke. *Generic Low-Rank Perturbations of Structured Regular Matrix Pencils and Structured Matrices*. Ph.D. thesis, Technische Universität Berlin, Germany, 2015.

[5] F. De Terán, F. Dopico, and J. Moro. Low rank perturbation of Weierstrass structure. *SIAM J. Matrix Anal. Appl.*, 30:538–547, 2008.

[6] J.H. Fourie, G.J. Groenewald, D.B. Janse van Rensburg, A.C.M. Ran. Rank one perturbations of H-positive real matrices. *Linear Algebra Appl.*, 439:653–674, 2013.

[7] I. Gohberg, P. Lancaster, and L. Rodman. *Matrices and Indefinite Scalar Products*. Birkhäuser, Basel, 1983.

[8] I. Gohberg, P. Lancaster, and L. Rodman. *Indefinite Linear Algebra and Applications*. Birkhäuser, Basel, 2005.

[9] L. Hörmander and A. Melin. A remark on perturbations of compact operators. *Math. Scand.* 75:255–262, 1994.

[10] D.B. Janse van Rensburg. *Structured matrices in indefinite inner product spaces: simple forms, invariant subspaces and rank-one perturbations.* Ph.D. thesis, North-West University, Potchefstroom, South Africa, 2012.

[11] P. Lancaster and L. Rodman. *The Algebraic Riccati Equation.* Oxford University Press, Oxford, 1995.

[12] P. Lancaster and L. Rodman. Canonical forms for Hermitian matrix pairs under strict equivalence and congruence. *SIAM Review*, 47:407–443, 2005.

[13] P. Lancaster and L. Rodman. Canonical forms for symmetric/skew-symmetric real matrix pairs under strict equivalence and congruence. *Linear Algebra Appl.*, 406:1–76, 2005.

[14] C. Mehl. On classification of normal matrices in indefinite inner product spaces. *Electron. J. Linear Algebra*, 15:50–83, 2006.

[15] C. Mehl, V. Mehrmann, A.C.M. Ran, and L. Rodman. Eigenvalue perturbation theory of classes of structured matrices under generic structured rank one perturbations, *Linear Algebra Appl.*, 435:687–716, 2011.

[16] C. Mehl, V. Mehrmann, A.C.M. Ran, and L. Rodman. Perturbation theory of selfadjoint matrices and sign characteristics under generic structured rank one perturbations. *Linear Algebra Appl.*, 436:4027–4042, 2012.

[17] C. Mehl, V. Mehrmann, A.C.M. Ran, and L. Rodman. Jordan forms of real and complex matrices under rank one perturbations. *Oper. Matrices*, 7:381–398, 2013.

[18] C. Mehl, V. Mehrmann, A.C.M. Ran, and L. Rodman. Eigenvalue perturbation theory of symplectic, orthogonal, and unitary matrices under generic structured rank one perturbations. *BIT*, 54:219–255, 2014.

[19] C. Mehl, V. Mehrmann, A.C.M. Ran, and L. Rodman. Eigenvalue perturbation theory of structured real matrices under generic structured rank-one perturbations. *Linear and Multilinear Algebra*, 64:527–556, 2016.

[20] J. Moro and F. Dopico. Low rank perturbation of Jordan structure. *SIAM J. Matrix Anal. Appl.*, 25:495–506, 2003.

[21] A.C.M. Ran and M. Wojtylak. Eigenvalues of rank one perturbations of unstructured matrices. *Linear Algebra Appl.*, 437: 589–600, 2012.

[22] L. Rodman. Similarity vs unitary similarity and perturbation analysis of sign characteristics: Complex and real indefinite inner products. *Linear Algebra Appl.*, 416:945–1009, 2006.

[23] S.V. Savchenko. Typical changes in spectral properties under perturbations by a rank-one operator. *Mat. Zametki*, 74:590–602, 2003 (Russian). Translation in Mathematical Notes. 74:557–568, 2003.

[24] S.V. Savchenko. On the change in the spectral properties of a matrix under a perturbation of a sufficiently low rank. *Funktsional. Anal. i Prilozhen*, 38:85–88, 2004 (Russian). Translation in Funct. Anal. Appl. 38:69–71, 2004.

[25] G.W. Stuart and J.-G. Sun. *Matrix Perturbation Theory.* Academic Press, Boston etc., 1990.

[26] R.C. Thompson. Pencils of complex and real symmetric and skew matrices. *Linear Algebra Appl.*, 147:323–371, 1991.

Leonhard Batzke and Christian Mehl
TU Berlin
Sekretariat MA 4-5
Straße des 17. Juni 136
D-10623 Berlin, Germany
e-mail: batzke@math.tu-berlin.de
mehl@math.tu-berlin.de

André C.M. Ran
Afdeling Wiskunde
Faculteit der Exacte Wetenschappen
Vrije Universiteit Amsterdam
De Boelelaan 1081a
NL-1081 HV Amsterdam, The Netherlands
and
Unit for BMI
North-West University
Potchefstroom, South Africa
e-mail: ran@few.vu.nl

Leiba Rodman †

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The Krein–von Neumann Realization of Perturbed Laplacians on Bounded Lipschitz Domains

Jussi Behrndt, Fritz Gesztesy, Till Micheler and Marius Mitrea

Abstract. In this paper we study the self-adjoint Krein–von Neumann realization A_K of the perturbed Laplacian $-\Delta + V$ in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. We provide an explicit and self-contained description of the domain of A_K in terms of Dirichlet and Neumann boundary traces, and we establish a Weyl asymptotic formula for the eigenvalues of A_K.

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1. Introduction

The main objective of this note is to investigate the self-adjoint Krein–von Neumann realization associated to the differential expression $-\Delta + V$ in $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$, $n > 1$, is assumed to be a bounded Lipschitz domain and V is a non-negative bounded potential. In particular, we obtain an explicit description of the domain of A_K in terms of Dirichlet and Neumann boundary traces, and we prove the Weyl asymptotic formula

$$N(\lambda, A_K) \underset{\lambda\to\infty}{=} (2\pi)^{-n} v_n |\Omega| \, \lambda^{n/2} + O\big(\lambda^{(n-(1/2))/2}\big). \tag{1.1}$$

Here $N(\lambda, A_K)$ denotes the number of nonzero eigenvalues of A_K not exceeding λ, v_n is the volume of the unit ball in $\mathbb{R}^n$, and $|\Omega|$ is the (n-dimensional) Lebesgue measure of Ω.

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Let us first recall the definition and some properties of the Krein–von Neumann extension in the abstract setting. Let S be a closed, densely defined, symmetric operator in a Hilbert space $\mathcal{H}$ and assume that S is strictly positive, that is, for some $c > 0$, $(Sf, f)_{\mathcal{H}} \geqslant c\|f\|_{\mathcal{H}}^2$ for all $f \in \operatorname{dom}(S)$. The Krein–von Neumann extension S_K of S is then given by

$$S_K f = S^* f, \quad f \in \operatorname{dom}(S_K) = \operatorname{dom}(S) \dot{+} \ker(S^*), \tag{1.2}$$

see the original papers Krein [48] and von Neumann [57]. It follows that S_K is a nonnegative self-adjoint extension of S and that for all other nonnegative self-adjoint extensions S_Θ of S the operator inequality $S_K \leqslant S_\Theta$ holds in the sense of quadratic forms. As $\ker(S_K) = \ker(S^*)$, it is clear that 0 is an eigenvalue of S_K (except if S is self-adjoint, in which case $S_K = S^* = S$). Furthermore, if the self-adjoint Friedrichs extension S_F of S has purely discrete spectrum then the same is true for the spectrum of S_K with the possible exception of the eigenvalue 0, which may have infinite multiplicity. For further developments, extensive references, and a more detailed discussion of the properties of the Krein–von Neumann extension of a symmetric operator we refer the reader to [2, Sect. 109], [3], [4]–[6], [7, Chs. 9, 10], [8]–[13], [14], [15], [16], [17], [19], [27], [28], [29], [32, Sect. 15], [33, Sect. 3.3], [36], [38], [39, Sect. 13.2], [40], [41], [50], [55], [58], [59, Ch. 13], [60], [61], [62], [64], [65], [66], and the references cited therein.

In the concrete case considered in this paper, the symmetric operator S above is given by the minimal operator $A_{\min}$ associated to the differential expression $-\Delta + V$ in the Hilbert space $L^2(\Omega)$, that is,

$$A_{\min} = -\Delta + V, \quad \operatorname{dom}(A_{\min}) = \mathring{H}^2(\Omega), \tag{1.3}$$

where $\mathring{H}^2(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the Sobolev space $H^2(\Omega)$, and $0 \leqslant V \in L^\infty(\Omega)$. It can be shown that $A_{\min}$ is the closure of the symmetric operator $-\Delta + V$ defined on $C_0^\infty(\Omega)$. We point out that here Ω is a bounded Lipschitz domain and no further regularity assumptions on $\partial\Omega$ are imposed. The adjoint $A_{\min}^*$ of $A_{\min}$ coincides with the maximal operator

$$\begin{aligned} A_{\max} &= -\Delta + V, \\ \operatorname{dom}(A_{\max}) &= \big\{ f \in L^2(\Omega) \,\big|\, \Delta f \in L^2(\Omega) \big\}, \end{aligned} \tag{1.4}$$

where Δf is understood in the sense of distributions. From (1.2) and (1.3) it is clear that the Krein–von Neumann extension A_K of $A_{\min}$ is then given by

$$A_K = -\Delta + V, \quad \operatorname{dom}(A_K) = \mathring{H}^2(\Omega) \dot{+} \ker(A_{\max}). \tag{1.5}$$

In the present situation $A_{\min}$ is a symmetric operator with infinite defect indices and therefore $\ker(A_{\min}^*) = \ker(A_{\max})$ is infinite-dimensional. In particular, 0 is an eigenvalue of A_K with infinite multiplicity, and hence belongs to the essential spectrum. It is also important to note that in general the functions in $\ker(A_K)$ do not possess any Sobolev regularity, that is, $\ker(A_K) \not\subset H^s(\Omega)$ for every $s > 0$. Moreover, since Ω is a bounded set, the Friedrichs extension of $A_{\min}$ (which coincides with the self-adjoint Dirichlet operator associated to $-\Delta + V$) has compact resolvent and hence its spectrum is discrete. The abstract considerations above then yield that with the exception of the eigenvalue 0 the spectrum of A_K consists of a sequence of positive eigenvalues with finite multiplicity which tend to $+\infty$.

The description of the domain of the Krein–von Neumann extension A_K in (1.5) is not satisfactory for applications involving boundary value problems. Instead, a more explicit description of $\mathrm{dom}(A_K)$ via boundary conditions seems to be natural and desirable. In the case of a bounded C^∞-smooth domain Ω, it is known that

$$\mathrm{dom}(A_K) = \bigl\{f \in \mathrm{dom}(A_{\max}) \,\big|\, \gamma_N f + M(0)\gamma_D f = 0\bigr\} \tag{1.6}$$

holds, where γ_D and γ_N denote the Dirichlet and Neumann trace operator, respectively, defined on the maximal domain $\mathrm{dom}(A_{\max})$, and $M(0)$ is the Dirichlet-to-Neumann map or Weyl–Titchmarsh operator at $z = 0$ for $-\Delta + V$. The description (1.6) goes back to Višik [67] and Grubb [37], where certain classes of elliptic differential operators with smooth coefficients are discussed in great detail. Note that in contrast to the Dirichlet and Neumann boundary conditions the boundary condition in (1.6) is nonlocal, as it involves $M(0)$ which, when Ω is smooth, is a boundary pseudodifferential operator of order 1. It is essential for the boundary condition (1.6) that both trace operators γ_D and γ_N are defined on $\mathrm{dom}(A_{\max})$. Even in the case of a smooth boundary $\partial\Omega$, the elements in $\mathrm{dom}(A_K)$, in general, do not possess any H^s-regularity for $s > 0$, and hence special attention has to be paid to the definition and the properties of the trace operators. In the smooth setting the classical analysis due to Lions and Magenes [49] ensures that $\gamma_D : \mathrm{dom}(A_{\max}) \to H^{-1/2}(\partial\Omega)$ and $\gamma_N : \mathrm{dom}(A_{\max}) \to H^{-3/2}(\partial\Omega)$ are well-defined continuous mappings when $\mathrm{dom}(A_{\max})$ is equipped with the graph norm.

Let us now turn again to the present situation, where Ω is assumed to be a bounded Lipschitz domain. Our first main objective is to extend the description of $\mathrm{dom}(A_K)$ in (1.6) to the nonsmooth setting. The main difficulty here is to define appropriate trace operators on the domain of the maximal operator. We briefly sketch the strategy from [18], which is mainly based and inspired by abstract extension theory of symmetric operators. For this denote by A_D and A_N the self-adjoint realizations of $-\Delta+V$ corresponding to Dirichlet and Neumann boundary conditions, respectively. Recall that by [43] and [31] their domains $\mathrm{dom}(A_D)$ and $\mathrm{dom}(A_N)$ are both contained in $H^{3/2}(\Omega)$. Now consider the boundary spaces

$$\begin{aligned} \mathscr{G}_D(\partial\Omega) &:= \bigl\{\gamma_D f \,\big|\, f \in \mathrm{dom}(A_N)\bigr\}, \\ \mathscr{G}_N(\partial\Omega) &:= \bigl\{\gamma_N f \,\big|\, f \in \mathrm{dom}(A_D)\bigr\}, \end{aligned} \tag{1.7}$$

equipped with suitable inner products induced by the Neumann-to-Dirichlet map and Dirichlet-to-Neumann map for $-\Delta+V-i$, see Section 3 for the details. It turns out that $\mathscr{G}_D(\partial\Omega)$ and $\mathscr{G}_N(\partial\Omega)$ are both Hilbert spaces which are densely embedded in $L^2(\partial\Omega)$. It was shown in [18] that the Dirichlet trace operator γ_D and Neumann trace operator γ_N can be extended by continuity to surjective mappings

$$\widetilde{\gamma}_D : \mathrm{dom}(A_{\max}) \to \mathscr{G}_N(\partial\Omega)^* \quad \text{and} \quad \widetilde{\gamma}_N : \mathrm{dom}(A_{\max}) \to \mathscr{G}_D(\partial\Omega)^*, \tag{1.8}$$

where $\mathscr{G}_D(\partial\Omega)^*$ and $\mathscr{G}_N(\partial\Omega)^*$ denote the adjoint (i.e., conjugate dual) spaces of $\mathscr{G}_D(\partial\Omega)$ and $\mathscr{G}_N(\partial\Omega)$, respectively. Within the same process also the Dirichlet-to-Neumann map $M(0)$ of $-\Delta + V$ (originally defined as a mapping from $H^1(\partial\Omega)$ to $L^2(\partial\Omega)$) admits an extension to a mapping $\widetilde{M}(0)$ from $\mathscr{G}_N(\partial\Omega)^*$ to $\mathscr{G}_D(\partial\Omega)^*$. With the help of the trace maps $\widetilde{\gamma}_D$ and $\widetilde{\gamma}_N$, and the extended Dirichlet-to-Neumann operator $\widetilde{M}(0)$ we are then able to extend the description of the domain of the Krein–von Neumann extension for smooth domains in (1.6) to the case of Lipschitz domains. More precisely, we show in Theorem 3.3 that the Krein–von Neumann extension A_K of $A_{\min}$ is defined on

$$\mathrm{dom}(A_K) = \big\{ f \in \mathrm{dom}(A_{\max}) \,\big|\, \widetilde{\gamma}_N f + \widetilde{M}(0)\widetilde{\gamma}_D f = 0 \big\}. \tag{1.9}$$

For an exhaustive treatment of boundary trace operators on bounded Lipschitz domains in $\mathbb{R}^n$ and applications to Schrödinger operators we refer to [17].

Our second main objective in this paper is to prove the Weyl asymptotic formula (1.1) for the nonzero eigenvalues of A_K. We mention that the study of the asymptotic behavior of the spectral distribution function of the Dirichlet Laplacian originates in the work of Weyl (cf. [68], [69], and the references in [70]), and that generalizations of the classical Weyl asymptotic formula were obtained in numerous papers – we refer the reader to [20], [21], [22], [23], [24], [25], [26], [56], [63], and the introduction in [16] for more details. There are relatively few papers available that treat the spectral asymptotics of the Krein Laplacian or the perturbed Krein Laplacian A_K. Essentially these considerations are inspired by Alonso and Simon who, at the end of their paper [3] posed the question if the asymptotics of the nonzero eigenvalues of the Krein Laplacian is given by Weyl's formula. In the case where Ω is bounded and C^∞-smooth, and $V \in C^\infty(\overline{\Omega})$, this has been shown to be the case three years later by Grubb [38], see also the more recent contributions [52], [53], and [40]. Following the ideas in [38] it was shown in [14] that for so-called quasi-convex domains (a nonsmooth subclass of bounded Lipschitz domains with the key feature that $\mathrm{dom}(A_D)$ and $\mathrm{dom}(A_N)$ are both contained in $H^2(\Omega)$) the Krein–von Neumann extension A_K is spectrally equivalent to the buckling of a clamped plate problem, which in turn can be reformulated with the help of the quadratic forms

$$\mathfrak{a}[f,g] := \big(A_{\min} f, A_{\min} g\big)_{L^2(\Omega)} \text{ and } \mathfrak{t}[f,g] := \big(f, A_{\min} g\big)_{L^2(\Omega)}, \tag{1.10}$$

defined on $\mathrm{dom}(A_{\min}) = \mathring{H}^2(\Omega)$. In the Hilbert space $(\mathring{H}^2(\Omega), \mathfrak{a}[\cdot,\cdot])$ the form $\mathfrak{t}$ can be expressed with the help of a nonnegative compact operator T, and it follows that

$$\lambda \in \sigma_p(A_K)\backslash\{0\} \text{ if and only if } \lambda^{-1} \in \sigma_p(T), \tag{1.11}$$

counting multiplicities. These considerations can be extended from quasi-convex domains to the more general setting of Lipschitz domains, see, for instance, Section 4 and Lemma 4.2. Finally, the main ingredient in the proof of the Weyl asymptotic formula (1.1) for the Krein–von Neumann extension A_K of $-\Delta + V$ on a bounded Lipschitz domain Ω is then a more general Weyl-type asymptotic formula due to Kozlov [46] (see also [45], [47]) which yields the asymptotics of the spectral distribution function of the compact operator T, and hence via (1.11) the asymptotics of the spectral distribution function of A_K. This reasoning in the proof of our second main result Theorem 4.1 is along the lines of [14, 15], where the special case of quasi-convex domains was treated. For perturbed Krein Laplacians this result completes work that started with Grubb more than 30 years ago and demonstrates that the question posed by Alonso and Simon in [3] regarding the validity of the Weyl asymptotic formula continues to have an affirmative answer for bounded Lipschitz domains – the natural end of the line in the development from smooth domains all the way to minimally smooth ones.

2. Schrödinger operators on bounded Lipschitz domains

This section is devoted to studying self-adjoint Schrödinger operators on a non-empty, bounded Lipschitz domain in $\mathbb{R}^n$ (which, by definition, is assumed to be open). We shall adopt the following background assumption.

Hypothesis 2.1. *Let $n \in \mathbb{N}\backslash\{1\}$, assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, and suppose that $0 \leqslant V \in L^\infty(\Omega)$.*

We consider operator realizations of the differential expression $-\Delta+V$ in the Hilbert space $L^2(\Omega)$. For this we define the *preminimal* realization A_p of $-\Delta+V$ by

$$A_p := -\Delta + V, \quad \operatorname{dom}(A_p) := C_0^\infty(\Omega). \tag{2.1}$$

It is clear that A_p is a densely defined, symmetric operator in $L^2(\Omega)$, and hence closable. The *minimal* realization $A_{\min}$ of $-\Delta + V$ is defined as the closure of A_p in $L^2(\Omega)$,

$$A_{\min} := \overline{A_p}. \tag{2.2}$$

It follows that $A_{\min}$ is a densely defined, closed, symmetric operator in $L^2(\Omega)$. The *maximal* realization $A_{\max}$ of $-\Delta + V$ is given by

$$A_{\max} := -\Delta + V, \quad \operatorname{dom}(A_{\max}) := \big\{f \in L^2(\Omega) \,\big|\, \Delta f \in L^2(\Omega)\big\}, \tag{2.3}$$

where the expression Δf, $f \in L^2(\Omega)$, is understood in the sense of distributions.

In the next lemma we collect some properties of the operators A_p, $A_{\min}$, and $A_{\max}$. The standard L^2-based Sobolev spaces of order $s \geqslant 0$ will be denoted by $H^s(\Omega)$; for the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ we write $\mathring{H}^s(\Omega)$.

Lemma 2.2. *Assume Hypothesis 2.1 and let A_p, $A_{\min}$, and $A_{\max}$ be as introduced above. Then the following assertions hold:*

(i) *$A_{\min}$ and $A_{\max}$ are adjoints of each other, that is,*

$$A_{\min}^* = A_p^* = A_{\max} \ \textit{and} \ A_{\min} = \overline{A_p} = A_{\max}^*. \tag{2.4}$$

(ii) $A_{\min}$ *is defined on* $\mathring{H}^2(\Omega)$, *that is,*

$$\operatorname{dom}(A_{\min}) = \mathring{H}^2(\Omega), \tag{2.5}$$

and the graph norm of $A_{\min}$ *and the* H^2*-norm are equivalent on the domain of* $A_{\min}$.

(iii) $A_{\min}$ *is strictly positive, that is, for some* $C > 0$ *we have*

$$(A_{\min} f, f)_{L^2(\Omega)} \geqslant C\|f\|^2_{L^2(\Omega)}, \quad f \in \mathring{H}^2(\Omega). \tag{2.6}$$

(iv) $A_{\min}$ *has infinite deficiency indices.*

One recalls that the *Friedrichs extension* A_F of $A_{\min}$ is defined by

$$A_F := -\Delta + V, \quad \operatorname{dom}(A_F) := \big\{ f \in \mathring{H}^1(\Omega) \,\big|\, \Delta f \in L^2(\Omega) \big\}. \tag{2.7}$$

It is well known that A_F is a strictly positive self-adjoint operator in $L^2(\Omega)$ with compact resolvent (see, e.g., [30, Sect. VI.1]).

In this note we are particularly interested in the *Krein–von Neumann extension* A_K of $A_{\min}$. According to (1.2), A_K is given by

$$A_K := -\Delta + V, \quad \operatorname{dom}(A_K) := \operatorname{dom}(A_{\min}) \dotplus \ker(A_{\max}). \tag{2.8}$$

In the following theorem we briefly collect some well-known properties of the Krein–von Neumann extension A_K in the present setting. For more details we refer the reader to the celebrated paper [48] by Krein and to [3], [4], [11], [14], [15], [16], [40], and [41] for further developments and references.

Theorem 2.3. *Assume Hypothesis* 2.1 *and let* A_K *be the Krein–von Neumann extension of* $A_{\min}$. *Then the following assertions hold:*

(i) A_K *is a nonnegative self-adjoint operator in* $L^2(\Omega)$ *and* $\sigma(A_K)$ *consists of eigenvalues only. The eigenvalue* 0 *has infinite multiplicity,*

$$\dim(\ker(A_K)) = \infty,$$

and the restriction $A_K|_{(\ker(A_K))^\perp}$ *is a strictly positive self-adjoint operator in the Hilbert space* $(\ker(A_K))^\perp$ *with compact resolvent.*

(ii) $\operatorname{dom}(A_K) \not\subset H^s(\Omega)$ *for every* $s > 0$.

(iii) *A nonnegative self-adjoint operator* B *in* $L^2(\Omega)$ *is a self-adjoint extension of* $A_{\min}$ *if and only if for some* (*and, hence for all*) $\mu < 0$,

$$(A_F - \mu)^{-1} \leqslant (B - \mu)^{-1} \leqslant (A_K - \mu)^{-1}. \tag{2.9}$$

We also mention that the Friedrichs extension A_F and the Krein–von Neumann extension A_K are relatively prime (or disjoint), that is,

$$\operatorname{dom}(A_F) \cap \operatorname{dom}(A_K) = \operatorname{dom}(A_{\min}) = \mathring{H}^2(\Omega). \tag{2.10}$$

For later purposes we briefly recall some properties of the Dirichlet and Neumann trace operator and the corresponding self-adjoint Dirichlet and Neumann realizations of $-\Delta + V$ in $L^2(\Omega)$. We consider the space

$$H^{3/2}_\Delta(\Omega) := \big\{ f \in H^{3/2}(\Omega) \,\big|\, \Delta f \in L^2(\Omega) \big\}, \tag{2.11}$$

equipped with the inner product

$$(f,g)_{H^{3/2}_\Delta(\Omega)} = (f,g)_{H^{3/2}(\Omega)} + (\Delta f, \Delta g)_{L^2(\Omega)}, \quad f,g \in H^{3/2}_\Delta(\Omega). \tag{2.12}$$

One recalls that the Dirichlet and Neumann trace operators γ_D and γ_N defined by

$$\gamma_D f := f\restriction_{\partial\Omega} \text{ and } \gamma_N f := \mathfrak{n}\cdot\nabla f\restriction_{\partial\Omega}, \quad f \in C^\infty(\overline{\Omega}), \tag{2.13}$$

admit continuous extensions to operators

$$\gamma_D : H^{3/2}_\Delta(\Omega) \to H^1(\partial\Omega) \text{ and } \gamma_N : H^{3/2}_\Delta(\Omega) \to L^2(\partial\Omega). \tag{2.14}$$

Here $H^1(\partial\Omega)$ denotes the usual L^2-based Sobolev space of order 1 on $\partial\Omega$; cf. [51, Chapter 3] and [54]. It is important to note that the extensions in (2.14) are both surjective, see [36, Lemma 3.1 and Lemma 3.2].

In the next theorem we collect some properties of the Dirichlet realization A_D and Neumann realization A_N of $-\Delta+V$ in $L^2(\Omega)$. We recall that the operators A_D and A_N are defined as the unique self-adjoint operators corresponding to the closed nonnegative forms

$$\begin{aligned} \mathfrak{a}_D[f,g] &:= (\nabla f, \nabla g)_{(L^2(\Omega))^n} + (Vf,g)_{L^2(\Omega)}, \quad \operatorname{dom}(\mathfrak{a}_D) := \mathring{H}^1(\Omega), \\ \mathfrak{a}_N[f,g] &:= (\nabla f, \nabla g)_{(L^2(\Omega))^n} + (Vf,g)_{L^2(\Omega)}, \quad \operatorname{dom}(\mathfrak{a}_N) := H^1(\Omega). \end{aligned} \tag{2.15}$$

In particular, one has $A_F = A_D$ by (2.7). In the next theorem we collect some well-known facts about the self-adjoint operators A_D and A_N. The $H^{3/2}$-regularity of the functions in their domains is remarkable, and a consequence of Ω being a bounded Lipschitz domain. We refer the reader to [35, Lemma 3.4 and Lemma 4.8] for more details, see also [42, 43] and [31].

Theorem 2.4. *Assume Hypothesis* 2.1 *and let A_D and A_N be the self-adjoint Dirichlet and Neumann realization of $-\Delta+V$ in $L^2(\Omega)$, respectively. Then the following assertions hold:*

(i) *The operator A_D coincides with the Friedrichs extension A_F and is given by*

$$A_D = -\Delta + V, \quad \operatorname{dom}(A_D) = \big\{ f \in H^{3/2}_\Delta(\Omega) \,\big|\, \gamma_D f = 0 \big\}. \tag{2.16}$$

The resolvent of A_D is compact, and the spectrum of A_D is purely discrete and contained in $(0,\infty)$.

(ii) *The operator A_N is given by*

$$A_N = -\Delta + V, \quad \operatorname{dom}(A_N) = \big\{ f \in H^{3/2}_\Delta(\Omega) \,\big|\, \gamma_N f = 0 \big\}. \tag{2.17}$$

The resolvent of A_N is compact, and the spectrum of A_N is purely discrete and contained in $[0,\infty)$.

3. Boundary conditions for the Krein–von Neumann realization

Our goal in this section is to obtain an explicit description of the domain of the Krein–von Neumann extension A_K in terms of Dirichlet and Neumann boundary traces. For this we describe an extension procedure of the trace maps γ_D and γ_N in (2.14) onto $\operatorname{dom}(A_{\max})$ from [18]. We recall that for $\varphi \in H^1(\partial\Omega)$ and $z \in \rho(A_D)$, the boundary value problem

$$-\Delta f + Vf = zf, \quad \gamma_D f = \varphi, \tag{3.1}$$

admits a unique solution $f_z(\varphi) \in H^{3/2}_\Delta(\Omega)$. Making use of this fact and the trace operators (2.14) we define the Dirichlet-to-Neumann operator $M(z)$, $z \in \rho(A_D)$, as follows:

$$M(z) : L^2(\partial\Omega) \supset H^1(\partial\Omega) \to L^2(\partial\Omega), \quad \varphi \mapsto -\gamma_N f_z(\varphi), \tag{3.2}$$

where $f_z(\varphi) \in H^{3/2}_\Delta(\Omega)$ is the unique solution of (3.1). It can be shown that $M(z)$ is an unbounded operator in $L^2(\partial\Omega)$. Moreover, if $z \in \rho(A_D) \cap \rho(A_N)$ then $M(z)$ is invertible and the inverse $M(z)^{-1}$ is a bounded operator defined on $L^2(\partial\Omega)$. Considering $z = i$, we set

$$\Sigma := \operatorname{Im}\left(-M(i)^{-1}\right). \tag{3.3}$$

The imaginary part $\operatorname{Im} M(i)$ of $M(i)$ is a densely defined bounded operator in $L^2(\partial\Omega)$ and hence it admits a bounded closure

$$\Lambda := \overline{\operatorname{Im}(M(i))} \tag{3.4}$$

in $L^2(\partial\Omega)$. Both operators Σ and Λ are self-adjoint and invertible with unbounded inverses. Next we introduce the boundary spaces

$$\mathscr{G}_D(\partial\Omega) := \big\{\gamma_D f \,\big|\, f \in \operatorname{dom}(A_N)\big\} \tag{3.5}$$

and

$$\mathscr{G}_N(\partial\Omega) := \big\{\gamma_N f \,\big|\, f \in \operatorname{dom}(A_D)\big\}. \tag{3.6}$$

It turns out that

$$\mathscr{G}_D(\partial\Omega) = \operatorname{ran}\big(\Sigma^{1/2}\big) \qquad \text{and} \qquad \mathscr{G}_N(\partial\Omega) = \operatorname{ran}\big(\Lambda^{1/2}\big), \tag{3.7}$$

and hence the spaces $\mathscr{G}_D(\partial\Omega)$ and $\mathscr{G}_N(\partial\Omega)$ can be equipped with the inner products

$$(\varphi,\psi)_{\mathscr{G}_D(\partial\Omega)} := \big(\Sigma^{-1/2}\varphi, \Sigma^{-1/2}\psi\big)_{L^2(\partial\Omega)}, \quad \varphi,\psi \in \mathscr{G}_D(\partial\Omega), \tag{3.8}$$

and

$$(\varphi,\psi)_{\mathscr{G}_N(\partial\Omega)} := \big(\Lambda^{-1/2}\varphi, \Lambda^{-1/2}\psi\big)_{L^2(\partial\Omega)}, \quad \varphi,\psi \in \mathscr{G}_N(\partial\Omega), \tag{3.9}$$

respectively. Then $\mathscr{G}_D(\partial\Omega)$ and $\mathscr{G}_N(\partial\Omega)$ both become Hilbert spaces which are dense in $L^2(\partial\Omega)$. The corresponding adjoint (i.e., conjugate dual) spaces will be denoted by $\mathscr{G}_D(\partial\Omega)^*$ and $\mathscr{G}_N(\partial\Omega)^*$, respectively. The following result can be found in [18, Section 4.1].

Theorem 3.1. *Assume Hypothesis* 2.1. *Then the Dirichlet trace operator* γ_D *and the Neumann trace operator* γ_N *in* (2.14) *can be extended by continuity to surjective mappings*

$$\widetilde{\gamma}_D : \mathrm{dom}(A_{\max}) \to \mathscr{G}_N(\partial\Omega)^* \quad \text{and} \quad \widetilde{\gamma}_N : \mathrm{dom}(A_{\max}) \to \mathscr{G}_D(\partial\Omega)^* \tag{3.10}$$

such that $\ker(\widetilde{\gamma}_D) = \ker(\gamma_D) = \mathrm{dom}(A_D)$ *and* $\ker(\widetilde{\gamma}_N) = \ker(\gamma_N) = \mathrm{dom}(A_N)$.

In a similar manner the boundary value problem (3.1) can be considered for all $\varphi \in \mathscr{G}_N(\partial\Omega)^*$ and the Dirichlet-to-Neumann operator $M(\cdot)$ in (3.2) can be extended. More precisely, the following statement holds.

Theorem 3.2. *Assume Hypothesis* 2.1 *and let* $\widetilde{\gamma}_D$ *and* $\widetilde{\gamma}_N$ *be the extended Dirichlet and Neumann trace operator from Theorem* 3.1. *Then the following are true:*

(i) *For* $\varphi \in \mathscr{G}_N(\partial\Omega)^*$ *and* $z \in \rho(A_D)$ *the boundary value problem*

$$-\Delta f + Vf = zf, \quad \widetilde{\gamma}_D f = \varphi, \tag{3.11}$$

admits a unique solution $f_z(\varphi) \in \mathrm{dom}(A_{\max})$.

(ii) *For* $z \in \rho(A_D)$ *the Dirichlet-to-Neumann operator* $M(z)$ *in* (3.2) *admits a continuous extension*

$$\widetilde{M}(z) : \mathscr{G}_N(\partial\Omega)^* \to \mathscr{G}_D(\partial\Omega)^*, \quad \varphi \mapsto -\widetilde{\gamma}_N f_z(\varphi), \tag{3.12}$$

where $f_z(\varphi) \in \mathrm{dom}(A_{\max})$ *is the unique solution of* (3.11).

Now we are able to state our main result in this section, amounting to a concrete description of the domain of the Krein–von Neumann extension A_K in terms of Dirichlet and Neumann boundary traces. The extended Dirichlet-to-Neumann map at $z = 0$ will enter as a regularization parameter in the boundary condition. For C^∞-smooth domains this result goes back to Grubb [37], where a certain class of elliptic differential operators with smooth coefficients is discussed systematically. For the special case of a so-called quasi-convex domains Theorem 3.3 reduces to [15, Theorem 5.5] and [36, Theorem 13.1]. In an abstract setting the Krein–von Neumann extension appears in a similar form in [18, Example 3.9].

Theorem 3.3. *Assume Hypothesis* 2.1 *and let* $\widetilde{\gamma}_D$, $\widetilde{\gamma}_N$ *and* $\widetilde{M}(0)$ *be as in Theorem* 3.1 *and Theorem* 3.2. *Then the Krein–von Neumann extension* A_K *of* $A_{\min}$ *is given by*

$$\begin{aligned} A_K &= -\Delta + V, \\ \mathrm{dom}(A_K) &= \big\{ f \in \mathrm{dom}(A_{\max}) \,\big|\, \widetilde{\gamma}_N f + \widetilde{M}(0)\widetilde{\gamma}_D f = 0 \big\}. \end{aligned} \tag{3.13}$$

Proof. We recall that the Krein–von Neumann extension A_K of $A_{\min}$ is defined on

$$\mathrm{dom}(A_K) = \mathrm{dom}(A_{\min}) \dotplus \ker(A_{\max}). \tag{3.14}$$

Thus, from Lemma 2.2 (ii) one concludes

$$\mathrm{dom}(A_K) = \mathring{H}^2(\Omega) \dotplus \ker(A_{\max}). \tag{3.15}$$

Next, we show the inclusion

$$\operatorname{dom}(A_K) \subseteq \big\{f \in \operatorname{dom}(A_{\max}) \,\big|\, \widetilde{\gamma}_N f + \widetilde{M}(0)\widetilde{\gamma}_D f = 0\big\}. \tag{3.16}$$

Fix $f \in \operatorname{dom}(A_K)$ and decompose f in the form $f = f_{\min} + f_0$, where $f_{\min} \in \mathring{H}^2(\Omega)$ and $f_0 \in \ker(A_{\max})$ (cf. (3.15)). Thus,

$$\gamma_D f_{\min} = \widetilde{\gamma}_D f_{\min} = 0 \quad \text{and} \quad \gamma_N f_{\min} = \widetilde{\gamma}_N f_{\min} = 0, \tag{3.17}$$

and hence it follows from Theorem 3.2 (ii) that

$$\widetilde{M}(0)\widetilde{\gamma}_D f = \widetilde{M}(0)\widetilde{\gamma}_D(f_{\min} + f_0) = \widetilde{M}(0)\widetilde{\gamma}_D f_0 = -\widetilde{\gamma}_N f_0 = -\widetilde{\gamma}_N f. \tag{3.18}$$

Thus, $\widetilde{\gamma}_N f + \widetilde{M}(0)\widetilde{\gamma}_D f = 0$ and the inclusion (3.16) holds.

Next we verify the opposite inclusion

$$\operatorname{dom}(A_K) \supseteq \big\{f \in \operatorname{dom}(A_{\max}) \,\big|\, \widetilde{\gamma}_N f + \widetilde{M}(0)\widetilde{\gamma}_D f = 0\big\}. \tag{3.19}$$

We use the direct sum decomposition

$$\operatorname{dom}(A_{\max}) = \operatorname{dom}(A_D) \dotplus \ker(A_{\max}), \tag{3.20}$$

which is not difficult to check. Assuming that $f \in \operatorname{dom}(A_{\max})$ satisfies the boundary condition

$$\widetilde{M}(0)\widetilde{\gamma}_D f + \widetilde{\gamma}_N f = 0, \tag{3.21}$$

according to the decomposition (3.20) we write f in the form $f = f_D + f_0$, where $f_D \in \operatorname{dom}(A_D)$ and $f_0 \in \ker(A_{\max})$. Thus, $\gamma_D f_D = \widetilde{\gamma}_D f_D = 0$ by Theorem 3.1 and with the help of Theorem 3.2 (ii) one obtains

$$\widetilde{M}(0)\widetilde{\gamma}_D f = \widetilde{M}(0)\widetilde{\gamma}_D(f_D + f_0) = \widetilde{M}(0)\widetilde{\gamma}_D f_0 = -\widetilde{\gamma}_N f_0. \tag{3.22}$$

Taking into account the boundary condition (3.21) one concludes

$$-\widetilde{\gamma}_N f = \widetilde{M}(0)\widetilde{\gamma}_D f = -\widetilde{\gamma}_N f_0, \tag{3.23}$$

and hence

$$0 = \widetilde{\gamma}_N(f - f_0) = \widetilde{\gamma}_N f_D. \tag{3.24}$$

Together with Theorem 3.1 this implies $f_D \in \ker(\widetilde{\gamma}_N) = \ker(\gamma_N) = \operatorname{dom}(A_N)$. Thus, one arrives at

$$f_D \in \operatorname{dom}(A_D) \cap \operatorname{dom}(A_N) = \operatorname{dom}(A_{\min}) = \mathring{H}^2(\Omega). \tag{3.25}$$

Summing up, one has

$$f = f_D + f_0 \in \mathring{H}^2(\Omega) \dotplus \ker(A_{\max}) = \operatorname{dom}(A_K), \tag{3.26}$$

which establishes (3.19) and completes the proof of Theorem 3.3. $\square$

4. Spectral asymptotics of the Krein–von Neumann extension

As the main result in this section we derive the following Weyl-type spectral asymptotics for the Krein–von Neumann extension A_K of $A_{\min}$.

Theorem 4.1. *Assume Hypothesis* 2.1. *Let* $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ *be the strictly positive eigenvalues of the Krein–von Neumann extension* A_K *enumerated in nondecreasing order counting multiplicity, and let*

$$N(\lambda, A_K) := \#\big\{j \in \mathbb{N} : 0 < \lambda_j \leqslant \lambda\big\}, \quad \lambda > 0, \tag{4.1}$$

be the eigenvalue distribution function for A_K. *Then the following Weyl asymptotic formula holds,*

$$N(\lambda, A_K) \underset{\lambda\to\infty}{=} \frac{v_n\,|\Omega|}{(2\pi)^n}\,\lambda^{n/2} + O\big(\lambda^{(n-(1/2))/2}\big), \tag{4.2}$$

where $v_n = \pi^{n/2}/\Gamma((n/2)+1)$ *denotes the* (*Euclidean*) *volume of the unit ball in* $\mathbb{R}^n$ (*with* $\Gamma(\cdot)$ *the classical Gamma function* [1, Sect. 6.1]) *and* $|\Omega|$ *represents the* (n*-dimensional*) *Lebesgue measure of* Ω.

The proof of Theorem 4.1 follows along the lines of [14, 15], where the case of quasi-convex domains was investigated. The main ingredients are a general Weyl type asymptotic formula due to Kozlov [46] (see also [45], [47] for related results) and the connection between the eigenvalues of the so-called buckling operator and the positive eigenvalues of the Krein–von Neumann extension A_K (cf. [15], [16]). We first consider the quadratic forms $\mathfrak{a}$ and $\mathfrak{t}$ on $\mathrm{dom}(A_{\min}) = \mathring{H}^2(\Omega)$ defined by

$$\mathfrak{a}[f,g] := \big(A_{\min}f, A_{\min}g\big)_{L^2(\Omega)}, \quad f,g \in \mathrm{dom}(\mathfrak{a}) := \mathring{H}^2(\Omega), \tag{4.3}$$

$$\mathfrak{t}[f,g] := \big(f, A_{\min}g\big)_{L^2(\Omega)}, \quad f,g \in \mathrm{dom}(\mathfrak{t}) := \mathring{H}^2(\Omega). \tag{4.4}$$

Since the graph norm of $A_{\min}$ and the H^2-norm are equivalent on $\mathrm{dom}(A_{\min}) = \mathring{H}^2(\Omega)$ by Lemma 2.2 (ii), it follows that $\mathcal{W} := (\mathrm{dom}(\mathfrak{a}); (\cdot,\cdot)_{\mathcal{W}})$, where the inner product is defined by

$$(f,g)_{\mathcal{W}} := \mathfrak{a}[f,g] = \big(A_{\min}f, A_{\min}g\big)_{L^2(\Omega)}, \quad f,g \in \mathrm{dom}(\mathfrak{a}), \tag{4.5}$$

is a Hilbert space. One observes that the embedding $\iota : \mathcal{W} \to L^2(\Omega)$ is compact; this is a consequence of Ω being bounded. Next, we consider for fixed $g \in \mathcal{W}$ the functional

$$\mathcal{W} \ni f \mapsto \mathfrak{t}[\iota f, \iota g], \tag{4.6}$$

which is continuous on the Hilbert space $\mathcal{W}$ and hence can be represented with the help of a bounded operator T in $\mathcal{W}$ in the form

$$(f, Tg)_{\mathcal{W}} = \mathfrak{t}[\iota f, \iota g], \quad f,g \in \mathcal{W}. \tag{4.7}$$

The nonnegativity of the form $\mathfrak{t}$ implies that T is a self-adjoint and nonnegative operator in $\mathcal{W}$. Furthermore, one obtains for $f,g \in \mathcal{W}$ from (4.4) that

$$(f, Tg)_{\mathcal{W}} = \mathfrak{t}[\iota f, \iota g] = \big(\iota f, A_{\min}\iota g\big)_{L^2(\Omega)} = \big(f, \iota^* A_{\min}\iota g\big)_{\mathcal{W}}, \tag{4.8}$$

and hence,

$$T = \iota^* A_{\min} \iota. \tag{4.9}$$

In particular, since $A_{\min}\iota : \mathcal{W} \to L^2(\Omega)$ is defined on the whole space $\mathcal{W}$ and is closed as an operator from $\mathcal{W}$ to $L^2(\Omega)$, it follows that $A_{\min}\iota$ is bounded and hence the compactness of $\iota^* : L^2(\Omega) \to \mathcal{W}$ implies that $T = \iota^* A_{\min}\iota$ is a compact operator in the Hilbert space $\mathcal{W}$.

The next lemma shows that the eigenvalues of T are precisely the reciprocals of the nonzero eigenvalues of A_K. Lemma 4.2 is inspired by the connection of the Krein–von Neumann extension to the buckling of a clamped plate problem (cf. [15, Theorem 6.2] and [14, 16, 38]).

Lemma 4.2. *Assume Hypothesis* 2.1 *and let T be the nonnegative compact operator in $\mathcal{W}$ defined by* (4.7). *Then*

$$\lambda \in \sigma_p(A_K)\backslash\{0\} \text{ if and only if } \lambda^{-1} \in \sigma_p(T), \tag{4.10}$$

counting multiplicities.

Proof. Assume first that $\lambda \neq 0$ is an eigenvalue of A_K and let g be a corresponding eigenfunction. We decompose g in the form

$$g = g_{\min} + g_0, \quad g_{\min} \in \operatorname{dom}(A_{\min}), \ g_0 \in \ker(A_{\max}) \tag{4.11}$$

(cf. (2.8)), where $g_{\min} \neq 0$ as $\lambda \neq 0$. Then one concludes

$$A_{\min} g_{\min} = A_K(g_{\min} + g_0) = A_K g, \tag{4.12}$$

and hence,

$$A_{\min} g_{\min} - \lambda g_{\min} = A_K g - \lambda g_{\min} = \lambda g - \lambda g_{\min} = \lambda g_0 \in \ker(A_{\max}), \tag{4.13}$$

so

$$A_{\max} A_{\min} g_{\min} = \lambda A_{\max} g_{\min} = \lambda A_{\min} g_{\min}. \tag{4.14}$$

This yields

$$\begin{aligned}(f, \lambda^{-1} g_{\min})_{\mathcal{W}} &= \mathfrak{a}[f, \lambda^{-1} g_{\min}] = \big(A_{\min} f, \lambda^{-1} A_{\min} g_{\min}\big)_{L^2(\Omega)} \\ &= \big(f, \lambda^{-1} A_{\max} A_{\min} g_{\min}\big)_{L^2(\Omega)} \\ &= \big(f, A_{\min} g_{\min}\big)_{L^2(\Omega)} \\ &= \mathfrak{t}[f, g_{\min}] = (f, T g_{\min})_{\mathcal{W}}, \quad f \in \mathcal{W},\end{aligned} \tag{4.15}$$

where, for simplicity, we have identified elements in $\mathcal{W}$ with those in $\operatorname{dom}(\mathfrak{a})$, and hence omitted the embedding map ι. From (4.15) we then conclude

$$T g_{\min} = \frac{1}{\lambda} g_{\min}, \tag{4.16}$$

which shows that $\lambda^{-1} \in \sigma_p(T)$.

Conversely, assume that $h \in \mathcal{W}\backslash\{0\}$ and $\lambda \neq 0$ are such that

$$Th = \frac{1}{\lambda} h \tag{4.17}$$

holds. Then it follows for $f \in \operatorname{dom}(\mathfrak{a})$ from (4.5) and (4.7) that

$$\mathfrak{a}[f, h] = \mathfrak{a}[f, \lambda T h] = (f, \lambda T h)_{\mathcal{W}} = \mathfrak{t}[f, \lambda h] = \big(f, \lambda A_{\min} h\big)_{L^2(\Omega)}. \tag{4.18}$$

As a consequence of the first representation theorem for quadratic forms [44, Theorem VI.2.1 (iii), Example VI.2.13], one concludes that $A_{\max} A_{\min}$ is the representing operator for $\mathfrak{a}$, and therefore,

$$h \in \operatorname{dom}(A_{\max} A_{\min}) \ \text{ and } \ A_{\max} A_{\min} h = \lambda A_{\min} h. \tag{4.19}$$

In particular, $h \in \operatorname{dom}(A_{\min})$ and

$$\begin{aligned} A_{\max}(A_{\min} - \lambda) h &= A_{\max} A_{\min} h - \lambda A_{\max} h \\ &= A_{\max} A_{\min} h - \lambda A_{\min} h = 0. \end{aligned} \tag{4.20}$$

Let us define

$$g := \frac{1}{\lambda} A_{\min} h = h + \frac{1}{\lambda}\big(A_{\min} - \lambda\big) h. \tag{4.21}$$

As $h \in \operatorname{dom}(A_{\min})$ and $(A_{\min} - \lambda) h \in \ker(A_{\max})$ by (4.20), we conclude from (2.8) that $g \in \operatorname{dom} A_K$. Moreover, $g \neq 0$ since $A_{\min}$ is positive. Furthermore,

$$A_K g = A_{\max} g = \frac{1}{\lambda} A_{\max} A_{\min} h = A_{\min} h = \lambda g, \tag{4.22}$$

shows that $\lambda \in \sigma_p(A_K)$. $\square$

Proof of Theorem 4.1. Let T be the nonnegative compact operator in $\mathcal{W}$ defined by (4.7). We order the eigenvalues of T in the form

$$0 \leqslant \cdots \leqslant \mu_{j+1}(T) \leqslant \mu_j(T) \leqslant \cdots \leqslant \mu_1(T), \tag{4.23}$$

listed according to their multiplicity, and set

$$\mathcal{N}(\lambda, T) := \#\big\{ j \in \mathbb{N} : \mu_j(T) \geqslant \lambda^{-1} \big\}, \quad \lambda > 0. \tag{4.24}$$

It follows from Lemma 4.2 that

$$\mathcal{N}(\lambda, T) = N(\lambda, A_K), \quad \lambda > 0, \tag{4.25}$$

and hence [46] yields the asymptotic formula,

$$N(\lambda, A_K) = \mathcal{N}(\lambda, T) \underset{\lambda \to \infty}{=} \omega\, \lambda^{n/2} + O\big(\lambda^{(n-(1/2))/2}\big), \tag{4.26}$$

with

$$\begin{aligned} \omega &:= \frac{1}{n(2\pi)^n} \int_\Omega \left(\int_{S^{n-1}} \left[\frac{\sum_{j=1}^n \xi_j^2}{\sum_{j,k=1}^n \xi_j^2 \xi_k^2} \right]^{\frac{n}{2}} d\omega_{n-1}(\xi) \right) d^n x \\ &= \frac{v_n\, |\Omega|}{(2\pi)^n}. \end{aligned} \tag{4.27}$$

$\square$

For bounds on $N(\,\cdot\,, A_K)$ in the case of $\Omega \subset \mathbb{R}^n$ open and of finite (n-dimensional) Lebesgue measure, and $V = 0$, we refer to [34].

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1972.

[2] N.I. Akhiezer and I.M. Glazman, *Theory of Linear Operators in Hilbert Space, Volume II*, Pitman, Boston, 1981.

[3] A. Alonso and B. Simon, *The Birman–Krein–Vishik theory of selfadjoint extensions of semibounded operators*, J. Operator Th. **4**, 251–270 (1980); Addenda: **6**, 407 (1981).

[4] T. Ando and K. Nishio, *Positive selfadjoint extensions of positive symmetric operators*, Tohoku Math. J. **22**, 65–75 (1970).

[5] Yu.M. Arlinskii, *On m-accretive extensions and restrictions*, Meth. Funct. Anal. Top. **4**, 1-26 (1998).

[6] Yu.M. Arlinskii, *Abstract boundary conditions for maximal sectorial extensions of sectorial operators*, Math. Nachr. **209**, 5–36 (2000).

[7] Yu. Arlinskii, S. Belyi, and E. Tsekanovskii, *Conservative Realizations of Herglotz–Nevanlinna Functions*, Operator Theory: Advances and Applications, Vol. 217, Birkhäuser, Springer, Basel, 2011.

[8] Yu.M. Arlinskii, S. Hassi, Z. Sebestyén, and H.S.V. de Snoo, *On the class of extremal extensions of a nonnegative operator*, in *Recent Advances in Operator Theory and Related Topics*, L. Kérchy, C. Foias, I. Gohberg, and H. Langer (eds.), Operator Theory: Advances and Applications, Vol. 127, Birkhäuser, Basel, 2001, pp. 41–81.

[9] Yu. Arlinskiĭ and Yu. Kovalev, *Operators in divergence form and their Friedrichs and Krein extensions*, Opusc. Math. **31**, 501–517 (2011).

[10] Yu.M. Arlinskii and E.R. Tsekanovskii, *On the theory of nonnegative selfadjoint extensions of a nonnegative symmetric operator*, Rep. Nat. Acad. Sci. Ukraine **2002**, no. 11, 30–37.

[11] Yu.M. Arlinskiĭ and E.R. Tsekanovskiĭ, *On von Neumann's problem in extension theory of nonnegative operators*, Proc. Amer. Math. Soc. **131**, 3143–3154 (2003).

[12] Yu.M. Arlinskiĭ and E.R. Tsekanovskiĭ, *The von Neumann problem for nonnegative symmetric operators*, Integr. Equ. Oper. Theory **51**, 319–356 (2005).

[13] Yu. Arlinskiĭ and E. Tsekanovskiĭ, *M. Kreĭn's research on semibounded operators, its contemporary developments, and applications*, in *Modern Analysis and Applications. The Mark Krein Centenary Conference*, Vol. 1, V. Adamyan, Y.M. Berezansky, I. Gohberg, M.L. Gorbachuk, V. Gorbachuk, A.N. Kochubei, H. Langer, and G. Popov (eds.), Operator Theory: Advances and Applications, Vol. 190, Birkhäuser, Basel, 2009, pp. 65–112.

[14] M.S. Ashbaugh, F. Gesztesy, M. Mitrea, and G. Teschl, *Spectral theory for perturbed Krein Laplacians in nonsmooth domains*, Adv. Math. **223**, 1372–1467 (2010).

[15] M.S. Ashbaugh, F. Gesztesy, M. Mitrea, R. Shterenberg, and G. Teschl, *The Krein–von Neumann extension and its connection to an abstract buckling problem*, Math. Nachr. **283**, 165–179 (2010).

[16] M.S. Ashbaugh, F. Gesztesy, M. Mitrea, R. Shterenberg, and G. Teschl, *A survey on the Krein–von Neumann extension, the corresponding abstract buckling problem, and Weyl-type spectral asymptotics for perturbed Krein Laplacians in non smooth*

domains, in *Mathematical Physics, Spectral Theory and Stochastic Analysis*, M. Demuth and W. Kirsch (eds.), Operator Theory: Advances and Applications, Vol. 232, Birkhäuser, Springer, Basel, 2013, pp. 1–106.

[17] J. Behrndt, F. Gesztesy, T. Micheler, and M. Mitrea, *Sharp boundary trace theory and Schrödinger operators on bounded Lipschitz domains*, in preparation.

[18] J. Behrndt and T. Micheler, *Elliptic differential operators on Lipschitz domains and abstract boundary value problems*, J. Funct. Anal. **267**, 3657–3709 (2014).

[19] M.S. Birman, *On the theory of self-adjoint extensions of positive definite operators*, Mat. Sbornik **38**, 431–450 (1956) (Russian).

[20] M.S. Birman and M.Z. Solomyak, *Leading term in the asymptotic spectral formula for "non-smooth" elliptic problems*, Funkcional. Anal. i Priložen **4**, no. 4, 1–13 (1970) (Russian); Engl. transl. in Funct. Anal. Appl. **4**, 265–275 (1970).

[21] M.S. Birman and M.Z. Solomyak, *On the asymptotic spectrum of "non-smooth" elliptic equations*, Funkcional. Anal. i Priložen **5**, no. 1, 69–70 (1971) (Russian); Engl. transl. in Funct. Anal. Appl. **5**, 56–57 (1971).

[22] M.S. Birman and M.Z. Solomjak, *Spectral asymptotics of nonsmooth elliptic operators. I*, Trans. Moscow Math. Soc. **27**, 1–52 (1972).

[23] M.S. Birman and M.Z. Solomjak, *Spectral asymptotics of nonsmooth elliptic operators. I*, Trans. Moscow Math. Soc. **28**, 1–32 (1973).

[24] M.S. Birman and M.Z. Solomyak, *Asymtotic behavior of the spectrum of differential equations*, Itogi Nauki i Tekhniki, Matematicheskii Analiz., **14**, 5–58 (1977) (Russian); Engl. transl. in J. Soviet Math. **12**, no. 3, 247–283 (1979).

[25] M.S. Birman and M.Z. Solomjak, *Quantitative Analysis in Sobolev Imbedding Theorems and Applications to Spectral Theory*, AMS Translations, Series 2, Vol. 114, Providence, RI, 1980, pp. 1–132.

[26] E.B. Davies, *L^p spectral theory of higher-order elliptic differential operators*, Bull. London Math. Soc. **29**, 513–546 (1997).

[27] V.A. Derkach and M.M. Malamud, *Weyl function of a Hermitian operator and its connection with characteristic function*, preprint Donetsk Phys. Tech. Inst. (1985), arXiv:1503.08956v1.

[28] V.A. Derkach and M.M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. **95**, 1–95 (1991).

[29] V.A. Derkach and M.M. Malamud, *The extension theory of Hermitian operators and the moment problem*, J. Math. Sci. **73**, 141–242 (1995).

[30] D.E. Edmunds and W.D. Evans, *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1989.

[31] E. Fabes, O. Mendez, and M. Mitrea, *Boundary layers on Sobolev–Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains*, J. Funct. Anal. **159**, 323–368 (1998).

[32] W.G. Faris, *Self-Adjoint Operators*, Lecture Notes in Mathematics, Vol. 433, Springer, Berlin, 1975.

[33] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, 2nd revised and extended ed., de Gruyter Studies in Math., Vol. 19, de Gruyter, Berlin, 2011.

[34] F. Gesztesy, A. Laptev, M. Mitrea, and S. Sukhtaiev, *A bound for the eigenvalue counting function for higher-order Krein Laplacians on open sets*, in *Mathematical Results in Quantum Mechanics, QMath*12 *Proceedings*, P. Exner, W. König, and H. Neidhardt (eds.), World Scientific, Singapore, 2015, pp. 3–29.

[35] F. Gesztesy and M. Mitrea, *Generalized Robin boundary conditions, Robin-to-Dirichlet maps, and Krein-type resolvent formulas for Schrödinger operators on bounded Lipschitz domains*, in *Perspectives in Partial Differential Equations, Harmonic Analysis and Applications: A Volume in Honor of Vladimir G. Maz'ya's* 70*th Birthday*, D. Mitrea and M. Mitrea (eds.), Proceedings of Symposia in Pure Mathematics, Vol. 79, Amer. Math. Soc., Providence, RI, 2008, pp. 105–173.

[36] F. Gesztesy and M. Mitrea, *A description of all self-adjoint extensions of the Laplacian and Krein-type resolvent formulas on non-smooth domains*, J. Analyse Math. **113**, 53–172 (2011).

[37] G. Grubb, *A characterization of the non-local boundary value problems associated with an elliptic operator*, Ann. Scuola Norm. Sup. Pisa (3), **22**, 425–513 (1968).

[38] G. Grubb, *Spectral asymptotics for the "soft" selfadjoint extension of a symmetric elliptic differential operator*, J. Operator Th. **10**, 9–20 (1983).

[39] G. Grubb, *Distributions and Operators*, Graduate Texts in Mathematics, Vol. 252, Springer, New York, 2009.

[40] G. Grubb, *Krein-like extensions and the lower boundedness problem for elliptic operators*, J. Diff. Eq. **252**, 852–885 (2012).

[41] S. Hassi, M. Malamud, and H. de Snoo, *On Krein's extension theory of nonnegative operators*, Math. Nachr. **274/275**, 40–73 (2004).

[42] D. Jerison and C. Kenig, *The Neumann problem in Lipschitz domains*, Bull. Amer. Math. Soc. (N.S.) **4**, 203–207 (1981).

[43] D. Jerison and C. Kenig, *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Funct. Anal. **130**, 161–219 (1995).

[44] T. Kato, *Perturbation Theory for Linear Operators*, corr. printing of the 2nd ed., Springer, Berlin, 1980.

[45] V.A. Kozlov, *Estimation of the remainder in a formula for the asymptotic behavior of the spectrum of nonsemibounded elliptic systems*, (Russian) Vestnik Leningrad. Univ. Mat. Mekh. Astronom. **125**, no 4., 112–113, (1979).

[46] V.A. Kozlov, *Estimates of the remainder in formulas for the asymptotic behavior of the spectrum for linear operator bundles*, Funktsional. Anal. i Prilozhen **17**, no. 2, 80–81 (1983). Engl. transl. in Funct. Anal. Appl. **17**, no. 2, 147–149 (1983).

[47] V.A. Kozlov, *Remainder estimates in spectral asymptotic formulas for linear operator pencils*, Linear and Nonlinear Partial Differential Equations. Spectral Asymptotic Behavior, pp. 34–56, Probl. Mat. Anal. **9**, Leningrad Univ., Leningrad, 1984; Engl. transl. in J. Sov. Math. **35**, 2180–2193 (1986).

[48] M.G. Krein, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. I*, Mat. Sbornik **20**, 431–495 (1947) (Russian).

[49] J.L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Vol. I, Die Grundlehren der mathematischen Wissenschaften, 1972.

[50] M.M. Malamud, *Certain classes of extensions of a lacunary Hermitian operator*, Ukrainian Math. J. **44**, no. 2, 190–204 (1992).

[51] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge Univ. Press, Cambridge, 2000.

[52] V.A. Mikhaĭlets, *Distribution of the eigenvalues of finite multiplicity of Neumann extensions of an elliptic operator*, Differentsial'nye Uravneniya **30**, 178–179 (1994) (Russian); Engl. transl. in Diff. Eq. **30**, 167–168 (1994).

[53] V.A. Mikhaĭlets, *Discrete spectrum of the extreme nonnegative extension of the positive elliptic differential operator*, in *Proceedings of the Ukrainian Mathematical Congress–*2001*, Section* 7*, Nonlinear Analysis*, Kyiv, 2006, pp. 80–94.

[54] I. Mitrea and M. Mitrea, *Multi-Layer Potentials and Boundary Problems for Higher-Order Elliptic Systems in Lipschitz Domains*, Lecture Notes in Mathematics, Vol. 2063, Springer, Berlin, 2013.

[55] G. Nenciu, *Applications of the Kreĭn resolvent formula to the theory of self-adjoint extensions of positive symmetric operators*, J. Operator Th. **10**, 209–218 (1983).

[56] Yu. Netrusov and Yu. Safarov, *Weyl asymptotic formula for the Laplacian on domains with rough boundaries*, Comm. Math. Phys. **253**, no. 2, 481–509 (2005).

[57] J. von Neumann, *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, Math. Ann. **102**, 49–131, (1929/1930).

[58] V. Prokaj and Z. Sebestyén, *On extremal positive operator extensions*, Acta Sci. Math. (Szeged) **62**, 485–491 (1996).

[59] K. Schmüdgen, *Unbounded Self-Adjoint Operators on Hilbert Space*, Graduate Texts in Mathematics, Vol. 265, Springer, Dordrecht, 2012.

[60] Z. Sebestyén and E. Sikolya, *On Krein–von Neumann and Friedrichs extensions*, Acta Sci. Math. (Szeged) **69**, 323–336 (2003).

[61] C.F. Skau, *Positive self-adjoint extensions of operators affiliated with a von Neumann algebra*, Math. Scand. **44**, 171-195 (1979).

[62] O.G. Storozh, *On the hard and soft extensions of a nonnegative operator*, J. Math. Sci. **79**, 1378–1380 (1996).

[63] Yu. Safarov and D. Vassiliev, *The Asymptotic Distribution of Eigenvalues of Partial Differential Operators*, Transl. Math. Monographs, Vol. 155, Amer. Math. Soc., Providence, RI, 1997.

[64] E.R. Tsekanovskii, *Non-self-adjoint accretive extensions of positive operators and theorems of Friedrichs–Krein–Phillips*, Funct. Anal. Appl. **14**, 156–157 (1980).

[65] E.R. Tsekanovskii, *Friedrichs and Krein extensions of positive operators and holomorphic contraction semigroups*, Funct. Anal. Appl. **15**, 308–309 (1981).

[66] E.R. Tsekanovskii, *Accretive extensions and problems on the Stieltjes operator-valued functions realizations*, in *Operator Theory and Complex Analysis*, T. Ando and I. Gohberg (eds.), Operator Theory: Advances and Applications, Vol. 59, Birkhäuser, Basel, 1992, pp. 328–347.

[67] M.I. Višik, *On general boundary problems for elliptic differential equations*, Trudy Moskov. Mat. Obsc. **1**, 187–246 (1952) (Russian); Engl. transl. in Amer. Math. Soc. Transl. (2), **24**, 107–172 (1963).

[68] H. Weyl, *Über die Abhängigkeit der Eigenschwingungen einer Membran und deren Begrenzung*, J. reine angew. Math. **141**, 1–11 (1912).

[69] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen* (*mit einer Anwendung auf die Theorie der Hohlraumstrahlung*), Math. Ann. **71**, 441–479 (1912).

[70] H. Weyl, *Ramifications, old and new, of the eigenvalue problem*, Bull. Amer. Math. Soc. **56**, 115–139 (1950).

Jussi Behrndt
Institut für Numerische Mathematik
Technische Universität Graz
Steyrergasse 30
A-8010 Graz, Austria
e-mail: behrndt@tugraz.at
URL: http://www.math.tugraz.at/∼behrndt/

Fritz Gesztesy and Marius Mitrea
Department of Mathematics
University of Missouri
Columbia, MO 65211, USA
e-mail: gesztesyf@missouri.edu
mitream@missouri.edu
URL: http://www.math.missouri.edu/personnel/faculty/gesztesyf.html
URL: http://www.math.missouri.edu/personnel/faculty/mitream.html

Till Micheler
Department of Mathematics
Technische Universität Berlin
D-10623 Berlin, Germany
e-mail: micheler@math.tu-berlin.de

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The Spectral Problem for the Dispersionless Camassa–Holm Equation

C. Bennewitz, B.M. Brown and R. Weikard

Abstract. We present a spectral and inverse spectral theory for the zero dispersion spectral problem associated with the Camassa–Holm equation. This is an alternative approach to that in [10] by Eckhardt and Teschl.

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1. Background

The Camassa–Holm (CH) equation

$$u_t - u_{xxt} + 3uu_x + 2\varkappa u_x = 2u_x u_{xx} + uu_{xxx}$$

was suggested as a model for shallow water waves by Camassa and Holm [5], although originally found by Fuchssteiner and Fokas [12]. Here $\varkappa$ is a constant related to dispersion. The equation has scaling properties such that one needs only study the cases $\varkappa = 1$ and the zero dispersion case $\varkappa = 0$.

There are compelling reasons to study the equation. Like the KdV equation it is an integrable system but, unlike the KdV equation, among its solutions are *breaking waves* (see Camassa and Holm [5] and Constantin [8]). These are solutions with smooth initial data that stay bounded, but where the wave front becomes vertical in finite time, so that the derivative blows up. A model for water waves displaying wave breaking was long sought after.

Since the CH equation is an integrable system it has an associated spectral problem, which is

$$-f'' + \tfrac{1}{4}f = \lambda w f, \tag{1.1}$$

where $w = u - u_{xx} + \varkappa$. At least two cases are particularly important, namely the periodic case and the case of decay at infinity. We only deal with the latter case here (see, *e.g.*, Constantin and Escher [9]), so in the zero dispersion case we should have w small at infinity. For the periodic case see Constantin and McKean [6].

In the zero dispersion case the solitons (here called *peakons*) give rise to w which is a Dirac measure, so one should clearly at least allow w to be a measure[1]. It is also important that one does not assume that w has a fixed sign, since no wave breaking will then take place (see Jiang, Ni and Zhou [15]).

In [3] we discussed scattering and inverse scattering in the case $\varkappa \neq 0$, which is the important case for shallow water waves. We did not discuss the zero dispersion case $\varkappa = 0$, which is relevant in some other situations, but this case was treated by Eckhardt and Teschl in [10], based on the results of Eckhardt [11].

The approach of [10] was based on the fact that in the zero dispersion case it is possible to define a Titchmarsh–Weyl type m-function for the whole line spectral problem. This approach does not work if $\varkappa \neq 0$. The fundamental reason behind this is that for corresponding half-line problems one gets a discrete spectrum in the zero dispersion case, but there is always a half-line of continuous spectrum if $\varkappa \neq 0$. More conceptually, the continuous spectrum is of multiplicity 2 which excludes the existence of a scalar m-function. For the inverse theory Eckhardt [11] uses de Branges' theory of Hilbert spaces of entire functions. Our approach is different and analogous to that in our paper [3].

It should be noted that the methods of this note combined with those of [3] allow one to prove a uniqueness theorem for inverse scattering in the case $\varkappa \neq 0$ for the case when w is a measure, extending the results of [3] where it was assumed that $w \in L^1_{\mathrm{loc}}$. These results do not appear to be accessible using de Branges' theory.

2. A Hilbert space

Instead of (1.1) we shall analyze the slightly more general spectral problem

$$-f'' + qf = \lambda w f, \tag{2.1}$$

where q is a positive measure not identically zero, since this presents few additional difficulties. A solution of (2.1), or more generally of $-f'' + qf = \lambda w f + g$, where g is a given measure, is a continuous function f satisfying the equation in the sense of distributions. Since $(\lambda w - q)f + g$ is then a measure it follows that a solution is locally absolutely continuous with a derivative of locally bounded variation. It is known that a unique solution exists with prescribed values of f and, say, its left derivative at a given point (this result may be found for example in Bennewitz [1, Chapter 1]), and we will occasionally use this. It follows that the solution space of the homogeneous equation is of dimension 2.

We will also have occasion to talk about the Wronskian $[f_1, f_2] = f_1 f_2' - f_1' f_2$ of two solutions f_1 and f_2 of (2.1). The main property is that such a Wronskian is constant, which easily follows on differentiation and use of the equation. Note that

[1] In this paper we use the word *measure* for a distribution of order 0.

the regularity of solutions is such that the product rule applies when differentiating the Wronskian in the sense of distributions. The unique solvability of the initial value problem shows that f_1 and f_2 are linearly dependent precisely if $[f_1, f_2] = 0$.

We shall consider (2.1) in a Hilbert space $\mathcal{H}_1$ with scalar product

$$\langle f, g\rangle = \int_{\mathbb{R}} (f'\overline{g'} + qf\overline{g}).$$

Thus we are viewing (2.1) as a 'left definite' equation. The space $\mathcal{H}_1$ consists of those locally absolutely continuous functions f which have derivative in $L^2(\mathbb{R})$ and for which $\int_{\mathbb{R}} q|f|^2 < \infty$, so it certainly contains the test functions $C_0^\infty(\mathbb{R})$. Some properties of the space $\mathcal{H}_1$ will be crucial for us.

Lemma 2.1. *Non-trivial solutions of $-u'' + qu = 0$ have at most one zero, and there is no non-trivial solution in $\mathcal{H}_1$.*

Proof. The real and imaginary parts of a solution u are also solutions and in $\mathcal{H}_1$ if u is, so it is enough to consider real-valued solutions. From the equation it is clear that such a solution is convex in any interval where it is positive, concave where it is negative.

The set of zeros of a real-valued non-trivial solution u is a closed set with no interior by the uniqueness of the initial value problem. Since u is continuous it keeps a fixed sign in any component of the complement. Convexity of $|u|$ in each component shows that any such component is unbounded, so u has at most one zero.

Since $|u|$ is convex and non-negative u' can only be in L^2 if u is constant. But this would imply $q = 0$, so the second claim follows. □

As we shall see there are, however, non-trivial solutions with $|u'|^2 + q|u|^2$ integrable on a half-line. We shall also need the following lemma (*cf.* Lemmas 2.1 and 2.2 of [3]).

Lemma 2.2. *Functions with square integrable (distributional) derivative for large $|x|$ are $o(\sqrt{|x|})$ as $x \to \pm\infty$ and point evaluations are bounded linear forms on $\mathcal{H}_1$. Furthermore, $C_0^\infty(\mathbb{R})$ is dense in $\mathcal{H}_1$,*

Proof. The first two claims are proved in [3, Lemma 2.1]). The final claim follows since clearly $C_0^\infty(\mathbb{R}) \subset \mathcal{H}_1$ and if $u \in \mathcal{H}_1$ is orthogonal to $C_0^\infty(\mathbb{R})$ an integration by parts shows that $\int u(-\varphi'' + q\varphi) = 0$ for all $\varphi \in C_0^\infty(\mathbb{R})$ so that u is a distributional solution of $-u'' + qu = 0$. By Lemma 2.1 it is therefore identically 0. □

We also need the following result.

Lemma 2.3. *For any $\lambda \in \mathbb{C}$ there can be at most one linearly independent solution of $-f'' + qf = \lambda wf$ with f' in L^2 near infinity. Similarly for f' in L^2 near $-\infty$.*

This means that (2.1) is in the 'limit-point case' at $\pm\infty$, with a terminology borrowed from the right definite case. The lemma is a consequence of general facts about left definite equations (see our paper [2]), but we will give a simple direct proof.

Proof. Suppose there are two linearly independent solutions f, g with f', g' in L^2 near ∞. We may assume the Wronskian $fg' - f'g = 1$. Now by Lemma 2.2 $f(x)/\sqrt{x}$ and $g(x)/\sqrt{x}$ are bounded for large x. It follows that $(fg' - f'g)/\sqrt{x} = 1/\sqrt{x}$ is in L^2 for large x, which is a contradiction.

Similar calculations may be made for x near $-\infty$. $\square$

Let $E(x)$ be the norm of the linear form $\mathcal{H}_1 \ni f \mapsto f(x)$. We can easily find an expression for $E(x)$, since the Riesz representation theorem tells us that there is an element $g_0(x, \cdot) \in \mathcal{H}_1$ such that $f(x) = \langle f, g_0(x, \cdot)\rangle$. Thus $|f(x)| \leq \|g_0(x, \cdot)\|\|f\|$, with equality for $f = g_0(x, \cdot)$ so that

$$E(x) = \|g_0(x, \cdot)\| = \sqrt{g_0(x, x)}.$$

If $\varphi \in C_0^\infty$ we have $\langle \varphi, g_0(x, \cdot)\rangle = \varphi(x)$, which after an integration by parts means

$$\int_{\mathbb{R}} (-\varphi'' + q\varphi)\overline{g_0(x, \cdot)} = \varphi(x)$$

so (in a distributional sense) $g_0(x, \cdot)$ is a solution of $-f'' + qf = \delta_x$, where δ_x is the Dirac measure at x. Since $g_0(x, y) = \langle g_0(x, \cdot), g_0(y, \cdot)\rangle$ we have a symmetry $g_0(x, y) = \overline{g_0(y, x)}$. Now g_0 is real-valued since $\operatorname{Im} g_0(x, \cdot)$ satisfies $-f'' + qf = 0$ and therefore vanishes according to Lemma 2.1. We may thus write

$$g_0(x, y) = F_+(\max(x, y))F_-(\min(x, y))$$

where $F_\pm$ are real-valued solutions of $-f'' + qf = 0$ small enough at $\pm\infty$ for $g_0(x, \cdot)$ to be in $\mathcal{H}_1$ and by Lemma 2.3 this determines $F_\pm$ up to real multiples. The equation satisfied by $g_0(x, \cdot)$ shows that the Wronskian $[F_+, F_-] = F_+F_-' - F_+'F_- = 1$. In particular, $E(x)$ is locally absolutely continuous. At any specified point of $\mathbb{R}$ there are elements of $\mathcal{H}_1$ that do not vanish, so that $E > 0$ and $F_\pm$ never vanish. Since $g_0(x, x) > 0$ we may therefore assume both to be strictly positive. Note that this still does not determine $F_\pm$ uniquely since multiplying F_+ and dividing F_- by the same positive constant does not change g_0.

However, $|F_\pm'|^2 + q|F_\pm|^2$ has finite integral near $\pm\infty$, although not, according to Lemma 2.3, over $\mathbb{R}$. If we can solve the equation $-f'' + qf = 0$ we can therefore determine $E(x)$. For example, if $q = 1/4$ we have $g_0(x, y) = \exp(-|x - y|/2)$ and $E(x) \equiv 1$.

We shall need some additional properties of $F_\pm$ and make the following definition.

Definition 2.4. Define $K = F_-/F_+$.

We have the following proposition.

Proposition 2.5.

- *$F_\pm$ are both convex,*
- $\lim_\infty F_+' = \lim_{-\infty} F_-' = 0$,
- *$F_\pm'$ as well as F_- are non-decreasing while F_+ is non-increasing,*
- *$F_+(x) \to \infty$ as $x \to -\infty$ and $F_-(x) \to \infty$ as $x \to \infty$,*

- $\lim_{-\infty} F_-$ *and* $\lim_{\infty} F_+$ *are finite,*
- $1/F_+ \in L^2$ *near* $-\infty$ *while* $1/F_- \in L^2$ *near* ∞*,*
- *The function* K *is strictly increasing with range* $\mathbb{R}_+$ *and of class* C^1 *with a* C^1 *inverse, and* $K' = 1/F_+^2$*.*

Proof. The convexity of $F_\pm$ follows from positivity and the differential equation they satisfy. Thus $F'_\pm$ has finite or infinite limits at $\pm\infty$, and since $F'_\pm$ is in L^2 near $\pm\infty$ we have $\lim_{-\infty} F'_- = \lim_{\infty} F'_+ = 0$ so $F'_- \geq 0$ while $F'_+ \leq 0$. It follows that $\lim_{\infty} F_+$ and $\lim_{-\infty} F_-$ are finite.

Neither of $F_\pm$ is constant so it follows that $\lim_{\infty} F_- = \lim_{-\infty} F_+ = +\infty$ and that the range of K is $\mathbb{R}_+$. Furthermore $K' = [F_+, F_-]/F_+^2 = 1/F_+^2$ so K' is continuous and > 0. Thus K has an inverse of class C^1.

Since $K(x) = \int_{-\infty}^x 1/F_+^2$ we have $1/F_+$ in L^2 near $-\infty$, and differentiating $1/K$ we similarly obtain $1/F_-$ in L^2 near ∞. □

3. Spectral theory

In addition to the scalar product, the Hermitian form $w(f,g) = \int_{\mathbb{R}} f\overline{g}w$ plays a role in the spectral theory of (2.1). We denote the total variation measure of w by $|w|$, and make the following assumption in the rest of the paper.

Assumption 3.1. w is a real-valued, non-zero measure (distribution of order zero) and $E^2|w|$ is a finite measure.

We then note the following.

Proposition 3.2. *If* $E^2|w|$ *is a finite measure the form* $w(f,g)$ *is bounded in* $\mathcal{H}_1$*.*

Proof. We have $|w(f,g)| \leq \|f\|\|g\| \int_{\mathbb{R}} E^2|w|$. □

As we shall soon see, the assumption actually implies that the form $w(f,g)$ is compact in $\mathcal{H}_1$. Note that if $q = 1/4$, or any other constant > 0, then the assumption is simply that $|w|$ is finite. It may be proved that this is the case also if $q - q_0$ is a finite signed measure for some constant $q_0 > 0$, and that it is in all cases enough if $(1+|x|)w(x)$ is finite.

Using Riesz' representation theorem Proposition 3.2 immediately shows that there is a bounded operator R_0 on $\mathcal{H}_1$ such that

$$\int_{\mathbb{R}} f\overline{g}w = \langle R_0 f, g\rangle, \tag{3.1}$$

where $\|R_0\| \leq \int_{\mathbb{R}} E^2|w|$. Since w is real-valued the operator R_0 is symmetric. We also have $R_0u(x) = \langle R_0u, g_0(x,\cdot)\rangle = \int_{\mathbb{R}} ug_0(x,\cdot)w$ so that R_0 is an integral operator.

It is clear that $R_0u = 0$ precisely if $uw = 0$, so unless[2] $\operatorname{supp} w = \mathbb{R}$ the operator R_0 has a nontrivial nullspace. We need the following definition.

[2]We always use supports in the sense of distributions.

Definition 3.3. The orthogonal complement of the nullspace of R_0 is denoted by $\mathcal{H}$.

The restriction of R_0 to $\mathcal{H}$, which we also denote by R_0, is an operator on $\mathcal{H}$ with dense range since the orthogonal complement of the range of $R_0^* = R_0$ is the nullspace of R_0. Thus the restriction of R_0 to $\mathcal{H}$ has a selfadjoint inverse T densely defined in $\mathcal{H}$ and R_0 is the resolvent of T at 0.

Lemma 3.4. *$f \in \mathcal{D}_T$ and $Tf = g$ precisely if $f, g \in \mathcal{H}$ and (in the sense of distributions) $-f'' + qf = wg$.*

Proof. $Tf = g$ means that $f = R_0 g$ which in turn means that $\langle f, \overline{\varphi} \rangle = \int g\varphi w$ for $\varphi \in C_0^\infty$ which may be written $\int f(-\varphi'' + q\varphi) = \int g\varphi w$ after an integration by parts. But this is the meaning of the equation $-f'' + qf = wg$.

The same calculation in reverse, using that according to Lemma 2.2 $C_0^\infty(\mathbb{R})$ is dense in $\mathcal{H}_1$, proves the converse. □

The complement of $\operatorname{supp} w$ is a countable union of disjoint open intervals. We shall call any such interval a *gap* in $\operatorname{supp} w$. We obtain the following characterization of the elements of $\mathcal{H}$.

Corollary 3.5.

- *The projection of $v \in \mathcal{H}_1$ onto $\mathcal{H}$ equals v in* $\operatorname{supp} w$, *and if (a, b) is a gap in the support of w the projection is determined in the gap as the solution of $-u'' + qu = 0$ which equals v in the endpoints a and b if these are finite.*

  If $a = -\infty$ the restriction of the projection to the gap is the multiple of F_- which equals v in b, and if $b = \infty$ it is the multiple of F_+ which equals v in a.
- *The support of an element of $\mathcal{H}$ can not begin or end inside a gap in the support of w.*
- *The reproducing kernel $g_0(x, \cdot) \in \mathcal{H}$ if and only if $x \in$* $\operatorname{supp} w$.

Proof. The difference between v and its projection onto $\mathcal{H}$ can be non-zero only in gaps of $\operatorname{supp} w$. Clearly $\varphi w = 0$ for any $\varphi \in C_0^\infty(a, b)$ so that $C_0^\infty(a, b)$ is orthogonal to $\mathcal{H}$. It follows that an element of $\mathcal{H}$ satisfies the equation $-u'' + qu = 0$ in any gap of the support of w.

The first two items are immediate consequences of this, that non-trivial solutions of $-u'' + qu = 0$ have at most one zero according to Lemma 2.1, and of the fact that elements of $\mathcal{H}$ are continuous.

The third item is an immediate consequence of the first two. □

Theorem 3.6. *Under Assumption* 3.1 *the operator R_0 is compact with simple spectrum, so T has discrete spectrum.*

Proof. Suppose $f_j \rightharpoonup 0$ weakly in $\mathcal{H}$. Since point evaluations are bounded linear forms we have $f_j \to 0$ pointwise, and $\{f_j\}_1^\infty$ is bounded in $\mathcal{H}$, as is $\{R_0 f_j\}_1^\infty$. We have

$$\|R_0 f_j\|^2 = \int_{\mathbb{R}} R_0 f_j \overline{f_j} w.$$

Here the coefficient of w tends pointwise to 0 and is bounded by $\|R_0\|\|f_j\|^2E^2$ which in turn is bounded by a multiple of E^2. It follows by dominated convergence that $\|R_0f_j\| \to 0$. Thus R_0 is compact, and the spectrum is simple by Lemma 2.3. □

Actually, R_0 is of trace class as is proved by Eckhardt and Teschl in [10] for the case $q = 1/4$, but we will not need this.

4. Jost solutions

In one-dimensional scattering theory *Jost solutions* play a crucial part. In the case of the Schrödinger equation these are solutions asymptotically equal at ∞ respectively $-\infty$ to certain solutions of the model equation $-f'' = \lambda f$. In the present case the model equation would be one where $w \equiv 0$, *i.e.*, $-f''+qf = 0$. We shall therefore look for solutions $f_\pm(\cdot,\lambda)$ of $-f''+qf = \lambda wf$ which are asymptotic to $F_\pm$ at $\pm\infty$.

Let us write $f_+(x,\lambda) = g(x,\lambda)F_+(x)$, so we are looking for g which tends to 1 at ∞. We shall see that if, with $K = F_-/F_+$ as in Definition 2.4, there is a bounded solution to the integral equation

$$g(x,\lambda) = 1 - \lambda \int_x^\infty (K - K(x))F_+^2 g(\cdot,\lambda)\, w, \tag{4.1}$$

then it will have the desired properties. For $x \le t$ Proposition 2.5 shows that

$$0 \le (K(t) - K(x))F_+^2(t) \le F_-(t)F_+(t) = E^2(t),$$

so that (4.1) implies that

$$|g(x,\lambda)| \le 1 + |\lambda| \int_x^\infty |g|E^2|w|. \tag{4.2}$$

Therefore successive approximations in (4.1) starting with 0 will lead to a bounded solution (see Bennewitz [1, Chapter 1]). The convergence is uniform in x and locally so in λ, so our 'Jost solution' $f_+(x,\lambda)$ exists for all complex λ and is an entire function of λ, locally uniformly in x and real-valued for real λ. Differentiating (4.1) we obtain

$$g'(x,\lambda) = \lambda F_+(x)^{-2} \int_x^\infty F_+^2 g(\cdot,\lambda)w, \tag{4.3}$$

so $f_+' = g'F_+ + gF_+' = \lambda F_+^{-1}\int_x^\infty F_+^2 gw + gF_+'$. Differentiating again shows that f_+ satisfies (2.1).

Since $F_+^2(t) = E^2(t)F_+(t)/F_-(t) \le E^2(t)F_+(x)/F_-(x)$ if $x \le t$ clearly f_+' is in L^2 near ∞, so since $g(\cdot,\lambda)$ is bounded the first term is $\mathcal{O}((F_-(x))^{-1}\int_x^\infty E^2|w|)$, and the second term is $\mathcal{O}(|F_+'|)$. By Lemma 2.3 there can be no linearly independent solution with derivative in L^2 near ∞. Since g is bounded in fact $|f_+'|^2+q|f_+|^2$ is integrable near ∞. Similar statements, with ∞ replaced by $-\infty$, are valid for f_-.

We summarize as follows.

Lemma 4.1. *The solutions $f_\pm$ have the following properties:*

- $f_+(x,\lambda) \sim F_+(x)$ *as* $x \to \infty$ *and* $f_-(x,\lambda) \sim F_-(x)$ *as* $x \to -\infty$.
- $f'_+(x,\lambda) \to 0$ *as* $x \to \infty$ *and* $f'_-(x,\lambda) \to 0$ *as* $x \to -\infty$.
- *Any solution* f *of* (2.1) *for which* $|f'|^2 + q|f|^2$ *is integrable near* ∞ *is a multiple of* f_+. *Similarly, integrability near* $-\infty$ *implies that* f *is a multiple of* f_-.
- λ_k *is an eigenvalue precisely if* $f_+(\cdot,\lambda_k)$ *and* $f_-(\cdot,\lambda_k)$ *are linearly dependent, and all eigenfunctions with eigenvalue* λ_k *are multiples of* $f_+(\cdot,\lambda_k)$.

Thus λ is an eigenvalue precisely if $f_\pm(\cdot,\lambda)$ are linearly dependent, the eigenvalues are simple, and the eigenfunctions are multiples of $f_+(\cdot,\lambda)$. Clearly

$$f'_+(x,\lambda) \to 0 \quad \text{as} \quad x \to \infty,$$

but in general one can not expect that $f'_+ \sim F'_+$. For $u \in \mathcal{H}$ and every eigenvalue λ_n we define the Fourier coefficients

$$u_\pm(\lambda_n) = \langle u, f_\pm(\cdot,\lambda_n)\rangle = \lambda_n \int_{\mathbb{R}} u f_\pm(\cdot,\lambda_n) w, \tag{4.4}$$

where the second equality follows from (3.1).

Applying Gronwall's inequality[3] to (4.2) gives

$$|g(x,\lambda)| \le \exp\Big(|\lambda| \int_x^\infty E^2|w|\Big),$$

$$|g'(x,\lambda)| \le E^{-2}(x)\Big(\exp\Big(|\lambda| \int_x^\infty E^2|w|\Big) - 1\Big),$$

where the second formula is easily obtained by inserting the first in (4.3). Thus $f_+(x,\cdot)$ and $f'_+(x,\cdot)$ are entire functions of exponential type $\int_x^\infty E^2|w|$ at most. This is easily sharpened to yield the following lemma.

Lemma 4.2. *As functions of* λ *and locally uniformly in* x, *the quantities* $f_\pm(x,\lambda)$ *and* $f'_\pm(x,\lambda)$ *are entire functions of zero exponential type*[4].

In fact, $\lambda \mapsto f_+(x,\lambda)/F_+(x)$ *is of zero exponential type uniformly for* x *in any interval bounded from below and* $f_-(x,\lambda)/F_-(x)$ *in any interval bounded from above. Also the Wronskian* $[f_+, f_-]$ *is an entire function of* λ *of zero exponential type.*

Proof. Consider first a solution f of (2.1) with initial data at some point a. Differentiating $H = |f'|^2 + |\lambda||f|^2$ we obtain

$$\begin{aligned} H' &= 2\,\mathrm{Re}((f'' + |\lambda| f)\overline{f'}) \\ &= 2\,\mathrm{Re}((q - \lambda w + |\lambda|) f \overline{f'}) \le \sqrt{|\lambda|}(|w| + 1 + |q|/|\lambda|)H. \end{aligned}$$

[3] A version of Gronwall's inequality valid when w is a measure may be found in [1, Lemma 1.3], and [1, Lemma 1.2] may be useful for the estimate of g'.

[4] Uniformity here means that one can for every $\varepsilon > 0$ find a constant C_ε so that the function may be estimated by $e^{\varepsilon|\lambda|}$ for $|\lambda| \ge C_\varepsilon$, independently of x.

By the use of Gronwall's inequality this shows that

$$H(x) \le H(a) \exp\Big(\sqrt{|\lambda|}\Big|\int_a^x (|w| + 1 + |q|/|\lambda|)\Big|\Big)$$

where the second factor contributes a growth of order $1/2$ and type locally bounded in x.

If now the initial data of f are entire functions of λ of exponential type then so are f and f', and at most of the same type as the initial data. It follows that locally uniformly in x the functions f_+ and f'_+ are entire of exponential type $\int_a^\infty E^2|w|$ for any a, and are thus of zero type. For f_+/F_+ the uniformity extends to intervals bounded from below.

Similar arguments may be carried out for f_- and f'_-, which immediately implies the result for the Wronskian. □

We shall need the following definition.

Definition 4.3. Let $\mathcal{H}(a,b) = \{u \in \mathcal{H} : \operatorname{supp} u \subset [a,b]\}$.

Clearly $\mathcal{H}(a,b)$ is a closed subspace of $\mathcal{H}$.

Corollary 4.4. *For every $u \in \mathcal{H}(a,\infty)$ with $a \in \mathbb{R}$ the generalized Fourier transform $\hat{u}_+$ extends to an entire function of zero exponential type vanishing at* 0 *and defined by*

$$\hat{u}_+(\lambda) = \lambda \int_{\mathbb{R}} u f_+(\cdot,\lambda) w.$$

A similar statement is valid for $\hat{u}_-$ given any $u \in \mathcal{H}(-\infty,a)$.

5. Inverse spectral theory

We shall give a uniqueness theorem for the inverse spectral problem. In order to avoid the trivial non-uniqueness caused by the fact that translating the coefficients of the equation by an arbitrary amount does not change the spectral properties of the corresponding operator, we normalize $F_\pm$, and thus $f_\pm$, by requiring $F_+(0) = F_-(0)$. This means that $F_+(0) = F_-(0) = E(0)$.

We will need the following lemma.

Lemma 5.1. *The Wronskian $W(\lambda) = [f_-(\cdot,\lambda), f_+(\cdot,\lambda)]$ is determined by the eigenvalues of T and if λ_k is an eigenvalue, then*

$$\lambda_k W'(\lambda_k) = \langle f_-(\cdot,\lambda_k), f_+(\cdot,\lambda_k)\rangle. \tag{5.1}$$

Proof. For any x we have

$$\begin{aligned} W(\lambda) - W(\lambda_k) = {} & [f_-(x,\lambda) - f_-(x,\lambda_k), f_+(x,\lambda) - f_+(x,\lambda_k)] \\ & + [f_-(x,\lambda), f_+(x,\lambda_k)] + [f_-(x,\lambda_k), f_+(x,\lambda)] \end{aligned}$$

since $W(\lambda_k) = 0$. Since $f_\pm(x,\cdot)$ and $f'_\pm(x,\cdot)$ are entire functions the first term is $\mathcal{O}(|\lambda - \lambda_k|^2)$ as $\lambda \to \lambda_k$.

The function $h(x) = [f_-(x,\lambda), f_+(x,\lambda_k)] \to 0$ as $x \to -\infty$ by Lemma 4.1 and since $f_\pm$ are proportional for $\lambda = \lambda_k$.

We have $h'(x) = (\lambda - \lambda_k) f_-(x,\lambda) f_+(x,\lambda_k) w$ so if w has no point mass at x,

$$\frac{[f_-(x,\lambda), f_+(x,\lambda_k)]}{\lambda - \lambda_k} \to \int_{-\infty}^{x} f_-(\cdot,\lambda_k) f_+(\cdot,\lambda_k) w$$

as $\lambda \to \lambda_k$, by Lemma 4.2. A similar calculation shows that interchanging λ and λ_k in the Wronskian the limit is the same integral, but taken over (x,∞), so we obtain $W'(\lambda_k) = \int_{\mathbb{R}} f_-(\cdot,\lambda_k) f_+(\cdot,\lambda_k) w$. Now, if $v \in \mathcal{H}$, then

$$\langle f_-(\cdot,\lambda_k), v\rangle = \lambda_k \langle R_0 f_-(\cdot,\lambda_k), v\rangle = \lambda_k \int_{\mathbb{R}} f_-(\cdot,\lambda_k) v w,$$

so we obtain (5.1).

The zeros of the Wronskian are located precisely at the eigenvalues, and by (5.1) the zeros of the Wronskian are all simple, so that the corresponding canonical product is determined by the eigenvalues.

However, if two entire functions with the same canonical product are both of zero exponential type, then their quotient is also entire of zero exponential type according to Lemma A.1 and has no zeros. It is therefore constant. It follows that the Wronskian, which equals -1 for $\lambda = 0$, is determined by the eigenvalues. $\square$

In addition to the eigenvalues we introduce, for each eigenvalue λ_n, the corresponding *matching constant* α_n defined by $f_+(\cdot,\lambda_n) = \alpha_n f_-(\cdot,\lambda_n)$. Together with the eigenvalues the matching constants will be our data for the inverse spectral theory. Instead of the matching constants one could use *normalization constants* $\|f_+(\cdot,\lambda_n)\|$ or $\|f_-(\cdot,\lambda_n)\|$. If λ_n is an eigenvalue, then by Lemma 5.1 the scalar product $\langle f_-(\cdot,\lambda_n), f_+(\cdot,\lambda_n)\rangle$ is determined by the Wronskian, in other words by the eigenvalues, and since

$$\langle f_-(\cdot,\lambda_n), f_+(\cdot,\lambda_n)\rangle = \alpha_n \|f_-(\cdot,\lambda_n)\|^2 = \alpha_n^{-1} \|f_+(\cdot,\lambda_n)\|^2$$

all three sets of data are equivalent if the eigenvalues are known. We therefore make the following definition.

Definition 5.2. By the *spectral data* of the operator T we mean the set of eigenvalues for T together with the corresponding matching constants and the two sets of normalization constants.

The spectral data of T are thus determined if the eigenvalues and for each eigenvalue either the matching constant or one of the normalization constants are known.

In our main result we will be concerned with two operators T and $\breve{T}$ of the type we have discussed. Connected with $\breve{T}$ there are then coefficients $\breve{q}, \breve{w}$ and solutions $\breve{F}_\pm, \breve{f}_\pm$, *etc.*

Theorem 5.3. *Suppose T and $\breve{T}$ have the same spectral data. Then there are continuous functions r, s defined on $\mathbb{R}$ such that r is strictly positive with a derivative* *of locally bounded variation, $s : \mathbb{R} \to \mathbb{R}$ is bijective and $s(x) = \int_0^x r^{-2}$. Moreover, $\breve{q} \circ s = r^3(-r'' + qr)$ and $\breve{w} \circ s = r^4 w$.*

Conversely, if the coefficients of T and $\breve{T}$ are connected in this way, then T and $\breve{T}$ have the same spectral data.

Given additional information one may even conclude that $T = \breve{T}$.

Corollary 5.4. *Suppose in addition to the operators T and $\breve{T}$ having the same spectral data that $\breve{q} = q$. Then $T = \breve{T}$.*

We postpone the proofs to the next section.

Remark 5.5. The spectral data of T, as we have defined them, are particularly appropriate for dealing with the Camassa–Holm equation, *i.e.*, the case $q = 1/4$, since if $w = u - u_{xx}$ where u is a solution of the Camassa–Holm equation for $\varkappa = 0$, then as w evolves with time the eigenvalues are unchanged while the other spectral data evolve in the following simple way:

- $\alpha_k(t) = e^{t/2\lambda_k}\alpha_k(0)$,
- $\|f_-(\cdot, \lambda_k; t)\|^2 = e^{-t/2\lambda_k}\|f_-(\cdot, \lambda_k; 0)\|^2$,
- $\|f_+(\cdot, \lambda_k; t)\|^2 = e^{t/2\lambda_k}\|f_+(\cdot, \lambda_k; 0)\|^2$.

6. Proofs of Theorem 5.3 and Corollary 5.4

We begin with the proof of the converse of Theorem 5.3, and then define $\varphi_\pm(\cdot, \lambda) = r\breve{f}_\pm(s(\cdot), \lambda)$. Using that $r^2 s' = 1$ one easily checks that $[\varphi_-, \varphi_+] = [\breve{f}_-, \breve{f}_+]$. If we can prove that $\varphi_\pm = f_\pm$ it follows that eigenvalues and matching constants agree for the two equations.

Now $\varphi_\pm(x, \lambda)/\varphi_\pm(x, 0) = \breve{f}_\pm(s(x), \lambda)/\breve{F}_\pm(s(x)) \to 1$ as $x \to \pm\infty$ so we only need to prove that $\varphi_\pm$ solve the appropriate equation and that $\varphi_\pm(\cdot, 0) = F_\pm$. The first property follows by an elementary computation, so it follows that $\varphi_\pm(\cdot, 0) = A_\pm F_+ + B_\pm F_-$ for constants $A_\pm$ and $B_\pm$. We have

$$\frac{A_- + B_- K}{A_+ + B_+ K} = \frac{\varphi_-(\cdot, 0)}{\varphi_+(\cdot, 0)} = \breve{K} \circ s,$$

so the Möbius transform $t \mapsto \frac{A_- + B_- t}{A_+ + B_+ t}$ has fixpoints 0, 1 and ∞ so that $A_- = B_+ = 0$ and $B_- = A_+ \neq 0$. Thus $\varphi_\pm(\cdot, 0) = AF_\pm$ for some constant A which is > 0 since $\varphi_\pm(\cdot, 0)$ and $F_\pm$ are all positive. But $1 = [\breve{F}_-, \breve{F}_+] = [\varphi_-(\cdot, 0), \varphi_+(\cdot, 0)] = A^2$ so $A = 1$ and the proof is finished.

Keys for proving our inverse result are the connections between the support of an element of $\mathcal{H}$ and the growth of its generalized Fourier transform. Such results are usually associated with the names of Paley and Wiener. We could easily prove a theorem of Paley–Wiener type for our equation, analogous to what is done in our paper [3], but shall not quite need this.

Lemma 6.1. *Suppose $\delta > 0$, $a \in \operatorname{supp} w$ and $u \in \mathcal{H}(a,\infty)$. Then*

$$\begin{aligned} &\hat{u}_+(\lambda)/\lambda f_+(a,\lambda) = \mathcal{O}(|\lambda/\operatorname{Im}\lambda|) \text{ as } \lambda \to \infty, \\ &\hat{u}_+(\lambda)/\lambda f_+(a,\lambda) = o(1) \qquad \text{ as } \lambda \to \infty \text{ in } |\operatorname{Im}\lambda| \geq \delta|\operatorname{Re}\lambda|. \end{aligned}$$

Similar estimates hold for $\hat{u}_-(\lambda)/\lambda f_-(a,\lambda)$ if $u \in \mathcal{H}(-\infty,a)$.

Proof. For $\operatorname{Im}\lambda \neq 0$ we have $f_+(x,\lambda) = \lambda f_+(a,\lambda)\frac{f_+(x,\lambda)}{\lambda f_+(a,\lambda)}$, where we denote the last factor by $\psi_{[a,\infty)}(x,\lambda)$, since this is the Weyl solution for the left definite Dirichlet problem (1.1) on $[a,\infty)$ (see our paper [2, Lemma 4.10]). Like in [2, Chapter 3] one may show that

$$\langle u, \overline{\psi_{[a,\infty)}(\cdot,\lambda)}\rangle = \int_{\mathbb{R}} \frac{\tilde{u}(t)}{t-\lambda}\, d\sigma(t)$$

with absolute convergence, where $\tilde{u}$ is the generalized Fourier transform of u associated with the Dirichlet problem on $[a,\infty)$ and $d\sigma$ the corresponding spectral measure. Thus

$$\hat{u}_+(\lambda) = \lambda f_+(a,\lambda) \int_{\mathbb{R}} \frac{\tilde{u}(t)}{t-\lambda}\, d\sigma(t),$$

so the statement for $\hat{u}_+$ follows by Lemma A.3. Similar calculations give the result for $\hat{u}_-$. □

We shall also need the following lemma.

Lemma 6.2. *Suppose $x \in \operatorname{supp} w$. Then*

$$\frac{f_-(x,\lambda)f_+(x,\lambda)}{[f_-,f_+]} = \mathcal{O}(|\lambda/\operatorname{Im}\lambda|) \text{ as } \lambda \to \infty.$$

Proof. Let $m_\pm(\lambda) = \pm f'_\pm(x,\lambda)/(\lambda f_\pm(x,\lambda))$. These are the Titchmarsh–Weyl m-functions (see [2, Chapter 3]) for the left definite problem (2.1) with Dirichlet boundary condition at x for the intervals $[x,\infty)$ and $(-\infty,x]$ respectively, and are thus Nevanlinna functions[5]. Setting $m = -1/(m_- + m_+)$ also m is a Nevanlinna function and

$$\frac{f_-(x,\lambda)f_+(x,\lambda)}{[f_-,f_+]} = -m(\lambda)/\lambda.$$

As a Nevanlinna function m may be represented as

$$m(\lambda) = A + B\lambda + \int_{\mathbb{R}} \frac{1+t\lambda}{t-\lambda}\, \frac{d\rho(t)}{t^2+1},$$

where $A \in \mathbb{R}$, $B \geq 0$ and $d\rho(t)/(t^2+1)$ is a finite positive measure. Thus

$$m(\lambda)/\lambda = A/\lambda + B + \frac{1}{\lambda}\int_{\mathbb{R}} \frac{1}{t-\lambda}\,\frac{d\rho(t)}{t^2+1} + \int_{\mathbb{R}} \frac{1}{t-\lambda}\,\frac{t\, d\rho(t)}{t^2+1}.$$

The lemma therefore follows by use of Lemma A.3. □

[5]That is, functions m analytic in $\mathbb{C}\setminus\mathbb{R}$ with $\operatorname{Im}\lambda \operatorname{Im} m(\lambda) \geq 0$ and $\overline{m(\lambda)} = m(\overline{\lambda})$.

We may expand every $u \in \mathcal{H}$ in a series $u(x) = \sum \hat{u}_\pm(\lambda_n) \frac{f_\pm(x,\lambda_n)}{\|f_\pm(\cdot,\lambda_n)\|^2}$ where $\{\hat{u}_\pm(\lambda_n)/\|f_\pm(\cdot,\lambda_n)\|\} \in \ell^2$. Conversely, any such series converges to an element of $\mathcal{H}$ and thus locally uniformly. Similarly for $\breve{u} \in \breve{\mathcal{H}}$. If the eigenvalues and normalization constants for T and $\breve{T}$ are the same we may therefore define a unitary map $\mathcal{U} : \mathcal{H} \to \breve{\mathcal{H}}$ by setting

$$\mathcal{U}u(s) = \breve{u}(s) = \sum \hat{u}_+(\lambda_n) \frac{\breve{f}_+(s,\lambda_n)}{\|\breve{f}_+(\cdot,\lambda_n)\|^2}.$$

Note that expanding with respect to $\{f_-(\cdot,\lambda_n)\}$ and defining $\mathcal{U}$ by use of these expansions we obtain the same operator $\mathcal{U}$. The following proposition is an immediate consequence of the definition of $\mathcal{U}$.

Proposition 6.3. *Suppose that $\breve{u} = \mathcal{U}u$, $\breve{v} = \mathcal{U}v$, λ_k is an eigenvalue and $\hat{u}_\pm(\lambda_k) = \langle u, f_\pm(\cdot,\lambda_k)\rangle$. Then $\hat{u}_\pm(\lambda_k) = \langle \breve{u}, \breve{f}_\pm(\cdot,\lambda_k)\rangle$, $\mathcal{U}f_\pm(\cdot,\lambda_k) = \breve{f}_\pm(\cdot,\lambda_k)$ and u is in the domain of T with $Tu = v$ if and only if $\breve{u}$ is in the domain of $\breve{T}$ with $\breve{T}\breve{u} = \breve{v}$.*

Assume now that the generalized Fourier transform $\hat{u}_\pm$ of $u \in \mathcal{H}$, which is defined on all eigenvalues λ_n, has an entire extension and define the auxiliary function

$$A_\pm(u,x,\lambda) = R_\lambda u(x) + \frac{\hat{u}_\pm(\lambda) f_\mp(x,\lambda)}{\lambda[f_-(\cdot,\lambda), f_+(\cdot,\lambda)]},$$

where R_λ is the resolvent at λ of T. Similar auxiliary functions $\breve{A}_\pm$ may be defined related to $\breve{T}$.

The next lemma is crucial.

Lemma 6.4. *Suppose $x \in \operatorname{supp} w$ and $y \in \operatorname{supp} \breve{w}$. Also suppose $u \in \mathcal{H}(x,\infty)$ and $\breve{v} \in \breve{\mathcal{H}}(y,\infty)$ and let $\breve{u} = \mathcal{U}u$, $v = \mathcal{U}^{-1}\breve{v}$. Then either $\breve{u} \in \breve{\mathcal{H}}(y,\infty)$ or $v \in \mathcal{H}(x,\infty)$.*

Similarly, if $u \in \mathcal{H}(-\infty,x)$ and $\breve{v} \in \breve{\mathcal{H}}(-\infty,y)$, then $\breve{u} \in \breve{\mathcal{H}}(-\infty,y)$ or $v \in \mathcal{H}(-\infty,x)$.

Proof. By Corollary 4.4 u and $\breve{v}$ have generalized Fourier transforms $\hat{u}_+$ and $\hat{v}_+$ which have entire extensions of zero exponential type. These are also extensions of the generalized Fourier transforms of $\breve{u}$ respectively v. We have

$$A_+(v,x,\lambda) = R_\lambda v(x) + \frac{\hat{v}_+(\lambda)}{\lambda \breve{f}_+(y,\lambda)} \frac{\breve{f}_+(y,\lambda)}{f_+(x,\lambda)} \frac{f_+(x,\lambda) f_-(x,\lambda)}{[f_-,f_+]}.$$

The first term is $\mathcal{O}(\|R_\lambda v\|)$ and therefore $\mathcal{O}(|\operatorname{Im}\lambda|^{-1})$, and by Lemmas 6.1 and 6.2 respectively both the first and last factors in the second term are $\mathcal{O}(|\lambda/\operatorname{Im}\lambda|)$ as $\lambda \to \infty$ while the first factor tends to 0 in any double sector $|\operatorname{Im}\lambda| \ge \delta|\operatorname{Re}\lambda|$. Adding similar considerations for $\breve{A}_+$ we therefore obtain

$$A_+(v,x,\lambda) = (|\lambda|/|\operatorname{Im}\lambda|)^2 \mathcal{O}\Big(1 + \Big|\frac{\breve{f}_+(y,\lambda)}{f_+(x,\lambda)}\Big|\Big) \text{ as } \lambda \to \infty,$$

$$\breve{A}_+(\breve{u},y,\lambda) = (|\lambda|/|\operatorname{Im}\lambda|)^2 \mathcal{O}\Big(1 + \Big|\frac{f_+(x,\lambda)}{\breve{f}_+(y,\lambda)}\Big|\Big) \text{ as } \lambda \to \infty,$$

$$A_+(v,x,\lambda) = o\Big(1 + \Big|\frac{\breve f_+(y,\lambda)}{f_+(x,\lambda)}\Big|\Big) \text{ as } \lambda\to\infty \text{ in } |\operatorname{Im}\lambda| \ge \delta|\operatorname{Re}\lambda|,$$
$$\breve A_+(\breve u,y,\lambda) = o\Big(1 + \Big|\frac{f_+(x,\lambda)}{\breve f_+(y,\lambda)}\Big|\Big) \text{ as } \lambda\to\infty \text{ in } |\operatorname{Im}\lambda| \ge \delta|\operatorname{Re}\lambda|.$$

Thus

$$\min(|A_+(v,x,\lambda)|, |\breve A_+(\breve u,y,\lambda)|) = \mathcal{O}(|\lambda/\operatorname{Im}\lambda|^2) \text{ as } \lambda\to\infty,$$
$$\min(|A_+(v,x,\lambda)|, |\breve A_+(\breve u,y,\lambda)|) = o(1) \qquad \text{as } \lambda\to\infty \text{ in } |\operatorname{Im}\lambda| \ge \delta|\operatorname{Re}\lambda|.$$

By Lemma 4.2 and Theorem A.4 the functions $A_+(v,x,\cdot)$ and $\breve A_+(\breve u,y,\cdot)$ are of zero exponential type, so by Lemma A.6 one of them vanishes.

If $A_+(v,x,\cdot)=0$ Lemma A.5 shows that $A_+(v,z,\cdot)=0$ for all $z\le x$. Thus inserting $f(z)=A_+(v,z,\lambda)$ in $-f''+(q-\lambda w)f$ shows that $wv=0$ in $(-\infty,x]$, so that $v=0$ in $(-\infty,x]$ except in gaps of $\operatorname{supp} w$. Since v vanishes at the endpoints of any gap with endpoints in $(-\infty,x]$ it follows by Corollary 3.5 that v vanishes in all such gaps. We conclude that $v\in\mathcal{H}(x,\infty)$. Similarly, if $\breve A_+(\breve u,y,\cdot)=0$ we conclude that $\breve u\in\breve{\mathcal{H}}(y,\infty)$.

Analogous considerations involving A_- and $\breve A_-$ prove the second statement. □

We next show how supports of elements of $\mathcal{H}$ are related to the supports of their images under $\mathcal{U}$. Note that $\dim\mathcal{H}$ equals the number of points in $\operatorname{supp} w$ if this is finite and is infinite otherwise.

Lemma 6.5. *Suppose* $\operatorname{supp} w$ *contains at least two points. Then so does* $\operatorname{supp}\breve w$ *and there are strictly increasing, bijective maps*

$$s_+ : \operatorname{supp} w\setminus\{\sup\operatorname{supp} w\} \to \operatorname{supp}\breve w\setminus\{\sup\operatorname{supp}\breve w\}$$
$$s_- : \operatorname{supp} w\setminus\{\inf\operatorname{supp} w\} \to \operatorname{supp}\breve w\setminus\{\inf\operatorname{supp}\breve w\}$$

such that $\breve{\mathcal{H}}(s_+(x),\infty)=\mathcal{U}\mathcal{H}(x,\infty)$ *and* $\breve{\mathcal{H}}(-\infty,s_-(x))=\mathcal{U}\mathcal{H}(-\infty,x)$ *for all x in the domains of s_+ respectively s_-.*

Proof. Suppose $u\in\mathcal{H}(x,\infty)$ where $x\in\operatorname{supp} w\setminus\{\sup\operatorname{supp} w\}$. There is at least one such $u\ne 0$ (obtained by subtracting an appropriate multiple of $g_0(z,\cdot)$ from $g_0(x,\cdot)$ where $x<z\in\operatorname{supp} w$). Therefore $\breve u\notin\breve{\mathcal{H}}(y,\infty)$ for some $y\in\operatorname{supp}\breve w$. By Lemma 6.4 this means that $v\in\mathcal{H}(x,\infty)$ for every $\breve v\in\breve{\mathcal{H}}(y,\infty)$. Now let $s_+(x)$ be the infimum of all $y\in\operatorname{supp}\breve w$ for which the last statement is true.

If $s_+(x)=-\infty$ the support of $\breve w$ is unbounded from below so that the projection onto $\breve{\mathcal{H}}$ of a compactly supported element of $\breve{\mathcal{H}}_1$ has a support bounded from below. Such elements of $\breve{\mathcal{H}}$ are dense, and consequently $\breve{\mathcal{H}}\subset\mathcal{U}\mathcal{H}(x,\infty)$. However, this would contradict the fact that $\mathcal{U}$ is unitary. Thus $s_+(x)$ is finite, so $s_+(x)\in\operatorname{supp}\breve w$.

Note that if $s_+(x)$ is the left endpoint of a gap in $\operatorname{supp}\breve{w}$, then the infimum defining $s_+(x)$ is attained. Thus, if it is not there are points of $\operatorname{supp}\breve{w}$ to the right of and arbitrarily close to $s_+(x)$. But then we may approximate elements of $\breve{\mathcal{H}}(s_+(x),\infty)$ arbitrarily well (see [3, Lemma 6.8]) by elements of $\breve{\mathcal{H}}(y,\infty)$ for some $y > s_+(x)$. It follows that $\breve{\mathcal{H}}(s_+(x),\infty) \subset \mathcal{U}\mathcal{H}(x,\infty)$.

On the other hand, if $y = -\infty$ or $\operatorname{supp}\breve{w} \ni y < s_+(x)$ there exists $\breve{v} \in \breve{\mathcal{H}}(y,\infty)$ such that $\mathcal{U}^{-1}\breve{v} \notin \mathcal{H}(x,\infty)$ and thus, by Lemma 6.4, $\mathcal{U}\mathcal{H}(x,\infty) \subset \breve{\mathcal{H}}(y,\infty)$. Since this is true for all $y \in \operatorname{supp}\breve{w}$ with $y < s_+(x)$ we have in fact $\mathcal{U}\mathcal{H}(x,\infty) \subset \breve{\mathcal{H}}(s_+(x),\infty)$ unless $s_+(x)$ is the right endpoint of a gap in $\operatorname{supp}\breve{w}$. In the latter case we may choose $y \geq -\infty$ so that $(y, s_+(x))$ is a gap in $\operatorname{supp}\breve{w}$.

Thus $\breve{\mathcal{H}}(y,\infty)$ is a one-dimensional extension of $\breve{\mathcal{H}}(s_+(x),\infty)$, so if there exists $u \in \mathcal{H}(x,\infty)$ with $\operatorname{supp}\mathcal{U}u$ intersecting $(y, s_+(x))$, then $\mathcal{U}^{-1}\breve{\mathcal{H}}(y,\infty) \subset \mathcal{H}(x,\infty)$. But this would mean that $s_+(x) \leq y$. It follows that $\mathcal{U}\mathcal{H}(x,\infty) = \breve{\mathcal{H}}(s_+(x),\infty)$ in all cases.

The function s_+ has range $\operatorname{supp}\breve{w} \setminus \{\sup\operatorname{supp}\breve{w}\}$, since if not let y be in this set but not in the range of s_+. An argument analogous to that defining s_+ determines $x \in \operatorname{supp} w$ such that $\breve{\mathcal{H}}(y,\infty) = \mathcal{U}\mathcal{H}(x,\infty)$. Since x can not be in the domain of s_+ we must have $x = \sup\operatorname{supp} w$, so that $\mathcal{H}(x,\infty) = \{0\}$ and thus also $\breve{\mathcal{H}}(y,\infty) = \{0\}$. This contradicts the choice of y.

Analogous reasoning proves the existence of the function s_-. □

We can now show that $\mathcal{U}$ is given by a so-called *Liouville transform.*

Lemma 6.6. *There exist real-valued maps r, s defined in* $\operatorname{supp} w$ *such that r does not vanish and $s : \operatorname{supp} w \to \operatorname{supp}\breve{w}$ is increasing and bijective and such that $u = r\mathcal{U}u \circ s$ on* $\operatorname{supp} w$ *for any $u \in \mathcal{H}$.*

Proof. If $\operatorname{supp} w = \{x\}$, then $\dim\mathcal{H} = 1$ so also $\dim\breve{\mathcal{H}} = 1$. It follows that also $\operatorname{supp}\breve{w}$ is a singleton, say $\{s\}$. It is clear that $\mathcal{H}$ consists of all multiples of $g_0(x,\cdot)$ and $\breve{\mathcal{H}}$ of all multiples of $\breve{g}_0(s,\cdot)$. It follows that for all $u \in \mathcal{H}$ we have $u(x) = r\breve{u}(s)$ where $r = g_0(x,x)/\breve{g}_0(s,s)$ which proves the lemma in this case, so now assume $\operatorname{supp} w$ has at least two points.

If $x \in \operatorname{supp} w$ and $v \in \mathcal{H}$ with $v(x) = 1$ we may, given any $u \in \mathcal{H}$, write $u = u_- + u_+ + u(x)v$ where $u_- \in \mathcal{H}(-\infty,x)$ and $u_+ \in \mathcal{H}(x,\infty)$. Applying $\mathcal{U}$ we obtain from Lemma 6.5 that $\breve{u} = \breve{u}_- + \breve{u}_+ + u(x)\breve{v}$ where $\breve{u}_- \in \breve{\mathcal{H}}(-\infty, s_-(x))$ unless $x = \inf\operatorname{supp} w$ in which case $u_- = 0$ and thus $\breve{u}_- = 0$. Similarly $\breve{u}_+ \in \breve{\mathcal{H}}(s_+(x),\infty)$ unless $x = \sup\operatorname{supp} w$ in which case $u_+ = 0$ and thus $\breve{u}_+ = 0$.

If $s_\pm$ are both defined at x we can not have $s_-(x) < s_+(x)$ since then the restrictions of elements of $\breve{\mathcal{H}}$ to $(s_-(x), s_+(x))$ would be a one-dimensional set, which implies that $(s_-(x), s_+(x))$ is an unbounded gap in $\operatorname{supp}\breve{w}$, contradicting the fact that $s_\pm(x)$ are in $\operatorname{supp}\breve{w}$.

A similar reasoning but starting from $\breve{u} \in \breve{\mathcal{H}}$ and using the inverses of $s_\pm$ shows that we can not have $s_-(x) > s_+(x)$ either, so that we define $s = s_+ = s_-$ whenever one of $s_\pm$ is defined. It now follows that $\breve{u}(s(x)) = \breve{v}(s(x))u(x)$, and

$\breve{v}(s(x)) \neq 0$ since not all elements of $\breve{\mathcal{H}}$ vanish at $s(x)$. We may now set $r(x) = 1/\breve{v}(s(x))$ and the proof is finished. □

Since $s : \operatorname{supp} w \to \operatorname{supp} \breve{w}$ is bijective and increasing it follows that (a, b) is a gap in $\operatorname{supp} w$ if and only if $(s(a), s(b))$ is a gap in $\operatorname{supp} \breve{w}$, and similarly if $a = -\infty$ or $b = \infty$. Thus gaps in $\operatorname{supp} w$ and $\operatorname{supp} \breve{w}$ are in a one-to-one correspondence. We now need to define the functions r, s also in gaps of $\operatorname{supp} w$ and prove the other claimed properties of these functions. The key to this is the following proposition.

Proposition 6.7. *If x and y are in* $\operatorname{supp} w$*, then*

$$g_0(x, y) = r(x)r(y)\breve{g}_0(s(x), s(y)).$$

Proof. Suppose $\breve{u} \in \breve{\mathcal{H}}$ and $u = \mathcal{U}^{-1}\breve{u}$. Since $s(x) \in \operatorname{supp} \breve{w}$ it follows that $\breve{g}_0(s(x), \cdot) \in \breve{\mathcal{H}}$ and, by Lemma 6.6, $u(x) = r(x)\breve{u}(s(x))$ so that

$$\langle \breve{u}, \mathcal{U} g_0(x, \cdot)\rangle = \langle u, g_0(x, \cdot)\rangle = u(x) = r(x)\breve{u}(s(x)) = r(x)\langle \breve{u}, \breve{g}_0(s(x), \cdot)\rangle.$$

Thus $\mathcal{U} g_0(x, \cdot) = r(x)\breve{g}_0(s(x), \cdot)$. Since $y \in \operatorname{supp} w$ Lemma 6.6 also shows that $g_0(x, y) = r(y)\mathcal{U} g_0(x, \cdot)(s(y))$, and combining these formulas completes the proof. □

The proposition has the following corollary.

Corollary 6.8. *If $x \in \operatorname{supp} w$, then*

$$F_\pm(x) = r(x)\breve{F}_\pm(s(x)). \tag{6.1}$$

Proof. Suppose $x, y \in \operatorname{supp} w$ and $y \leq x$. Then, by Proposition 6.7,

$$\frac{F_+(x)}{r(x)\breve{F}_+(s(x))} = \frac{r(y)\breve{F}_-(s(y))}{F_-(y)}.$$

This implies that both sides are independent of x and y and thus equal a constant C. The corollary is proved if we can prove that $C = 1$.

Now let λ be an eigenvalue of $\breve{T}$ so that $\breve{f}_+(\cdot, \lambda)$ is an eigenfunction and according to Proposition 6.3 $f_+(\cdot, \lambda)$, given by $f_+(x, \lambda) = r(x)\breve{f}_+(s(x), \lambda)$ for $x \in \operatorname{supp} w$, the corresponding eigenfunction for T. We then have

$$C\frac{f_+(x, \lambda)}{F_+(x)} = \frac{\breve{f}_+(s(x), \lambda)}{\breve{F}_+(s(x))}$$

for all $x \in \operatorname{supp} w$. If $\operatorname{supp} w$ is bounded above, choose $x = \sup \operatorname{supp} w$. Then we have $f_+(x, \lambda) = F_+(x)$ and $\breve{f}_+(s(x), \lambda) = \breve{F}_+(s(x))$ so that $C = 1$. If $\operatorname{supp} w$ is not bounded above we take a limit as $x \to \infty$ in $\operatorname{supp} w$ and arrive at the same conclusion. □

If we can extend the definitions of r and s to continuous functions such that (6.1) continues to hold for all x it follows that $u = r\mathcal{U}u \circ s$ for all $u \in \mathcal{H}$ even in gaps of $\operatorname{supp} w$. This is a consequence of two facts. Firstly, the formula $u = r\breve{u} \circ s$ then gives a bijective map of the solutions of $-u'' + qu = 0$ to the solutions of

$-\breve{u}''+\breve{q}\breve{u}=0$ and, secondly, elements of $\mathcal{H}$ and $\breve{\mathcal{H}}$ are determined in gaps of $\operatorname{supp} w$ respectively $\operatorname{supp} \breve{w}$ as described in Corollary 3.5.

With K as in Definition 2.4 and $\breve{K}$ defined similarly we must define s so that $K=\breve{K}\circ s$, so Proposition 2.5 and the normalization of $F_\pm$ and $\breve{F}_\pm$ show that $s(0)=0$ and we have $s=\breve{K}^{-1}\circ K$. Thus s is strictly increasing of class C^1 with range $\mathbb{R}$ and a strictly positive derivative $s'=(\breve{F}_+\circ s/F_+)^2$, which is locally absolutely continuous. Furthermore we must define $r=F_+/\breve{F}_+\circ s$. This gives $r>0$ and shows that r is locally absolutely continuous with a derivative of locally bounded variation as well as $r^2s'=1$ so that $s(x)=\int_0^x r^{-2}$. With these definitions (6.1) holds for all x.

Differentiating $F_+=r\breve{F}_+\circ s$ we obtain $F_+'=rs'\breve{F}_+'\circ s+r'\breve{F}_+\circ s=\breve{F}_+'\circ s/r+r'\breve{F}_+\circ s$. Differentiating once more we obtain

$$\begin{aligned} qF_+=F_+''&=s'\breve{F}_+''\circ s/r-r'\breve{F}_+'\circ s/r^2+r's'\breve{F}_+'\circ s+r''\breve{F}_+\circ s\\ &=r^{-3}\breve{q}\circ s\breve{F}_+\circ s+r''\breve{F}_+\circ s=r^{-4}\breve{q}\circ sF_++r''F_+/r. \end{aligned}$$

It follows that

$$\breve{q}\circ s=r^3(-r''+qr).$$

A similar calculation, using that according to Proposition 6.3 $Tu=v$ precisely if $\breve{T}\breve{u}=\breve{v}$, shows that we also have

$$\breve{w}\circ s=r^4w.$$

This uses that the range of T is $\mathcal{H}$, so that there always are choices of v different from 0 in a neighborhood of any given point.

This completes the proof of Theorem 5.3. To prove Corollary 5.4 we need only note that if $q=\breve{q}$, then $K=\breve{K}$ so that s is the identity and $r\equiv 1$. Thus $\breve{w}=w$.

Appendix: Some technical lemmas

We begin by quoting a standard fact.

Lemma A.1. *Suppose f, g are entire functions of zero exponential type such that f/g is entire. Then f/g is also of zero exponential type.*

The lemma is a special case of the corollary to Theorem 12 in Chapter I of Levin [14]. We shall also need the following lemma.

Lemma A.2. *Suppose f is entire and for every $\varepsilon>0$ satisfies*

$$\operatorname{Im}(z)f(z)=\mathcal{O}(e^{\varepsilon|z|})$$

for large $|z|$. Then f is of zero exponential type.

Proof. Put $u = \log^+ |f|$. Then, with $z = re^{i\theta}$,

$$0 \leq u(r,\theta) \leq \varepsilon r + \mathcal{O}(1) + \log(|\sin\theta|^{-1})$$

for large r. The last term is locally integrable, so we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\,d\theta \leq \varepsilon r + \mathcal{O}(1).$$

Now, since u is subharmonic and non-negative we have, by the Poisson integral formula,

$$0 \leq u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}\, u(re^{i\theta})\,d\theta \leq \frac{3}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta$$

if $|z| \leq r/2$, since then

$$0 \leq \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} \leq \frac{r^2 - |z|^2}{(r - |z|)^2} = \frac{r + |z|}{r - |z|} \leq 3.$$

It follows that $0 \leq u(z) \leq 6\varepsilon|z| + \mathcal{O}(1)$ if $|z| = r/2$, so $|f(z)| = \mathcal{O}(e^{6\varepsilon|z|})$ for large $|z|$. Thus f is of zero exponential type. □

Our next lemma estimates the Stieltjes transform of certain measures.

Lemma A.3. *Suppose $d\mu$ is a (signed) Lebesgue–Stieltjes measure and that $h(\lambda) = \int_{\mathbb{R}} \frac{d\mu(t)}{t-\lambda}$ is absolutely convergent for $\operatorname{Im}\lambda \neq 0$. As $\lambda \to \infty$ we then have $h(\lambda) = \mathcal{O}(|\lambda|/|\operatorname{Im}\lambda|)$ and for any $\delta > 0$ we have $h(\lambda) = o(1)$ as $\lambda \to \infty$ in the double sector $|\operatorname{Im}\lambda| \geq \delta|\operatorname{Re}\lambda|$.*

Proof. We have

$$|h(\lambda)| \leq \int_{\mathbb{R}} \left|\frac{t-i}{t-\lambda}\right| \frac{|d\mu|(t)}{|t-i|}.$$

Here the first factor may be easily estimated by $(2|\lambda| + 1)/|\operatorname{Im}\lambda|$ so[6] the first statement follows. Furthermore, the first factor tends boundedly to 0 as $\lambda \to \infty$ in the sector $|\operatorname{Im}\lambda| \geq \delta|\operatorname{Re}\lambda|$, so the second statement follows. □

We now turn to the auxiliary functions of the previous section.

Theorem A.4. *If $\hat{u}_+(\lambda)/\lambda$ is entire so is $A_+(u,x,\cdot)$, and if $\hat{u}_+$ is also of zero exponential type so is $A_+(u,x,\cdot)$. Similarly for $A_-(u,x,\cdot)$, depending on properties of $\hat{u}_-$.*

Proof. Let A denote the function $A_+(u,x,\cdot)$, *i.e.*,

$$A(\lambda) = (R_\lambda u)(x) + \frac{\hat{u}_+(\lambda) f_-(x,\lambda)}{\lambda W(\lambda)}$$

[6]The best possible t-independent estimate is $(|\lambda + i| + |\lambda - i|)/(2|\operatorname{Im}\lambda|)$.

where $W(\lambda) = [f_-(\cdot,\lambda), f_+(\cdot,\lambda)]$. Thus A is meromorphic with poles possible at the eigenvalues of T, which are also the zeros of W. There is no pole at 0, since this is no eigenvalue and $\hat{u}_+$ vanishes there. We have

$$R_\lambda u(x) = \sum \frac{\hat{u}_+(\lambda_n) f_+(x,\lambda_n)}{(\lambda_n - \lambda)\|f_+(\cdot,\lambda_n)\|^2},$$

so the residue at $\lambda = \lambda_n$ is

$$-\hat{u}_+(\lambda_n)\frac{f_+(x,\lambda_n)}{\|f_+(\cdot,\lambda_n)\|^2} = -\hat{u}_+(\lambda_n)\frac{f_-(x,\lambda_n)}{\langle f_-(\cdot,\lambda_n), f_+(\cdot,\lambda_n)\rangle}.$$

Since $\lambda_n W'(\lambda_n) = \langle f_-(\cdot,\lambda_n), f_+(\cdot,\lambda_n)\rangle$ by Lemma 5.1 the residues of the two terms in A cancel and A is entire.

It is also clear that $f(\lambda) = R_\lambda u(x) W(\lambda)$ is entire, and since $\operatorname{Im}(\lambda) R_\lambda$ is bounded we obtain the same growth estimates for $\operatorname{Im}(\lambda) f$ as for W. Since W is of zero exponential type, so is f by Lemma A.2. It follows that A is the quotient of two functions of zero exponential type if $\hat{u}_+$ is of zero exponential type. Thus A is itself of zero exponential type by Lemma A.1.

Similarly one proves the statements about $A_-(u,x,\cdot)$. □

We shall also need the following result.

Lemma A.5. *Suppose $\lambda \mapsto A_+(u,z,\lambda)$ is an entire function of zero exponential type for every $z \le x$ and that it vanishes identically for $z = x$. Then it vanishes identically for all $z \le x$.*

Similarly, if $\lambda \mapsto A_-(u,z,\lambda)$ is an entire function of zero exponential type for every $z \ge x$ and vanishes identically for $z = x$, then it vanishes identically for all $z \ge x$.

Proof. Suppose $A_+(u,x,\cdot) = 0$. Then

$$A_+(u,z,\lambda) = R_\lambda u(z) - \psi_{(-\infty,x]}(z,\lambda)\lambda R_\lambda u(x),$$

where $\psi_{(-\infty,x]}(z,\lambda) = f_-(z,\lambda)/(\lambda f_-(x,\lambda))$ is the Weyl solution for (2.1) on $(-\infty,x]$ with a Dirichlet condition at x. This function tends to 0 as $\lambda \to \infty$ along the imaginary axis (see [2, Corollary 3.12]), while the operator λR_λ stays bounded, so it is clear that $A_+(v,z,\lambda) \to 0$ as $\lambda \to \infty$ on the imaginary axis. Since $A_+(v,z,\cdot)$ is entire of zero exponential type it follows by the theorems of Phragmén–Lindelöf and Liouville that $A_+(v,z,\cdot) = 0$.

Similar arguments apply in the case of A_-. □

The next lemma is crucial but a very slight extension of a lemma by de Branges. We shall give a full proof, however, since there is an oversight in the proof by de Branges which will be corrected below. We are not aware of the oversight being noted in the literature, but a correct proof may also be found in the Diplomarbeit of Koliander [13].

Lemma A.6. *Suppose F_j are entire functions of zero exponential type, and assume that for some $\alpha \geq 0$ we have*

$$\min(|F_1(\lambda)|, |F_2(\lambda)|) = o(|\lambda|^\alpha)$$

uniformly in $\operatorname{Re}\lambda$ *as* $|\operatorname{Im}\lambda| \to \infty$*, as well as* $\min(|F_1(i\nu)|, |F_2(i\nu)|) = o(1)$ *as* $\nu \to \pm\infty$*. Then F_1 or F_2 vanishes identically.*

This is a simple consequence of the following lemma, which is essentially de Branges' [4, Lemma 8, p. 108].

Lemma A.7. *Let F_j be entire functions of zero exponential type, and assume that* $\min(|F_1(z)|, |F_2(z)|) = o(1)$ *uniformly in* $\operatorname{Re} z$ *as* $|\operatorname{Im} z| \to \infty$*. Then F_1 or F_2 is identically zero.*

Proof of Lemma A.6. Suppose first that F_1 is a polynomial not identically zero. Then, by assumption, $F_2(i\nu) = o(1)$ as $\nu \to \pm\infty$, so by the theorems of Phragmén–Lindelöf and Liouville it follows that F_2 vanishes identically. Similarly if F_2 is a polynomial.

In all other cases F_1, F_2 both have infinitely many zeros, so if $n \geq \alpha$ and $z_1, \dots, z_n$ are zeros of F_1 we put $G_1(\lambda) = F_1(\lambda)/\prod_1^n(\lambda - z_j)$. Defining G_2 similarly we now have $\min(|G_1(\lambda)|, |G_2(\lambda)|) = o(1)$ uniformly in $\operatorname{Re}\lambda$ as $\operatorname{Im}\lambda \to \pm\infty$, while G_1, G_2 are still entire of zero exponential type. By Lemma A.7 it follows that G_1 or G_2 is identically zero, and the lemma follows. □

To prove Lemma A.7 we need some additional lemmas.

Lemma A.8. *Suppose F is entire of exponential type. If there is a constant C and a sequence $r_j \to \infty$ such that $|F(z)| = \mathcal{O}(1)$ as $j \to \infty$ for $|\operatorname{Im} z| \geq C$ and $|z| = r_j$, then F is constant.*

Proof. Setting $u = \log^+ |F|$ we have $u(z) = \mathcal{O}(1)$ if $|\operatorname{Im} z| \geq C$ and $|z| = r_j$. If $z = r_j e^{i\theta}$ the condition $|\operatorname{Im} z| \leq C$ means $|\sin\theta| \leq C/r_j$, and the measure of the set of $\theta \in [0, 2\pi]$ satisfying this is $\mathcal{O}(1/r_j)$ as $j \to \infty$, whereas $|F(z)| \leq e^{\mathcal{O}(|z|)}$ so that $u(r_j e^{i\theta}) = \mathcal{O}(r_j)$. Thus $\int_0^{2\pi} u(r_j e^{i\theta})\, d\theta = \mathcal{O}(1)$ as $j \to \infty$.

It follows that F is bounded, using the Poisson integral formula in much the same way as in the proof of Lemma A.2, so that F is constant. □

Next we prove a version of de Branges' Lemma 7 on p. 108 of [4], with the added assumption that $0 < p < 1$, with p as below. Without the extra assumption the lemma is not true[7]. If F is an entire function we define u as before and

$$V(r) = \int_0^{2\pi} (u(re^{i\theta}))^2\, d\theta.$$

Furthermore, let $x = \log r$ so that $u(re^{i\theta}) = u(e^{x+i\theta})$ is a continuous, subharmonic and non-negative function of (x, θ), with period 2π in θ, and put $v(x) = V(e^x)$.

[7] The original statement of de Branges is correct if one defines $p(x) = \infty$ whenever $p(x) = 1$ according to de Branges. This is not an unnatural definition, but will not help in proving his Theorem 35 nor our Lemma A.7.

Let $M = \{(x,\theta) : u(e^{x+i\theta}) > 0\}$. The set M has period 2π in θ, and we define $p(x)$ so that $2\pi p(x)$ is the measure of the trace

$$M(x) = \{\theta \in [0, 2\pi) : (x,\theta) \in M\}.$$

The function p is lower semi-continuous, and we have $p(x) \leq 1$. Now assume one may choose a so that $p(x) > 0$ for $x \geq a$. Thus p is locally in $[a, \infty)$ bounded away from 0, so that $1/p$ is upper semi-continuous, positive and locally bounded. We may therefore define the strictly increasing function

$$s(x) = \int_a^x \exp\Big(\int_a^t 1/p\Big)\, dt.$$

Lemma A.9. *Suppose* $0 < p(x) < 1$ *for all* $x \geq a$*. Then the quantity* v *is a convex function of* $s > 0$*.*

Proof. We may think of u as defined on a cylindrical manifold $\mathcal{C}$ with coordinates $(x,\theta) \in \mathbb{R} \times [0, 2\pi)$ of which M is an open subset. In M the function u is harmonic, and the boundary ∂M is a level set of $|F|$. The boundary is therefore of class C^1 except where the gradient of $|F|$ vanishes. However, the length of the gradient equals $|F'|$, as is easily seen, and the exceptional points are therefore locally finite in number. We may therefore use integration by parts (the divergence theorem or the general Stokes theorem) for the set M.

Assuming $\varphi \in C_0^\infty(\mathcal{C})$ and integrating by parts we obtain

$$\int_M \Delta\varphi u^2 = \int_{\partial M} \Big(u^2 \frac{\partial \varphi}{\partial n} - 2\varphi u \frac{\partial u}{\partial n}\Big) + \int_M \varphi \Delta u^2 = 2 \int_M \varphi |\operatorname{grad} u|^2,$$

since u vanishes on ∂M and is harmonic in M. Now suppose φ is independent of θ. Then we may write the above as

$$\int_{\mathbb{R}} \varphi'' v = \int_{\mathbb{R}} \varphi(x)\Big(2 \int_{M(x)} (u_x^2 + u_\theta^2)\Big)\, dx,$$

so that (in the sense of distributions) $v''(x) = 2\int_{M(x)}(u_x^2 + u_\theta^2)$. A similar calculation shows that $v'(x) = \int_{M(x)} 2uu_x$.

The function s has a C^1 inverse, so we may think of x, and thus v, as a function of s. We obtain $v' = s' \frac{dv}{ds}$ and $v'' = (s')^2 \frac{d^2 v}{ds^2} + s'' \frac{dv}{ds}$. Thus $(s')^2 \frac{d^2v}{ds^2} = v'' - v's''/s' = v'' - v'/p$. We need to prove the positivity of this. Now

$$\begin{aligned} v''(x) - v'(x)/p(x) &= 2 \int_{M(x)} (u_x^2 + u_\theta^2 - uu_x/p) \\ &= 2 \int_{M(x)} ((u_x - u/2p)^2 + u_\theta^2 - u^2/4p^2)\, d\theta \\ &\geq 2\Big(\int_{M(x)} u_\theta^2 - \frac{1}{4p^2} \int_{M(x)} u^2\Big). \end{aligned}$$

Positivity therefore follows if we have the inequality

$$\int_{M(x)} u_\theta^2 \geq \frac{1}{4p^2(x)} \int_{M(x)} u^2. \tag{A.2}$$

Since $p(x) < 1$ the function $\theta \mapsto u(e^{x+i\theta})$ has a zero, so that u vanishes at the endpoints of all components of the open set $M(x)$. If I is such a component we therefore have $\int_I (u_\theta)^2 \geq (\pi/|I|)^2 \int_I u^2$ where $|I|$ is the length of I.

This just expresses the fact that the smallest eigenvalue of $-u'' = \lambda u$ with Dirichlet boundary conditions on I is $(\pi/|I|)^2$. We have $(\pi/|I|)^2 \geq (2p)^{-2}$ since $|I| \leq 2\pi p$, so adding up the inequalities for the various components of $M(x)$ we obtain (A.2), and the proof is finished. $\square$

Proof of Lemma A.7. Suppose first that F_1 is bounded and therefore constant. If this constant is not zero the assumption implies that $F_2(i\nu) \to 0$ as $\nu \to \pm\infty$. Since F_2 is of zero exponential type the Phragmén-Lindelöf principle shows that F_2 is bounded and has limit zero along the imaginary axis and therefore is the constant 0. Similarly if F_2 is bounded. We may thus assume that F_1 and F_2 are both unbounded.

If there is a sequence $r_j \to \infty$ such that $F_1(z)$ satisfies the assumptions of Lemma A.8, then F_1 is constant according to Lemma A.8. Similarly for F_2.

We may thus also assume that for $k = 1, 2$ and every large r the inequality $|F_k(z)| \leq 1$ is violated for some z with $|z| = r$ and $|\operatorname{Im} z| > C$. Since F_k is analytic and thus continuous, the opposite inequalities must hold on some open θ-sets for $z = re^{i\theta}$ and every large r.

But if $|F_1(z)| > 1$ we must have $|F_2(z)| \leq 1$ for large $|z|$ and $|\operatorname{Im} z| > C$ and *vice versa*. It follows that for some a we have $0 < p_k(x) < 1$, $k = 1, 2$, for $x \geq a$.

By Cauchy–Schwarz $\frac{1}{2\pi}\int_0^{2\pi} u_1(re^{i\theta})\,d\theta \leq \left(\frac{1}{2\pi}\int_0^{2\pi} u_1^2(re^{i\theta})\,d\theta\right)^{1/2}$, so it follows that if v_1 is bounded, then so is F_1, using the Poisson integral formula in much the same way as in the proof of Lemma A.2. Thus v_1 must be unbounded, and since it is non-negative and convex as a function of s_1 there is a constant $c > 0$ such that $v_1(x) \geq cs_1(x)$ for large x. Similarly we may assume $v_2(x) \geq cs_2(x)$ for large x. We shall show that this contradicts the assumption of order for F_1, F_2.

Using the convexity of the exponential function we obtain for large $x > a$

$$\big(V_1(r(x)) + V_2(r(x))\big)/2 \geq c\int_a^x \exp\Big(\int_a^t (1/p_1 + 1/p_2)/2\Big)\,dt. \tag{A.3}$$

Now, by assumption $\min(u_1(re^{i\theta}), u_2(re^{i\theta})) = 0$ for large r and $C \leq r|\sin\theta|$ so that then u_1 or u_2 equal zero. The measure of the θ-set not satisfying $r|\sin\theta| \geq C$ for a given r is less than $2\pi C/r$. It follows that $p_1 + p_2 \leq 1 + C/r$. Since

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{p_1 + p_2}{p_1 p_2} \geq \frac{4}{p_1 + p_2} \geq \frac{4r}{r + C} = \frac{4e^x}{e^x + C}$$

the integral in (A.3) is at least $\frac{1}{2}(e^{2x}-e^{2a})/(e^a+C)^2$. Thus $V_1(r)+V_2(r) \geq c'r^2$ for some constant $c'>0$ and large r. The assumption of order for F_k means, however, that $V_k(r)=o(r^2)$. This contradiction proves the lemma. □

References

[1] Christer Bennewitz. Spectral asymptotics for Sturm–Liouville equations. *Proc. London Math. Soc.* (3), 59(2):294–338, 1989.

[2] C. Bennewitz, B.M. Brown, and R. Weikard. Inverse spectral and scattering theory for the half-line left-definite Sturm–Liouville problem. *SIAM J. Math. Anal.*, 40(5):2105–2131, 2009.

[3] Bennewitz, C., Brown, B.M. and Weikard, R. Scattering and inverse scattering for a left-definite Sturm-Liouville problem. *J. Differential Equations* 253 (2012), *no.* 8, 2380–2419.

[4] Louis de Branges Hilbert spaces of entire functions Prentice-Hall, Englewood Cliffs, 1968.

[5] Camassa, Roberto and Holm, Darryl D. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.* 71 (1993), *no.* 11, 1661–1664.

[6] A. Constantin and H.P. McKean, *A shallow water equation on the circle*, Comm. Pure Appl. Math., 52 (1999), pp. 949–982.

[7] Constantin, A. On the scattering problem for the Camassa–Holm equation. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.,* 457 (2001), *pp.* 953–970.

[8] Adrian Constantin. Existence of permanent and breaking waves for a shallow water equation: a geometric approach. *Ann. Inst. Fourier* (*Grenoble*), 50(2):321–362, 2000.

[9] Adrian Constantin and Joachim Escher. Wave breaking for nonlinear nonlocal shallow water equations. *Acta Math.*, 181(2):229–243, 1998.

[10] Eckhardt, Jonathan and Teschl, Gerald. On the isospectral problem of the dispersionless Camassa–Holm equation. *Adv. Math.* 235 (2013), 469–495.

[11] Eckhardt, Jonathan. Direct and inverse spectral theory of singular left-definite Sturm–Liouville operators. *J. Differential Equations* 253 (2012), *no.* 2, 604–634.

[12] B. Fuchssteiner and A.S. Fokas. Symplectic structures, their Bäcklund transformations and hereditary symmetries. *Phys. D*, 4(1):47–66, 1981/82.

[13] Koliander, G. Hilbert Spaces of Entire Functions in the Hardy Space Setting. *Diplomarbeit, Technische Universität Wien*, 2011.

[14] B.Ja. Levin. *Distribution of zeros of entire functions*, volume 5 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, R.I., revised edition, 1980. Translated from the Russian by R.P. Boas, J.M. Danskin, F.M. Goodspeed, J. Korevaar, A.L. Shields and H.P. Thielman.

[15] Z. Jiang, L. Ni, and Y. Zhou. *Wave breaking of the Camassa–Holm equation*, J. Nonlinear Sci., 22(2): 235–245, 2012.

C. Bennewitz
Centre for Mathematical Sciences
Box 118
SE-221 00 Lund, Sweden
e-mail: christer.bennewitz@telia.com

B.M. Brown
School of Computer Science
Cardiff University, Cardiff
CF24 3XF, UK
e-mail: malcolm@cs.cf.ac.uk

R. Weikard
Department of Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170, USA
e-mail: rudi@math.uab.edu

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Schatten Class Integral Operators Occurring in Markov-type Inequalities

Albrecht Böttcher, Holger Langenau and Harold Widom

Abstract. This paper is motivated by the search for best constants in Markov-type inequalities with different weights on both sides. It is known that in a large range of cases these constants involve the operator norm of certain Volterra integral operators. The proofs are based on the happy circumstance that these operators are Hilbert-Schmidt. The conjecture is that in the remaining cases the same operators occur, but a proof is still outstanding. We here show that in these cases the operators are Schatten class operators, and hence in particular compact, having hopes this will be of use in future efforts towards a confirmation of the conjecture on the best constants.

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1. Introduction and result

This paper is devoted to the problem of finding the smallest constant C in a Markov-type inequality of the form

$$\|D^\nu f\|_\beta \le C\|f\|_\alpha \text{ for all } f \in \mathcal{P}_n. \tag{1}$$

Here $\mathcal{P}_n$ stands for the linear space of the algebraic polynomials of degree at most n with complex coefficients, D^ν is the operator of taking the νth derivative, and $\|\cdot\|_\gamma$ is the norm given by

$$\|f\|_\gamma^2 = \int_0^\infty |f(t)|^2 t^\gamma \mathrm{e}^{-t}\mathrm{d}t, \tag{2}$$

where $\gamma > -1$ is a real parameter. Thus, with $\mathcal{P}_n(\gamma)$ denoting the space $\mathcal{P}_n$ with the norm (2), the best constant C in (1) is just the operator norm (= spectral norm) of the linear operator $D^\nu : \mathcal{P}_n(\alpha) \to \mathcal{P}_n(\beta)$. We denote this best constant C by $\lambda_n^{(\nu)}(\alpha, \beta)$.

The original inequalities by the Markov brothers had the maximum norm on both sides. Erhard Schmidt was the first to consider such inequalities in Hilbert space norms. In [8], he proved in particular that $\lambda_n^{(1)}(0,0) \sim \frac{2}{\pi}n$, where here and in what follows $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$. Subsequently, Turán [11] found the exact formula

$$\lambda_n^{(1)}(0,0) = \left(2\sin\frac{\pi}{4n+2}\right)^{-1}.$$

Shampine [9], [10] made the first step towards higher derivatives. He established the asymptotic formula $\lambda_n^2(0,0) \sim \frac{1}{\mu_0^2}n^2$, where μ_0 is the smallest positive solution of the equation $1+\cos\mu\cosh\mu = 0$. Dörfler [5] went further to $\nu \geq 3$ and proved that

$$\frac{1}{2\nu!}\sqrt{\frac{4}{2\nu+1}} \leq \liminf_{n\to\infty}\frac{\lambda_n^{(\nu)}(0,0)}{n^\nu} \leq \limsup_{n\to\infty}\frac{\lambda_n^{(\nu)}(0,0)}{n^\nu} \leq \frac{1}{2\nu!}\sqrt{\frac{2\nu}{2\nu-1}}.$$

It had not been known until [2] whether or not $\lambda_n^{(\nu)}(0,0)/n^\nu$ possesses a limit as $n \to \infty$ if $\nu \geq 3$. The results of [2] ($\alpha = 0$) and [3] ($\alpha > -1$) say that this limit exists and that, moreover, this limit is the operator norm of a certain Volterra integral operator:

$$\lambda_n^{(\nu)}(\alpha,\alpha) \sim \|L^*_{\nu,\alpha,\alpha}\|_\infty n^\nu, \tag{3}$$

where $\|\cdot\|_\infty$ is the operator norm and $L^*_{\nu,\alpha,\alpha}$ acts on $L^2(0,1)$ by the rule

$$(L^*_{\nu,\alpha,\alpha}f)(x) = \frac{1}{(\nu-1)!}\int_0^x x^{-\alpha/2}y^{\alpha/2}(x-y)^{\nu-1}f(y)\mathrm{d}y.$$

First results on the case $\alpha \neq \beta$ are in [1], [6], where it is in particular shown that

$$\lambda_n^{(\nu)}(\alpha,\alpha+\nu) = \sqrt{\frac{n!}{(n-\nu)!}} \sim n^{\nu/2}. \tag{4}$$

The situation for different α and β is best understood by looking at the number $\omega := \beta - \alpha - \nu$. Thus, (4) settles the case $\omega = 0$ while (3) disposes of the case $\omega = -\nu$. For general ω, a conjecture was raised in [4]. The following is a more precise and stronger version of that conjecture.

Conjecture 1.1. *Let $\alpha, \beta > -1$ be real numbers, let ν be a positive integer, and put $\omega = \beta - \alpha - \nu$. Then*

$$\lambda_n^{(\nu)}(\alpha,\beta) \sim C_\nu(\alpha,\beta)n^{(\nu+|\omega|)/2}$$

with

$$C_\nu(\alpha,\beta) = \begin{cases} 2^\omega & \text{for} \quad \omega \geq 0, \\ \|L^*_{\nu,\alpha,\beta}\|_\infty & \text{for} \quad \omega < 0, \end{cases}$$

where $L^*_{\nu,\alpha,\beta}$ is the Volterra integral operator on $L^2(0,1)$ given by

$$(L^*_{\nu,\alpha,\beta}f)(x) = \frac{1}{\Gamma(-\omega)}\int_0^x x^{-\alpha/2}y^{\beta/2}(x-y)^{-\omega-1}f(y)\mathrm{d}y. \tag{5}$$

This conjecture was confirmed in [4] in the case where $\omega \geq 0$ is an integer and then in [7] in the case where $\omega \geq 0$ is an arbitrary real number. The conjecture was also proved in [4] under the assumption that $\omega < -1/2$. Moreover, for $\omega = -1$, the norm $\|L^*_{\nu,\alpha,\beta}\|_\infty$ was shown to be $2/(\nu+1)$ times the reciprocal of the smallest positive zero of the Bessel function $J_{(\alpha-1)/(\nu+1)}$, which yields in particular such nice formulas as

$$\lambda_n^{(2)}\left(-\frac{1}{2},\frac{1}{2}\right) \sim \frac{4}{3\pi}n^{3/2}, \quad \lambda_n^{(4)}\left(\frac{7}{2},\frac{13}{2}\right) \sim \frac{2}{5\pi}n^{5/2}, \quad \lambda_n^{(5)}(4,8) \sim \frac{1}{3\pi}n^3.$$

Thus, what remains open is the case $-1/2 \leq \omega < 0$.

The method used in [2], [3], [4] to prove the conjecture for $\omega < -1/2$ is as follows. Finding the best constant in (1) comes down to determining the operator norm (= spectral norm) of the matrix representation of D^ν in an appropriate pair of orthonormal bases. In the case of the Laguerre norms we choose the normalized Laguerre polynomials with respect to the parameters α and β. The kth Laguerre polynomial associated with the norm (2) is

$$P_k(t,\gamma) = \frac{1}{\Gamma(k+1)} t^{-\gamma} \mathrm{e}^t \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(t^{k+\gamma}\mathrm{e}^{-t}\right) = \sum_{\ell=0}^{k} (-1)^\ell \binom{k+\gamma}{k-\ell} \frac{t^\ell}{\ell!}.$$

The kth normalized Laguerre polynomial is then given by

$$\hat{P}_k(t,\gamma) = w_k(\gamma) P_k(t,\gamma), \quad w_k(\gamma) := \sqrt{\frac{\Gamma(k+1)}{\Gamma(k+\gamma+1)}}.$$

The resulting matrix is upper-triangular and the first ν diagonals are zero. Thus, for a fixed n we have the following entries for the upper nonzero $N \times N$ matrix block A_N, $N = n - \nu + 1$:

$$(A_N)_{jk} = (-1)^{k+\nu-j} \frac{w_{k+\nu}(\alpha)}{w_j(\beta)} \binom{\beta-\alpha-\nu}{k-j}, \quad 0 \leq j \leq k \leq N-1.$$

Now let L_N be the integral operator on $L^2(0,1)$ that is defined by

$$(L_N f)(x) = \int_0^1 \ell_N(x,y) f(y) \mathrm{d}y$$

with the piecewise constant kernel $\ell_N(x,y) = (A_N^*)_{[Nx],[Ny]}$, where $[\cdot]$ denotes the integral part and A_N^* is the Hermitian adjoint of A_N. A result established in [12], [13] and independently also rediscovered by Shampine [9], [10] says that

$$\|A_N^*\|_\infty = N \|L_N\|_\infty.$$

Taking a closer look on the scaled operators $N^{1-(\nu+|\omega|)/2} L_N$ leads to the guess that these should converge in the operator norm to the integral operator (5) on $L^2(0,1)$, which will be temporarily abbreviated to L, that is, $L = L^*_{\nu,\alpha,\beta}$. This would give

$$\|A_N^*\|_\infty = N\|L_N\|_\infty \sim N\, N^{-(1-(\nu+|\omega|)/2)} \|L\|_\infty = N^{(\nu+|\omega|)/2} \|L\|_\infty$$

and hence prove Conjecture 1.1 for $\omega < 0$. In the papers cited above this was shown to work under the assumption that $\omega < -1/2$. The technically most difficult part was to prove the convergence of $N^{1-(\nu+|\omega|)/2}L_N$ to L in the operator norm. Fortunately, L can be shown to be a Hilbert–Schmidt operator if $\omega < -1/2$ and it can also be shown that $N^{1-(\nu+|\omega|)/2}L_N$ converges to L in the Hilbert–Schmidt norm for $\omega < -1/2$.

If $\omega \geq -1/2$, the operator L is no longer Hilbert–Schmidt. The result of this paper, stated below as Theorem 1.2, tells us that L is still a compact operator for $\omega < 0$. This is not of immediate help for proving Conjecture 1.1 but could be of use for further attempts towards accomplishing that goal. Namely, since L is compact, it follows that $P_N L P_N$ converges to L in the operator norm whenever $\{P_N\}$ is a sequence of operators such that P_N and the adjoints P_N^* converge strongly (= pointwise) to the identity operator. Our hope is that one can find a clever sequence $\{P_N\}$ which enables one to prove that

$$\|N^{1-(\nu+|\omega|)/2}L_N - P_N L P_N\|_\infty \to 0,$$

which together with the fact that $\|P_N L P_N - L\|_\infty \to 0$ implies the desired uniform convergence of $N^{1-(\nu+|\omega|)/2}L_N$ to L. Our second result, Theorem 1.3, says that L is in the 2^nth Schatten class if $\omega < -1/2^n$. This has again no immediate consequences for a proof of Conjecture 1.1, but we consider this fact as noteworthy, because any additional piece of information about L might be of use when approaching Conjecture 1.1.

Here are the necessary notions and notations. Let T be a bounded operator acting on some separable Hilbert space H and let $\{s_k(T)\}_{k\in\mathbb{N}}$ denote the sequence of singular values of T in non-increasing order. The operator T is said to belong to the pth Schatten class if $\{s_k(T)\}_{k\in\mathbb{N}} \in \ell^p(\mathbb{N})$. We write $\mathcal{S}_p$ for the set of these operators and define the norm by $\|T\|_{\mathcal{S}_p} = \|\{s_k(T)\}_{k\in\mathbb{N}}\|_{\ell^p}$. In the following we only consider values of p that are powers of two and just write $\|T\|_{\mathcal{S}_p} = \|T\|_{2^n}$ for $p = 2^n$. Clearly, $\|T\|_2 \geq \|T\|_{2^2} \geq \cdots \geq \|T\|_{2^n} \geq \cdots \geq \|T\|_\infty$. All we need is the equality $\|T\|_{2^n} = \|T^*T\|_{2^{n-1}}^{1/2}$, which holds for all $n \geq 1$, and the fact that the Hilbert-Schmidt norm $\|T\|_2$ of an integral operator T is equal to the L^2 norm of the kernel of T.

Herewith the results of this paper.

Theorem 1.2. *Let α, β, ω be real numbers and suppose $\beta > -1$, $\omega < (\beta - \alpha)/2$, $\omega < 0$. Then the operator given by the right-hand side of* (5) *is compact.*

Theorem 1.3. *Let $\alpha > -1, \beta > -1$, $\nu \geq 1$ be real numbers and put $\omega = \beta - \alpha - \nu$. If n is a positive integer and $\omega < -1/2^n$, then the operator* (5) *belongs to the 2^nth Schatten class.*

Theorem 1.2 will be proved in Section 2. The proof of Theorem 1.3 will be given in Sections 3 and 4.

2. Proof of Theorem 1.2

The factor $1/\Gamma(-\omega)$ is irrelevant for the compactness of the operator (5). Thus, we consider the operator M defined on $L^2(0,1)$ by

$$(Mf)(x) = \int_0^x x^{-\alpha/2} y^{\beta/2} (x-y)^{-\omega-1} f(y) \mathrm{d}y.$$

For $0 < r < 1$, let M_r be the operator on $L^2(0,1)$ that is given by

$$(M_r f)(x) = \int_0^{rx} x^{-\alpha/2} y^{\beta/2} (x-y)^{-\omega-1} f(y) \mathrm{d}y.$$

The square of the Hilbert–Schmidt norm of M_r is

$$\int_0^1 \int_0^{rx} x^{-\alpha} y^{\beta} (x-y)^{-2\omega-2} \mathrm{d}y \mathrm{d}x = \int_0^1 \int_0^r x^{\beta-\alpha-2\omega-1} y^{\beta} (1-y)^{-2\omega-2} \mathrm{d}y \mathrm{d}x.$$

This is finite if $\beta > -1$ and $\omega < (\beta - \alpha)/2$. Consequently, these two assumptions ensure that M_r is compact. We have

$$\begin{aligned} ((M - M_r)f)(x) &= \int_{rx}^x x^{-\alpha/2} y^{\beta/2} (x-y)^{-\omega-1} f(y) \mathrm{d}y \\ &= \int_r^1 x^{(\beta-\alpha)/2-\omega} y^{\beta/2} (1-y)^{-\omega-1} f(xy) \mathrm{d}y, \end{aligned}$$

and since $\omega < (\beta - \alpha)/2$, it follows that

$$|((M - M_r)f)(x)| \le \int_r^1 y^{\beta/2} (1-y)^{-\omega-1} |f(xy)| \mathrm{d}y.$$

We therefore obtain

$$\begin{aligned} \|(M - M_r)f\| &= \left(\int_0^1 |((M - M_r)f)(x)|^2 \mathrm{d}x \right)^{1/2} \\ &\le \left(\int_0^1 \left(\int_r^1 y^{\beta/2} (1-y)^{-\omega-1} |f(xy)| \mathrm{d}y \right)^2 \mathrm{d}x \right)^{1/2}, \end{aligned}$$

and by virtue of Minkowski's inequality for integrals, this is not larger than

$$\begin{aligned} &\int_r^1 \left(\int_0^1 y^{\beta} (1-y)^{-2\omega-2} |f(xy)|^2 \mathrm{d}x \right)^{1/2} \mathrm{d}y \\ &\quad = \int_r^1 y^{\beta/2} (1-y)^{-\omega-1} \left(\int_0^1 |f(xy)|^2 \mathrm{d}x \right)^{1/2} \mathrm{d}y. \end{aligned} \tag{6}$$

Taking into account that $\int_0^1 |f(xy)|^2 \mathrm{d}x = y^{-1} \int_0^y |f(t)|^2 \mathrm{d}t \le y^{-1} \|f\|^2$, we see that (6) does not exceed

$$\int_r^1 y^{\beta/2-1/2} (1-y)^{-\omega-1} \|f\| \mathrm{d}y.$$

In summary, we have shown that

$$\|(M-M_r)f\| \le \left(\int_r^1 y^{\beta/2-1/2}(1-y)^{-\omega-1}\mathrm{d}y\right)\|f\|. \tag{7}$$

The assumption that $\omega < 0$ guarantees that the integral occurring in (7) goes to zero as $r \to 1$. This implies that $\|M-M_r\|_\infty \to 0$ as $r \to 1$, which proves that M is compact.

3. Auxiliary results and an example

Let T be an integral operator on $L^2(0,1)$ with a real-valued kernel $k(\cdot,\cdot)$ and T^* its adjoint. These are then given by

$$(Tf)(x) = \int_0^1 k(x,y)f(y)\mathrm{d}y, \quad (T^*f)(x) = \int_0^1 k(y,x)f(y)\mathrm{d}y,$$

and thus,

$$((T^*T)f)(x) = \int_0^1 \left(\int_0^1 k(z,x)k(z,y)\mathrm{d}z\right) f(y)\mathrm{d}y.$$

We define a sequence of kernel functions $\{k_{2^n}\}_{n\ge 0}$ making up the integral operators K_{2^n}, respectively. We set

$$k_1(x,y) = \begin{cases} y^{-\alpha/2}x^{\beta/2}(y-x)^{-\omega-1} & \text{for } x<y, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, K_1 is just $\Gamma(-\omega)$ times the operator (5). Next, we set

$$k_{2^n}(x,y) = \int_0^1 k_{2^{n-1}}(z,x)k_{2^{n-1}}(z,y)\mathrm{d}z.$$

It follows that $K_{2^n} = K^*_{2^{n-1}}K_{2^{n-1}}$. We want to show that $\|K_1\|_{2^n} < \infty$. This is the same as $\|(K_1^*K_1)^{n-1}\|_2 = \|K_{2^{n-1}}\|_2 < \infty$. So we reduce the estimation of the 2^nth Schatten norm of the operator K_1 to the estimation of the Hilbert–Schmidt norm of the operator $K_{2^{n-1}}$, which is given by

$$\|K_{2^{n-1}}\|_2^2 = \int_0^1\int_0^1 k_{2^{n-1}}(x,y)k_{2^{n-1}}(x,y)\mathrm{d}x\mathrm{d}y.$$

To anticipate the arguments that will be used in the proof in the general case, we start with considering the case $n=2$. Thus, suppose $-1/2 \le \omega < -1/4$. Our aim is to show that K_2 is a Hilbert–Schmidt operator. Since $k_2(x,y) = k_2(y,x)$, we have

$$\begin{aligned}\|K_2\|_2^2 &= \int_0^1\int_0^1 k_2(x_2,x_0)k_2(x_0,x_2)\mathrm{d}x_0\mathrm{d}x_2 \\ &= \int_0^1\int_0^1\int_0^1\int_0^1 k_1(x_1,x_2)k_1(x_1,x_0)k_1(x_3,x_0)k_1(x_3,x_2)\mathrm{d}x_0\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3.\end{aligned}$$

The indexing of the variables might seem strange at the first glance, but it will turn out to be perfect when treating the general case. Notice also that all these kernels are non-negative, which implies that the integral over the cube is equal to the iterated integrals and that we can change the order of integration.

We have to distinguish between the cases $x_i < x_j$ and $x_i > x_j$. To this end we split the area of integration, that is, the cube $[0,1]^4$, into 4! disjoint simplices

$$\Omega_\pi = \{(x_0, x_1, x_2, x_3) \in [0,1]^4 : x_{\pi(0)} < x_{\pi(1)} < x_{\pi(2)} < x_{\pi(3)}\},$$

where π is a permutation of the numbers $0, 1, 2, 3$. The integral for $\|K_2\|_2^2$ then splits into 4! integrals over the areas Ω_π. In all but four cases one of the kernels $k_1(x_i, x_j)$ is zero. These four cases are the permutations which send $(0,1,2,3)$ to $(1,3,0,2)$, $(1,3,2,0)$, $(3,1,0,2)$, $(3,1,2,0)$. We are therefore left with showing that each of these four integrals is finite. Let us consider the integral corresponding to the last permutation, that is, the simplex given by $x_3 < x_1 < x_2 < x_0$. This integral equals

$$I_4 := \int_0^1 \int_0^{x_0} \int_0^{x_2} \int_0^{x_1} \varphi_4(x) \mathrm{d}x_3 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_0$$

with

$$\varphi_4(x) = x_0^{-\alpha} x_2^{-\alpha} x_1^{\beta} x_3^{\beta} (x_2 - x_1)^{-\sigma} (x_0 - x_1)^{-\sigma} (x_2 - x_3)^{-\sigma} (x_0 - x_3)^{-\sigma},$$

where here and in the following $\sigma := \omega + 1$. The inner integration in I_4 gives

$$\int_0^{x_1} \varphi_4(x) \mathrm{d}x_3 = x_0^{-\alpha} x_2^{-\alpha} x_1^{\beta} (x_2 - x_1)^{-\sigma} (x_0 - x_1)^{-\sigma} \int_0^{x_1} x_3^{\beta} (x_2 - x_3)^{-\sigma} (x_0 - x_3)^{-\sigma} \mathrm{d}x_3.$$

Now a first lemma comes into the game. Recall that $0 < \sigma = \omega + 1 < 1$.

Lemma 3.1. *Let $a > -1$, $\tau > 0$, $\sigma > 0$ be real numbers and let $k \geq 0$ and $\ell \geq 0$ be integers. Suppose $(k + \ell + 1)\tau < 1$ and $(1 + \tau)\sigma < 1$. Assume further that $0 < s \leq y < x$. Then*

$$\int_0^s t^a (x - t)^{-(1-k\tau)\sigma} (y - t)^{-(1-\ell\tau)\sigma} \mathrm{d}t \leq C (x - y)^{-(1-(k+\ell+1)\tau)\sigma} s^{a-(1+\tau)\sigma+1}$$

with some constant $C < \infty$.

Proof. We write $(x-t)^{-(1-k\tau)\sigma} = (x-t)^{-(1-(k+\ell+1)\tau)\sigma} (x-t)^{-(\ell+1)\tau\sigma}$, and since

$$(x - t)^{-(1-(k+\ell+1)\tau)\sigma} \leq (x - y)^{-(1-(k+\ell+1)\tau)\sigma},$$
$$(x - t)^{-(\ell+1)\tau\sigma} \leq (s - t)^{-(\ell+1)\tau\sigma}, \quad (y - t)^{-(1-\ell\tau)\sigma} \leq (s - t)^{-(1-\ell\tau)\sigma},$$

we obtain that the integral is not larger than

$$(x - y)^{-(1-(k+\ell+1)\tau)\sigma} \int_0^s t^a (s - t)^{-(1+\tau)\sigma} \mathrm{d}t.$$

The last integral equals

$$s^{a-(1+\tau)\sigma+1}\int_0^1 t^a(1-t)^{-(1+\tau)\sigma}\mathrm{d}t = s^{a-(1+\tau)\sigma+1}\cdot C,$$

where $C := \Gamma(a+1)\Gamma(1-(1+\tau)\sigma)/\Gamma(a+2-(1+\tau)\sigma) < \infty$. □

Now choose $\tau = 1/3$. Since $\sigma = 1+\omega < 1-1/4$, we have $(1+\tau)\sigma < 1$. Applying the lemma with $k=\ell=0$ to the above integral $\int_0^{x_1}\varphi_4(x)\mathrm{d}x_3$ we get

$$\begin{aligned}\int_0^{x_1}\varphi_4(x)\mathrm{d}x_3 &\le Cx_0^{-\alpha}x_2^{-\alpha}x_1^{2\beta-(1+\tau)\sigma+1}(x_2-x_1)^{-\sigma}(x_0-x_1)^{-\sigma}(x_0-x_2)^{-(1-\tau)\sigma}\\ &=: \varphi_3(x).\end{aligned}$$

Next we perform the inner integration in

$$I_4 \le \int_0^1\int_0^{x_0}\int_0^{x_2}\varphi_3(x)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_0.$$

We obtain

$$\begin{aligned}\int_0^{x_2}\varphi_3(x)\mathrm{d}x_1 &= Cx_0^{-\alpha}x_2^{-\alpha}(x_0-x_2)^{-(1-\tau)\sigma}\\ &\quad\times\int_0^{x_2}x_1^{2\beta-(1+\tau)\sigma+1}(x_2-x_1)^{-\sigma}(x_0-x_1)^{-\sigma}\mathrm{d}x_1,\end{aligned}$$

and hence may again use Lemma 3.1 with $k=\ell=0$. The only question is whether $a = 2\beta-(1+\tau)\sigma+1 > -1$. This problem is disposed of by the following lemma.

Lemma 3.2. *Let $\alpha > -1$, $\beta > -1$, $\nu \ge 1$ be real numbers. Put $\omega = \beta-\alpha-\nu$ and suppose $-1/2^{n-1} \le \omega < -1/2^n$. If k and ℓ are integers satisfying $0 \le \ell \le k \le 2^{n-1}$ and τ is defined as $\tau = 1/(2^n-1)$, then*

$$k\beta-\ell\alpha-(k+\ell-1)(1+\tau)(\omega+1)+(k+\ell-1) > \ell-1.$$

Proof. Since $(1+\tau)(\omega+1) < 1$, we have $-(k+\ell-1)(1+\tau)(\omega+1)+(k+\ell-1) > 0$. Hence

$$\begin{aligned}k\beta-\ell\alpha-(k+\ell-1)(1+\tau)(\omega+1)+(k+\ell-1) &> k\beta-\ell\alpha\\ = k(\beta-\alpha)+(k-\ell)\alpha = k(\omega+\nu)+(k-\ell)\alpha &> k(\omega+1)-(k-\ell)\\ = k\omega+\ell \ge \ell-k/2^{n-1} &\ge \ell-1.\end{aligned}$$

□

In the present case, $n=2$ and accordingly $\tau = 1/3$, as above. Lemma 3.2 with $k=2$ and $\ell=0$ shows that indeed $a = 2\beta-(1+\tau)\sigma+1 > -1$. We may therefore use Lemma 3.1 with $k=\ell=0$ to conclude that

$$\int_0^{x_2}\varphi_3(x)\mathrm{d}x_1 \le Cx_0^{-\alpha}x_2^{2\beta-\alpha-2(1+\tau)\sigma+2}(x_0-x_2)^{-(1-\tau)\sigma}(x_0-x_2)^{-(1-\tau)\sigma} =: \varphi_2(x),$$

where here and throughout what follows C denotes a finite constant, but not necessarily the same at each occurrence. Thus,

$$I_4 \le \int_0^1 \int_0^{x_0} \varphi_2(x) \mathrm{d}x_2 \mathrm{d}x_0.$$

We have

$$\begin{aligned} \int_0^{x_0} \varphi_2(x) \mathrm{d}x_2 &= C x_0^{-\alpha} \int_0^{x_0} x_2^{2\beta-\alpha-2(1+\tau)\sigma+2} (x_0 - x_2)^{-2(1-\tau)\sigma} \mathrm{d}x_2 \\ &= x_0^b \int_0^1 t^c (1-t)^{-2(1-\tau)\sigma} \mathrm{d}t =: x_0^b \cdot \tilde{C} \end{aligned}$$

with $b = 2\beta - 2\alpha - 2(1+\tau)\sigma + 2 - 2(1-\tau)\sigma + 1$ and $c = 2\beta - \alpha - 2(1+\tau)\sigma + 2$. Clearly, $2(1-\tau)\sigma < 2(1-1/3)(1-1/4) = 1$, Lemma 3.2 with $k = 2$ and $\ell = 1$ shows that $c > -1$, and finally,

$$\begin{aligned} b &= 2\beta - 2\alpha - 2(1+\tau)\sigma + 2 - 2(1-\tau)\sigma + 1 \\ &\ge 2(\omega+1) - 2(1+\tau)(\omega+1) - 2(1-\tau)(\omega+1) + 3 = 3 - 2(\omega+1) > 1. \end{aligned}$$

This proves that $I_4 \le \tilde{C} \int_0^1 x_0^b \mathrm{d}x_0 < \infty$.

4. Proof of Theorem 1.3

We now turn to the general case. The case $n = 1$ is a simple computation. So suppose $n \ge 2$ and $-1/2^{n-1} \le \omega < -1/2^n$. Put $\sigma = 1 + \omega$. We have to show that

$$\|K_{2^{n-1}}\|_2^2 = \int_0^1 \int_0^1 k_{2^{n-1}}(x_0, x_{2^{n-1}}) k_{2^{n-1}}(x_{2^{n-1}}, x_{2^n}) \mathrm{d}x_{2^{n-1}} \mathrm{d}x_0 \quad (x_{2^n} := x_0)$$

is finite; notice that $k_{2^{n-1}}(x_{2^{n-1}}, x_0) = k_{2^{n-1}}(x_0, x_{2^{n-1}})$ for $n \ge 2$. We write

$$k_{2^{n-1}}(x_i, x_j) = \int_0^1 k_{2^{n-2}}(x_\ell, x_i) k_{2^{n-2}}(x_\ell, x_j) \mathrm{d}x_\ell$$

where $\ell = (i+j)/2$ and so on until we have only the kernels $k_1(\cdot,\cdot)$. For example, if $n = 4$, then

$$\|K_8\|_2^2 = \int_0^1 \int_0^1 k_8(x_0, x_8) k_8(x_8, x_{16}) \mathrm{d}x_8 \mathrm{d}x_0 \quad (x_{16} := x_0)$$

with

$$\begin{aligned} k_8(x_0, x_8) &= \int_0^1 k_4(x_4, x_0) k_4(x_4, x_8) \mathrm{d}x_4 \\ &= \int_0^1 \int_0^1 \int_0^1 k_2(x_2, x_4) k_2(x_2, x_0) k_2(x_6, x_4) k_2(x_6, x_8) \mathrm{d}x_2 \mathrm{d}x_6 \mathrm{d}x_4 \\ &= \int_0^1 \cdots \int_0^1 k_1(x_3, x_2) k_1(x_3, x_4) k_1(x_1, x_2) k_1(x_1, x_0) \\ &\qquad \times k_1(x_5, x_6) k_1(x_5, x_4) k_1(x_7, x_6) k_1(x_7, x_8) \mathrm{d}x_1 \cdots \mathrm{d}x_7 \end{aligned}$$

and a similar expression for $k_8(x_8, x_{16})$. In this way the integral for $\|K_{2^{n-1}}\|_2^2$ becomes an integral over $\Omega = [0,1]^{2^n}$. We divide Ω into $(2^n)!$ disjoint simplices

$$\Omega_\pi = \{(x_0, \dots, x_{2^n-1}) \in [0,1]^{2^n} : x_{\pi(0)} < x_{\pi(1)} < \dots < x_{\pi(2^n-1)}\},$$

labelled by the permutations π of the numbers $0, 1, \dots, 2^{n-1}$. The result is

$$\|K_{2^{n-1}}\|_2^2 = \sum_\pi \int_0^1 \int_0^{x_{\pi(2^n-1)}} \int_0^{x_{\pi(2^n-2)}} \cdots \int_0^{x_{\pi(1)}} \left(\prod_{j=0}^{2^{n-1}-1} k_1(x_{2j+1}, x_{2j}) k_1(x_{2j+1}, x_{2j+2}) \right) \mathrm{d}x_{\pi(0)} \dots \mathrm{d}x_{\pi(2^n-1)}.$$

We perform the integrations from the inside to the outside and may restrict ourselves to the permutations π for which we never meet a kernel whose first variable is greater than the second. Thus, take such a permutation and consider

$$I_{2^n} = \int_0^1 \int_0^{x_{\pi(2^n-1)}} \int_0^{x_{\pi(2^n-2)}} \cdots \int_0^{x_{\pi(1)}} \varphi_{2^n}(x) \mathrm{d}x_{\pi(0)} \dots \mathrm{d}x_{\pi(2^n-1)}$$

with

$$\begin{aligned} \varphi_{2^n}(x) &= \prod_{j=0}^{2^{n-1}-1} k_1(x_{2j+1}, x_{2j}) k_1(x_{2j+1}, x_{2j+2}) \\ &= \prod_{j=0}^{2^{n-1}-1} x_{2j}^{-\alpha} x_{2j+1}^{\beta} [(x_{2j} - x_{2j+1})(x_{2j+2} - x_{2j+1})]^{-\sigma}. \end{aligned}$$

We put $\tau = 1/(2^n - 1)$. Then

$$(1+\tau)\sigma < \left(1 + \frac{1}{2^n - 1}\right)\left(1 - \frac{1}{2^n}\right) = 1.$$

The first integral is an integral as in Lemma 3.1 with $a = \beta$ and $k = \ell = 0$. We estimate this integral from above exactly as in this lemma and obtain a function $\varphi_{2^n-1}(x)$. Integrating this function, we have again an integral as in Lemma 3.1 with $k = 1$ and $\ell = 0$, and we estimate as in this lemma to get a function $\varphi_{2^n-2}(x)$. In this way we perform $2^n - 2$ integrations and estimates. In the end we have a function $\varphi_2(x)$.

In each step, we use Lemma 3.1 with some a and some k and ℓ. Let us first describe the evolution of the exponents a. After the first integration it equals $2\beta - (1+\tau)\sigma + 1$. Each further integration adds $-(1+\tau)\sigma + 1$ to the exponent, and from outside the integral we still have to add the values β or $-\alpha$ in dependence on whether the j in the integral $\int_0^{x_j}$ is odd or even. Thus, each time we add $\beta - (1+\tau)\sigma + 1$ or $-\alpha - (1+\tau)\sigma + 1$, and after $k + \ell$ integrations the exponent is $(k+1)\beta - \ell\alpha - (k+\ell)(1+\tau) + (k+\ell)$. Since we do not meet kernels which are identically zero, at each place in the sequence $\pi(0) < \dots < \pi(2^n - 1)$ the number of predecessors with odd subscript is at least as large as the number of predecessors with even subscript. This implies that always $k + 1 \geq \ell$. The first integration is over a variable with odd subscript. It follows that the number of integrals $\int_0^{x_j}$ with odd j is at most $2^{n-1}-1$, so that always $k+1 \le 2^{n-1}$. We therefore obtain from Lemma 3.2 (with k replaced by $k+1$) that the exponent a is greater than $\ell - 1 \ge -1$.

Our next objective is the evolution of the numbers k and ℓ occurring in Lemma 3.1. For this purpose, we associate weighted graphs $G_{2^n}, \ldots, G_2$ with the functions $\varphi_{2^n}(x), \ldots, \varphi_2(x)$. The graph G_{2^n} has 2^n vertices, which are labeled from x_0 to x_{2^n-1}, and 2^n edges, which join x_j and x_{j+1} and will be denoted by $[x_j, x_{j+1}]$. Each edge gets the weight 0. This is because in $\varphi_{2^n}(x)$ each $|x_j - x_{j+1}|$ has the exponent $-\sigma$, which may be written as $-(1-m\tau)\sigma$ with $m=0$. The function $\varphi_{2^n-1}(x)$ results from $\varphi_{2^n}(x)$ via an estimate of the form

$$\int_0^{x_j} x_i^a (x_{i-1}-x_i)^{-\sigma}(x_{i+1}-x_i)^{-\sigma}\mathrm{d}x_i \le C x_j^{a-(1+\tau)\sigma+1}|x_{i-1}-x_{i+1}|^{-(1-\tau)\sigma};$$

we write the differences in absolute values, since this dispenses us from distinguishing the cases $x_{i-1} < x_{i+1}$ and $x_{i+1} < x_{i-1}$. Thus, the differences $x_{i-1} - x_i$ and $x_{i+1} - x_i$ are no longer present in $\varphi_{2^n-1}(x)$. Instead, $\varphi_{2^n-1}(x)$ contains $|x_{i-1} - x_{i+1}|$ with the exponent $-(1-\tau)\sigma$, which is $-(1-m\tau)\sigma$ with $m=1$. Accordingly, G_{2^n-1} results from G_{2^n} by deleting the edges $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ and introducing a new edge $[x_{i-1}, x_{i+1}]$ with the weight $m=1$. We proceed in this way. If $\varphi_{h-1}(x)$ is obtained from $\varphi_h(x)$ by an estimate

$$\begin{aligned}&\int_0^{x_j} x_i^a (x_p - x_i)^{-(1-k\tau)\sigma}(x_q - x_i)^{-(1-\ell\tau)\sigma}\mathrm{d}x_i \\ &\quad \le C x_j^{a-(1+\tau)\sigma+1}|x_p - x_q|^{-(1-(k+\ell+1)\tau)\sigma},\end{aligned} \tag{8}$$

then G_h contained the edge $[x_p, x_i]$ with the weight k and the edge $[x_i, x_q]$ with the weight ℓ, we delete these two edges, and replace them by the edge $[x_p, x_q]$ with the weight $k+\ell+1$ to obtain G_{h-1}.

The graph G_2 consists of two edges which both join $x_{\pi(2^n-2)}$ and $x_{\pi(2^n-1)}$. Let r and s be the weights of these edges. The sum of all weights in G_{2^n} is zero, and in each step the sum of the weights increases by $-k-\ell+(k+\ell+1)=1$. As we made 2^n-2 steps, it follows that $r+s = 2^n-2$. We see in particular that in (8) we always have $k+\ell < 2^n-2$, whence $(k+\ell+1)\tau < (2^n-1)/(2^n-1) = 1$. This (together with the inequality $a > -1$ shown above) justifies the application of Lemma 3.1 in each step.

Figure 1 shows the graphs for the introductory example considered in Section 3, while Figure 2 presents the sequence of graphs for $n=3$ and the simplex associated with the permutation $x_5 < x_1 < x_3 < x_2 < x_4 < x_7 < x_6 < x_0$.

We abbreviate $x_{\pi(2^n-2)}$ and $x_{\pi(2^n-1)}$ to x_p and x_q. What we are left with is to prove that

$$\int_0^1 \int_0^{x_q} \varphi_2(x)\mathrm{d}x_p \mathrm{d}x_q < \infty$$

with

$$\varphi_2(x) = C x_q^{-\alpha} x_p^a (x_q - x_p)^{-(1-r\tau)\sigma}(x_q - x_p)^{-(1-s\tau)\sigma}.$$

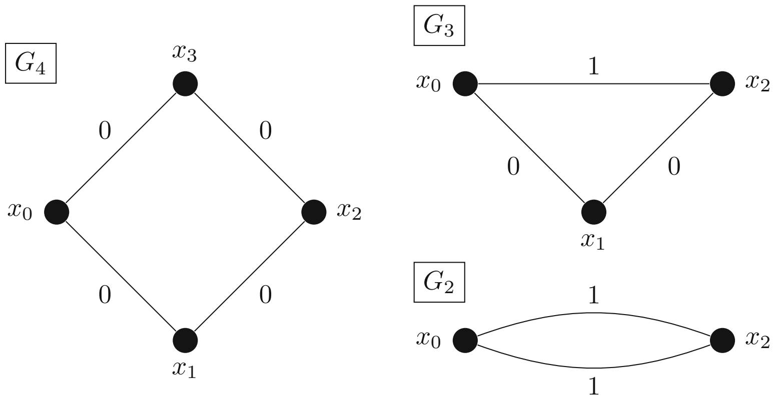

FIGURE 1. The sequence of graphs for $n = 2$ and $x_3 < x_1 < x_2 < x_0$.

The exponent a comes from $k = 2^{n-1} - 1$ integrals $\int_0^{x_j}$ with odd subscript j and $\ell = 2^{n-1} - 1$ integrals $\int_0^{x_j}$ with even j. (Notice that p and q are necessarily even.) Hence $a = (k+1)\beta - \ell\alpha - (k+\ell)(1+\tau)\sigma + (k+\ell)$, and from Lemma 3.2 we infer that $a > -1$. It follows that

$$\begin{aligned}\int_0^{x_q} \varphi_2(x)\mathrm{d}x_p &= Cx_q^{-\alpha} \int_0^{x_q} x_p^a (x_q - x_p)^{-(2-(r+s)\tau)\sigma} \mathrm{d}x_p \\ &= Cx_q^{-\alpha} x_q^{a-(2-(r+s)\tau)\sigma+1} \int_0^1 t^a (1-t)^{-(2-(r+s)\tau)\sigma} \mathrm{d}t.\end{aligned} \tag{9}$$

Obviously,

$$(2-(r+s)\tau)\sigma = \left(2 - \frac{2^n - 2}{2^n - 1}\right)(1+\omega) = \frac{2^n}{2^n - 1}(1+\omega) < \frac{2^n}{2^n - 1}\left(1 - \frac{1}{2^n}\right) = 1,$$

and hence (9) is finite. It remains to consider the integral $\int_0^1 x_q^b \mathrm{d}x_q$ with the exponent $b = -\alpha + a - (2-(r+s)\tau)\sigma + 1$. We just proved that $1 - (2-(r+s)\tau)\sigma > 0$. We also have

$$\begin{aligned}-\alpha + a &= (k+1)\beta - (k+1)\alpha - 2k(1+\tau)\sigma + 2k \\ &= (k+1)\beta - (k+1)\alpha - (2k+1)(1+\tau)\sigma + (2k+1) + (1+\tau)\sigma - 1 \\ &> k + 1 - 1 + (1+\tau)\sigma - 1 \quad \text{(Lemma 3.2)} \\ &= k - 1 + (1+\tau)\sigma > k - 1 = 2^{n-1} - 2 \geq 0.\end{aligned}$$

This shows that $b > 0$ and thus that $\int_0^1 x_q^b \mathrm{d}x_q < \infty$. The proof of Theorem 1.3 is complete.

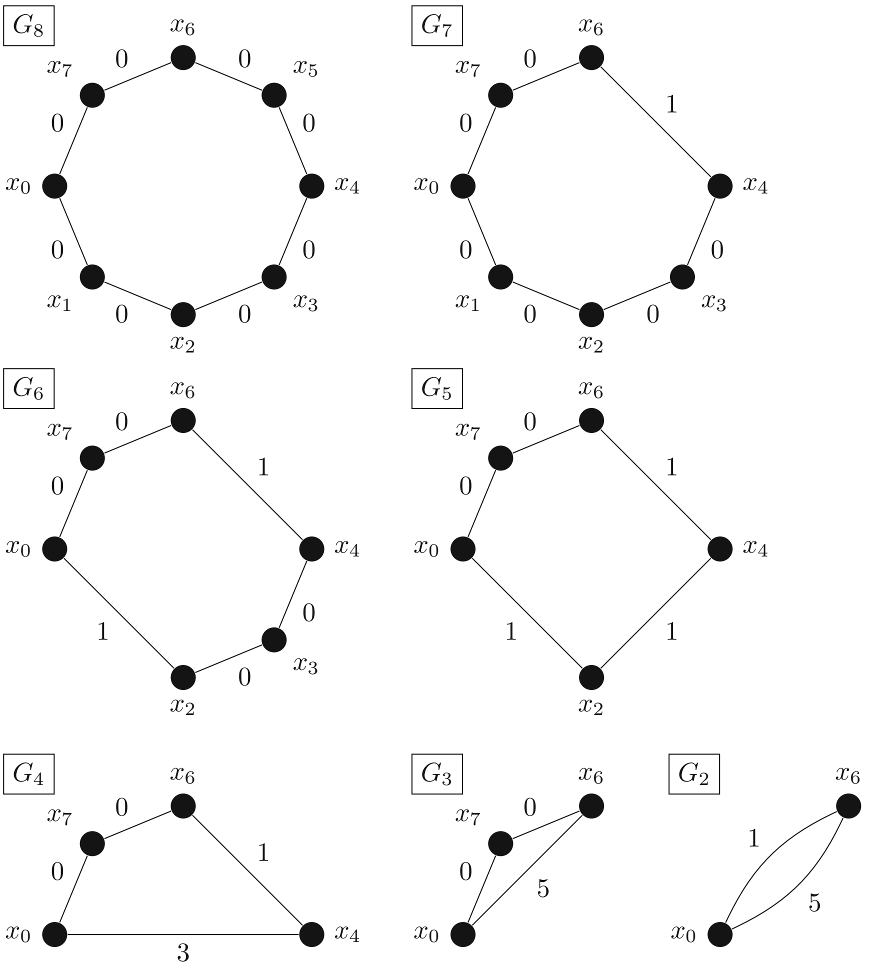

FIGURE 2. The sequence of graphs obtained for $n = 3$ and the permutation $x_5 < x_1 < x_3 < x_2 < x_4 < x_7 < x_6 < x_0$.

References

[1] R.P. Agarwal and G.V. Milovanović, *Extremal problems, inequalities, and classical orthogonal polynomials*. Appl. Math. Comput. 128, 151–166 (2002).

[2] A. Böttcher and P. Dörfler, *On the best constants in inequalities of the Markov and Wirtinger types for polynomials on the half-line*. Linear Algebra Appl. 430, 1057–1069 (2009).

[3] A. Böttcher and P. Dörfler, *Weighted Markov-type inequalities, norms of Volterra operators, and zeros of Bessel functions*. Math. Nachr. 283, 40–57 (2010).

[4] A. Böttcher and P. Dörfler, *On the best constants in Markov-type inequalities involving Laguerre norms with different weights*. Monatsh. Math. 161, 357–367 (2010).

[5] P. Dörfler, *A Markov type inequality for higher derivatives of polynomials*. Monatsh. Math. 109, 113-122 (1990).

[6] A. Guessab and G.V. Milovanović, *Weighted L^2 analogues of Bernstein's inequality and classical orthogonal polynomials*. J. Math. Anal. Appl. 182, 244–249 (1994).

[7] H. Langenau, *Asymptotically sharp inequalities for polynomials involving mixed Laguerre norms*. Linear Algebra Appl. 458, 116–127 (2014).

[8] E. Schmidt, *Über die nebst ihren Ableitungen orthogonalen Polynomensysteme und das zugehörige Extremum*. Math. Ann. 119, 165–204 (1944).

[9] L.F. Shampine, *Some L_2 Markov inequalities*. J. Res. Nat. Bur. Standards 69B, 155–158 (1965).

[10] L.F. Shampine, *An inequality of E. Schmidt*. Duke Math. J. 33, 145–150 (1966).

[11] P. Turán, *Remark on a theorem by Ehrhard Schmidt*. Mathematica (Cluj) 2 (25), 373–378 (1960).

[12] H. Widom, *Rapidly increasing kernels*. Proc. Amer. Math. Soc. 14, 501–506 (1963).

[13] H. Widom, *Hankel matrices*. Trans. Amer. Math. Soc. 121, 1–35 (1966).

Albrecht Böttcher and Holger Langenau
Fakultät für Mathematik
Technische Universität Chemnitz
D-09107 Chemnitz, Germany
e-mail: aboettch@mathematik.tu-chemnitz.de
holger.langenau@mathematik.tu-chemnitz.de

Harold Widom
Department of Mathematics
University of California
Santa Cruz, CA 95064, USA
e-mail: widom@ucsc.edu

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Twenty Years After

Harry Dym

Abstract. A glimpse backwards at the twenty plus years since I met and began to collaborate with Dima Arov. Some highlights of the research that began ever so briefly with prediction for multivariate stationary processes, quickly evolved into the study of direct and inverse spectral problems for canonical integral and differential systems and Dirac–Krein systems, and a number of bitangential interpolation and extension problems and circled back to prediction a couple of years ago will be presented.

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1. Genesis, or, how it all began

I met Dima for the first time in June/July 1991 in Japan. That year IWOTA was held in Sapporo (organized by Professor Ando) and MTNS was held the week following in Kobe. This was shortly after Perestroika and Glasnost and a number of Russian mathematicians attended. The list of participants included Vadim Adamjan, Damir Arov, Adolf Nudelman, Lev Sakhnovich, Edward Tsekanovskii and possibly others. It was probably the first time that they were permitted to attend a conference outside the former Soviet Bloc. We almost did not meet, since Dima got on the wrong plane. This is difficult to do. But, as many of you know, Dima is very clever.

Since Dima's English at the time was rather limited and my knowledge of Russian was limited to *Da* and *Nyet*, and neither of us spoke Japanese, we did not communicate with each until almost the end of the second week. Then one evening, at a barbecue organized during the MTNS week, Dima conveyed an interest in visiting the Weizmann Institute through our mutual friend and colleague Israel Gohberg. Fortunately, I was able to arrange this, and in the autumn of 1992, Dima and his wife Natasha, came to the Institute for the first time.

2. Autumn 1992

At the Institute Dima gave two series of lectures, one on system theory and one on J theory. Each lecture was on the order of 2 hours. Dima just got up there and spoke, without notes. A truly impressive performance. (I often wondered if paper was very expensive in the FSU.)

We also started to look for a problem of mutual interest that we could work on together and began to investigate the analytic counterpart of the problem of prediction for vector-valued stationary stochastic processes, given a finite segment of the past. Thus, as of the date of this IWOTA conference, we have been working together for twenty two years; unfortunately, the title of Dumas novel [Du45] that was borrowed for this talk refers to only twenty years, but that was the closest that I could find.

3. A version of the 1992 problem

Given: a $p\times p$ measurable mvf $\Delta(\mu)$ on $\mathbb{R}$ that meets the following three conditions

$$\begin{aligned} &\Delta(\mu) \quad \text{is positive definite a.e. on } \mathbb{R}, \quad \int_{-\infty}^{\infty} \frac{\operatorname{trace}\Delta(\mu)}{1+\mu^2}d\mu < \infty \\ &\text{and } \int_{-\infty}^{\infty} \frac{\ln\{\det\, \Delta(\mu)\}}{1+\mu^2}d\mu > -\infty. \end{aligned} \tag{1}$$

Let

$$\varphi_t(\mu) = i\int_0^t e^{i\mu s}dsI_p = \frac{e^{it\mu}-1}{\mu}I_p \tag{2}$$

and

$$Z^{[0,a]}(\Delta) = \text{closed linear span}\{\varphi_t\xi : \ t \in [0,a] \text{ and } \xi \in \mathbb{C}^p\} \tag{3}$$

in $L_2^p(\Delta)$, for $0 < a < \infty$

Objective: Compute the orthogonal projection of $f \in L_2^p(\Delta)$ onto $Z^{[0,a]}(\Delta)$.

More precisely, the objective was to identify Δ as the spectral density of a system of integral or differential equations and then use the transforms based on the fundamental solution of this system to compute the projection, in much the same way as had already been done for the case $p = 1$, following a program that was envisioned by M.G. Krein [Kr54] and completed in the 1976 monograph [DMc76].

Although some progress was made, it became clear that in order to penetrate further, it was necessary to develop a deeper understanding of direct and inverse problems for canonical systems of integral and differential equations and the associated families of RKHS's (reproducing kernel Hilbert spaces). Accordingly, we decided to postpone the study of the prediction problem for a while, and to focus on canonical systems.

That was about a 20 year detour.

4. Reproducing Kernel Hilbert Spaces

A Hilbert space $\mathcal{H}$ of $p \times 1$ vvf's (vector-valued functions) defined on $\Omega \subset \mathbb{C}$ is said to be a RKHS if there exists a $p \times p$ mvf (matrix-valued function) $K_\omega(\lambda)$ on $\Omega \times \Omega$ such that

(1) $K_\omega u \in \mathcal{H}$ for every $\omega \in \Omega$ and $u \in \mathbb{C}^p$.

(2) $\langle f, K_\omega u\rangle_{\mathcal{H}} = u^* f(\omega)$ for every $f \in \mathcal{H}$, $\omega \in \Omega$ and $u \in \mathbb{C}^p$.

A $p \times p$ mvf that meets these two conditions is called a RK (reproducing kernel).

It is well known (and not hard to check) that

(1) A RKHS has exactly one RK.

(2) $K_\omega(\lambda)^* = K_\lambda(\omega)$ for all points λ, ω in $\Omega \times \Omega$.

(3) $K_\omega(\lambda)$ is positive in the sense that

$$\sum_{i,j=1}^{n} u_i^* K_{\omega_i}(\omega_j) u_j \geq 0 \quad \begin{array}{l}\text{for any set of points } \omega_1, \dots, \omega_n \text{ in } \Omega \\ \text{and vectors } u_1, \dots, u_n \text{ in } \mathbb{C}^p.\end{array} \tag{4}$$

(4) Point evaluation is a bounded vector-valued functional

$$\|f(\omega)\| \leq \|f\|_{\mathcal{H}} \left\{\|K_\omega(\omega)\|\right\}^{1/2} :$$

for $\omega \in \Omega$ and $f \in \mathcal{H}$.

Conversely, by the matrix version of a theorem of Aronszajn (see, e.g., Theorem 5.2 in [ArD08b]) each $p \times p$ kernel $K_\omega(\lambda)$ that is positive on $\Omega \times \Omega$ in the sense of (3) can be identified as the RK of exactly one RKHS of $p \times 1$ vv.'s on Ω. There is also a converse to item (4): If e_j, $j = 1, \dots, p$, denotes the standard basis for $\mathbb{C}^p$, $\mathcal{H}$ is a Hilbert space of $p \times 1$ vvf's and

$$|e_j^* f(\omega)| \leq \|f\|_{\mathcal{H}} M_\omega \quad \text{for } j = 1, \dots, p,\ \omega \in \Omega \text{ and } f \in \mathcal{H},$$

then, by the Riesz representation theorem, there exists vectors $q_\omega^j \in \mathcal{H}$ such that

$$e_j^* f(\omega) = \langle f, q_\omega^j\rangle_{\mathcal{H}} \quad \text{for } j = 1, \dots, p.$$

Thus, if Q_ω denotes the array $\begin{bmatrix} q_\omega^1 & \cdots & q_\omega^p \end{bmatrix}$ and $u = \sum_{j=1}^p u_j e_j$, then

$$u^* f(\omega) = \sum_{j=1}^{p} \overline{u_j} (e_j^* f)(\omega) = \sum_{j=1}^{p} \overline{u_j} \langle f, q_\omega^j\rangle_{\mathcal{H}} = \langle f, Q_\omega u\rangle_{\mathcal{H}},$$

i.e., the $p \times p$ mvf $Q_\omega(\lambda)$ on $\Omega \times \Omega$ is a RK for $\mathcal{H}$.

5. Examples of RKHS's

The Hardy space H_2^p of $p \times 1$ vvf's that are

(1) holomorphic in the open upper half-plane $\mathbb{C}_+$;

(2) meet the constraint

$$\sup_{b>0}\int_{-\infty}^{\infty} f(a+ib)^* f(a+ib)da < \infty$$

(3) and are endowed with the standard inner product (applied to the nontangential boundary limits)

$$\langle f, g\rangle_{st} = \int_{-\infty}^{\infty} g(\mu)^* f(\mu)d\mu$$

is a RKHS with RK

$$K_\omega(\lambda) = I_p/\rho_\omega(\lambda),$$

where

$$\rho_\omega(\lambda) = -2\pi i(\lambda - \overline{\omega}) \quad \text{for } \omega \in \mathbb{C}_+. \tag{5}$$

The verification is Cauchy's theorem for H_2^p:

If $p = 1$, then

$$\langle f, 1/\rho_\omega\rangle_{st} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(\mu)}{\mu - \omega} d\mu = f(\omega) \quad \text{for } \omega \in \mathbb{C}_+.$$

If $p > 1$ and $v \in \mathbb{C}^p$, then

$$\langle f, v/\rho_\omega\rangle_{st} = v^* \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(\mu)}{\mu - \omega} d\mu = v^* f(\omega) \quad \text{for } \omega \in \mathbb{C}_+.$$

If $b(\lambda)$ is a $p \times p$ inner mvf, then $H_2^p \ominus bH_2^p$ is a RKHS with RK

$$K_\omega^b(\lambda) = \frac{I_p - b(\lambda)b(\omega)^*}{\rho_\omega(\lambda)} \quad for\ \lambda, \omega \in \mathbb{C}_+.$$

6. Entire de Branges matrices

An entire $p \times 2p$ mvf

$$\mathfrak{E}(\lambda) = \begin{bmatrix} E_-(\lambda) & E_+(\lambda) \end{bmatrix} \quad \text{with } p \times p \text{ blocks } E_\pm(\lambda)$$

is said to be a de Branges matrix if

(1) $\det E_+(\lambda) \not\equiv 0$ in $\mathbb{C}_+$, the open upper half-plane.

(2) $E_+^{-1}E_-$ is a $p \times p$ inner mvf with respect to $\mathbb{C}_+$, i.e.,

$$\|(E_+^{-1}E_-)(\lambda)\| \leq 1 \quad \text{if } \lambda \in \mathbb{C}_+$$

and

$$(E_+^{-1}E_-)(\mu) \quad \text{is unitary for } \mu \in \mathbb{R}.$$

7. de Branges spaces $\mathcal{B}(\mathfrak{E})$

The de Branges space $\mathcal{B}(\mathfrak{E})$ associated with an entire de Branges matrix $\mathfrak{E}$ is

$$\mathcal{B}(\mathfrak{E}) = \{\text{entire } p \times 1 \text{ vvf's: } E_+^{-1} f \in H_2^p \text{ and } E_-^{-1} f \in (H_2^p)^\perp\}$$

endowed with the inner product

$$\langle f, g \rangle_{\mathcal{B}(\mathfrak{E})} = \int_{-\infty}^{\infty} g(\mu)^* \{E_+(\mu)E_+(\mu)^*\}^{-1} f(\mu) d\mu$$

$\mathcal{B}(\mathfrak{E})$ is a RKHS with RK

$$K_\omega^{\mathfrak{E}}(\lambda) = \begin{cases} \dfrac{E_+(\lambda)E_+(\omega)^* - E_-(\lambda)E_-(\omega)^*}{\rho_\omega(\lambda)} & \text{if } \lambda \neq \overline{\omega} \\ \dfrac{E_+'(\overline{\omega})E_+(\omega)^* - E_-'(\overline{\omega})E_-(\omega)^*}{-2\pi i} & \text{if } \lambda = \overline{\omega} \end{cases},$$

with $\rho_\omega(\lambda)$ as in (5) (and $E_\pm'(\lambda)$ denotes the derivative of $E_\pm(\lambda)$ with respect to λ). This again may be verified by Cauchy's theorem.

8. A special subclass of de Branges matrices

We shall restrict attention to entire de Branges matrices with the extra property that

$$(\rho_i E_-^\#)^{-1} \in H_2^{p\times p} \quad \text{and} \quad (\rho_i E_+)^{-1} \in H_2^{p\times p}, \tag{6}$$

where

$$f^\#(\lambda) = f(\overline{\lambda})^*.$$

Condition (6) is equivalent to other conditions that are formulated in terms of the generalized backwards shift operator

$$(R_\alpha f)(\lambda) = \begin{cases} \dfrac{f(\lambda) - f(\alpha)}{\lambda - \alpha} & \text{when} \quad \lambda \neq \alpha \\ f'(\alpha) & \text{when} \quad \lambda = \alpha \end{cases}:$$

The following three conditions are equivalent for entire de Branges matrices $\mathfrak{E} = \begin{bmatrix} E_- & E_+ \end{bmatrix}$*:*

(1) $\mathfrak{E}$ *meets the constraints in* (6).
(2) $\mathcal{B}(\mathfrak{E})$ *is invariant under* R_α *for at least one point* $\alpha \in \mathbb{C}$.
(3) $\mathcal{B}(\mathfrak{E})$ *is invariant under* R_α *for every point* $\alpha \in \mathbb{C}$.

Additional equivalences are discussed on pp. 145–146 of [ArD12].

Moreover, under the constraint (6),

$$(E_-^\#)^{-1} = b_3\varphi_3 \quad \text{and} \quad E_+^{-1} = \varphi_4 b_4,$$

where b_3 and b_4 are entire inner $p \times p$ mvf's and $\rho_i^{-1}\varphi_3$ and $\rho_i^{-1}\varphi_4$ are outer $p \times p$ mvf's in $H_2^{p\times p}$.

The mvf's b_3 and b_4 are uniquely determined by $E_-^\#$ and E_+ up to a right constant unitary factor for b_3 and a left constant unitary factor for b_4. They are entire mvf's of exponential type. The set

$$ap(\mathfrak{E}) \stackrel{\text{def}}{=} \{(b_3u, vb_4) : u, v \in \mathbb{C}^{p\times p} \text{ and } u^*u = v^*v = I_p\}$$

is called the set of **associated pairs** of $\mathfrak{E}$.

9. de Branges spaces are of interest

de Branges spaces play a central role in prediction problems because if $\Delta(\mu)$ is subject to the constraints in (1), then the spaces

$$Z^{[0,a]}(\Delta) = \text{closed linear span}\{\varphi_t\xi : t \in [0, a] \text{ and } \xi \in \mathbb{C}^p\}$$

in $L_2^p(\Delta)$ with

$$\varphi_t(\mu) = i\int_0^t e^{i\mu s} ds I_p = \frac{e^{it\mu} - 1}{\mu} I_p$$

can be identified as de Branges spaces. Then $\Pi_{Z^{[0,a]}}$ can be calculated via the RK of this space, as will be illustrated in a number of examples below.

10. Example 1, $\Delta(\mu) = I_p$

Let

$$\widehat{f}(\mu) = \int_{-\infty}^{\infty} e^{i\mu s} f(s)ds \quad \text{and} \quad f^\vee(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu t} f(\mu)d\mu$$

denote the Fourier transform and inverse Fourier transforms respectively in L_2^p.

With the help of the Paley–Wiener theorem, it is not hard to show that

$$Z^{[0,a]}(I_p) = \left\{\int_0^a e^{i\lambda s} g(s)ds : g \in L_2^p([0, a])\right\}.$$

Correspondingly, $f \in Z^{[0,a]}(I_p)$ if and only if

$$f(\lambda) = \int_0^a e^{i\lambda s} f^\vee(s)ds = \int_0^a e^{i\lambda s}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu s} f(\mu)d\mu\right\} ds$$

$$= \int_{-\infty}^{\infty}\left\{\frac{1}{2\pi}\int_0^a e^{i\lambda s} e^{-i\mu s} ds\right\} f(\mu)d\mu,$$

i.e., for each $v \in \mathbb{C}^p$,

$$v^* f(\lambda) = \langle f, \mathcal{Z}_\lambda^{[0,a]} v\rangle_{st} \quad \text{the standard inner product}$$

with

$$\mathcal{Z}_\lambda^{[0,a]}(\mu) = \left\{\frac{1}{2\pi}\int_0^a e^{i(\mu-\overline{\lambda})s} ds\right\} I_p = \left\{\frac{1 - e^{i(\mu-\overline{\lambda})a}}{\rho_\lambda(\mu)}\right\} I_p.$$

Thus, $Z^{[0,a]}(I_p)$ is a de Branges space with RK

$$\mathcal{Z}_\mu^{[0,a]}(\lambda) = \frac{E_+(\lambda)E_+(\mu)^* - E_-(\lambda)E_-(\mu)^*}{\rho_\mu(\lambda)},$$

in which $E_-(\lambda) = e^{i\lambda a}I_p$ and $E_+(\lambda) = I_p$.

Moreover, the orthogonal projection $\Pi_{Z^{[0,a]}}$ of $f \in L_2^p$ onto $Z^{[0,a]}(I_p)$ is given by the formula

$$(\Pi_{Z^{[0,a]}} f)(\lambda) = \int_0^a e^{i\lambda s} f^\vee(s)ds = \int_{-\infty}^{\infty} \mathcal{Z}_\mu^{[0,a]}(\lambda)f(\mu)d\mu.$$

An analogous set of calculations for the space $Z^{[-a,a]}$ leads to the formula

$$\mathcal{Z}_\lambda^{[-a,a]}(\mu) = \left\{\frac{1}{2\pi}\int_{-a}^{a} e^{i(\mu-\overline{\lambda})s}ds\right\} I_p = \left\{\frac{e^{-i(\mu-\overline{\lambda})a} - e^{i(\mu-\overline{\lambda})a}}{\rho_\lambda(\mu)}\right\} I_p,$$

i.e., $Z^{[-a,a]}(I_p)$ is a de Branges space with $E_-(\lambda) = e^{i\lambda a}I_p$ and $E_+(\lambda) = e^{-i\lambda a}I_p$.

11. Example 2, $\alpha I_p \le \Delta(\mu) \le \beta I_p$ for some $\beta \ge \alpha > 0$

If, in addition to (1), the density $\Delta(\mu)$ is subject to the constraints

$$0 < \alpha I_p \le \Delta(\mu) \le \beta I_p \quad \text{a.e. on } \mathbb{R} \text{ (and } \beta < \infty), \tag{7}$$

then

$$f \in Z^{[0,a]}(\Delta) \Longleftrightarrow f \in Z^{[0,a]}(I_p)$$

and point evaluation is a bounded vector-valued functional in both. Thus, both spaces are RKHS's.

Let $K_\omega^a(\lambda)$ denote the RK for $Z^{[0,a]}(\Delta)$ and let $\mathcal{Z}_\omega^a(\lambda)$ continue to denote the RK for $Z^{[0,a]}(I_p)$. Then

$$v^*f(\omega) = \langle f, \mathcal{Z}_\omega^a v\rangle_{st} \quad \forall\ f \in Z^{[0,a]}(I_p),\ v \in \mathbb{C}^p \ \text{ and } \omega \in \mathbb{C}$$

and

$$v^*f(\omega) = \langle f, K_\omega^a v\rangle_{\Delta} \quad \forall\ f \in Z^{[0,a]}(\Delta),\ v \in \mathbb{C}^p \ \text{ and } \omega \in \mathbb{C}.$$

Therefore, since

$$Z^{[0,a]}(I_p) = Z^{[0,a]}(\Delta) \quad \text{(as vector spaces)}$$

when (7) is in force,

$$\langle f, \mathcal{Z}_\omega^a v\rangle_{st} = \langle f, K_\omega^a v\rangle_\Delta = \langle f, \Delta K_\omega^a v\rangle_{st} = \langle f, \Pi_a \Delta K_\omega^a v\rangle_{st}$$

for every choice of $f \in Z^{[0,a]}(I_p)$, $v \in \mathbb{C}^p$ and $\omega \in \mathbb{C}$, where

$$\Pi_a \quad \text{denotes the orthogonal projection of } L_2^p \text{ onto } Z^{[0,a]}(I_p).$$

Thus,

$$\Pi_a \Delta K_\omega^a v = \mathcal{Z}_\omega^a v. \tag{8}$$

12. Spectral densities in the Wiener algebra

If

$$\Delta(\mu) = I_p + \widehat{h}(\mu) \quad \text{with } h \in L_1^p \tag{9}$$

and $\Delta(\mu) > 0$ for $\mu \in \mathbb{R}$, then, since $\Delta(\mu)$ is continuous on $\mathbb{R}$ and, by the Riemann–Lebesgue lemma, $\Delta(\pm\infty) = I_p$, Δ meets the constraints in (7) for some choice of $\beta > \alpha > 0$. Consequently, in view of (8),

$$\begin{aligned}\Pi_a \Delta K_\omega^a v &= \Pi_a (I_p + \widehat{h}) K_\omega^a v \\ &= K_\omega^a v + \Pi_a \widehat{h} K_\omega^a v = \mathcal{Z}_\omega^a v.\end{aligned}$$

Thus, as

$$K_\omega^a(\lambda) = \int_0^a e^{i\lambda s} \varphi_\omega(s) ds \quad \text{and} \quad \mathcal{Z}_\omega^a(\lambda) = \frac{1}{2\pi} \int_0^a e^{i\lambda s} e^{-i\overline{\omega} s} I_p ds,$$

the formula

$$K_\omega^a v + \Pi_a \widehat{h} K_\omega^a v = \mathcal{Z}_\omega^a v$$

can be reexpressed **in the time domain** as

$$\varphi_\omega(s) + \int_0^a h(t-s) \varphi_\omega(s) ds = \frac{1}{2\pi} e^{-i\overline{\omega} s} I_p \quad \text{for } s \in [0, a]. \tag{10}$$

If it is also assumed that $h(t)$ is continuous, then the solution of (10) can be expressed explicitly as

$$\varphi_\omega(t) = \frac{1}{2\pi} e^{-i\overline{\omega} t} I_p + \frac{1}{2\pi} \int_0^a \gamma_a(t,s) e^{-i\overline{\omega} s} ds I_p$$

in which $\gamma_a(t,s)$ is the kernel of an integral operator and the RK of $Z^{[0,a]}(\Delta)$

$$\begin{aligned}K_\omega^a(\lambda) &= \int_0^a e^{i\lambda t} \varphi_\omega(t) dt \\ &= \frac{1}{2\pi} \int_0^a e^{i\lambda t} \left\{ e^{-i\overline{\omega} t} I_p + \int_0^a e^{-i\overline{\omega} s} \gamma_a(t,s) ds \right\} dt.\end{aligned}$$

With the help of the Krein–Sobolev formula (see, e.g., [GK85] for a clear discussion of this formula)

$$\frac{\partial}{\partial a} \gamma_a(t,s) = \gamma_a(t,a) \gamma_a(a,s)$$

and a variant thereof

$$\frac{\partial}{\partial a} \gamma_a(a-t, a-s) = \gamma_a(a-t, 0) \gamma_a(0, a-s)$$

it can be checked by brute force calculation that

$$\frac{\partial}{\partial a} K_\omega^a(\lambda) = \frac{1}{2\pi} E_-^a(\lambda) E_-^a(\omega)^*$$

where

$$E_-^a(\lambda) = e^{i\lambda a} I_p + \int_0^a e^{i\lambda t} \gamma_a(t,a) dt. \tag{11}$$

Thus, as $K_\omega^0(\lambda)=0$,

$$K_\omega^a(\lambda)=\int_0^a \frac{\partial}{\partial s}K_\omega^s(\lambda)ds=\frac{1}{2\pi}\int_0^a E_-^s(\lambda)E_-^s(\omega)^*ds. \tag{12}$$

The $p\times 2p$ mvf $\mathfrak{E}_a(\lambda)=\begin{bmatrix}E_-^a(\lambda) & E_+^a(\lambda)\end{bmatrix}$ with

$$E_+^a(\lambda)=I_p+\int_0^a e^{i\lambda s}\gamma_a(s,0)ds$$

is a de Branges matrix and

$$\frac{\partial}{\partial t}\mathfrak{E}_t(\lambda)=i\lambda\mathfrak{E}_t(\lambda)\begin{bmatrix}I_p & 0\\ 0 & 0\end{bmatrix}+\mathfrak{E}_t(\lambda)\begin{bmatrix}0 & \gamma_t(t,0)\\ \gamma_t(0,t) & 0\end{bmatrix}.$$

The assumption that $h(t)$ is continuous on $\mathbb{R}$ can be relaxed to the weaker assumption that $h(t)$ is continuous on $(-\infty,0)\cup(0,\infty)$ with left and right limits at 0. This is shown in a recent paper of Alpay, Gohberg, Kaashoek, Lerer and Alexander Sakhnovich [AGKLS10].

If $h=0$ in formula (9), then formulas (11) and (12) reduce to

$$E_-^a(\lambda)=e^{i\lambda a}I_p \quad \text{and} \quad K_\omega^a(\lambda)=\mathcal{Z}_\omega^a(\lambda)=\left\{\frac{1}{2\pi}\int_0^a e^{i(\lambda-\overline{\omega})s}ds\right\}I_p,$$

respectively.

13. 1993–2011

The formulas referred to in the previous section for $\Delta(\mu)$ of the form (9) are attractive and were accessible in 1992. However, this class of spectral densities is far too restrictive. It does not even include the simple case

$$\Delta(\mu)=\frac{1}{1+\mu^2}.$$

Thus, it was clear that it was essential to develop analogous projection formulas for a wider class of spectral densities. This lead us to investigate:

(1) Direct and inverse problems for canonical integral and differential systems and Dirac–Krein systems.
(2) Bitangential interpolation and extension problems.

The exploration of these two topics **and the interplay between them** before we returned to reconsider multivariate prediction took almost twenty years. The conclusions from these studies were presented in a lengthy series of articles that culminated in due course in the two volumes [ArD08b] and [ArD12]. A small sample of some of the major themes are surveyed briefly in the remaining sections of this paper. The focus will be on spectral densities $\Delta(\mu)$ that meet the constraints in (3).

14. Entire J-inner mvf's

A matrix $J \in \mathbb{C}^{m\times m}$ is said to be a **signature matrix**, if it is both self-adjoint and unitary with respect to the standard inner product, i.e., if

$$J = J^* \quad \text{and} \quad J^*J = I_m.$$

The main choices of J are

$$\pm I_m, \quad j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad j_p = j_{pp} \quad \text{and} \quad J_p = \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix}.$$

The signature matrix j_{pq} is most appropriate for problems concerned with contractive mvf's, whereas J_p is most appropriate for problems concerned with mvf's having a nonnegative real part, since:

$$\text{if } \varepsilon \in \mathbb{C}^{p\times q}, \text{ then} \quad I_p - \varepsilon^*\varepsilon \geq 0 \Longleftrightarrow \begin{bmatrix} \varepsilon^* & I_p \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \begin{bmatrix} \varepsilon \\ I_p \end{bmatrix} \leq 0;$$

$$\text{if } \varepsilon \in \mathbb{C}^{p\times p}, \text{ then} \quad \varepsilon + \varepsilon^* \geq 0 \Longleftrightarrow \begin{bmatrix} \varepsilon^* & I_p \end{bmatrix} \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix} \begin{bmatrix} \varepsilon \\ I_p \end{bmatrix} \leq 0.$$

The signature matrices J_p and j_p are unitarily equivalent:

$$\mathfrak{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} -I_p & I_p \\ I_p & I_p \end{bmatrix} \Longrightarrow \mathfrak{V} J_p \mathfrak{V} = j_p \quad \text{and} \quad \mathfrak{V} j_p \mathfrak{V} = J_p.$$

An $m \times m$ mvf $U(\lambda)$ is said to belong to the class $\mathcal{E} \cap \mathcal{U}(J)$ of entire J-inner mvf's with respect to an $m \times m$ signature matrix J if

(1) $U(\lambda)$ is an entire mvf.
(2) $J - U(\lambda)JU(\lambda)$ is positive semidefinite for every point $\lambda \in \mathbb{C}_+$.
(3) $J - U(\lambda)JU(\lambda) = 0$ for every point $\lambda \in \mathbb{R}$.

The last equality extends by analytic continuation to

$$U(\lambda)JU^{\#}(\lambda) = J \quad \text{for every point } \lambda \in \mathbb{C}$$

and thus implies further that

(4) $U(\lambda)$ is invertible for every point $\lambda \in \mathbb{C}$.
(5) $U(\lambda)^{-1} = JU^{\#}(\lambda)J$ for every point $\lambda \in \mathbb{C}$.
(6) $J - U(\lambda)JU(\lambda)$ is negative semidefinite for every point $\lambda \in \mathbb{C}_-$.

15. Canonical systems

A **canonical integral system** is a system of integral equations of the form

$$u(t,\lambda) = u(0,\lambda) + i\lambda \int_0^t u(s,\lambda)dM(s)J, \tag{13}$$

where $M(s)$ is a continuous nondecreasing $m \times m$ mvf on $[0,d]$ or $[0,\infty)$ with $M(0) = 0$ and signature matrix J.

In many problems $M(t) = \int_0^t H(s)ds$ with $H(s) \geq 0$ a.e. and at least locally summable. Then, the integral system can be written as

$$u(t, \lambda) = u(0, \lambda) + i\lambda \int_0^t u(s, \lambda) H(s) ds J$$

and the fundamental solution of this system is the $m \times m$ continuous solution of the integral system

$$U(t, \lambda) = I_m + i\lambda \int_0^t U(s, \lambda) H(s) ds J.$$

Then, by iterating the inequality

$$\|U(t, \lambda)\| \leq 1 + |\lambda| \int_0^t \|U(s, \lambda)\| \, \|H(s)\| ds,$$

it is readily checked that

$$\|U(t, \lambda)\| \leq \exp\left\{ |\lambda| \int_0^t \|H(s)\| ds \right\},$$

and hence that $U(t, \lambda)$ is an entire mvf of exponential type in the variable λ. Moreover,

$$\frac{J - U(t, \lambda) J U(t, \omega)^*}{\rho_\omega(\lambda)} = \frac{1}{2\pi} \int_0^t U(s, \lambda) H(s) U(s, \omega)^* ds. \tag{14}$$

Formula (14) implies that the kernel

$$K_\omega^{U_t}(\lambda) = \begin{cases} \dfrac{J - U(t, \lambda) J U(t, \omega)^*}{\rho_\omega(\lambda)} & \text{for } \lambda \neq \overline{\omega} \\ \dfrac{1}{2\pi i} \left(\dfrac{\partial U_t}{\partial \lambda} \right) (\overline{\omega}) & \text{for } \lambda = \overline{\omega} \end{cases}$$

is positive and hence, by the matrix version of a theorem of Aronszajn (see, e.g., Theorem 5.2 in [ArD08b]), there exists exactly one RKHS of $m \times 1$ vvf's with $K_\omega^{U_t}(\lambda)$ as its RK. We shall denote this space by $\mathcal{H}(U_t)$.

Formula (14) also implies that

$$J - U(t, \lambda) J U(t, \omega)^* = -i(\lambda - \overline{\omega}) \int_0^a U(s, \lambda) H(s) U(s, \omega)^* ds \tag{15}$$

and hence that

$$J - U(t, \omega) J U(t, \omega)^* \geq 0 \quad \text{if } \omega \in \mathbb{C}_+ \text{ with equality if } \omega \in \mathbb{R}.$$

Thus, $U_t(\lambda) = U(t, \lambda)$ belongs to the class

$$\mathcal{E} \cap \mathcal{U}^\circ(J) \quad \text{of } \textbf{entire } \boldsymbol{J}\textbf{-inner} \text{ mvf's } U \text{ with } U(0) = I_m$$

(in the variable λ).

Formula (15) also implies that

$$J - U(t, \overline{\omega}) J U(t, \omega)^* = 0$$

and hence that $U_t(\omega)$ is invertible for every point $\omega \in \mathbb{C}$.

The spaces $\mathcal{H}(U_t)$ are nested:

$$\mathcal{H}(U_{t_1}) \subseteq \mathcal{H}(U_{t_2}) \quad \text{if } 0 \le t_1 \le t_2,$$

but the inclusions are not necessarily isometric.

In particular, if $A_t(\lambda)$ denotes the fundamental solution of (13) when $J = J_p$ and

$$\begin{bmatrix} E_-^t(\lambda) & E_+^t(\lambda) \end{bmatrix} = \sqrt{2}\begin{bmatrix} 0 & I_p \end{bmatrix} A_t(\lambda)\mathfrak{V},$$

then

$$\begin{aligned} &\sqrt{2}\begin{bmatrix} 0 & I_p \end{bmatrix}\left\{\frac{J_p - A_t(\lambda)J_pA_t(\omega)^*}{\rho_\omega(\lambda)}\right\}\sqrt{2}\begin{bmatrix} 0 \\ I_p \end{bmatrix} \\ &= \sqrt{2}\begin{bmatrix} 0 & I_p \end{bmatrix}\left\{\frac{J_p - A_t(\lambda)\mathfrak{V}j_p\mathfrak{V}A_t(\omega)^*}{\rho_\omega(\lambda)}\right\}\sqrt{2}\begin{bmatrix} 0 \\ I_p \end{bmatrix} \\ &= \frac{E_+^t(\lambda)E_+^t(\omega)^* - E_-^t(\lambda)E_-^t(\omega)^*}{\rho_\omega(\lambda)}, \end{aligned}$$

where

$$\rho_\omega(\lambda) = -2\pi i(\lambda - \overline{\omega}).$$

The point is that the positivity of the first kernel implies the positivity of the second kernel and

$$A_t(0) = I_m \Longrightarrow \begin{bmatrix} E_-^t(0) & E_+^t(0) \end{bmatrix} = \begin{bmatrix} I_p & I_p \end{bmatrix}.$$

Thus, $\begin{bmatrix} E_-^t(\lambda) & E_+^t(\lambda) \end{bmatrix}$ is an entire de Branges matrix with $E_-^t(0) = E_+^t(0) = I_p$.

16. Linear fractional transformations

Let

$$\mathcal{S}^{p\times p} = \{\varepsilon : \varepsilon \text{ is holomorphic in } \mathbb{C}_+ \text{ and } \|s(\lambda)\| \le 1 \text{ in } \mathbb{C}_+\},$$

denote the **Schur class** and

$$\mathcal{C}^{p\times p} = \{\tau : \tau \text{ is holomorphic in } \mathbb{C}_+ \text{ and } \Re c(\lambda) \ge 0 \text{ in } \mathbb{C}_+\}$$

denote the **Carathéodory class**.

If $W \in \mathcal{U}(j_p)$, then the linear fractional transformation

$$T_W[\varepsilon] = (w_{11}\varepsilon + w_{12})(w_{21}\varepsilon + w_{22})^{-1} \quad \text{maps} \quad \varepsilon \in \mathcal{S}^{p\times p} \mapsto \mathcal{S}^{p\times p},$$

whereas, if $A \in \mathcal{U}(J_p)$, then

$$T_A[\varepsilon] = (a_{11}\varepsilon + a_{12})(a_{21}\varepsilon + a_{22})^{-1} \quad \text{maps} \quad \varepsilon \in \mathcal{C}^{p\times p} \mapsto \mathcal{C}^{p\times p},$$

when $\det\{a_{21}\varepsilon + a_{22}\} \not\equiv 0$ in $\mathbb{C}_+$.

If $A \in \mathcal{E} \cap \mathcal{U}^\circ(J_p)$ and $B(\lambda) = A(\lambda)\mathfrak{V} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, then

$$T_B[\varepsilon] = (b_{11}\varepsilon + b_{12})(b_{21}\varepsilon + b_{22})^{-1} \quad \text{maps} \quad \varepsilon \in \mathcal{S}^{p\times p} \mapsto \mathcal{C}^{p\times p}, \tag{16}$$

when $\det\{b_{21}\varepsilon + b_{22}\} \not\equiv 0$ in $\mathbb{C}_+$.

17. Subclasses of $\mathcal{E} \cap \mathcal{U}^\circ(J)$ with $J \neq \pm I_m$

A mvf $U \in \mathcal{E} \cap \mathcal{U}^\circ(J)$ with $J \neq \pm I_m$ belongs to the class

$\mathcal{U}_S(J)$ of **singular** J-inner mvf's if it is of minimal exponential type

$\mathcal{U}_{rR}(J)$ of **right regular** J-inner mvf's if it has no singular right divisors

$\mathcal{U}_{rsR}(J)$ of **strongly right regular** J-inner mvf's if it is unitarily equivalent to a mvf $W \in \mathcal{U}(j_{pq})$ in the class

$\mathcal{U}_{rsR}(j_{pq})$ of **strongly right regular** j_{pq}-inner mvf's if there exists a mvf $\varepsilon \in \mathcal{S}^{p\times q}$ such that $\|T_W[\varepsilon]\| \le \delta < 1$.

18. A pleasing RK result

A pleasing result that was obtained early in this period (in [ArD97]) is that a mvf $U \in \mathcal{E} \cap \mathcal{U}(J)$ with $J \neq \pm I_m$ belongs to the class

$$\mathcal{U}_S(J) \Longleftrightarrow \mathcal{H}(U) \cap L_2^p = \{0\}$$

$$\mathcal{U}_{rR}(J) \Longleftrightarrow \mathcal{H}(U) \cap L_2^p \text{ is dense in } \mathcal{H}(U)$$

$$\mathcal{U}_{rsR}(J) \Longleftrightarrow \mathcal{H}(U) \subset L_2^p.$$

Some years later (in [ArD01]) it was discovered that if $U \in \mathcal{E} \cap \mathcal{U}(J)$, $J \neq \pm I_m$ and $P_\pm = (I_m \pm J)/2$, then

$$U \in \mathcal{E} \cap \mathcal{U}_{rsR}(J) \quad \text{if and only if the mvf} \quad P_+ + U(\mu)P_-U(\mu)^* \tag{17}$$

satisfies the matrix Muckenhoupt (A_2) condition formulated by Treil and Volberg in [TV97]. Chapter 10 of [ArD08b] contains characterizations of the class $\mathcal{U}_{rsR}(J)$ of J-inner mvf's that are not necessarily entire.

This characterization of the class $\mathcal{E} \cap \mathcal{U}_{rsR}(J)$ has a nice reformulation ([ArD??]) that rests on the observation that

$$\mathfrak{F}(\lambda) = \begin{bmatrix} F_-(\lambda) & F_+(\lambda) \end{bmatrix} = \begin{bmatrix} U(\lambda)P_+ + P_- & U(\lambda)P_- + P_+ \end{bmatrix},$$

is a de Branges matrix that is related to the mvf in (17) by the formula

$$F_+(\mu)F_+(\mu)^* = P_+ + U(\mu)P_-U(\mu)^*.$$

Moreover,

$$f \in \mathcal{H}(U) \Longleftrightarrow f \in \mathcal{B}(\mathfrak{F}) \Longleftrightarrow F_+^{-1} f \in H_2^m \quad \text{and} \quad (F_-^{-1} f) \in (H_2^m)^\perp$$

and

$$\|f\|_{\mathcal{H}(U)}^2 = \int_{-\infty}^{\infty} f(\mu)^* \{F_+(\mu) F_+(\mu)^*\}^{-1} f(\mu) d\mu,$$

which exhibits the role of the mvf $P_+ + U(\mu) P_- U(\mu)^*$ in the calculation of the norm in $\mathcal{H}(U)$.

19. A simple inverse monodromy problem

The given data for the inverse monodromy problem is a mvf $U \in \mathcal{E} \cap \mathcal{U}(J)$ with $U(0) = I_m$.

The objective is to find an $m \times m$ mvf $H(t)$ on $[0, d]$ such that

(1) $H(t) \geq 0$, $H \in L_1^{m \times m}([0, d])$ and $\operatorname{trace} H(t) = 1$ a.e. on $[0, d]$

(2) $U(\lambda) = U_d(\lambda)$, where

$$U_t(\lambda) = I_m + i\lambda \int_0^d U_s(\lambda) H(s) ds J. \tag{18}$$

The existence of a solution to this problem is guaranteed by a theorem of Potapov (see, e.g., pp. 182–184 in [ArD08b]). Moreover, it follows easily from (18) that

$$U_t(0) = I_m \quad \text{and} \quad \frac{U_d(\lambda) - I_m}{i\lambda} J = \int_0^d U_s(\lambda) H(s) ds$$

and hence that

$$d = \operatorname{trace} \left\{ -i \left(\frac{dU_d}{d\lambda} \right) (0) J \right\}.$$

In general $H(t)$ is not unique unless other constraints are imposed.

If, for example, $m = p + q$ and $p = q = 1$,

$$J = j_{11}, \quad W(\lambda) = \begin{bmatrix} e^{i\lambda a_1} & 0 \\ 0 & e^{-i\lambda a_2} \end{bmatrix} \quad \text{with } a_1 \geq 0,\ a_2 \geq 0,\ a_1 + a_2 > 0,$$

then $d = a_1 + a_2$. Thus, if $H(t)$ is a solution of the inverse monodromy problem for the given W, then the fundamental solution

$$W_t(\lambda) = I_m + i\lambda \int_0^t W_s(\lambda) H(s) ds j_{11} \quad \text{for } t \in [0, d],$$

must be of the form

$$W_t(\lambda) = \begin{bmatrix} e^{i\lambda \varphi_1(t)} & 0 \\ 0 & e^{-i\lambda \varphi_2(t)} \end{bmatrix}.$$

Consequently,

$$-i \left(\frac{\partial}{\partial \lambda} W_t \right) (0) j_{11} = \begin{bmatrix} \varphi_1(t) & 0 \\ 0 & \varphi_2(t) \end{bmatrix} = \int_0^t H(s) ds.$$

Thus, φ_1 and φ_2 are absolutely continuous and

$$H(t) = \begin{bmatrix} \varphi_1'(t) & 0 \\ 0 & \varphi_2'(t) \end{bmatrix}.$$

These functions are subject to the constraint

$$\varphi_1'(t) + \varphi_2'(t) = \operatorname{trace} H(t) = 1, \tag{19}$$

but are otherwise completely arbitrary, i.e., no uniqueness.

A theorem of de Branges (see [Br68] for the original proof, [DMc76] for an adaptation of de Branges' proof and Theorem 8.3 in [ArD12]) guarantees there is exactly one real-valued solution $H(t)$ under the added assumption that given monodromy matrix $W(\lambda)$ is **symplectic**, i.e.,

$$\begin{bmatrix} w_{11} & w_{21} \\ w_{12} & w_{22} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

This extra assumption forces $\varphi_1(t) = \varphi_2(t)$ and hence (19) reduces to

$$\varphi_1'(t) = \varphi_2'(t) = 1/2$$

and thus, there is only one solution $H(t)$ of this problem.

de Branges' theorem is not applicable if $m > 2$. To at least sense the added complexity when $m = p + q$, $q \geq 1$ and $p > 1$, it is perhaps helpful to observe that if $a \geq 0$, then $e^{ia\lambda}$ is the only entire inner function of exponential type a, whereas if $p = 2$, then

$$\begin{bmatrix} e^{ia\lambda} & 0 \\ 0 & e^{ib\lambda} \end{bmatrix}$$

is an entire inner mvf of exponential type a for every $b \in [0, a]$.

To overcome this difficulty, in our formulation of the inverse monodromy problem with Dima, we associate a pair of inner functions b_1 of size $p \times p$ and b_2 of size $q \times q$, $p + q = m$, with the given monodromy matrix $W \in \mathcal{E} \cap \mathcal{U}(j_{pq})$ and specify a chain of inner divisors $\{b_1^t, b_2^t\}$ of $\{b_1, b_2\}$ for $t \in [0, d]$ in addition to W; see Chapter 8 of [ArD12] for details.

20. Helical extension problems

If $\alpha = \alpha^* \in \mathbb{C}^{p \times p}$ and $\Delta(\mu)$ meets the constraints in (3), then it is readily checked that the mvf

$$g_\Delta^{(\alpha)}(t) = \begin{cases} it\alpha + \dfrac{1}{\pi} \displaystyle\int_{-\infty}^{\infty} \left\{ e^{-i\mu t} - 1 + \dfrac{i\mu t}{1 + \mu^2} \right\} \dfrac{\Delta(\mu)}{\mu^2} d\mu & \text{for } t \neq 0 \\ 0 & \text{for } t = 0 \end{cases}$$

enjoys the following properties:

(1) $g_\Delta^{(\alpha)}(t)$ is continuous on $\mathbb{R}$.

(2) $g_\Delta^{(\alpha)}(-t) = \left\{g_\Delta^{(\alpha)}(t)\right\}^*$.

(3) $g_\Delta^{(\alpha)}(t-s) - g_\Delta^{(\alpha)}(t) - g_\Delta^{(\alpha)}(-s) + g_\Delta^{(\alpha)}(0)$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{e^{-i\mu t}-1}{\mu}\right)\Delta(\mu)\left(\frac{e^{i\mu s}-1}{\mu}\right) I_p ds$$

Consequently, $g_\Delta^{(\alpha)}(t)$ belongs to the class

$\mathcal{G}_\infty^{p\times p}(0)$ of continuous $p\times p$ mvf's $g(t)$ on $\mathbb{R}$ with $g(0)\le 0$ and $g(-t)=g(t)^*$ such that the kernel

$$k(t,s) = g(t-s) - g(t) - g(-s) + g(0)$$

is positive in the sense of (4). The mvf's in $\mathcal{G}_\infty^{p\times p}(0)$ are called **helical mvf's**.

The **helical extension problem**

HEP$(g_\Delta^{(\alpha)}; a)$ is to describe the set

$$\{g \in \mathcal{G}_\infty^{p\times p}(0):\ g(t) = g_\Delta^{(\alpha)}(t) \quad \text{for } |t|\le a\}.$$

Because of the constraints imposed on Δ in (3), this is a **completely indeterminate** extension problem. This means that for each vector $v\in\mathbb{C}^p$ there exists at least one mvf $g\in\mathcal{G}_\infty^{p\times p}(0)$ such that

$$(g(t) - g_\Delta^{(\alpha)}(t))v \not\equiv 0.$$

Moreover, the fact that this problem is completely indeterminate guarantees that for each $a\in(0,\infty)$ there is a natural way to specify a mvf $A_a\in\mathcal{E}\cap\mathcal{U}(J_p)$ such that the linear fractional transformation $T_{B_a}[\varepsilon]$ based on $B_a(\lambda) = A_a(\lambda)\mathfrak{V}$ that is defined by formula (16) maps $\{\varepsilon:\ \varepsilon\in\mathcal{S}^{p\times p}\}$ onto a set of mvf's in the Carathéodory class $\mathcal{C}^{p\times p}$ which is in one to one correspondence with the set of solutions of the HEP$(g_\Delta^{(\alpha)}; a)$. Moreover, if $A_d\in\mathcal{E}\cap\mathcal{U}_{rsR}(J_p)$ for some $d\in(0,\infty)$, then:

(1) The family $\{A_a\}$, $0\le a\le d$, is the fundamental solution of a canonical integral system.

(2) The de Branges spaces $\mathcal{B}(\mathfrak{E}_a)$ based on the de Branges matrices

$$\mathfrak{E}_a(\lambda) = \sqrt{(}2)\begin{bmatrix}0 & I_p\end{bmatrix} A_a(\lambda)\mathfrak{V}$$

coincide with the spaces $Z^{[0,a]}(\Delta)$ for $0\le a\le d$.

(3) The orthogonal projection $\Pi_{Z^{[0,a]}} f$ of $f\in Z^{[0,d]}(\Delta)$ onto $Z^{[0,a]}(\Delta)$ is given by the formula

$$(\Pi_{Z^{[0,a]}} f)(\omega) = \int_{-\infty}^{\infty} K_\mu^{\mathfrak{E}_a}(\omega)\Delta(\mu)f(\mu). \tag{20}$$

21. A reformulation in the Carathéodory class

The formula

$$c(\lambda) = \lambda^2 \int_0^\infty e^{i\lambda t} g(t)dt \quad \text{for } \lambda \in \mathbb{C}_+$$

defines a 1:1 transformation from the class of helical mvf's $g \in \mathcal{G}_\infty^{p\times p}(0)$ onto mvf's c in the Carathéodory class $\mathcal{C}^{p\times p}$, i.e., from mvf's of the form

$$g(t) = \begin{cases} -\beta + it\alpha + \frac{1}{\pi}\int_{-\infty}^\infty \left\{e^{-i\mu t} - 1 + \dfrac{i\mu t}{1+\mu^2}\right\} \frac{d\sigma(\mu)}{\mu^2} & \text{for } t \in \mathbb{R}\setminus\{0\} \\ 0 & \text{for } t = 0 \end{cases}$$

with $\alpha = \alpha^* \in \mathbb{C}^{p\times p}$, $\beta \in \mathbb{C}^{p\times p}$, $\beta \geq 0$ and a nondecreasing $p \times p$ mvf σ that is subject to the constraint

$$\int_{-\infty}^\infty \frac{d\,\text{trace}\sigma(\mu)}{1+\mu^2} < \infty \tag{21}$$

onto mvf's of the form

$$c(\lambda) = i\alpha - i\lambda\beta + \frac{1}{\pi i}\int_{-\infty}^\infty \left\{\frac{1}{\mu - \lambda} - \frac{\mu}{1+\mu^2}\right\} d\sigma(\mu) \quad \text{for } \lambda \in \mathbb{C}_+,$$

with the same α, β and σ as for $g(t)$. Consequently, helical extension problems can be reformulated as extension problems in the Carathéodory class. See [ArD12] for additional details and generalizations, and [ArD08a] for a short survey. (The latter may be downloaded free from MSRI.) The general strategy of identifying the resolvent matrices of appropriately defined extension problems with the fundamental solutions of integral or differential systems originates with M.G. Krein.

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References

[AGKLS10] Daniel Alpay, Israel Gohberg, Marinus A. Kaashoek, Leonid Lerer and Alexander L. Sakhnovich, Krein systems and canonical systems on a finite interval: accelerants with a jump discontinuity at the origin and continuous potentials, Integral Equations Operator Theory **68** (2010), no. 1, 115–150.

[ArD97] Damir Z. Arov and Harry Dym, J-inner matrix functions, interpolation and inverse problems for canonical systems, I: Foundations, Integral Equations Operator Theory **29** (1997), 373–454.

[ArD01] Damir Z. Arov and Harry Dym, Matricial Nehari problems, J-inner matrix functions and the Muckenhoupt condition, J. Funct. Anal.,**181** (2001), 227–299.

[ArD08a] Damir Z. Arov and Harry Dym, Bitangential direct and inverse problems for systems of differential equations, in *Probability, Geometry and Integrable Systems*, Cambridge University Press, Cambridge, England, 2008.

[ArD08b] Damir Z. Arov and Harry Dym, *J-Contractive Matrix Valued Functions and Related Topics*, Cambridge University Press, Cambridge, England, 2008, pp. 1–28.

[ArD12] Damir Z. Arov and Harry Dym, *Bitangential Direct and Inverse Problems for Systems of Integral and Differential Equations*, Cambridge University Press, Cambridge, England, 2012.

[ArD??] Damir Z. Arov and Harry Dym, *Multivariate Prediction, de Branges Spaces and Related Extension and Inverse Problems*, in preparation.

[Br68] Louis de Branges, *Hilbert Spaces of Entire Functions*, Prentice Hall, Englewood Cliffs, 1968.

[Du45] Alexander Dumas, Vingt ans après, 1845.

[DMc76] Harry Dym and Henry P. McKean, *Gaussian Processes, Function Theory and the Inverse Spectral Problem*, Academic Press, New York, 1976.

[GK85] Israel Gohberg and Israel Koltracht, Numerical solution of integral equations, fast algorithms and Krein–Sobolev equation, Numer. Math. **47** (1985), no. 2, 237–288.

[Kr54] Mark G. Krein, On a fundamental approximation problem in the theory of extrapolation and filtration of stationary random processes, Dokl. Akad. Nauk SSSR **94** (1954), 13–16 [English transl.: Amer. Math. Soc. Selected Transl. Math. Statist. Prob. **4** (1964), 127–131.

[TV97] Sergei Treil and Alexander Volberg, Wavelets and the angle between past and future, J. Funct. Anal., **143** (1997), 73–131.

Harry Dym
Department of Mathematics
The Weizmann Institute of Science
Rehovot 7610001, Israel
e-mail: harry.dym@weizmann.ac.il

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Matrix-valued Hermitian Positivstellensatz, Lurking Contractions, and Contractive Determinantal Representations of Stable Polynomials

Anatolii Grinshpan, Dmitry S. Kaliuzhnyi-Verbovetskyi, Victor Vinnikov and Hugo J. Woerdeman

Dedicated to Leiba Rodman, a dear friend and wonderful colleague, who unfortunately passed away too soon

Abstract. We prove that every matrix-valued rational function F, which is regular on the closure of a bounded domain $\mathcal{D}_{\mathbf{P}}$ in $\mathbb{C}^d$ and which has the associated Agler norm strictly less than 1, admits a finite-dimensional contractive realization

$$F(z) = D + C\mathbf{P}(z)_n(I - A\mathbf{P}(z)_n)^{-1}B.$$

Here $\mathcal{D}_{\mathbf{P}}$ is defined by the inequality $\|\mathbf{P}(z)\| < 1$, where $\mathbf{P}(z)$ is a direct sum of matrix polynomials $\mathbf{P}_i(z)$ (so that an appropriate Archimedean condition is satisfied), and $\mathbf{P}(z)_n = \bigoplus_{i=1}^k \mathbf{P}_i(z) \otimes I_{n_i}$, with some k-tuple n of multiplicities n_i; special cases include the open unit polydisk and the classical Cartan domains. The proof uses a matrix-valued version of a Hermitian Positivstellensatz by Putinar, and a lurking contraction argument. As a consequence, we show that every polynomial with no zeros on the closure of $\mathcal{D}_{\mathbf{P}}$ is a factor of $\det(I - K\mathbf{P}(z)_n)$, with a contractive matrix K.

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1. Introduction

It is well known (see [5, Proposition 11]) that every rational matrix function that is contractive on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ can be realized as

$$F(z) = D + zC(I - zA)^{-1}B, \tag{1.1}$$

with a contractive (in the spectral norm) colligation matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. In several variables, a celebrated result of Agler [1] gives the existence of a realization of the form

$$F(z) = D + CZ_{\mathcal{X}}(I - AZ_{\mathcal{X}})^{-1}B, \qquad Z_{\mathcal{X}} = \bigoplus_{i=1}^{d} z_i I_{\mathcal{X}_i}, \tag{1.2}$$

where $z = (z_1, \dots, z_d) \in \mathbb{D}^d$ and the colligation $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a Hilbert-space unitary operator (with A acting on the orthogonal direct sum of Hilbert spaces $\mathcal{X}_1, \dots, \mathcal{X}_d$), for F an operator-valued function analytic on the unit polydisk $\mathbb{D}^d$ whose Agler norm

$$\|F\|_{\mathcal{A}} = \sup_{T \in \mathcal{T}} \|F(T)\| \le 1.$$

Here $\mathcal{T}$ is the set of d-tuples $T = (T_1, \dots, T_d)$ of commuting strict contractions on a Hilbert space. Such functions constitute the Schur–Agler class.

Agler's result was generalized to polynomially defined domains in [3, 6]. Given a d-variable $\ell \times m$ matrix polynomial $\mathbf{P}$, let

$$\mathcal{D}_{\mathbf{P}} = \{z \in \mathbb{C}^d \colon \|\mathbf{P}(z)\| < 1\},$$

and let $\mathcal{T}_{\mathbf{P}}$ be the set of d-tuples T of commuting bounded operators on a Hilbert space satisfying $\|\mathbf{P}(T)\| < 1$. Important special cases are:

1. When $\ell = m = d$ and $\mathbf{P}(z) = \operatorname{diag}[z_1, \dots, z_d]$, the domain $\mathcal{D}_{\mathbf{P}}$ is the unit polydisk $\mathbb{D}^d$, and $\mathcal{T}_{\mathbf{P}} = \mathcal{T}$ is the set of d-tuples of commuting strict contractions.
2. When $d = \ell m$, $z = (z_{rs})$, $r = 1, \dots, \ell$, $s = 1, \dots, m$, $\mathbf{P}(z) = [z_{rs}]$, the domain $\mathcal{D}_{\mathbf{P}}$ is a matrix unit ball a.k.a. Cartan's domain of type I. In particular, if $\ell = 1$, then $\mathcal{D}_{\mathbf{P}} = \mathbb{B}^d = \{z \in \mathbb{C}^d \colon \sum_{i=1}^d |z_i|^2 < 1\}$ and $\mathcal{T}_{\mathbf{P}}$ consists of commuting strict row contractions $T = (T_1, \dots, T_d)$.
3. When $\ell = m$, $d = m(m+1)/2$, $z = (z_{rs})$, $1 \le r \le s \le m$, $\mathbf{P}(z) = [z_{rs}]$, where for $r > s$ we set $z_{rs} = z_{sr}$, and the domain $\mathcal{D}_{\mathbf{P}}$ is a (complex) symmetric matrix unit ball a.k.a. Cartan's domain of type II.
4. When $\ell = m$, $d = m(m-1)/2$, $z = (z_{rs})$, $1 \le r < s \le m$, $\mathbf{P}(z) = [z_{rs}]$, where for $r > s$ we set $z_{rs} = -z_{sr}$, and $z_{rr} = 0$ for all $r = 1, \dots, m$. The domain $\mathcal{D}_{\mathbf{P}}$ is a (complex) skew-symmetric matrix unit ball a.k.a. Cartan's domain of type III.

   We notice that Cartan domains of types IV-VI can also be represented as $\mathcal{D}_{\mathbf{P}}$, with a linear $\mathbf{P}$.

For $T \in \mathcal{T}_{\mathbf{P}}$, the Taylor joint spectrum $\sigma(T)$ [20] lies in $\mathcal{D}_{\mathbf{P}}$ (see [3, Lemma 1]), and therefore for an operator-valued function F analytic on $\mathcal{D}_{\mathbf{P}}$ one defines

$F(T)$ by means of Taylor's functional calculus [21] and

$$\|F\|_{\mathcal{A},\mathbf{P}} := \sup_{T \in \mathcal{T}_{\mathbf{P}}} \|F(T)\|.$$

We say that F belongs to the operator-valued Schur–Agler class associated with $\mathbf{P}$, denoted by $\mathcal{SA}_{\mathbf{P}}(\mathcal{U}, \mathcal{Y})$ if F is analytic on $\mathcal{D}_{\mathbf{P}}$, takes values in the space $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ of bounded linear operators from a Hilbert space $\mathcal{U}$ to a Hilbert space $\mathcal{Y}$, and $\|F\|_{\mathcal{A},\mathbf{P}} \leq 1$.

The generalization of Agler's theorem mentioned above that has appeared first in [3] for the scalar-valued case and extended in [6] to the operator-valued case, says that a function F belongs to the Schur–Agler class $\mathcal{SA}_{\mathbf{P}}(\mathcal{U}, \mathcal{Y})$ if and only if there exists a Hilbert space $\mathcal{X}$ and a unitary colligation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : (\mathbb{C}^m \otimes \mathcal{X}) \oplus \mathcal{U} \to (\mathbb{C}^\ell \otimes \mathcal{X}) \oplus \mathcal{Y}$$

such that

$$F(z) = D + C(\mathbf{P}(z) \otimes I_{\mathcal{X}})\Big(I - A(\mathbf{P}(z) \otimes I_{\mathcal{X}})\Big)^{-1} B. \tag{1.3}$$

If the Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$ are finite-dimensional, F can be treated as a matrix-valued function (relative to a pair of orthonormal bases for $\mathcal{U}$ and $\mathcal{Y}$). It is natural to ask whether every rational $\alpha \times \beta$ matrix-valued function in the Schur–Agler class $\mathcal{SA}_{\mathbf{P}}(\mathbb{C}^\beta, \mathbb{C}^\alpha)$ has a realization (1.3) with a contractive colligation matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. This question is open, unless when $d = 1$ or F is an inner (i.e., taking unitary boundary values a.e. on the unit torus $\mathbb{T}^d = \{z = (z_1, \ldots, z_d) \in \mathbb{C}^d \colon |z_i| = 1,\ i = 1, \ldots, d\}$) matrix-valued Schur–Agler function on $\mathbb{D}^d$. In the latter case, the colligation matrix can be chosen unitary; see [13] for the scalar-valued case, and [7, Theorem 2.1] for the matrix-valued generalization. We notice here that not every inner function is Schur–Agler; see [9, Example 5.1] for a counterexample.

In the present paper, we show that finite-dimensional contractive realizations of a rational matrix-valued function F exist when F is regular on the closed domain $\overline{\mathcal{D}_{\mathbf{P}}}$ and the Agler norm $\|F\|_{\mathcal{A},\mathbf{P}}$ is strictly less than 1 if $\mathbf{P} = \bigoplus_{i=1}^k \mathbf{P}_i$ and the matrix polynomials $\mathbf{P}_i$ satisfy a certain natural Archimedean condition. The proof has two ingredients: a matrix-valued version of a Hermitian Positivstellensatz [17] (see also [12, Corollary 4.4]), and a lurking contraction argument. For the first ingredient, we introduce the notion of a matrix system of Hermitian quadratic modules and the Archimedean property for them, and use the hereditary functional calculus for evaluations of a Hermitian symmetric matrix polynomial on d-tuples of commuting operators on a Hilbert space. For the second ingredient, we proceed similarly to the lurking isometry argument [1, 8, 3, 6], except that we are constructing a contractive matrix colligation instead of a unitary one.

We then apply this result to obtain a determinantal representation $\det(I - K\mathbf{P}_n)$, where K is a contractive matrix and $\mathbf{P}_n = \bigoplus_{i=1}^k (\mathbf{P}_i \otimes I_{n_i})$, with some k-tuple $n = (n_1, \ldots, n_k)$ of nonnegative integers[1], for a multiple of every polynomial

[1]We use the convention that if $n_i = 0$ then the corresponding direct summand for $\mathbf{P}_n$ is void.

which is strongly stable on $\mathcal{D}_{\mathbf{P}}$. (We recall that a polynomial is called stable with respect to a given domain if it has no zeros in the domain, and strongly stable if it has no zeros in the domain closure.) The question of existence of such a representation for a strongly stable polynomial (without multiplying it with an extra factor) on a general domain $\mathcal{D}_{\mathbf{P}}$ is open.

When $\mathcal{D}_{\mathbf{P}}$ is the open unit polydisk $\mathbb{D}^d$, the representation takes the form $\det(I - KZ_n)$, where $Z_n = \bigoplus_{i=1}^d z_i I_{n_i}$, $n = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$ (see our earlier work [9, 10]). In the cases of $\mathbb{D}$ and $\mathbb{D}^2$, a contractive determinantal representation of a given stable polynomial always exists; see [16, 10]. It also exists in the case of multivariable linear functions that are stable on $\mathbb{D}^d$, $d = 1, 2, \dots$ [9]. In addition, we showed recently in [11] that in the matrix poly-ball case (a direct sum of Cartan domains of type I) a strongly stable polynomial always has a strictly contractive realization.

The paper is organized as follows. In Section 2, we prove a matrix-valued version of a Hermitian Positivstellensatz. We then use it in Section 3 to establish the existence of contractive finite-dimensional realizations for rational matrix functions from the Schur–Agler class. In Section 4, we study contractive determinantal representations of strongly stable polynomials.

2. Positive matrix polynomials

In this section, we extend the result [12, Corollary 4.4] to matrix-valued polynomials. We will write $A \geq 0$ ($A > 0$) when a Hermitian matrix (or a self-adjoint operator on a Hilbert space) A is positive semidefinite (resp., positive definite). For a polynomial with complex matrix coefficients

$$P(w, z) = \sum_{\lambda,\, \mu} P_{\lambda\mu} w^\lambda z^\mu,$$

where $w = (w_1, \dots, w_d)$, $z = (z_1, \dots, z_d)$, and $w^\lambda = w_1^{\lambda_1} \cdots w_d^{\lambda_d}$, we define

$$P(T^*, T) := \sum_{\lambda,\, \mu} P_{\lambda\mu} \otimes T^{*\lambda} T^\mu,$$

where $T = (T_1, \dots, T_d)$ is a d-tuple of commuting operators on a Hilbert space. We will prove that P belongs to a certain Hermitian quadratic module determined by matrix polynomials $P_1, \dots, P_k$ in w and z when the inequalities $P_j(T^*, T) \geq 0$ imply that $P(T^*, T) > 0$.

We denote by $\mathbb{C}[z]$ the algebra of d-variable polynomials with complex coefficients, and by $\mathbb{C}^{\beta\times\gamma}[z]$ the module over $\mathbb{C}[z]$ of d-variable polynomials with the coefficients in $\mathbb{C}^{\beta\times\gamma}$. We denote by $\mathbb{C}^{\gamma\times\gamma}[w, z]_{\mathrm{h}}$ the vector space over $\mathbb{R}$ consisting of polynomials in w and z with coefficients in $\mathbb{C}^{\gamma\times\gamma}$ satisfying $P_{\lambda\mu} = P_{\mu\lambda}^*$, i.e., those whose matrix of coefficients is Hermitian. If we denote by $P^*(w, z)$ a polynomial in w and z with the coefficients $P_{\lambda\mu}$ replaced by their adjoints $P_{\lambda\mu}^*$, then the last property means that $P^*(w, z) = P(z, w)$.

We will say that $\mathcal{M} = \{\mathcal{M}_\gamma\}_{\gamma\in\mathbb{N}}$ is a matrix system of Hermitian quadratic modules over $\mathbb{C}[z]$ if the following conditions are satisfied:

1. For every $\gamma \in \mathbb{N}$, $\mathcal{M}_\gamma$ is an additive subsemigroup of $\mathbb{C}^{\gamma\times\gamma}[w,z]_{\mathrm{h}}$, i.e., $\mathcal{M}_\gamma + \mathcal{M}_\gamma \subseteq \mathcal{M}_\gamma$.
2. $1 \in \mathcal{M}_1$.
3. For every $\gamma,\gamma' \in \mathbb{N}$, $P \in \mathcal{M}_\gamma$, and $F \in \mathbb{C}^{\gamma\times\gamma'}[z]$, one has $F^*(w)P(w,z)F(z) \in \mathcal{M}_{\gamma'}$.

We notice that $\{\mathbb{C}^{\gamma\times\gamma}[w,z]_{\mathrm{h}}\}_{\gamma\in\mathbb{N}}$ is a trivial example of a matrix system of Hermitian quadratic modules over $\mathbb{C}[z]$.

Remark 2.1. We first observe that $A \in \mathcal{M}_\gamma$ if $A \in \mathbb{C}^{\gamma\times\gamma}$ is such that $A = A^* \geq 0$. Indeed, using (2) and letting $P = 1 \in \mathcal{M}_1$ and F be a constant row of size γ in (3), we obtain that $0_{\gamma\times\gamma} \in \mathcal{M}_\gamma$ and that every constant positive semidefinite $\gamma\times\gamma$ matrix of rank 1 belongs to $\mathcal{M}_\gamma$, and then use (1). In particular, we obtain that $I_\gamma \in \mathcal{M}_\gamma$. Together with (2) and (3) with $\gamma' = \gamma$, this means that $\mathcal{M}_\gamma$ is a Hermitian quadratic module (see, e.g., [19] for the terminology).

We also observe that, for each γ, $\mathcal{M}_\gamma$ is a cone, i.e., it is invariant under addition and multiplication with positive scalars.

Finally, we observe that $\mathcal{M}$ respects direct sums, i.e., $\mathcal{M}_\gamma \oplus \mathcal{M}_{\gamma'} \subseteq \mathcal{M}_{\gamma+\gamma'}$. In order to see this we first embed $\mathcal{M}_\gamma$ and $\mathcal{M}_{\gamma'}$ into $\mathcal{M}_{\gamma+\gamma'}$ by using (3) with $P \in \mathcal{M}_\gamma$, $F = [I_\gamma \ 0_{\gamma\times\gamma'}]$ and $P' \in \mathcal{M}_{\gamma'}$, $F' = [0_{\gamma'\times\gamma} \ I_{\gamma'}]$, and then use (1).

The following lemma generalizes [19, Lemma 6.3].

Lemma 2.2. *Let $\mathcal{M}$ be a matrix system of Hermitian quadratic modules over $\mathbb{C}[z]$. The following statements are equivalent:*

(i) *For every $\gamma \in \mathbb{N}$, I_γ is an algebraic interior point of $\mathcal{M}_\gamma$, i.e., $\mathbb{R}I_\gamma + \mathcal{M}_\gamma = \mathbb{C}^{\gamma\times\gamma}[w,z]_{\mathrm{h}}$.*

(ii) *1 is an algebraic interior point of $\mathcal{M}_1$, i.e., $\mathbb{R} + \mathcal{M}_1 = \mathbb{C}[w,z]_{\mathrm{h}}$.*

(iii) *For every $i = 1,\dots,d$, one has $-w_iz_i \in \mathbb{R} + \mathcal{M}_1$.*

A matrix system $\mathcal{M} = \{\mathcal{M}_\gamma\}_{\gamma\in\mathbb{N}}$ of Hermitian quadratic modules over $\mathbb{C}[z]$ that satisfies any (and hence all) of properties (i)–(iii) in Lemma 2.2 is called Archimedean.

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i). Let $\mathcal{A}_\gamma = \{F \in \mathbb{C}^{\gamma\times\gamma}[z]\colon -F^*(w)F(z) \in \mathbb{R}I_\gamma + \mathcal{M}_\gamma\}$. It suffices to prove that $\mathcal{A}_\gamma = \mathbb{C}^{\gamma\times\gamma}[z]$ for all $\gamma \in \mathbb{N}$. Indeed, any $P \in \mathbb{C}^{\gamma\times\gamma}[w,z]_{\mathrm{h}}$ can be written as

$$\begin{aligned} P(w,z) &= \sum_{\lambda,\mu} P_{\lambda\mu} w^\lambda z^\mu = \mathrm{row}_\lambda[w^\lambda I_\gamma][P_{\lambda\mu}]\,\mathrm{col}_\mu[z^\mu I_\gamma] \\ &= \mathrm{row}_\lambda[w^\lambda I_\gamma][A_\lambda^* A_\mu - B_\lambda^* B_\mu]\,\mathrm{col}_\mu[z^\mu I_\gamma] = A^*(w)A(z) - B^*(w)B(z), \end{aligned}$$

where
$$A(z) = \sum_\mu A_\mu z^\mu \in \mathbb{C}^{\gamma\times\gamma}[z], \quad B(z) = \sum_\mu B_\mu z^\mu \in \mathbb{C}^{\gamma\times\gamma}[z].$$
If $-B^*(w)B(z) \in \mathbb{R}I_\gamma + \mathcal{M}_\gamma$, then so is $P(w,z) = A^*(w)A(z) - B^*(w)B(z)$.

By the assumption, $z_i \in \mathcal{A}_1$ for all $i = 1,\dots,d$. We also have that $\mathbb{C}^{\gamma\times\gamma} \in \mathcal{A}_\gamma$ for every $\gamma \in \mathbb{N}$. Indeed, given $B \in \mathbb{C}^{\gamma\times\gamma}$, we have that $\|B\|^2 I_\gamma - B^*B \geq 0$. By Remark 2.1 we obtain that $\|B\|^2 I_\gamma - B^*B \in \mathcal{M}_\gamma$, therefore $-B^*B \in \mathbb{R}I_\gamma + \mathcal{M}_\gamma$. It follows that $\mathcal{A}_\gamma = \mathbb{C}^{\gamma\times\gamma}[z]$ for all $\gamma \in \mathbb{N}$ if $\mathcal{A}_1$ is a ring over $\mathbb{C}$ and $\mathcal{A}_\gamma$ is a module over $\mathbb{C}[z]$. We first observe from the identity
$$\begin{aligned}(F^*(w) + G^*(w))(F(z) + G(z)) + (F^*(w) - G^*(w))(F(z) - G(z))\\ = 2(F^*(w)F(z) + G^*(w)G(z))\end{aligned}$$
for $F, G \in \mathcal{A}_\gamma$ that
$$\begin{aligned}&-(F^*(w) + G^*(w))(F(z) + G(z))\\ &= -2(F^*(w)F(z) + G^*(w)G(z)) + (F^*(w) - G^*(w))(F(z) - G(z)) \in \mathbb{R}I_\gamma + \mathcal{M}_\gamma,\end{aligned}$$
hence $F + G \in \mathcal{A}_\gamma$. Next, for $F \in \mathcal{A}_\gamma$ and $g \in \mathcal{A}_1$ we can find positive scalars a and b such that $aI_\gamma - F^*(w)F(z) \in \mathcal{M}_\gamma$ and $b - g^*(w)g(z) \in \mathcal{M}_1$. Then we have
$$\begin{aligned}&abI_\gamma - (g^*(w)F^*(w))(g(z)F(z))\\ &\quad = b(aI_\gamma - F^*(w)F(z)) + F^*(w)\Big((b - g(w)^*g(z))I_\gamma\Big)F(z) \in \mathcal{M}_\gamma,\end{aligned}$$
Therefore $gF \in \mathcal{A}_\gamma$. Setting $\gamma = 1$, we first conclude that $\mathcal{A}_1$ is a ring over $\mathbb{C}$, thus $\mathcal{A}_1 = \mathbb{C}[z]$. Then, for an arbitrary $\gamma \in \mathbb{N}$, we conclude that $\mathcal{A}_\gamma$ is a module over $\mathbb{C}[z]$, thus $\mathcal{A}_\gamma = \mathbb{C}^{\gamma\times\gamma}[z]$. ☐

Starting with polynomials $P_j \in \mathbb{C}^{\gamma_j\times\gamma_j}[w,z]_{\mathrm{h}}$, we introduce the sets $\mathcal{M}_\gamma$, $\gamma \in \mathbb{N}$, consisting of polynomials $P \in \mathbb{C}^{\gamma\times\gamma}[w,z]_{\mathrm{h}}$ for which there exist $H_j \in \mathbb{C}^{\gamma_j n_j\times\gamma}[z]$, for some $n_j \in \mathbb{N}$, $j = 0,\dots,k$, such that
$$P(w,z) = H_0^*(w)H_0(z) + \sum_{j=1}^k H_j^*(w)(P_j(w,z) \otimes I_{n_j})H_j(z). \tag{2.1}$$
Here $\gamma_0 = 1$. We also assume that there exists a constant $c > 0$ such that $c - w_i z_i \in \mathcal{M}_1$ for every $i = 1,\dots,d$. Then $\mathcal{M} = \mathcal{M}_{P_1,\dots,P_k} = \{\mathcal{M}_\gamma\}_{\gamma\in\mathbb{N}}$ is an Archimedean matrix system of Hermitian quadratic modules generated by $P_1,\dots,P_k$. It follows from Lemma 2.2 that each $\mathcal{M}_\gamma$ is a convex cone in the real vector space $\mathbb{C}^{\gamma\times\gamma}[w,z]_{\mathrm{h}}$ and I_γ is an interior point in the finite topology (where a set is open if and only if its intersection with any finite-dimensional subspace is open; notice that a Hausdorff topology on a finite-dimensional topological vector space is unique).

We can now state the main result of this section.

Theorem 2.3. *Let $P_j \in \mathbb{C}^{\gamma_j\times\gamma_j}[w,z]$, $j = 1,\dots,k$. Suppose there exists $c > 0$ such that $c^2 - w_i z_i \in \mathcal{M}_1$, for all $i = 1,\dots,d$. Let $P \in \mathbb{C}^{\gamma\times\gamma}[w,z]$ be such that for every d-tuple $T = (T_1,\dots,T_d)$ of Hilbert space operators satisfying $P_j(T^*,T) \geq 0$, $j = 1,\dots,k$, we have that $P(T^*,T) > 0$. Then $P \in \mathcal{M}_\gamma$.*

Proof. Suppose that $P \notin \mathcal{M}_\gamma$. By Lemma 2.2, $I_\gamma \pm \epsilon P \in \mathcal{M}_\gamma$ for $\epsilon > 0$ small enough. By the Minkowski–Eidelheit–Kakutani separation theorem (see, e.g., [14, Section 17]), there exists a linear functional L on $\mathbb{C}^{\gamma\times\gamma}[w,z]_{\mathrm{h}}$ nonnegative on $\mathcal{M}_\gamma$ such that $L(P) \le 0 < L(I_\gamma)$. For $A \in \mathbb{C}^{1\times\gamma}[z]$ we define

$$\langle A, A\rangle = L(A^*(w)A(z)).$$

We extend the definition by polarization:

$$\langle A, B\rangle = \frac{1}{4}\sum_{r=0}^{3} i^r \langle A + i^r B, A + i^r B\rangle.$$

We obtain that $(\mathbb{C}^{1\times\gamma}[z], \langle\cdot,\cdot\rangle)$ is a pre-Hilbert space. Let $\mathcal{H}$ be the Hilbert space completion of the quotient space $\mathbb{C}^{1\times\gamma}[z]/\{A\colon \langle A, A\rangle = 0\}$. Note that $\mathcal{H}$ is non-trivial since $L(I_\gamma) > 0$.

Next we define multiplication operators M_{z_i}, $i = 1,\dots,d$, on $\mathcal{H}$. We define M_{z_i} first on the pre-Hilbert space via $M_{z_i}(A(z)) = z_i A(z)$. Suppose that $\langle A, A\rangle = 0$. Since $c^2 - w_i z_i \in \mathcal{M}_1$, it follows that $A^*(w)(c^2 - w_i z_i)A(z) \in \mathcal{M}_\gamma$. Since L is nonnegative on the cone $\mathcal{M}_\gamma$, we have

$$\begin{aligned}0 &\le L(A^*(w)(c^2 - w_i z_i)A(z))\\ &= c^2\langle A, A\rangle - \langle M_{z_i}(A), M_{z_i}(A)\rangle = -\langle M_{z_i}(A), M_{z_i}(A)\rangle.\end{aligned}$$

Thus, $\langle M_{z_i}(A), M_{z_i}(A)\rangle = 0$, yielding that M_{z_i} can be correctly defined on the quotient space. The same computation as above also shows that $\|M_{z_i}\| \le c$ on the quotient space, and then by continuity this is true on $\mathcal{H}$. Thus we obtain commuting bounded multiplication operators M_{z_i}, $i = 1,\dots,d$, on $\mathcal{H}$.

Next, let us show that

$$P_j(M^*, M) \ge 0, \quad j = 1,\dots,k,$$

where $M = (M_{z_1},\dots,M_{z_d})$. Let $h = [h_r]_{r=1}^{\gamma_j} \in \mathbb{C}^{\gamma_j}\otimes\mathcal{H}$, and moreover assume that h_r are elements of the quotient space $\mathbb{C}^{1\times\gamma}[z]/\{A\colon \langle A, A\rangle = 0\}$ (which is dense in $\mathcal{H}$). We will denote a representative of the coset h_r in $\mathbb{C}^{1\times\gamma}[z]$ by $h_r(z)$ with a hope that this will not cause a confusion. Let us compute $\langle P_j(M^*, M)h, h\rangle$. We have

$$P_j(w,z) = \sum_{\lambda,\,\mu} P_{\lambda\mu}^{(j)} w^\lambda z^\mu, \quad P_{\lambda\mu}^{(j)} = [P_{\lambda\mu}^{(j;r,s)}]_{r,s=1}^{\gamma_j}.$$

Then

$$P_j(M^*, M) = \sum_{\lambda,\,\mu} [P_{\lambda\mu}^{(j;r,s)} M^{*\lambda} M^\mu]_{r,s=1}^{\gamma_j}.$$

Now

$$\Big\langle P_j(M^*, M)h, h\Big\rangle = \sum_{r,\,s=1}^{\gamma_j} \Big\langle \sum_{\lambda,\,\mu} P_{\lambda\mu}^{(j;r,s)} M^{*\lambda} M^\mu h_r, h_s\Big\rangle$$

$$
\begin{aligned}
&= \sum_{r,\,s=1}^{\gamma_j} \sum_{\lambda,\,\mu} P_{\lambda\mu}^{(j;r,s)} \left\langle M^{\mu} h_r, M^{\lambda} h_s \right\rangle \\
&= \sum_{r,\,s=1}^{\gamma_j} \sum_{\lambda,\,\mu} P_{\lambda\mu}^{(j;r,s)} L\Big(h_s^*(w) w^{\lambda} z^{\mu} h_r(z) \Big) \\
&= L\Big(\sum_{r,s=1}^{\gamma_j} h_s^*(w) \Big(\sum_{\lambda,\,\mu} P_{\lambda\mu}^{(j;r,s)} w^{\lambda} z^{\mu} \Big) h_r(z) \Big) \\
&= L(h^*(w) P_j(w,z) h(z)),
\end{aligned}
$$

which is nonnegative since $h^*(w)P_j(w,z)h(z) \in \mathcal{M}_\gamma$.

By the assumption on P we now have that $P(M^*, M) > 0$. By a calculation similar to the one in the previous paragraph, we obtain that

$$L(h^*(w)P(w,z)h(z)) > 0 \quad \text{for all} \quad h \neq 0.$$

Choose now $h(z) \equiv I_\gamma \in \mathbb{C}^{\gamma\times\gamma}[z]$, and we obtain that $L(P) > 0$. This contradicts the choice of L. □

3. Finite-dimensional contractive realizations

In this section, we assume that $\mathbf{P}(z) = \bigoplus_{i=1}^{k} \mathbf{P}_i(z)$, where $\mathbf{P}_i$ are polynomials in $z = (z_1, \ldots, z_d)$ with $\ell_i \times m_i$ complex matrix coefficients, $i = 1, \ldots, k$. Then, clearly, $\mathcal{D}_{\mathbf{P}}$ is a cartesian product of the domains $\mathcal{D}_{\mathbf{P}_i}$. Next, we assume that every d-tuple T of commuting bounded linear operators on a Hilbert space, satisfying $\|\mathbf{P}(T)\| \leq 1$ is a norm limit of elements of $\mathcal{T}_{\mathbf{P}}$. We also assume that the polynomials $P_i(w,z) = I_{m_i} - \mathbf{P}_i^*(w)\mathbf{P}_i(z)$, $i = 1, \ldots, k$, generate an Archimedean matrix system of Hermitian quadratic modules over $\mathbb{C}[z]$. This in particular means that the domain $\mathcal{D}_{\mathbf{P}}$ is bounded, because for some $c > 0$ we have $c^2 - w_i z_i \in \mathcal{M}_1$ which implies that $c^2 - |z_i| \geq 0$, $i = 1, \ldots, d$, when $z \in \mathcal{D}_{\mathbf{P}}$. We notice that in the special cases (1)–(4) in Section 1, the Archimedean condition holds.

We recall that a polynomial convex hull of a compact set $K \subseteq \mathbb{C}^d$ is defined as the set of all points $z \in \mathbb{C}^d$ such that $|p(z)| \leq \max_{w\in K} |p(w)|$ for every polynomial $p \in \mathbb{C}[z]$. A set in $\mathbb{C}^d$ is called polynomially convex if it agrees with its polynomial convex hull.

Lemma 3.1. *$\overline{\mathcal{D}_{\mathbf{P}}}$ is polynomially convex.*

Proof. We first observe that $\overline{\mathcal{D}_{\mathbf{P}}}$ is closed and bounded, hence compact. Next, if $z \in \mathbb{C}^d$ is in the polynomial convex hull of $\overline{\mathcal{D}_{\mathbf{P}}}$, then for all unit vectors $g \in \mathbb{C}^\ell$ and $h \in \mathbb{C}^m$, one has

$$|g^*\mathbf{P}(z)h| \leq \max_{w\in\overline{\mathcal{D}_{\mathbf{P}}}} |g^*\mathbf{P}(w)h| \leq \max_{w\in\overline{\mathcal{D}_{\mathbf{P}}}} \|\mathbf{P}(w)\| \leq 1.$$

Then
$$\|\mathbf{P}(z)\| = \max_{\|g\|=\|h\|=1} |g^*\mathbf{P}(z)h| \leq 1,$$
therefore $z \in \overline{\mathcal{D}_{\mathbf{P}}}$. $\square$

Lemma 3.2. *There exists a d-tuple $T_{\max}$ of commuting bounded linear operators on a separable Hilbert space satisfying $\|\mathbf{P}(T_{\max})\| \leq 1$ and such that*
$$\|q(T_{\max})\| = \|q\|_{\mathcal{A},\mathbf{P}}$$
for every polynomial $q \in \mathbb{C}[z]$.

Proof. The proof is exactly the same as the one suggested in [18, Page 65 and Exercise 5.6] for the special case of commuting contractions and the Agler norm $\|\cdot\|_{\mathcal{A}}$ associated with the unit polydisk, see the first paragraph of Section 1. Notice that the boundedness of $\mathcal{D}_{\mathbf{P}}$ guarantees that $\|q\|_{\mathcal{A},\mathbf{P}} < \infty$ for every polynomial q. $\square$

Lemma 3.3. *Let F be an $\alpha \times \beta$ matrix-valued function analytic on $\overline{\mathcal{D}_{\mathbf{P}}}$. Then $\|F\|_{\mathcal{A},\mathbf{P}} < \infty$.*

Proof. Since F is analytic on some open neighborhood of the set $\overline{\mathcal{D}_{\mathbf{P}}}$ which by Lemma 3.1 is polynomially convex, by the Oka–Weil theorem (see, e.g., [2, Theorem 7.3]) for each scalar-valued function F_{ij} there exists a sequence of polynomials $Q_{ij}^{(n)} \in \mathbb{C}^{\alpha\times\beta}[z]$, $n \in \mathbb{N}$, which converges to F_{ij} uniformly on $\overline{\mathcal{D}_{\mathbf{P}}}$. Therefore the sequence of matrix polynomials $Q^{(n)} = [Q_{ij}^{(n)}]$, $n \in \mathbb{N}$, converges to F uniformly on $\overline{\mathcal{D}_{\mathbf{P}}}$. Let T be any d-tuple of commuting bounded linear operators on a Hilbert space with the Taylor joint spectrum in $\overline{\mathcal{D}_{\mathbf{P}}}$. By [3, Lemma 1], the Taylor joint spectrum of T lies in the closed domain $\overline{\mathcal{D}_{\mathbf{P}}}$ where F is analytic. By the continuity of Taylor's functional calculus [21], we have that
$$F(T) = \lim_n Q^{(n)}(T).$$
Using Lemma 3.2, we obtain that the limit
$$\lim_n \|Q^{(n)}\|_{\mathcal{A},\mathbf{P}} = \lim_n \|Q^{(n)}(T_{\max})\| = \|F(T_{\max})\|$$
exists and
$$\begin{aligned}\|F\|_{\mathcal{A},\mathbf{P}} &= \sup_{T\in\mathcal{T}_{\mathbf{P}}} \|F(T)\| = \sup_{T\in\mathcal{T}_{\mathbf{P}}} \lim_n \|Q^{(n)}(T)\| \\ &\leq \lim_n \|Q^{(n)}(T_{\max})\| = \|F(T_{\max})\| < \infty.\end{aligned}$$
$\square$

Theorem 3.4. *Let F be a rational $\alpha \times \beta$ matrix function regular on $\overline{\mathcal{D}_{\mathbf{P}}}$ and with $\|F\|_{\mathcal{A},\mathbf{P}} < 1$. Then there exists $n = (n_1, \ldots, n_k) \in \mathbb{Z}_+^k$ and a contractive colligation matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ of size $(\sum_{i=1}^k n_i m_i + \alpha) \times (\sum_{i=1}^k n_i \ell_i + \beta)$ such that*
$$F(z) = D + C\mathbf{P}(z)_n(I - A\mathbf{P}(z)_n)^{-1}B, \qquad \mathbf{P}(z)_n = \bigoplus_{i=1}^k (\mathbf{P}_i(z) \otimes I_{n_i}).$$

Proof. Let $F = QR^{-1}$ with $\det R$ nonzero on $\overline{\mathcal{D}_{\mathbf{P}}}$ and let $\|F\|_{\mathcal{A},\mathbf{P}} < 1$. Then we have

$$R(T)^*R(T) - Q(T)^*Q(T) \geq (1 - \|F\|^2_{\mathcal{A},\mathbf{P}})R(T)^*R(T) \geq \epsilon^2 I \tag{3.1}$$

for every $T \in \mathcal{T}_{\mathbf{P}}$ with some $\epsilon > 0$. Indeed, the rational matrix function R^{-1} is regular on $\overline{\mathcal{D}_{\mathbf{P}}}$. By Lemma 3.3 $\|R^{-1}\|_{\mathcal{A},\mathbf{P}} < \infty$. Since $\|R(T)\|\|R^{-1}(T)\| \geq 1$, we obtain

$$\|R(T)\| \geq \|R^{-1}(T)\|^{-1} \geq \|R^{-1}\|^{-1}_{\mathcal{A},\mathbf{P}} > 0,$$

which yields the estimate (3.1).

By Theorem 2.3 there exist $n_0, \dots, n_k \in \mathbb{Z}_+$ and polynomials H_i with coefficients in $\mathbb{C}^{n_i m_i \times \beta}$, $i = 0, \dots, k$, (where we set $m_0 = 1$) such that by (2.1) we obtain

$$\begin{aligned} &R^*(w)R(z) - Q^*(w)Q(z) \\ &\quad = H_0^*(w)H_0(z) + \sum_{i=1}^k H_i^*(w)\Big((I - \mathbf{P}_i^*(w)\mathbf{P}_i(z)) \otimes I_{n_i}\Big)H_i(z). \end{aligned} \tag{3.2}$$

Denote

$$v(z) = \begin{bmatrix} (\mathbf{P}_1(z) \otimes I_{n_1})H_1(z) \\ \vdots \\ (\mathbf{P}_k(z) \otimes I_{n_k})H_k(z) \\ R(z) \end{bmatrix} \in \mathbb{C}^{(\sum_{i=1}^k \ell_i n_i + \beta) \times \beta}[z],$$

$$x(z) = \begin{bmatrix} H_1(z) \\ \vdots \\ H_k(z) \\ Q(z) \end{bmatrix} \in \mathbb{C}^{(\sum_{i=1}^k m_i n_i + \alpha) \times \beta}[z].$$

Then we may rewrite (3.2) as

$$v^*(w)v(z) = H_0^*(w)H_0(z) + x^*(w)x(z). \tag{3.3}$$

Let us define

$$\mathcal{V} = \operatorname{span}\{v(z)y \colon z \in \mathbb{C}^d, y \in \mathbb{C}^\beta\}, \quad \mathcal{X} = \operatorname{span}\{x(z)y \colon z \in \mathbb{C}^d, y \in \mathbb{C}^\beta\},$$

and let $\{v(z^{(1)})y^{(1)}, \dots, v(z^{(\nu)})y^{(\nu)}\}$ be a basis for $\mathcal{V} \subseteq \mathbb{C}^{\sum_{i=1}^k \ell_i n_i + \beta}$.

Claim 1. If $v(z)y = \sum_{i=1}^\nu a_i v(z^{(i)})y^{(i)}$, then

$$x(z)y = \sum_{i=1}^\nu a_i x(z^{(i)})y^{(i)}.$$

Indeed, this follows from

$$0 = \begin{bmatrix} y \\ -a_1 y^{(1)} \\ \vdots \\ -a_\nu y^{(\nu)} \end{bmatrix}^* \begin{bmatrix} v(z)^* \\ v(z^{(1)})^* \\ \vdots \\ v(z^{(\nu)})^* \end{bmatrix} \begin{bmatrix} v(z) & v(z^{(1)}) & \dots & v(z^{(\nu)}) \end{bmatrix} \begin{bmatrix} y \\ -a_1 y^{(1)} \\ \vdots \\ -a_\nu y^{(\nu)} \end{bmatrix}$$

$$= \begin{bmatrix} y \\ -a_1 y^{(1)} \\ \vdots \\ -a_\nu y^{(\nu)} \end{bmatrix}^* \begin{bmatrix} H_0(z)^* & x(z)^* \\ H_0(z^{(1)})^* & x(z^{(1)})^* \\ \vdots & \vdots \\ H_0(z^{(\nu)})^* & x(z^{(\nu)})^* \end{bmatrix}$$

$$\times \begin{bmatrix} H_0(z) & H_0(z^{(1)}) & \cdots & H_0(z^{(\nu)}) \\ x(z) & x(z^{(1)}) & \dots & x(z^{(\nu)}) \end{bmatrix} \begin{bmatrix} y \\ -a_1 y^{(1)} \\ \vdots \\ -a_\nu y^{(\nu)} \end{bmatrix},$$

where we used (3.3). This yields

$$\begin{bmatrix} H_0(z) & H_0(z^{(1)}) & \cdots & H_0(z^{(\nu)}) \\ x(z) & x(z^{(1)}) & \dots & x(z^{(\nu)}) \end{bmatrix} \begin{bmatrix} y \\ -a_1 y^{(1)} \\ \vdots \\ -a_\nu y^{(\nu)} \end{bmatrix} = 0,$$

and thus in particular $x(z)y = \sum_{i=1}^{\nu} a_i x(z^{(i)})y^{(i)}$.

We now define $S\colon \mathcal{V} \to \mathcal{X}$ via $Sv(z^{(i)})y^{(i)} = x(z^{(i)})y^{(i)}, i = 1, \dots, \nu$. By Claim 1,

$$Sv(z)y = x(z)y \text{ for all } z \in \bigoplus_{j=1}^{k} \mathbb{C}^{\ell_j \times m_j} \text{ and } y \in \mathbb{C}^{\beta}. \tag{3.4}$$

Claim 2. S is a contraction. Indeed, let $v = \sum_{i=1}^{\nu} a_i v(z^{(i)})y^{(i)} \in \mathcal{V}$. Then $Sv = \sum_{i=1}^{\nu} a_i x(z^{(i)})y^{(i)}$, and we compute, using (3.3) in the second equality,

$$\begin{aligned} \|v\|^2 - \|Sv\|^2 &= \begin{bmatrix} a_1 y^{(1)} \\ \vdots \\ a_\nu y^{(\nu)} \end{bmatrix}^* \begin{bmatrix} v(z^{(1)})^* \\ \vdots \\ v(z^{(\nu)})^* \end{bmatrix} \begin{bmatrix} v(z^{(1)}) & \dots & v(z^{(\nu)}) \end{bmatrix} \begin{bmatrix} a_1 y^{(1)} \\ \vdots \\ a_\nu y^{(\nu)} \end{bmatrix} \\ &\quad - \begin{bmatrix} a_1 y^{(1)} \\ \vdots \\ a_\nu y^{(\nu)} \end{bmatrix}^* \begin{bmatrix} x(z^{(1)})^* \\ \vdots \\ x(z^{(\nu)})^* \end{bmatrix} \begin{bmatrix} x(z) & x(z^{(1)}) & \dots & x(z^{(\nu)}) \end{bmatrix} \begin{bmatrix} a_1 y^{(1)} \\ \vdots \\ a_\nu y^{(\nu)} \end{bmatrix} \\ &= \begin{bmatrix} a_1 y^{(1)} \\ \vdots \\ a_\nu y^{(\nu)} \end{bmatrix}^* \begin{bmatrix} H_0(z^{(1)})^* \\ \vdots \\ H_0(z^{(\nu)})^* \end{bmatrix} \begin{bmatrix} H_0(z^{(1)}) & \cdots & H_0(z^{(\nu)}) \end{bmatrix} \begin{bmatrix} a_1 y^{(1)} \\ \vdots \\ a_\nu y^{(\nu)} \end{bmatrix} \geq 0, \end{aligned}$$

proving Claim 2.

Extending S to the contraction $S_{ext} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \mathbb{C}^{\sum_{i=1}^{k} \ell_i n_i + \beta} \to \mathbb{C}^{\sum_{i=1}^{k} m_i n_i + \alpha}$ by setting $S_{ext}|_{\mathcal{V}^\perp} = 0$, we now obtain from (3.4) that

$$A\mathbf{P}(z)_n H(z) + BR(z) = H(z), \quad C\mathbf{P}(z)_n H(z) + DR(z) = Q(z).$$

Eliminating $H(z)$, we arrive at

$$(D + C\mathbf{P}(z)_n(I - A\mathbf{P}(z)_n)^{-1}B)R(z) = Q(z),$$

yielding the desired realization for $F = QR^{-1}$. □

The following statement is a special case of Theorem 3.4.

Corollary 3.5. *Let F be a rational matrix function regular on the closed bidisk $\overline{\mathbb{D}^2}$ such that*

$$\|F\|_\infty = \sup_{(z_1,z_2)\in\mathbb{D}^2} \|F(z_1, z_2)\| < 1.$$

Then F has a finite-dimensional contractive realization (1.2)*, that is, there exist $n_1, n_2 \in \mathbb{Z}_+$ such that $\mathcal{X}_i = \mathbb{C}^{n_i}$, $i = 1, 2$, and $Z_\mathcal{X} = z_1 I_{n_1} \oplus z_2 I_{n_2} = Z_n$.*

Proof. One can apply Theorem 3.4 after observing that on the bidisk the Agler norm and the supremum norm coincide, a result that goes back to [4]. □

4. Contractive determinantal representations

Let a polynomial $\mathbf{P} = \bigoplus_{i=1}^k \mathbf{P}_i$ and a domain $\mathcal{D}_\mathbf{P}$ be as in Section 3. We apply Theorem 3.4 to obtain a contractive determinantal representation for a multiple of every polynomial strongly stable on $\mathcal{D}_\mathbf{P}$. Please notice the analogy with the main result in [15], where a similar result is obtained in the setting of definite determinantal representation for hyperbolic polynomials.

Theorem 4.1. *Let p be a polynomial in d variables $z = (z_1, \ldots, z_d)$, which is strongly stable on $\mathcal{D}_\mathbf{P}$. Then there exists a polynomial q, nonnegative integers $n_1, \ldots, n_k$, and a contractive matrix K of size $\sum_{i=1}^k m_i n_i \times \sum_{i=1}^k \ell_i n_i$ such that*

$$p(z)q(z) = \det(I - K\mathbf{P}(z)_n), \qquad \mathbf{P}(z)_n = \bigoplus_{i=1}^k (\mathbf{P}_i(z) \otimes I_{n_i}).$$

Proof. Since p has no zeros in $\overline{\mathcal{D}_\mathbf{P}}$, the rational function $g = 1/p$ is regular on $\overline{\mathcal{D}_\mathbf{P}}$. By Lemma 3.3, $\|g\|_{\mathcal{A},\mathbf{P}} < \infty$. Thus we can find a constant $c > 0$ so that $\|cg\|_{\mathcal{A},\mathbf{P}} < 1$. Applying now Theorem 3.4 to $F = cg$, we obtain a k-tuple $n = (n_1, \ldots, n_k) \in \mathbb{Z}_+^k$ and a contractive colligation matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ so that

$$\begin{aligned} cg(z) = \frac{c}{p(z)} &= D + C\mathbf{P}(z)_n(I - A\mathbf{P}(z)_n)^{-1}B \\ &= \frac{\det \begin{bmatrix} I - A\mathbf{P}(z)_n & B \\ -C\mathbf{P}(z)_n & D \end{bmatrix}}{\det(I - A\mathbf{P}(z)_n)}. \end{aligned} \tag{4.1}$$

This shows that

$$\frac{\det(I - A\mathbf{P}(z)_n)}{p(z)}$$

is a polynomial. Let $K = A$. Then K is a contraction, and

$$q(z) = \frac{\det(I - K\mathbf{P}(z)_n)}{p(z)}$$

is a polynomial. □

Remark 4.2. Since the polynomial $\det(I - K\mathbf{P}(z)_n)$ in Theorem 4.1 is stable on $\mathcal{D}_\mathbf{P}$, so is q.

References

[1] J. Agler. On the representation of certain holomorphic functions defined on a polydisc, In Topics in operator theory: Ernst D. Hellinger Memorial Volume, *Oper. Theory Adv. Appl.*, Vol. **48**, pp. 47–66, Birkhäuser, Basel, 1990.

[2] H. Alexander and J. Wermer. *Several complex variables and Banach algebras.* Third edition. Graduate Texts in Mathematics, 35. Springer-Verlag, New York, 1998.

[3] C.-G. Ambrozie and D.A. Timotin. Von Neumann type inequality for certain domains in $\mathbb{C}^n$. *Proc. Amer. Math. Soc.* 131 (2003), no. 3, 859–869 (electronic).

[4] T. Ando. On a pair of commutative contractions. *Acta Sci. Math.* (*Szeged*) 24 (1963), 88–90.

[5] D.Z. Arov. Passive linear steady-state dynamical systems. (Russian) *Sibirsk. Mat. Zh.* 20 (1979), no. 2, 211–228, 457.

[6] J.A. Ball and V. Bolotnikov. Realization and interpolation for Schur–Agler-class functions on domains with matrix polynomial defining function in $\mathbb{C}^n$. *J. Funct. Anal.* 213 (2004), no. 1, 45–87.

[7] J.A. Ball and D.S. Kaliuzhnyi-Verbovetskyi. Rational Cayley inner Herglotz–Agler functions: Positive-kernel decompositions and transfer-function realizations. *Linear Algebra Appl.* 456 (2014), 138–156.

[8] J.A. Ball and T.T. Trent. Unitary colligations, reproducing kernel Hilbert spaces, and Nevanlinna–Pick interpolation in several variables. *J. Funct. Anal.* 157 (1998), no. 1, 1–61.

[9] A. Grinshpan, D.S. Kaliuzhnyi-Verbovetskyi, and H.J. Woerdeman. Norm-constrained determinantal representations of multivariable polynomials. *Complex Anal. Oper. Theory* 7 (2013), 635–654.

[10] A. Grinshpan, D.S. Kaliuzhnyi-Verbovetskyi, V. Vinnikov, and H. J. Woerdeman. Stable and real-zero polynomials in two variables. *Multidim. Syst. Sign. Process.* 27 (2016), 1–26.

[11] A. Grinshpan, D.S. Kaliuzhnyi-Verbovetskyi, V. Vinnikov, and H. J. Woerdeman. Contractive determinantal representations of stable polynomials on a matrix polyball. *Math. Z.* 283 (2016), 25–37.

[12] J.W. Helton and M. Putinar. Positive polynomials in scalar and matrix variables, the spectral theorem, and optimization. In *Operator theory, structured matrices, and dilations*, 229–306, Theta Ser. Adv. Math., 7, Theta, Bucharest, 2007.

[13] G. Knese. Rational inner functions in the Schur–Agler class of the polydisk. *Publ. Mat.*, 55 (2011), 343–357.

[14] G. Köthe. Topologische lineare Räume. I. (German) Zweite verbesserte Auflage. Die Grundlehren der Mathematischen Wissenschaften, Band 107 Springer-Verlag, Berlin-New York 1966.

[15] M. Kummer, Determinantal Representations and the Bézout Matrix, arXiv:1308.5560.

[16] A. Kummert. Synthesis of two-dimmensional lossless m-ports with prescribed scattering matrix. *Circuits Systems Signal Processing* 8 (1989), no. 1, 97–119.

[17] M. Putinar. On Hermitian polynomial optimization. *Arch. Math.* (*Basel*) 87 (2006), no. 1, 41–51.

[18] V. Paulsen. *Completely bounded maps and operator algebras.* Cambridge Studies in Advanced Mathematics, 78. Cambridge University Press, Cambridge, 2002.

[19] M. Putinar and C. Scheiderer. Quillen property of real algebraic varieties. *Münster J. of Math.* 7 (2014), 671–696.

[20] J.L. Taylor. A joint spectrum for several commuting operators. *J. Functional Analysis* 6 (1970), 172–191.

[21] J.L. Taylor. The analytic-functional calculus for several commuting operators. *Acta Math.* 125 (1970), 1–38.

Anatolii Grinshpan, Dmitry S. Kaliuzhnyi-Verbovetskyi
and Hugo J. Woerdeman
Department of Mathematics
Drexel University
3141 Chestnut St.
Philadelphia, PA, 19104
e-mail: tolya@math.drexel.edu
dmitryk@math.drexel.edu
hugo@math.drexel.edu

Victor Vinnikov
Department of Mathematics
Ben-Gurion University of the Negev
Beer-Sheva, Israel, 84105
e-mail: vinnikov@math.bgu.ac.il

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Form Inequalities for Symmetric Contraction Semigroups

Markus Haase

Abstract. Consider – for the generator $-A$ of a symmetric contraction semigroup over some measure space X, $1 \le p < \infty$, q the dual exponent and given measurable functions F_j, $G_j : \mathbb{C}^d \to \mathbb{C}$ – the statement:

$$\operatorname{Re} \sum_{j=1}^{m} \int_{\mathrm{X}} AF_j(\mathbf{f}) \cdot G_j(\mathbf{f}) \;\ge\; 0$$

for all $\mathbb{C}^d$*-valued measurable functions* $\mathbf{f}$ *on* X *such that* $F_j(\mathbf{f}) \in \operatorname{dom}(A_p)$ *and* $G_j(\mathbf{f}) \in \mathrm{L}^q(\mathrm{X})$ *for all* j.

It is shown that this statement is valid in general if it is valid for X being a two-point Bernoulli $(\frac{1}{2}, \frac{1}{2})$-space and A being of a special form. As a consequence we obtain a new proof for the optimal angle of L^p-analyticity for such semigroups, which is essentially the same as in the well-known sub-Markovian case.

The proof of the main theorem is a combination of well-known reduction techniques and some representation results about operators on $\mathrm{C}(K)$-spaces. One focus of the paper lies on presenting these auxiliary techniques and results in great detail.

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1. Introduction

In the recent preprint [2], A. Carbonaro and O. Dragičević consider symmetric contraction semigroups $(S_t)_{t\ge0}$ over some measure space $\mathrm{X} = (X, \Sigma, \mu)$ and prove so-called spectral multiplier results (= functional calculus estimates) for A_p, where $-A_p$ is the generator of $(S_t)_{t\ge0}$ on $\mathrm{L}^p(\mathrm{X})$, $1 \le p < \infty$.

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Their proof consists of three major steps. In the first one, the authors show how to generate functional calculus estimates for the operator $A = A_p$ from *form inequalities* of the type

$$\sum_{j=1}^{m} \mathrm{Re} \int_X [AF_j(f_1, \dots, f_d)] \cdot G_j(f_1, \dots, f_d)\, \mathrm{d}\mu \geq 0, \tag{1.1}$$

where F_j and G_j are measurable functions $\mathbb{C}^d \to \mathbb{C}$ with certain properties and $(f_1, \dots, f_d)$ varies over a suitable subset of measurable functions on X. This first step is based on the so-called *heat-flow method.* In the second step, the authors show how to find functions F_j and G_j with the desired properties by employing a so-called *Bellman function.* Their third step consists in establishing the inequality (1.1) by reducing the problem to the case that $A = \mathrm{I} - E_\lambda$ on $\mathbb{C}^2$, where

$$E_\lambda = \begin{pmatrix} 0 & \overline{\lambda} \\ \lambda & 0 \end{pmatrix}, \qquad (\lambda \in \mathbb{T}).$$

The underlying reduction procedure is actually well known in the literature, but has been used mainly for symmetric *sub-Markovian* semigroups, i.e., under the additional assumption that all $S_t \geq 0$. Here, the last step becomes considerably simpler, since then one need only consider the cases $A = \mathrm{I} - E_1$ and $A = \mathrm{I}$.

One intention with the present paper is to look more carefully at the employed reduction techniques (Section 3) and prove a general theorem (Theorem 2.2) that puts the above-mentioned "third step" on a formal basis. Where the authors of [2] confine their arguments to their specific case of Bellman functions, here we treat general functions F_j and G_j and hence pave the way for further applications.

It turns out that the heart of the matter are results about representing bilinear forms $(f, g) \mapsto \int_L Tf \cdot g\, \mathrm{d}\mu$ as integrals over product spaces like

$$\int_L Tf \cdot g\, \mathrm{d}\mu = \int_{K\times L} f(x)g(y)\, \mathrm{d}\mu_T(x, y).$$

(Here, K and L are compact spaces, μ is a positive regular Borel measure on L and $T : \mathrm{C}(K) \to \mathrm{L}^1(L, \mu)$ is a linear operator.) These results go back to Grothendieck's work on tensor products and "integral" bilinear forms [9]. They are "well known" in the sense that they could – on a careful reading – be obtained from standard texts on tensor products and Banach lattices, such as [21, Chap. IV]. However, it seems that the communities of those people who are familiar with these facts in their abstract form and those who would like to apply them to more concrete situations are almost disjoint. Our exposition, forming the contents of Section 4, can thus be viewed as an attempt to increase the intersection of these two communities.

After this excursion into abstract operator theory, in Section 5 we turn back to the proof of Theorem 2.2. Then, as an application, we consider the question about the optimal *angle of analyticity* on L^p of a symmetric contraction semigroup $(S_t)_{t\geq 0}$. For the sub-Markovian case this question has been answered long ago, in fact, by the very methods which we just mentioned and which form the core content of this paper. The general symmetric case has only recently been settled by Kriegler in [16]. Kriegler's proof rests on arguments from non-commutative operator theory, but Carbonaro and Dragičević show in [2] that the result can also be derived as a corollary from their results involving Bellman functions. We shall point out in Section 6 below that the Bellman function of Carbonaro and Dragičević is not really needed here, and that one can prove the general case by essentially the same arguments as used in the sub-Markovian case.

Terminology and Notation. In this paper, $\mathrm{X} := (X, \Sigma, \mu)$ denotes a general measure space. (Sometimes we shall suppose in addition that μ is a finite measure, but we shall always make this explicit.) Integration with respect to μ is abbreviated by

$$\int_{\mathrm{X}} f := \int_X f \,\mathrm{d}\mu$$

whenever it is convenient. The corresponding L^p-space for $0 < p \le \infty$ is denoted by $\mathrm{L}^p(\mathrm{X})$, but if the underlying measure space is understood, we shall simply write L^p. Whenever $1 \le p \le \infty$ is fixed we denote by q the *dual exponent*, i.e., the unique number $q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

With the symbol $\mathcal{M}(\mathrm{X}; \mathbb{C}^d)$ ($\mathcal{M}(\mathrm{X})$ in the case d=1) we denote the space of $\mathbb{C}^d$-valued measurable functions on X, modulo equality almost everywhere. We shall tacitly identify $\mathcal{M}(\mathrm{X}; \mathbb{C}^d)$ with $\mathcal{M}(\mathrm{X})^d$ and use the notation

$$\mathbf{f} = (f_1, \ldots, f_d)$$

to denote functions into $\mathbb{C}^d$.

For a set $M \subseteq \mathbb{C}^d$ and $\mathbf{f} = (f_1, \ldots, f_d) \in \mathcal{M}(\mathrm{X}; \mathbb{C}^d)$ as above, we write "$(f_1, \ldots, f_d) \in M$ almost everywhere" shorthand for: "$(f_1(x), \ldots, f_d(x)) \in M$ for μ-almost all $x \in X$." By abuse of notation, if $F : \mathbb{C}^d \to \mathbb{C}$ is measurable and $\mathbf{f} \in \mathcal{M}(\mathrm{X}; \mathbb{C}^d)$ we write $F(\mathbf{f})$ to denote the function $F \circ \mathbf{f}$, i.e., $F(\mathbf{f})(x) = F(f_1(x), \ldots, f_d(x))$.

The letters $K, L, \ldots$ usually denote compact and sometimes locally compact Hausdorff spaces. We abbreviate this by simply saying that K, $L, \ldots$ are (locally) compact. If K is locally compact, then $\mathrm{C_c}(K)$ denotes the space of continuous functions on K with compact support, and $\mathrm{C}_0(K)$ is the sup-norm closure of $\mathrm{C_c}(K)$ within the Banach space of all bounded continuous functions. If K is compact, then of course $\mathrm{C_c}(K) = \mathrm{C}_0(K) = \mathrm{C}(K)$.

If K is (locally) compact then, by the Riesz representation theorem, the dual space of $\mathrm{C}(K)$ ($\mathrm{C}_0(K)$) is isometrically and lattice isomorphic to $\mathrm{M}(K)$, the space of complex regular Borel measures on K, with the total variation (norm) as absolute value (norm). A (*locally*) *compact measure space* is a pair (K, ν) where K is (locally) compact and ν is a positive regular Borel measure on K. (If K is locally compact, the measure ν need not be finite.)

We work with complex Banach spaces by default. In particular, L^p-spaces have to be understood as consisting of complex-valued functions. For an operator T with domain and range being spaces of complex-valued functions, the *conjugate operator* is defined by $\overline{T}f := \overline{T\overline{f}}$, and the *real part* and *imaginary part* are defined by

$$\operatorname{Re} T := \tfrac{1}{2}(T + \overline{T}) \quad \text{and} \quad \operatorname{Im} T := \tfrac{1}{2\mathrm{i}}(T - \overline{T}),$$

respectively. For Banach spaces E and F we use the symbol $\mathcal{L}(E;F)$ to denote the space of bounded linear operators from E to F and $E' = \mathcal{L}(E;\mathbb{C})$ for the dual space. The dual of an operator $T \in \mathcal{L}(E;F)$ is denoted by $T' \in \mathcal{L}(F',E')$.

If K is locally compact, $\mathrm{X} = (X, \Sigma, \mu)$ is a measure space and $T : \mathrm{C_c}(K) \to \mathrm{L}^1(\mathrm{X})$ is a linear operator, then $T'\mu$ denotes the linear functional on $\mathrm{C_c}(K)$ defined by

$$\langle f\,, T'\mu \rangle := \int_X Tf \,\mathrm{d}\mu \qquad (f \in \mathrm{C_c}(K)).$$

If T is bounded for the uniform norm on $\mathrm{C_c}(K)$ then $T'\mu$ is bounded too, and we identify it with a complex regular Borel measure in $\mathrm{M}(K)$. If T is not bounded but positive, then, again by the Riesz representation theorem, $T'\mu$ can be identified with a positive (but infinite) regular Borel measure on K.

At some places we use some basic notions of Banach lattice theory (e.g., lattice homomorphism, ideal, order completeness). The reader unfamiliar with this terminology can consult [5, Chap. 7] for a brief account. However, the only Banach lattice that appears here and is not a function space will be $\mathrm{M}(K)$, where K is locally compact.

2. Main results

An **absolute contraction**, or a **Dunford–Schwartz operator**, over a measure space X is an operator $T : \mathrm{L}^1 \cap \mathrm{L}^\infty \to \mathrm{L}^1 + \mathrm{L}^\infty$ satisfying $\|Tf\|_p \le \|f\|_p$ for $p = 1$ and $p = \infty$. It is then well known that T extends uniquely and consistently to linear contraction operators $T_p : \mathrm{L}^p \to \mathrm{L}^p$ for $1 \le p < \infty$, and $T_\infty : \mathrm{L}^{(\infty)} \to \mathrm{L}^{(\infty)}$, where $\mathrm{L}^{(\infty)}$ is the closed linear hull of $\mathrm{L}^1 \cap \mathrm{L}^\infty$ within L^∞. It is common to use the single symbol T for each of the operators T_p.

An absolute contraction T is **sub-Markovian** if it is **positive**, i.e., if $Tf \ge 0$ whenever $f \ge 0$, $f \in \mathrm{L}^1 \cap \mathrm{L}^\infty$. (Then also its canonical extension T_p to L^p, $1 \le p < \infty$ and $\mathrm{L}^{(\infty)}$, $p = \infty$, is positive.) This terminology is coherent with [20, Def. 2.12].

An absolute contraction T is called **Markovian**, if it satifies

$$f \le b\mathbf{1} \quad \Longrightarrow \quad Tf \le b\mathbf{1}$$

for every $b \in \mathbb{R}$ and $f \in \mathrm{L}^1 \cap \mathrm{L}^\infty$. (Here, $\mathbf{1}$ is the constant function with value equal to 1.) In particular, T is positive, i.e., sub-Markovian. If the measure space X is finite, an absolute contraction is Markovian if and only if T is positive and $T\mathbf{1} = \mathbf{1}$. This is easy to see, cf. [10, Lemma 3.2].

An operator $T : \mathrm{L}^1 \cap \mathrm{L}^\infty \to \mathrm{L}^1 + \mathrm{L}^\infty$ is **symmetric** if

$$\int_\mathrm{X} Tf \cdot \overline{g} = \int_\mathrm{X} f \cdot \overline{Tg}$$

for all $f, g \in \mathrm{L}^1 \cap \mathrm{L}^\infty$. A symmetric operator is an absolute contraction if and only if it is L^∞-contractive if and only if it is L^1-contractive; and in this case the canonical extension to L^2 is a bounded self-adjoint operator.

A (strongly continuous) **absolute contraction semigroup** over X is a family $(S_t)_{t\geq 0}$ of absolute contractions on X such that $S_0 = \mathrm{I}$, $S_{t+s} = S_t S_s$ for all $t, s \geq 0$ and

$$\|f - S_t f\|_p \to 0 \quad \text{as} \quad t \searrow 0 \tag{2.1}$$

for all $f \in \mathrm{L}^1 \cap \mathrm{L}^\infty$ and all $1 \leq p < \infty$. It follows that the operator family $(S_t)_{t\geq 0}$ can be considered a strongly continuous semigroup on each space L^p, $1 \leq p < \infty$. *We shall always assume this continuity property even when it is not explicitly mentioned.* An absolute contraction semigroup $(S_t)_{t\geq 0}$ is called a **symmetric contraction semigroup** (**symmetric (sub-)Markovian semigroup**) if each operator S_t, $t \geq 0$, is symmetric (symmetric and (sub-)Markovian).

Remarks 2.1.

1) A symmetric sub-Markovian semigroup is called a "symmetric diffusion semigroup" in the classical text [23]. It appears that the "diffusion semigroups" of operator space theory [16, Def. 2] lack the property of positivity, and hence do not specialize to Stein's concept in the commutative case, but rather to what we call "symmetric contraction semigroups" here.
2) As Voigt [25] has shown, the strong continuity assumption (2.1) for $p \neq 2$ is a consequence of the case $p = 2$ together with the requirement that all operators S_t are L^p-contractions.

Given an absolute contraction semigroup $(S_t)_{t\geq 0}$ one can consider, for $1 \leq p < \infty$, the negative generator $-A_p$ of the strongly continuous semigroup $(S_t)_{t\geq 0}$ on L^p, defined by

$$\begin{aligned} \operatorname{dom}(A_p) &= \{f \in \mathrm{L}^p : \lim_{t\searrow 0} \tfrac{1}{t}(f - S_t f) \text{ exists in } \mathrm{L}^p\}, \\ A_p f &= \lim_{t\searrow 0} \tfrac{1}{t}(f - S_t f). \end{aligned}$$

The operators A_p are compatible for different indices p, a fact which is easily seen by looking at the resolvent of A_p

$$(\mathrm{I} + A_p)^{-1} f = \int_0^\infty \mathrm{e}^{-t} S_t f \,\mathrm{d}t \qquad (f \in \mathrm{L}^p,\ 1 \leq p < \infty).$$

Hence, it is reasonable to drop the index p and simply write A instead of A_p.

In order to formulate the main result, we first look at the very special case that the underlying measure space consists of two atoms with equal mass. Let this (probability) space be denoted by Z_2, i.e.,

$$\mathrm{Z}_2 := (\{0, 1\}, 2^{\{0,1\}}, \zeta_2).$$

Then, for $1 \le p < \infty$, $\mathrm{L}^p(\mathrm{Z}_2) = \mathbb{C}^2$ with norm

$$\left\| \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\|_p^p = \tfrac{1}{2}(|z_1|^p + |z_2|^p).$$

The scalar product on the Hilbert space $H = \mathrm{L}^2(\mathrm{Z}_2)$ is

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \cdot_{\mathrm{Z}_2} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \tfrac{1}{2}(z_1\overline{w_1} + z_2\overline{w_2}).$$

Symmetric operators on $\mathrm{L}^2(\mathrm{Z}_2)$ are represented by matrices

$$T = \begin{pmatrix} a & \overline{w} \\ w & b \end{pmatrix}$$

with $a, b \in \mathbb{R}$. The property that T is an absolute contraction is equivalent with the conditions $|a| + |w| \le 1$ and $|b| + |w| \le 1$. Thus, the absolute contractions on Z_2 form a closed convex set

$$C_2 := \left\{ \begin{pmatrix} a & \overline{w} \\ w & b \end{pmatrix} \;\middle|\; a, b \in \mathbb{R},\, w \in \mathbb{C},\, \max\{|a|, |b|\} \le 1 - |w| \right\},$$

and it is easy to see that each matrix

$$E_\lambda := \begin{pmatrix} 0 & \overline{\lambda} \\ \lambda & 0 \end{pmatrix}, \qquad \lambda \in \mathbb{T},$$

is an extreme point of C_2. We can now formulate the desired (meta-)theorem.

Theorem 2.2 (Symmetric Contraction Semigroups). *Let m, $d \in \mathbb{N}$, $1 \le p < \infty$ and let, for each $1 \le j \le m$, F_j, $G_j : \mathbb{C}^d \to \mathbb{C}$ be measurable functions. For any generator $-A$ of a symmetric contraction semigroup over a measure space* X *consider the following statement:*
"For all measurable functions $\mathbf{f} \in \mathcal{M}(\mathrm{X}; \mathbb{C}^d)$ such that $F_j(\mathbf{f}) \in \operatorname{dom}(A_p)$ and $G_j(\mathbf{f}) \in \mathrm{L}^q(\mathrm{X})$ for all $1 \le j \le m$:

$$\sum_{j=1}^{m} \operatorname{Re} \int_{\mathrm{X}} AF_j(\mathbf{f}) \cdot G_j(\mathbf{f}) \;\ge\; 0."$$

Then this statement holds true provided it holds true whenever X *is replaced by* Z_2 *and A is replaced by* $\mathrm{I} - E_\lambda$, $\lambda \in \mathbb{T}$.

If, in addition, the semigroup is sub-Markovian, we have an even better result. In slightly different form (but with more or less the same method), this result has been obtained by Huang in [12, Theorem 2.2].

Theorem 2.3 (Sub-Markovian Semigroups). *Let m, $d \in \mathbb{N}$, $1 \le p < \infty$ and let, for each $1 \le j \le m$, F_j, $G_j : \mathbb{C}^d \to \mathbb{C}$ be measurable functions. For any generator $-A$ of a symmetric sub-Markovian semigroup over a measure space* X *consider the following statement:*

"For all measurable functions $\mathbf{f} \in \mathcal{M}(\mathrm{X}; \mathbb{C}^d)$ *such that* $F_j(\mathbf{f}) \in \mathrm{dom}(A_p)$ *and* $G_j(\mathbf{f}) \in \mathrm{L}^q(\mathrm{X})$ *for all* $1 \le j \le m$*:*

$$\sum_{j=1}^{m} \mathrm{Re} \int_{\mathrm{X}} AF_j(\mathbf{f}) \cdot G_j(\mathbf{f}) \ \ge \ 0."$$

Then this statement holds true provided it holds true whenever X *is replaced by* Z_2 *and* A *is replaced by* $\mathrm{I} - E_1$ *and by* I.

The second condition here (that the statement holds for Z_2 and $A = \mathrm{I}$) just means that the scalar inequality

$$\sum_{j=1}^{m} \mathrm{Re}\, F_j(x) G_j(x) \ge 0$$

holds for all $x \in \mathbb{C}^d$, cf. Lemma 5.1 below.

Finally, we suppose that the measure space X is finite and the semigroup is *Markovian*, i.e., $S_t \ge 0$ and $S_t\mathbf{1} = \mathbf{1}$ for each $t \ge 0$. Then we have an even simpler criterion.

Theorem 2.4 (Markovian Semigroups). *Let* $m,\, d \in \mathbb{N}$, $1 \le p < \infty$ *and let, for each* $1 \le j \le m$, F_j, $G_j : \mathbb{C}^d \to \mathbb{C}$ *be measurable functions. For any generator* $-A$ *of a symmetric Markovian semigroup over a measure space* X *consider the following statement:*
"For all measurable functions $\mathbf{f} \in \mathcal{M}(\mathrm{X}; \mathbb{C}^d)$ *such that* $F_j(\mathbf{f}) \in \mathrm{dom}(A_p)$ *and* $G_j(\mathbf{f}) \in \mathrm{L}^q(\mathrm{X})$ *for all* $1 \le j \le m$*:*

$$\sum_{j=1}^{m} \mathrm{Re} \int_{\mathrm{X}} AF_j(\mathbf{f}) \cdot G_j(\mathbf{f}) \ \ge \ 0."$$

Then this statement holds true provided it holds true whenever X *is replaced by* Z_2 *and* A *is replaced by* $\mathrm{I} - E_1$*.*

The proofs of Theorems 2.2–2.4 are completed in Section 5 below after we have performed some preparatory reductions (Section 3) and provided some results from abstract operator theory (Section 4).

3. Reduction steps

In this section we shall formulate and prove three results that, when combined, reduce the proof of Theorem 2.2 to the case when $\mathrm{X} = (K, \mu)$ is a compact measure space, μ has full support, $\mathrm{L}^\infty(\mathrm{X}) = \mathrm{C}(K)$, and $A = \mathrm{I} - T$, where T is a single symmetric absolute contraction on X. These steps are, of course, well known, but for the convenience of the reader we discuss them in some detail.

3.1. Reduction to bounded operators

Suppose that $(S_t)_{t\geq 0}$ is an absolute contraction semigroup on X with generator $-A$. Then each operator $-(\mathrm{I}-S_\varepsilon)$ is itself the (bounded) generator of a (uniformly continuous) absolute contraction semigroup $\big(\mathrm{e}^{-t(\mathrm{I}-S_\varepsilon)}\big)_{t\geq 0}$ on X. By definition of A,

$$\frac{1}{\varepsilon}(\mathrm{I}-S_\varepsilon)g \to Ag \quad \text{as} \quad \varepsilon \searrow 0$$

in L^p for $g \in \mathrm{dom}(A_p)$. We thus have the following first reduction result.

Proposition 3.1. *Let m, $d \in \mathbb{N}$, $1 \leq p < \infty$ and let, for each $1 \leq j \leq m$, F_j, $G_j : \mathbb{C}^d \to \mathbb{C}$ be measurable functions. For any generator $-A$ of an absolute contraction semigroup $(S_t)_{t\geq 0}$ over a measure space* X *consider the following statement:*
"For all measurable functions $\mathbf{f} \in \mathcal{M}(\mathrm{X};\mathbb{C}^d)$ such that $F_j(\mathbf{f}) \in \mathrm{dom}(A_p)$ and $G_j(\mathbf{f}) \in \mathrm{L}^q(\mathrm{X})$ for all $1 \leq j \leq m$:

$$\sum_{j=1}^{m} \mathrm{Re} \int_{\mathrm{X}} AF_j(\mathbf{f}) \cdot G_j(\mathbf{f}) \;\geq\; 0.\text{"}$$

Then this statement holds true provided it holds true whenever A is replaced by $\mathrm{I}-S_\varepsilon$, $\varepsilon > 0$.

Note that in the case $A = \mathrm{I}-T$, the condition $F_j(\mathbf{f}) \in \mathrm{dom}(A_p)$ just asserts that $F_j(\mathbf{f}) \in \mathrm{L}^p$.

3.2. Reduction to a finite measure space

Now it is shown that one may confine to finite measure spaces. For a given measure space $\mathrm{X} = (X, \Sigma, \mu)$, the set

$$\Sigma_{\mathrm{fin}} := \{B \in \Sigma \,:\, \mu(B) < \infty\}$$

is directed with respect to set inclusion. For asymptotic statements with respect to this directed set we use the abbreviation "$B \to X$". The multiplication operators

$$M_B : \mathcal{M}(\mathrm{X};\mathbb{C}^d) \to \mathcal{M}(\mathrm{X};\mathbb{C}^d), \qquad M_B\mathbf{f} := \mathbf{1}_B \cdot \mathbf{f}$$

form a net, with $M_B \to \mathrm{I}$ strongly on L^p as $B \to X$ and $1 \leq p < \infty$. It follows that for a given absolute contraction T on X and functions $f \in \mathrm{L}^p(\mathrm{X})$ and $g \in \mathrm{L}^q(\mathrm{X})$

$$\int_{\mathrm{X}} (\mathrm{I}-T)M_B f \cdot (M_B g) \to \int_{\mathrm{X}} (\mathrm{I}-T)f \cdot g \quad \text{as } B \to X.$$

For given $B \in \Sigma_{\mathrm{fin}}$ we form the finite measure space (B, Σ_B, μ_B), where $\Sigma_B := \{C \in \Sigma \,:\, C \subseteq B\}$ and $\mu_B := \mu|_{\Sigma_B}$. Then we have the extension operator

$$\mathrm{Ext}_B : \mathcal{M}(B;\mathbb{C}^d) \to \mathcal{M}(\mathrm{X};\mathbb{C}^d), \qquad \mathrm{Ext}_B\mathbf{f} = \begin{cases} \mathbf{f} & \text{on } B \\ 0 & \text{on } X \setminus B, \end{cases}$$

and the restriction operator

$$\mathrm{Res}_B : \mathcal{M}(\mathrm{X};\mathbb{C}^d) \to \mathcal{M}(B;\mathbb{C}^d), \qquad \mathrm{Res}_B f := f|_B.$$

Note that $\mathrm{Ext}_B \, \mathrm{Res}_B = M_B$ and $\mathrm{Res}_B \mathrm{Ext}_B = \mathrm{I}$ and

$$\int_B \mathrm{Res}_B f \, \mathrm{d}\mu_B = \int_X M_B f \, \mathrm{d}\mu \qquad (f \in \mathrm{L}^1(\mathrm{X})).$$

A short computation yields that $\mathrm{Res}_B^* = \mathrm{Ext}_B$ between the respective L^2-spaces. Hence, if T is a (symmetric) absolute contraction on $\mathrm{X} = (X, \Sigma, \mu)$, then the operator

$$T_B := \mathrm{Res}_B \, T \, \mathrm{Ext}_B$$

is a (symmetric) absolute contraction on (B, Σ_B, μ_B). Another short computation reveals that

$$\int_X (\mathrm{I} - T) M_B f \cdot (M_B g) \, \mathrm{d}\mu = \int_B (\mathrm{I}_{\mathrm{L}^p(B)} - T_B)(\mathrm{Res}_B f) \cdot (\mathrm{Res}_B g) \, \mathrm{d}\mu_B$$

whenever $f \in \mathrm{L}^p(\mathrm{X})$ and $g \in \mathrm{L}^q(\mathrm{X})$. Finally, suppose that $F : \mathbb{C}^d \to \mathbb{C}$ is measurable and suppose that $\mathbf{f} \in \mathcal{M}(\mathrm{X}; \mathbb{C}^d)$ is such that $F(\mathbf{f}) \in \mathrm{L}^p(\mathrm{X})$. Then

$$\mathrm{Res}_B[F(\mathbf{f})] = F(\mathrm{Res}_B \mathbf{f}) \in \mathrm{L}^p(B).$$

Combining all these facts yields our second reduction result.

Proposition 3.2. *Let m, $d \in \mathbb{N}$, $1 \le p < \infty$ and let, for each $1 \le j \le m$, F_j, $G_j : \mathbb{C}^d \to \mathbb{C}$ be measurable functions. For any absolute contraction T over a measure space $\mathrm{X} = (X, \Sigma, \mu)$ consider the following statement:*
"For all measurable functions $\mathbf{f} \in \mathcal{M}(\mathrm{X}; \mathbb{C}^d)$ such that $F_j(\mathbf{f}) \in \mathrm{L}^p(\mathrm{X})$ and $G_j(\mathbf{f}) \in \mathrm{L}^q(\mathrm{X})$ for all $1 \le j \le m$:

$$\sum_{j=1}^m \mathrm{Re} \int_{\mathrm{X}} (\mathrm{I} - T) F_j(\mathbf{f}) \cdot G_j(\mathbf{f}) \ \ge \ 0."$$

Then this statement holds true provided it holds true whenever $\mathrm{X} = (X, \Sigma, \mu)$ is replaced by (B, Σ_B, μ_B) and T is replaced by T_B, where $B \in \Sigma_{\mathrm{fin}}$.

Finally, we observe that if T is sub-Markovian (=positive) or Markovian, then so is each of the operators $T_B = \mathrm{Res}_B \, T \, \mathrm{Ext}_B$, $B \in \Sigma_{\mathrm{fin}}$.

3.3. Reduction to a compact measure space

In the next step we pass from general finite measure spaces to *compact* spaces with a finite positive Borel measure on it.

Let $\mathrm{X} = (X, \Sigma, \mu)$ be a *finite* measure space. The space $\mathrm{L}^\infty(\mathrm{X})$ is a commutative, unital C^*-algebra, hence by the Gelfand–Naimark theorem there is a compact space K, the Gelfand space, and an isomorphism of unital C^*-algebras

$$\Phi : \mathrm{L}^\infty(\mathrm{X}) \to \mathrm{C}(K).$$

In particular, Φ is an isometry. Since the order structure is determined by the C^*-algebra structure (an element f is ≥ 0 if and only if there is g such that $f = g\overline{g}$), Φ is also an isomorphism of complex Banach lattices. The following auxiliary result is, essentially, a consequence of the Stone–Weierstrass theorem.

Lemma 3.3. *In the situation from above, let $M \subseteq \mathbb{C}^d$ be compact and let $f_1, \dots, f_d \in \mathrm{L}^\infty(\mathrm{X})$ be such that $(f_1, \dots, f_d) \in M$ μ-almost everywhere. Then $(\Phi f_1, \dots, \Phi f_d) \in M$ everywhere on K and*

$$\Phi\bigl(F(f_1, \dots, f_d)\bigr) = F(\Phi f_1, \dots, \Phi f_d) \tag{3.1}$$

for all continuous functions $F \in \mathrm{C}(M)$.

Proof. Suppose first that $M = \mathrm{B}[0, r] := \{x \in \mathbb{C}^d : \|x\|_\infty \le r\}$ for some $r > 0$. Then the condition "$(f_1, \dots, f_d) \in M$ almost everywhere" translates into the inequalities $|f_j| \le r\mathbf{1}$ (almost everywhere) for all $j = 1, \dots, d$, and hence one has also $|\Phi f_j| \le r\Phi\mathbf{1} = r\mathbf{1}$ (pointwise everywhere) for all $j = 1, \dots, d$. It follows that $F(\Phi f_1, \dots, \Phi f_d)$ is well defined.

Now, the set of functions $F \in \mathrm{C}(M)$ such that (3.1) holds is a closed conjugation-invariant subalgebra of $\mathrm{C}(M)$ that separate the points and contains the constants. Hence, by the Stone–Weierstrass theorem, it is all of $\mathrm{C}(M)$.

For general M one can proceed in the same way provided one can assure that $(\Phi f_1, \dots, \Phi f_d) \in M$ everywhere on K. Let $y \in \mathbb{C}^d \setminus M$ and let F be any continuous function with compact support on $\mathbb{C}^d$ such that $F = 0$ on M and $F(y) = 1$. Let $r > 0$ by so large that $M \subseteq \mathrm{B}[0, r]$ and consider F as a function on $\mathrm{B}[0, r]$. Then $0 = \Phi(0) = \Phi(F(f_1, \dots, f_d)) = F(\Phi f_1, \dots, \Phi f_d)$, hence y cannot be in the image of $(\Phi f_1, \dots, \Phi f_d)$. $\square$

By the Riesz–Markov representation theorem, there is a unique regular Borel measure ν on K such that

$$\int_\mathrm{X} f = \int_K \Phi f \, \mathrm{d}\nu$$

for all $f \in \mathrm{L}^\infty(\mathrm{X})$. It follows from Lemma 3.3 that $|\Phi f|^p = \Phi(|f|^p)$ for every $1 \le p < \infty$ and every $f \in \mathrm{L}^\infty(\mathrm{X})$. Therefore, Φ is an isometry with respect to each p-norm. It follows that Φ extends to an isometric (lattice) isomorphism

$$\Phi : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(K, \nu).$$

It is shown in the Appendix that Φ, furthermore, extends canonically (and uniquely) to a unital $*$-algebra and lattice isomorphism

$$\Phi : \mathcal{M}(\mathrm{X}) \to \mathcal{M}(K, \nu).$$

The compact measure space (K, ν) (together with the mapping Φ) is called the **Stone model** of the probability space X. Note that under the lattice isomorphism Φ the respective L^∞-spaces must correspond to each other, whence it follows that $\mathrm{L}^\infty(K, \mu) = \mathrm{C}(K)$ in the obvious sense.

We use the canonical extension to vector-valued functions $\Phi : \mathcal{M}(\mathrm{X}; \mathbb{C}^d) \to \mathcal{M}(K, \nu; \mathbb{C}^d)$ of the Stone model. By Theorem A.3,

$$\Phi\bigl(F(\mathbf{f})\bigr) = F(\Phi\mathbf{f}) \qquad \nu\text{-almost everywhere}$$

for all measurable functions $\mathbf{f} = (f_1, \dots, f_d) \in \mathcal{M}(\mathrm{X}; \mathbb{C}^d)$ and all measurable functions $F : \mathbb{C}^d \to \mathbb{C}$. Hence, we arrive at the next reduction result.

Proposition 3.4. *Let m, $d \in \mathbb{N}$, $1 \le p < \infty$ and let, for each $1 \le j \le m$, F_j, $G_j : \mathbb{C}^d \to \mathbb{C}$ be measurable functions. For any absolute contraction T over a probability space* X *consider the following statement:*
"For all measurable functions $\mathbf{f} \in \mathcal{M}(\mathrm{X};\mathbb{C}^d)$ such that $F_j(\mathbf{f}) \in \mathrm{L}^p(\mathrm{X})$ and $G_j(\mathbf{f}) \in \mathrm{L}^q(\mathrm{X})$ for all $1 \le j \le m$:

$$\sum_{j=1}^{m} \mathrm{Re} \int_{\mathrm{X}} [(\mathrm{I} - T)F_j(\mathbf{f})] \cdot G_j(\mathbf{f}) \ \geq \ 0."$$

Then this statement holds true provided it holds true if X *is replaced by (K,ν) and T is replaced by $\Phi T \Phi^{-1}$, where (K,ν) and*

$$\Phi : \mathcal{M}(\mathrm{X}) \to \mathcal{M}(K,\nu)$$

is the Stone model of X*.*

As in the reduction step before, we observe that the properties of being symmetric, sub-Markovian or Markovian are preserved during the reduction process, i.e., in passing from T to $\Phi^{-1}T\Phi$.

Remark 3.5. In the late 1930s and beginning 1940s, several representation results for abstract structures were developed first by Stone [24] (for Boolean algebras), then by Gelfand [7, 8] (for normed algebras) and Kakutani [13, 14] (for *AM*- and *AL*-spaces). However, it is hard to determine when for the first time there was made effective use of these results in a context similar to ours. Halmos in his paper [11] on a theorem of Dieudonné on measure disintegration employs the idea but uses Stone's original theorem. A couple of years later, Segal [22, Thm. 5.4] revisits Dieudonné's theorem and gives a proof based on algebra representations. (He does not mention Gelfand–Naimark, but only says "by well-known results".)

In our context, the idea – now through the Gelfand–Naimark theorem – was employed by Nagel and Voigt [19] in order to simplify arguments in the proof of Liskevich and Perelmuter [17] on the optimal angle of analyticity in the sub-Markovian case, see Section 6 below. Through Ouhabaz' book [20] it has become widely known in the field, and also Carbonaro and Dragičević [2, p. 19] use this idea.

4. Operator theory

In order to proceed with the proof of the main theorem (Theorem 2.2) we need to provide some results from the theory of operators of the form $T : \mathrm{C}(K) \to \mathrm{L}^1(L,\mu)$, where K and L are compact.[1] For the application to symmetric contraction semigroups as considered in the previous sections, we only need the case that $\mathrm{C}(L) = \mathrm{L}^\infty(L,\mu)$, and this indeed would render simpler some of the proofs below. However, a restriction to this case is artificial, and we develop the operator theory in reasonable generality.

[1]The case that K and L are locally compact is touched upon in some additional remarks.

4.1. The linear modulus

In this section we introduce the linear modulus of an order-bounded operator $T : \mathrm{C}(K) \to \mathrm{L}^1(\mathrm{X})$. This can be treated in the framework of general Banach lattices, see [21, Chapter IV, §1], but due to our concrete situation, things are a little easier than in an abstract setting.

Let $\mathrm{X} = (X, \Sigma, \mu)$ be a measure space and let K be compact. A linear operator $T : \mathrm{C}(K) \to \mathrm{L}^1(\mathrm{X})$ is called **order-bounded** if for each $0 \le f \in \mathrm{C}(K)$ there is $0 \le h \in \mathrm{L}^1(\mathrm{X})$ such that

$$|Tu| \le h \quad \text{for all } u \in \mathrm{C}(K) \text{ with } |u| \le f.$$

And T is called **regular** if it is a linear combination of positive operators. It is clear that each regular operator is order-bounded. The converse also holds, by the following construction.

Suppose that $T : \mathrm{C}(K) \to \mathrm{L}^1(\mathrm{X})$ is order-bounded. Then, for $0 \le f \in \mathrm{C}(K)$ let

$$|T|f := \sup\{|Tg| \,:\, g \in \mathrm{C}(K), |g| \le f\} \tag{4.1}$$

as a supremum in the lattice sense. (This supremum exists since the set on the right-hand side is order-bounded by hypothesis and L^1 is order-complete, see [5, Cor. 7.8].)

Lemma 4.1. *Suppose that $T : \mathrm{C}(K) \to \mathrm{L}^1(\mathrm{X})$ is order-bounded. Then the mapping $|T|$ defined by* (4.1) *extends uniquely to a positive operator*

$$|T| : \mathrm{C}(K) \to \mathrm{L}^1(\mathrm{X}).$$

Moreover, the following assertions hold:

a) $|Tf| \le |T|\,|f|$ *for all* $f \in \mathrm{C}(K)$.

b) $\|T\| \le \||T|\|$,

c) $\overline{T}$ *is order-bounded and* $\left|\overline{T}\right| = |T|$.

d) *If* $S : \mathrm{C}(K) \to \mathrm{L}^1(\mathrm{X})$ *is order-bounded, then* $S + T$ *is also order-bounded, and* $|S + T| \le |S| + |T|$.

The operator $|T| : \mathrm{C}(K) \to \mathrm{L}^1(\mathrm{X})$ whose existence is asserted in the theorem is called the **linear modulus** of T.

Proof. For the first assertion, it suffices to show that $|T|$ is additive and positively homogeneous. The latter is straightforward, so consider additivity. Fix $0 \le f, g \in \mathrm{C}(K)$ and let $u \in \mathrm{C}(K)$ with $|u| \le f + g$. Define

$$u_1 = \frac{fu}{f+g}, \quad u_2 = \frac{gu}{f+g},$$

where $u_1 = u_2 = 0$ on the set $[\, f + g = 0\,]$. Then u_1, $u_2 \in \mathrm{C}(K)$, $|u_1| \le f$, $|u_1| \le g$ and $u_1 + u_2 = u$. Hence

$$|Tu| \le |Tu_1| + |Tu_2| \le |T|f + |T|g$$

and taking the supremum with respect to u we obtain $|T|(f+g) \leq |T|f + |T|g$. Conversely, let u, $v \in \mathrm{C}(K)$ with $|u| \leq f$ and $|v| \leq g$. Then, for any $\alpha \in \mathbb{C}^2$ with $|\alpha_1|, |\alpha_2| \leq 1$ we have $|\alpha_1 u + \alpha_2 v| \leq f + g$, and hence

$$|Tu| + |Tv| = \sup_\alpha |\alpha_1 Tu + \alpha_2 Tv| = \sup_\alpha |T(\alpha_1 u + \alpha_2 v)| \leq |T|(f+g).$$

Taking suprema with respect to u and v we arrive at $|T|f + |T|g \leq |T|(f+g)$. The remaining statements are now easy to establish. □

Suppose that $T : \mathrm{C}(K) \to \mathrm{L}^1(\mathrm{X})$ is order-bounded, so that $|T|$ exists. Then, by Lemma 4.1, also $\operatorname{Re} T$ and $\operatorname{Im} T$ are order-bounded. If T is real, i.e., if $T = \overline{T}$, then clearly $T \leq |T|$, and hence $T = |T| - (|T| - T)$ is regular. It follows that every order-bounded operator is regular. (See also [21, IV.1, Props. 1.2 and 1.6].)

Let us turn to another characterization of order-boundedness. If $T : \mathrm{C}(K) \to \mathrm{L}^1(\mathrm{X})$ is order-bounded and $|T|$ is its linear modulus, we denote by $|T|'\mu$ the unique element $\nu \in \mathrm{M}(K)$ such that

$$\int_K f \,\mathrm{d}\nu = \int_\mathrm{X} |T|f \quad \text{for all } f \in \mathrm{C}(K).$$

It is then easy to see that T extends to a contraction $T : \mathrm{L}^1(K,\nu) \to \mathrm{L}^1(\mathrm{X})$. We shall see that the existence of a positive regular Borel measure ν on K with this property characterizes the order-boundedness. The key is the following general result, which has (probably) been established first by Grothendieck [9, p. 67, Corollaire].

Lemma 4.2. *Let* X, Y *be measure spaces and let* $T : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ *be a bounded operator. Then for any finite sequence* $f_1, \ldots, f_n, \in \mathrm{L}^1(\mathrm{Y})$

$$\int_\mathrm{X} \sup_{1 \leq j \leq n} |Tf_j| \leq \|T\| \int_\mathrm{Y} \sup_{1 \leq j \leq n} |f_j|\,.$$

Proof. By approximation, we may suppose that all the functions f_j are integrable step functions with respect to one finite partition $(A_k)_k$. We use the variational form

$$\sup_{1 \leq j \leq n} |z_j| = \sup \Big\{ \Big| \sum\nolimits_j^n \alpha_j z_j \Big| \,:\, \alpha \in \ell_n^1, \|\alpha\|_1 \leq 1 \Big\}$$

for complex numbers $z_1, \ldots, z_n$. Then, with $f_j = \sum_k c_{jk} \mathbf{1}_{A_k}$,

$$\begin{aligned} \sup_{1 \leq j \leq n} |Tf_j| &= \sup_\alpha \Big| \sum\nolimits_j^n \sum\nolimits_k \alpha_j c_{jk} T\mathbf{1}_{A_k} \Big| \\ &\leq \sup_\alpha \sum\nolimits_k \|\alpha\|_1 \big(\sup_{1 \leq j \leq n} |c_{jk}| \big) \, |T\mathbf{1}_{A_k}| \\ &= \sum\nolimits_k \big(\sup_{1 \leq j \leq n} |c_{jk}| \big) \, |T\mathbf{1}_{A_k}|\,. \end{aligned}$$

Integrating yields

$$\begin{aligned}\int_{\mathrm{X}} \sup_{1\le j\le n} |Tf_j| &\le \sum\nolimits_k \big(\sup_{1\le j\le n} |c_{jk}| \big) \, \|T\mathbf{1}_{A_k}\|_1 \\ &\le \|T\| \sum\nolimits_k \big(\sup_{1\le j\le n} |c_{jk}| \big) \, \|\mathbf{1}_{A_k}\|_1 \\ &= \|T\| \int_{\mathrm{Y}} \sum\nolimits_k \big(\sup_{1\le j\le n} |c_{jk}| \big) \mathbf{1}_{A_k} = \|T\| \int_{\mathrm{Y}} \sup_{1\le j\le n} |f_j| \, . \qquad \square\end{aligned}$$

We can now formulate the main result of this section.

Theorem 4.3. *Let* $\mathrm{X} = (X, \Sigma, \mu)$ *be any measure space and* $T : \mathrm{C}(K) \to \mathrm{L}^1(\mathrm{X})$ *a linear operator. Then the following assertions are equivalent:*

(i) *T is order-bounded.*
(ii) *T is regular.*
(iii) *There is a positive regular Borel measure $\nu \in \mathrm{M}(K)$ such that T extends to a contraction $\mathrm{L}^1(K, \nu) \to \mathrm{L}^1(\mathrm{X})$.*

If (i)–(iii) *hold, then*

$$|T|'\mu = \min\big\{\nu \in \mathrm{M}_+(K) \, : \, \|Tf\|_{\mathrm{L}^1(\mathrm{X})} \le \|f\|_{\mathrm{L}^1(K,\nu)} \text{ for all } f \in \mathrm{C}(K)\big\}.$$

In particular, if $0 \le \nu \in \mathrm{M}(K)$ is such that T extends to a contraction $\mathrm{L}^1(K, \nu) \to \mathrm{L}^1(\mathrm{X})$, then so does $|T|$.

Proof. The implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) have already been established. Moreover, if (i) holds then it follows from the inequality $|Tf| \le |T||f|$ that $\|Tf\|_1 \le \|f\|_{\mathrm{L}^1(K,\nu)}$ with $\nu = |T|'\mu$.

On the other hand, suppose (iii) holds and that $0 \le \nu \in \mathrm{M}(K)$ is such that $\int_{\mathrm{X}} |Tf| \le \int_K |f| \, \mathrm{d}\nu$ for all $f \in \mathrm{C}(K)$. Let $0 \le f \in \mathrm{C}(K)$, $n \in \mathbb{N}$ and $u_j \in \mathrm{C}(K)$ with $|u_j| \le f$ $(1 \le j \le n)$. Then, by Lemma 4.2,

$$\int_{\mathrm{X}} \sup_{1\le j\le n} |Tu_j| \le \int_K \sup_{1\le j\le n} |u_j| \, \mathrm{d}\nu \le \int_K f \, \mathrm{d}\nu.$$

Now, any upwards directed and norm bounded net in L^1_+ is order-bounded and converges in L^1-norm towards its supremum, see [5, Thm. 7.6]. It follows that T is order-bounded, and

$$\int_{\mathrm{X}} |T|f \le \int_K f \, \mathrm{d}\nu.$$

Consequently, $|T|'\mu \le \nu$, as claimed. $\square$

Remarks 4.4.

1) Suppose that (i)–(iii) of Theorem 4.3 hold. Then $|T'\mu| \le |T|'\mu$, and equality holds if and only if T extends to a contraction $T : \mathrm{L}^1(K, |T'\mu|) \to \mathrm{L}^1(\mathrm{X})$.

2) The modulus mapping $T \mapsto |T|$ turns $\mathcal{L}^r(\mathrm{C}(K), \mathrm{L}^1(\mathrm{X}))$, the set of regular operators, into a complex Banach lattice with the norm $\|T\|_r := \||T|\|$, see [21, Chap. IV, §1].

3) All the results of this section hold *mutatis mutandis* for linear operators $T : \mathrm{C_c}(Y) \to \mathrm{L}^1(\mathrm{X})$, where Y is a locally compact space and $\mathrm{C_c}(Y)$ is the space of continuous functions on Y with compact support.

The modulus of a linear operator appears already in the seminal work of Kantorovich [15] on operators on linear ordered spaces. For operators on an L^1-space the linear modulus was (re-)introduced in [3] by Chacon and Krengel who probably were not aware of Kantorovich's work. Later on, their construction was generalized to order-bounded operators between general Banach lattices by Luxemburg and Zaanen in [18] and then incorporated by Schaefer in his monograph [21].

The equivalence of order-bounded and regular operators is of course a standard lemma from Banach lattice theory. Lemma 4.2 is essentially equivalent to saying that every bounded operator between L^1-spaces is order-bounded. This has been realized by Grothendieck in [9, p. 66, Prop. 10]. (Our proof differs considerably from the original one.) The equivalence of (i)–(iii) in Theorem 4.3 can also be derived from combining Theorem IV.1.5 and Corollary 1 of Theorem II.8.9 of [21]. However, the remaining part of Theorem 4.3 might be new.

4.2. Integral representation of bilinear forms

In this section we aim for yet another characterization of order-bounded operators $T : \mathrm{C}(K) \to \mathrm{L}^1(\mathrm{X})$ in the case that $\mathrm{X} = (L, \mu)$ is a compact measure space. We shall see that an operator T is order-bounded if, and only if, there is a (necessarily unique) complex regular Borel measure μ_T on $K \times L$ such that

$$\int_{K\times L} f \otimes g \,\mathrm{d}\mu_T = \int_L (Tf)\cdot g \,\mathrm{d}\mu \quad \text{for all } f \in \mathrm{C}(K) \text{ and } g \in \mathrm{C}(L). \tag{4.2}$$

This result goes essentially back to Grothendieck's characterization of "integral" operators in [9, p. 141, Thm. 11], but we give ad hoc proofs avoiding the tensor product theory. The following simple lemma is the key result here.

Lemma 4.5. *Let K, L be compact spaces. Then, for any bounded operator $T : \mathrm{C}(K) \to \mathrm{C}(L)$ and any $\mu \in \mathrm{M}(L)$ there is a unique complex regular Borel measure $\mu_T \in \mathrm{M}(K \times L)$ such that (4.2) holds. Moreover, $\mu_T \geq 0$ whenever $\mu \geq 0$ and $T \geq 0$.*

Proof. The uniqueness is clear since $\mathrm{C}(K) \otimes \mathrm{C}(L)$ is dense in $\mathrm{C}(K \times L)$. For the existence, let $S : \mathrm{C}(K \times L) \to \mathrm{C}(L)$ be given by composition of all of the operators in the following chain:

$$\mathrm{C}(K \times L) \cong \mathrm{C}(L; \mathrm{C}(K)) \xrightarrow{T^{\otimes}} \mathrm{C}(L; \mathrm{C}(L)) \cong \mathrm{C}(L \times L) \xrightarrow{D} \mathrm{C}(L).$$

Here, $T^{\otimes}$ denotes the operator $G \mapsto T \circ G$ and D denotes the "diagonal contraction", defined by $DG(x) := G(x, x)$ for $x \in L$ and $G \in \mathrm{C}(L \times L)$. Then $\mu_T := S'\mu$ satisfies the requirements, as a short argument reveals. □

Remarks 4.6.

1) The formula (4.2) stays true for all choices of $f \in \mathrm{C}(K)$ and g a bounded measurable function on L.
2) Our proof of Lemma 4.5 yields a formula for the integration of a general $F \in \mathrm{C}(K \times L)$ with respect to μ_T:
$$\int_{K\times L} F(x,y)\,\mathrm{d}\mu_T(x,y) = \int_L \big(TF(\cdot,y)\big)(y)\,\mathrm{d}\mu(y).$$
This means: fix $y \in L$, apply T to the function $F(\cdot, y)$ and evaluate this at y; then integrate this function in y with respect to μ.
3) Compare this proof of Lemma 4.5 with the one given in [20, pp. 90/91].
4) Lemma 4.5 remains valid if K and L are merely locally compact, and $\mathrm{C}(\,\cdot\,)$ is replaced by $\mathrm{C}_0(\,\cdot\,)$ at each occurrence.

Combining Lemma 4.5 with a Stone model leads to the desired general theorem.

Theorem 4.7. *Let K be compact, (L,μ) a compact measure space, and $T : \mathrm{C}(K) \to \mathrm{L}^1(L,\mu)$ a linear operator. Then the following assertions are equivalent:*

(i) *T is order-bounded.*
(ii) *T is regular.*
(iii) *T extends to a contraction $\mathrm{L}^1(K,\nu) \to \mathrm{L}^1(L,\mu)$ for some $0 \le \nu \in \mathrm{M}(K)$.*
(iv) *There is a complex regular Borel measure $\mu_T \in \mathrm{M}(K \times L)$ such that* (4.2) *holds.*

In this case, μ_T from (iv) *is unique, and if ν is as in* (iii), *then $|T|'\mu \le \nu$.*

Proof. It was shown in Theorem 4.3 that (i)–(iii) are pairwise equivalent.

Denote by $\pi_K : K \times L \to K$ the canonical projection. Suppose that (iv) holds and let $\nu = (\pi_K)_*\,|\mu_T|$, i.e.,
$$\int_K f\,\mathrm{d}\nu = \int_{K\times L} f \otimes \mathbf{1}\,\mathrm{d}|\mu_T| \qquad (f \in \mathrm{C}(K)).$$
Then, for $f \in \mathrm{C}(K)$ and $g \in \mathrm{C}(L)$ with $|g| \le 1$,
$$\Big|\int_L Tf \cdot g\,\mathrm{d}\mu\Big| \le \int_{K\times L} |f| \otimes |g|\,\mathrm{d}|\mu_T| \le \int_{K\times L} |f| \otimes \mathbf{1}\,\mathrm{d}|\mu_T| = \int_K |f|\,\mathrm{d}\nu.$$
This implies that T extends to a contraction $\mathrm{L}^1(K,\nu) \to \mathrm{L}^1(L,\mu)$, whence we have (iii).

Now suppose that (i)–(iii) hold. In order to prove (iv) define the operator $S : \mathrm{C}(K) \to \mathrm{L}^\infty(L,\mu)$ by
$$Sf := \begin{cases} \frac{Tf}{|T|\mathbf{1}} & \text{on } [\,|T|\mathbf{1} > 0\,],\\ 0 & \text{on } [\,|T|\mathbf{1} = 0\,].\end{cases}$$

Let $\Phi : \mathrm{L}^1(L,\mu) \to \mathrm{L}^1(\Omega,\tilde{\mu})$ be the Stone model of (L,μ) (see Section 3.3 above), and let us identify $\mathrm{L}^\infty(L,\mu)$ with $\mathrm{C}(\Omega)$ via Φ. Then $S : \mathrm{C}(K) \to \mathrm{C}(\Omega)$ is a positive operator. Hence we can apply Lemma 4.5 to S and the positive measure $(|T|\mathbf{1})\tilde{\mu}$ to obtain a positive measure ρ on $K \times \Omega$ such that

$$\begin{aligned}\int_{K\times\Omega} f \otimes g \,\mathrm{d}\rho &= \int_\Omega Sf \cdot g \,\mathrm{d}(|T|\mathbf{1})\tilde{\mu} = \int_\Omega Sf \cdot |T|\mathbf{1} \cdot g \,\mathrm{d}\tilde{\mu} \\ &= \int_\Omega Tf \cdot g \,\mathrm{d}\tilde{\mu} = \int_L Tf \cdot g \,\mathrm{d}\mu.\end{aligned}$$

Finally, let μ_T be the pull-back of ρ to $K \times L$ via the canonical inclusion map $\mathrm{C}(L) \to \mathrm{L}^\infty(L,\mu) = \mathrm{C}(\Omega)$. □

Remark 4.8. With a little more effort one can extend Theorem 4.7 to the case of *locally compact* (and not necessarily finite) measure spaces (K,ν) and (L,μ) instead of compact ones, cf. Remarks 4.4 and 4.6 above. Then the decisive implication (ii)⇒(iv) is proved by passing first to open and relatively compact subsets $U \subseteq K$ and $V \subseteq L$ and considering the operator $T_{U,V} : \mathrm{C}_0(U) \to \mathrm{L}^1(V,\mu)$. By modifying our proof, one then obtains a measure $\mu_T^{U,V}$ on $U \times V$, and finally μ_T as an inductive limit. (Of course, one has to speak of Radon measures here.) Compare this to the ad hoc approach in [6, Lemma 1.4.1].

Theorem 4.7 can also be generalized to the case that K and L are *Polish* (but not necessarily locally compact) spaces and μ is a finite positive Borel measure on L. In this case the decisive implication (ii)⇒(iv) is proved as follows: first, one chooses compact metric models (K',ν') and (L',μ') for the finite Polish measure spaces (K,ν) and (L,μ), respectively, see [5, Sec. 12.3]; by a theorem of von Neumann [5, App. F.3], the isomorphisms between the original measure spaces and their models are induced by measurable maps $\varphi : K' \to K$ and $\psi : L' \to L$, say. Theorem 4.7 yields – for the transferred operator – a representing measure on $K' \times L'$, and this is mapped by $\varphi \times \psi$ to a representing measure on $K \times L$ for the original operator.

We now combine the integral Theorem 4.7 with the construction of the modulus. We employ the notation $\pi_L : K \times L \to L$ for the canonical projection, and identify

$$\mathrm{L}^1(L,\mu) = \{\lambda \in \mathrm{M}(L) \,:\, |\lambda| \ll \mu\}$$

with a closed ideal in $\mathrm{M}(L)$ via the Radon–Nikodým theorem.

Theorem 4.9. *Suppose that K and L are compact spaces and $0 \le \mu \in \mathrm{M}(L)$. Then, for any order-bounded operator $T : \mathrm{C}(K) \to \mathrm{L}^1(L,\mu)$,*

$$|\mu_T| = \mu_{|T|}.$$

The mapping

$$\mathcal{L}^r(\mathrm{C}(K), \mathrm{L}^1(L,\mu)) \to \mathrm{M}(K \times L), \qquad T \mapsto \mu_T$$

is an isometric lattice homomorphism onto the closed ideal

$$\{\rho \in \mathrm{M}(K \times L) \,:\, \pi_{L*}|\rho| \in \mathrm{L}^1(L,\mu)\}$$

of $\mathrm{M}(K \times L)$.

Proof. It is clear that the mapping $T \mapsto \mu_T$ is linear, injective and positive. Hence $|\mu_T| \le \mu_{|T|}$, and therefore $\pi_{L*}|\mu_T| \le \pi_{L*}\mu_{|T|} = (|T|\mathbf{1})\mu \in \mathrm{L}^1(L,\mu)$. Conversely, suppose that $\rho \in \mathrm{M}(K \times L)$ such that $\pi_{L*}|\rho| \in \mathrm{L}^1(L,\mu)$. For $f \in \mathrm{C}(K)$ consider the linear mapping

$$T : \mathrm{C}(K) \to \mathrm{M}(L), \qquad (Tf)g := \int_{K\times L} f \otimes g \,\mathrm{d}\rho.$$

Then $|Tf| \le \|f\|_\infty \pi_{L*}|\rho|$, hence $Tf \in \mathrm{L}^1(L,\mu)$. Therefore, by construction,

$$\int_{K\times L} f \otimes g \,\mathrm{d}\rho = \int_L Tf \cdot g \,\mathrm{d}\mu$$

for $f \in \mathrm{C}(K)$ and $g \in \mathrm{C}(L)$. By Theorem 4.7, T is regular. If ρ is positive, then T is positive, too.

The proof of the converse inequality $\mu_{|T|} \le |\mu_T|$ would now follow immediately if we used the fact (from Remark 4.4) that the modulus map turns $\mathcal{L}^r$, the set of regular operators, into a complex vector lattice. However, we want to give a different proof here.

By a standard argument, it suffices to establish the inequality

$$\int_L |T|\mathbf{1} \,\mathrm{d}\mu \le \int_{K\times L} \mathbf{1} \otimes \mathbf{1} \,\mathrm{d}|\mu_T|.$$

To this end, define the positive measure ν on K by

$$\int_K f \,\mathrm{d}\nu := \int_{K\times L} f \otimes \mathbf{1} \,\mathrm{d}|\mu_T| \quad (f \in \mathrm{C}(K)).$$

Given $f \in \mathrm{C}(K)$ there is a bounded measurable function h on L such that $|Tf| = (Tf)h$ and $|h| \le 1$. Hence,

$$\int_L |Tf| \,\mathrm{d}\mu = \int_L Tf \cdot h \,\mathrm{d}\mu = \int_{K\times L} f \otimes h \,\mathrm{d}\mu_T \le \int_{K\times L} |f| \otimes \mathbf{1} \,\mathrm{d}\mu_T = \int_K |f| \,\mathrm{d}\nu.$$

This means that T extends to a contraction $\mathrm{L}^1(K,\nu) \to \mathrm{L}^1(L,\mu)$. By Theorem 4.3, it follows that $|T|'\mu \le \nu$, hence in particular

$$\int_L |T|\mathbf{1} \,\mathrm{d}\mu = \int_K \mathbf{1}\mathrm{d}(|T|'\mu) \le \int_K \mathbf{1} \,\mathrm{d}\nu = \int_{K\times L} \mathbf{1} \otimes \mathbf{1} \,\mathrm{d}|\mu_T|.$$

This concludes the proof. $\square$

Remark 4.10. One can avoid the use of the bounded measurable function h in the second part of the proof of Theorem 4.9 by passing to the Stone model of $\mathrm{L}^1(L,\mu)$.

In case that T has additional properties, one can extend the defining formula for the measure μ_T to some non-continuous functions.

Theorem 4.11. *Let (K,ν) and (L,μ) be compact measure spaces, and let $T : \mathrm{C}(K) \to \mathrm{L}^\infty(L,\mu)$ be a bounded operator that extends to a bounded operator $\mathrm{L}^1(K,\nu) \to \mathrm{L}^1(L,\mu)$. Then the formula*

$$\int_L Tf \cdot g \,\mathrm{d}\mu = \int_{K\times L} f \otimes g \,\mathrm{d}\mu_T \tag{4.3}$$

holds for all $f \in \mathrm{L}^p(K,\nu)$, $g \in \mathrm{L}^q(L,\mu)$ and $1 \le p,\, q \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We may suppose that $T : \mathrm{L}^1(K,\nu) \to \mathrm{L}^1(L,\mu)$ (and hence also $|T|$) is a contraction. In a first step, we claim that the formula 4.3 holds for all bounded Baire measurable functions f, g on K, L, respectively. Indeed, this follows from a standard argument by virtue of the dominated convergence theorem and the fact that the bounded Baire-measurable functions on a compact space form the smallest set of functions that contains the continuous ones and is closed under pointwise convergence of uniformly bounded sequences, see [5, Thm. E.1].

Replacing T by $|T|$ in 4.3 we then can estimate for bounded Baire-measurable functions f and g and $1 < p < \infty$

$$\begin{aligned}
\int_{K\times L} |f \otimes g| \,\mathrm{d}\mu_{|T|} &= \int_{K\times L} (|f| \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes |g|) \,\mathrm{d}\mu_{|T|} \\
&\le \Big(\int_{K\times L} |f|^p \otimes \mathbf{1} \,\mathrm{d}\mu_{|T|}\Big)^{\frac{1}{p}} \cdot \Big(\int_{K\times L} \mathbf{1} \otimes |g|^q \,\mathrm{d}\mu_{|T|}\Big)^{\frac{1}{q}} \\
&= \Big(\int_L |T||f|^p \,\mathrm{d}\mu\Big)^{\frac{1}{p}} \cdot \Big(\int_L (|T|\mathbf{1}) \cdot |g|^q \,\mathrm{d}\mu\Big)^{\frac{1}{q}} \\
&\le \Big(\int_K |f|^p \,\mathrm{d}\nu\Big)^{\frac{1}{p}} \cdot \Big(\int_L (|T|\mathbf{1}) \cdot |g|^q \,\mathrm{d}\mu\Big)^{\frac{1}{q}} \\
&= \|f\|_{\mathrm{L}^p(\nu)} \left\|(|T|\mathbf{1})^{\frac{1}{q}} g\right\|_{\mathrm{L}^q(\mu)}.
\end{aligned}$$

It follows that if A is a ν-null Baire set of K and B is a μ-null Baire set of L, then the sets $A \times L$ and $K \times B$ are $\mu_{|T|}$-null Baire sets of $K \times L$. Moreover, the bilinear mapping $(f,g) \mapsto f \otimes g$ extends to a bounded bilinear mapping

$$\mathrm{L}^p(K,\nu) \times \mathrm{L}^q(L,\mu) \to \mathrm{L}^1(K \times L, \mu_{|T|}).$$

By interpolation, T is L^p-bounded, and hence the bilinear mapping $(f,g) \mapsto Tf \cdot g$ is a bounded bilinear mapping $\mathrm{L}^p(K,\nu) \times \mathrm{L}^q(L,\mu) \to \mathrm{L}^1(L,\mu)$. Now (4.3) holds for bounded Baire-measurable functions f and g, hence by approximation for all $f \in \mathrm{L}^p(K,\nu)$ and $g \in \mathrm{L}^q(L,\mu)$. (Choose sequences that approximate in norm and almost everywhere. Observe that from the reasoning above it follows that if $f_n \to f$ ν-a.e. and $g_n \to g$ μ-a.e., then $f_n \otimes g_n \to f \otimes g$ $\mu_{|T|}$-a.e.)

Finally, consider $p = 1$ (the case $q = 1$ being similar). If $g \in \mathrm{L}^\infty(L,\mu)$ then, by choosing a Baire-measurable representative for g such that $\|g\|_\infty = \|g\|_{\mathrm{L}^\infty(L,\nu)}$ and using the results from above, we can estimate for each $f \in \mathrm{L}^\infty(K,\nu)$,

$$\int_{K\times L} |f \otimes g| \,\mathrm{d}\mu_{|T|} = \int_{K\times L} (|f| \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes |g|) \,\mathrm{d}\mu_{|T|}$$

$$\leq \int_{K\times L} |f| \otimes \mathbf{1} \cdot \|g\|_\infty \, d\mu_{|T|} = \||T|\,|f|\|_{\mathrm{L}^1(L,\nu)} \|g\|_\infty$$
$$\leq \|f\|_{\mathrm{L}^1(K,\nu)} \|g\|_{\mathrm{L}^\infty(L,\mu)} \, .$$

The assertion then follows by approximation (almost everywhere and in norm) as before. □

Remark 4.12. If an operator $T : \mathrm{C}(K) \to \mathrm{L}^1(L,\mu)$ factors through $\mathrm{L}^\infty(L,\mu)$, it is of course order-bounded, and hence its modulus exists. If, in addition, it factors even through $\mathrm{C}(K)$, then the existence of μ_T follows from Lemma 4.5 directly and one does not have to pass through the Stone model. If (L,μ) *is* already its own Stone model (as is the case in the proof of Theorem 2.2 after the reduction step in Section 3.3) then also $|T|$ factors through $\mathrm{C}(L)$, and hence Lemma 4.5 is completely sufficient to construct the measures μ_T and $\mu_{|T|}$.

Using modern tensor product terminology, we have

$$\mathrm{C}(K \times L) = \mathrm{C}(K) \otimes_\varepsilon \mathrm{C}(L) \subseteq \mathrm{C}(K) \otimes_\varepsilon \mathrm{L}^\infty(L,\mu) = \mathrm{C}(K) \otimes_\varepsilon \mathrm{L}^1(L,\mu)'.$$

This implies (via the Stone model of (L,μ)) that an operator $T : \mathrm{C}(K) \to \mathrm{L}^1(L,\mu)$ is "integral" (in the sense of Grothendieck) if and only if there is $\mu_T \in \mathrm{M}(K \times L)$ such that (4.2) holds. Hence, the decisive equivalence of (ii) and (iv) in Theorem 4.9 is essentially [9, p. 141, Thm. 11]. Schaefer incorporates these results in his systematic study of operators between Banach lattices, see [21, IV, Theorem 5.6]. However, the property $|\mu_T| = \mu_{|T|}$, essential for our application below, does not appear there. It has been stated and proved explicitly in [2, Lemma 30], but our proof is different.

4.3. The disintegration theorem

In this section we develop further the results of the previous section. The endpoint will be a "disintegration" theorem for operators of the form $\mathrm{I} - T$, where T is a symmetric absolute contraction over a compact measure space.

We start with some auxiliary results.

Proposition 4.13. *Let (K,ν) and (L,μ) be compact measure spaces and let $T : \mathrm{C}(K) \to \mathrm{L}^1(L,\mu)$ and $S : \mathrm{C}(L) \to \mathrm{L}^1(K,\nu)$ be linear operators such that*

$$\int_L Tf \cdot g \, d\mu = \int_K f \cdot Sg \, d\nu \qquad (f \in \mathrm{C}(K),\ g \in \mathrm{C}(L)). \tag{4.4}$$

If one of the operators T and S is order-bounded, then so is the other and (4.4) holds with T and S replaced by $|T|$ and $|S|$, respectively. Moreover, $\mu_T = r_*\nu_S$, where $r : L \times K \to K \times L$ is the swapping map defined by $r(x,y) = (y,x)$.

Proof. Suppose that S is order-bounded. Then, for $f \in \mathrm{C}(K)$ and $g \in \mathrm{C}(L)$ with $|g| \leq 1$,

$$\Big| \int_L Tf \cdot g \, d\mu \Big| \leq \int_K |f| \cdot |Sg| \, d\nu \leq \int_K |f| \, (|S|\mathbf{1}) d\nu.$$

It follows that T extends to a contraction $\mathrm{L}^1(K,(|S|\mathbf{1})\nu) \to \mathrm{L}^1(L,\mu)$, hence, by Theorem 4.3, T is order-bounded and $|T|'\mu \le (|S|\mathbf{1})\nu$. (Recall that the unit ball of $\mathrm{C}(L)$ is L^1-dense in the unit ball of $\mathrm{L}^\infty(L,\mu)$.)

In order to prove the first of the two remaining claims, fix $0 \le g \in \mathrm{C}(L)$, and let $f \in \mathrm{C}(K)$ and $u \in \mathrm{C}(K)$ with $|u| \le 1$. Then

$$\Big|\int_L Tf \cdot (gu)\,\mathrm{d}\mu\Big| = \Big|\int_K f \cdot S(gu)\,\mathrm{d}\nu\Big| \le \int_K |f|\,|S|g\,\mathrm{d}\nu.$$

Taking the supremum over all these functions u, we obtain

$$\int_L |Tf| \cdot g\,\mathrm{d}\mu \le \int_K |f|\,|S|g\,\mathrm{d}\nu.$$

This means that T extends to a contraction $T : \mathrm{L}^1(K,(|S|g)\nu) \to \mathrm{L}^1(L,g\mu)$. It follows that $|T|_g'(g\mu) \le (|S|g)\nu$, where $|T|_g$ denotes the modulus of T considered as an operator $\mathrm{C}(K) \to \mathrm{L}^1(L,g\mu)$. However, since $\mathrm{L}^1(L,\mu)$ "embeds" onto an ideal of $\mathrm{L}^1(L,g\mu)$, it follows that $|T|_g = |T|$. Putting things together we obtain

$$\int_L |T|f \cdot g\,\mathrm{d}\nu = \int_K f\,\mathrm{d}\,|T|_g'(g\mu) \le \int_K f \cdot |S|g\,\mathrm{d}\nu$$

for $0 \le f \in \mathrm{C}(K)$. The converse inequality holds by symmetry, and the last remaining statement is obtained by integrating both measures against functions of the form $f \otimes g$. $\square$

Suppose that $T : \mathrm{C}(K) \to \mathrm{L}^1(L,\mu)$ is order-bounded. Then $|\mu_T| = \mu_{|T|}$ by Theorem 4.9, hence by standard integration theory there is a $\mu_{|T|}$-almost everywhere unique $\lambda \in \mathrm{L}^\infty(K \times L;\mu_{|T|})$ with $|\lambda| = 1$ almost everywhere and

$$\int_{K\times L} F(x,y)\,\mathrm{d}\mu_T = \int_{K\times L} F(x,y)\lambda(x,y)\,\mathrm{d}\mu_{|T|} \tag{4.5}$$

for all $F \in \mathrm{L}^1(K \times L;\mu_{|T|})$. This leads to the following corollary for the case that $K = L$ and $\mu = \nu$.

Corollary 4.14. *Let (K,μ) be a compact measure space, let $T : \mathrm{C}(K) \to \mathrm{L}^1(K,\mu)$ be an order-bounded operator, and let $\lambda \in \mathrm{L}^\infty(K \times K,\mu_{|T|})$ with $|\lambda| = 1$ almost everywhere and such that (4.5) holds for $L = K$ and all $F \in \mathrm{L}^1(K \times L;\mu_{|T|})$. Suppose, in addition, that T is symmetric, i.e., T satisfies*

$$\int_K Tf \cdot \overline{g}\,\mathrm{d}\mu = \int_K f \cdot \overline{Tg}\,\mathrm{d}\mu \qquad (f,g \in \mathrm{C}(K)).$$

Then $|T|$ is symmetric, too, and

$$\lambda(x,y) = \overline{\lambda(y,x)} \quad \textit{for } \mu_{|T|}\textit{-almost all } (x,y) \in K^2.$$

Proof. Note that, by hypothesis, (4.4) holds with $S = \overline{T}$, hence it holds for T and S replaced by $|T|$ and $|S| = |T|$, respectively. It follows that $|T|$ is symmetric and that $r_*\mu_{|T|} = \mu_{|T|}$. The last assertion is now straightforward. $\square$

The following is the main result of this section. It has essentially been proved by Carbonaro and Dragičević [2, pp. 22/23].

Theorem 4.15 (Disintegration). *Let (K, μ) be a compact measure space, and let T be a symmetric absolute contraction on $\mathrm{L}^1(K, \mu)$. Then*

$$\int_K (\mathrm{I} - T)f \cdot g \,\mathrm{d}\mu = \int_K (\mathrm{I} - M_{|T|\mathbf{1}})f \cdot g \,\mathrm{d}\mu \\ + \int_{K\times K} \int_{\mathrm{Z}_2} \left[\mathrm{I} - \begin{pmatrix} 0 & \overline{\lambda(x,y)} \\ \lambda(x,y) & 0 \end{pmatrix} \right] \begin{pmatrix} f(x) \\ f(y) \end{pmatrix} \cdot \begin{pmatrix} g(x) \\ g(y) \end{pmatrix} \mathrm{d}\zeta_2 \,\mathrm{d}\mu_{|T|}(x,y)$$

for all $f \in \mathrm{L}^p(K, \mu)$, $g \in \mathrm{L}^q(K, \mu)$, $1 \le p \le \infty$.

Proof. We first write $\mathrm{I} - T = (\mathrm{I} - M_{|T|\mathbf{1}}) + (M_{|T|\mathbf{1}} - T)$ and then compute

$$\int_K (M_{|T|\mathbf{1}} - T)f \cdot g \,\mathrm{d}\mu = \int_K (|T|\,\mathbf{1})f \cdot g \,\mathrm{d}\mu - \int_K Tf \cdot g \,\mathrm{d}\mu \\ = \int_{K^2} \mathbf{1} \otimes fg \,\mathrm{d}\mu_{|T|} - \int_{K^2} f \otimes g \,\mathrm{d}\mu_T = \int_{K^2} \mathbf{1} \otimes fg - (f \otimes g)\lambda \,\mathrm{d}\mu_{|T|}.$$

Since T is symmetric and $\left|\overline{T}\right| = |T|$, also $|T|$ is symmetric and $\mu_{|T|}$ is a symmetric positive measure. Therefore, by a change of variable $(x, y) \mapsto (y, x)$ in the formula from above,

$$\int_K (M_{|T|\mathbf{1}} - T)f \cdot g \,\mathrm{d}\mu = \int_{K^2} fg \otimes \mathbf{1} - (g \otimes f)\overline{\lambda} \,\mathrm{d}\mu_{|T|}.$$

Taking the arithmetic average of this and the previous form we obtain the claimed formula. □

Corollary 4.16. *Let (K, μ) be a compact measure space, and let T be a symmetric sub-Markovian operator on $\mathrm{L}^1(K, \mu)$. Then*

$$\int_K (\mathrm{I} - T)f \cdot g \,\mathrm{d}\mu \\ = \int_K (\mathbf{1} - T\mathbf{1})f \cdot g \,\mathrm{d}\mu + \int_{K\times K} \int_{\mathrm{Z}_2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} f(x) \\ f(y) \end{pmatrix} \cdot \begin{pmatrix} g(x) \\ g(y) \end{pmatrix} \mathrm{d}\zeta_2 \,\mathrm{d}\mu_T(x,y)$$

for all $f \in \mathrm{L}^p(K, \mu)$, $g \in \mathrm{L}^q(K, \mu)$, $1 \le p \le \infty$.

5. Proof of the main results

Let us return to the proof of the main result, Theorem 2.2. By the reduction steps from Section 3, one can suppose from the start that $\mathrm{X} = (K, \mu)$ is a compact measure space, $A = \mathrm{I} - T$ for some symmetric absolute contraction on $\mathrm{L}^1(K, \mu)$. In particular, the Disintegration Theorem 4.15 is applicable.

Let, as in the hypothesis of Theorem 2.2, $1 \le p < \infty$, d, $m \in \mathbb{N}$ and $F_j, G_j : K \to \mathbb{C}^d$ be measurable functions for $1 \le j \le m$. The assertion to prove is:

For all measurable functions $\mathbf{f} \in \mathcal{M}(K,\mu;\mathbb{C}^d)$ *such that* $F_j(\mathbf{f}) \in \mathrm{L}^p(K,\mu)$ *and* $G_j(\mathbf{f}) \in \mathrm{L}^q(K,\mu)$ *for all* $1 \le j \le m$*:*

$$\sum_{j=1}^{m} \mathrm{Re} \int_K (\mathrm{I}-T)F_j(\mathbf{f}) \cdot G_j(\mathbf{f})\,\mathrm{d}\mu \;\ge\; 0.$$

and we may suppose that this assertion holds when (K,μ) is replaced by Z_2, and T is replaced by E_λ for each $\lambda \in \mathbb{T}$.

Lemma 5.1. *Under the given hypotheses,*

$$\mathrm{Re} \sum_{j=1}^{m} F_j(x)G_j(x) \ge 0 \quad \textit{for all } x \in \mathbb{C}^d. \tag{5.1}$$

Proof. Note that the integral inequality is *convex* in T, and that it holds trivially for $T = \mathrm{I}$. Since it holds for each $T = E_\lambda$, $\lambda \in \mathbb{T}$, it also holds for $T = \frac{1}{2}E_1 + \frac{1}{2}E_{-1} = 0$. Given $(x_1,\ldots,x_d) \in \mathbb{C}^d$, let $f_j := (x_j,x_j)^t \in \mathcal{M}(\mathrm{Z}_2)$ and inserting this into the inequality with $T = 0$ on Z_2 yields the claim. □

Suppose now that $\mathbf{f} \in \mathcal{M}(K,\mu;\mathbb{C}^d)$ such that $F_j(\mathbf{f}) \in \mathrm{L}^p(K,\mu)$ and $G_j(\mathbf{f}) \in \mathrm{L}^q(K,\mu)$. We can apply the Disintegration Theorem 4.15 and obtain, for each $j = 1,\ldots,m$

$$\begin{aligned}\int_K (\mathrm{I}-T)F_j(\mathbf{f}) \cdot G_j(\mathbf{f})\,\mathrm{d}\mu &= \int_K (\mathbf{1} - |T|\,\mathbf{1})F_j(\mathbf{f})G_j(\mathbf{f})\,\mathrm{d}\mu \\ &+ \int_{K\times K}\int_{\mathrm{Z}_2} \big(\mathrm{I} - E_{\lambda(x,y)}\big)\begin{pmatrix} F_j(\mathbf{f}(x)) \\ F_j(\mathbf{f}(y))\end{pmatrix} \cdot \begin{pmatrix} G_j(\mathbf{f}(x)) \\ G_j(\mathbf{f}(y))\end{pmatrix} \mathrm{d}\zeta_2\,\mathrm{d}\mu_{|T|}(x,y).\end{aligned}$$

Now sum over j and take the real part. Finally, apply Lemma 5.1 for the first summand and the hypothesis over $E_{\lambda(x,y)}$ for the second to conclude that the result has to be ≥ 0. Hence, Theorem 2.2 is completely proved.

The corresponding results for symmetric sub-Markovian and Markovian semigroups (Theorem 2.3, Theorem 2.4) are proved similarly. (Note that by the reduction steps in Section 3 one only needs to show the assertion for the case that $A = \mathrm{I} - T$ where T is a symmetric absolute contraction on a compact measure space (K,ν), and T is sub-Markovian or Markovian, respectively.)

In the sub-Markovian case (Theorem 2.3), the hypothesis tells in particular that the statement is true for $T = 0$ on Z_2, hence (5.1) holds. Now apply Corollary 4.16 and proceed as before.

In the Markovian case, one has $T\mathbf{1} = \mathbf{1}$ and the first summand in the disintegration formula of Corollary 4.16 vanishes. This leads to Theorem 2.4. (Note that in the Markovian case, (5.1) is not a necessary condition any more.)

6. Application: The sector of analyticity

Let $(S_t)_{t\ge 0}$ be an absolute contraction semigroup over a measure space X, and let $1 < p < \infty$. As a consequence of the Lumer–Phillips theorem, the semigroup

$(S_t)_{t\geq 0}$ extends to an analytic contraction semigroup on $\mathrm{L}^p(\mathrm{X})$ defined on the sector

$$\Sigma_\varphi := \{z \in \mathbb{C} \setminus 0 \,:\, |\arg z| < \varphi\}$$

(where $0 < \varphi \leq \frac{\pi}{2}$) if and only if

$$\mathrm{Re} \int_{\mathrm{X}} \mathrm{e}^{\pm\varphi\mathrm{i}} (Af) \cdot \overline{f}\, |f|^{p-2} \geq 0 \tag{6.1}$$

for all $f \in \mathrm{dom}(A_p)$. For some time it had been an open question whether, in the case that $(S_t)_t$ is a *symmetric* contraction semigroup, inequality (6.1) must hold for the angle $\varphi = \varphi_p$, where

$$\varphi_p := \arccos \left|1 - \tfrac{2}{p}\right| = \arctan \frac{2\sqrt{p-1}}{|p-2|} \tag{6.2}$$

for $1 < p < \infty$. Such a result had been first established by Bakry [1] for a certain subclass of sub-Markovian symmetric semigroups and later extended to all sub-Markovian symmetric semigroups by Liskevich and Perelmuter [17]. That proof was subsequently improved by Nagel and Voigt [19] and in that form became part of Chapter 3 in Ouhabaz' book [20]. The best general result for all symmetric contraction semigroups had for a long time been the one by Cowling [4], when Kriegler finally settled the case with a positive answer in [16]. Carbonaro and Dragičević showed in [2, Remark 35] that the optimal angle can be obtained also from their results.

We shall see in this section that the general symmetric case reduces to the same scalar inequality as the sub-Markovian case. We apply Theorem 2.2 with $d = m = 1$, $F(x) = x$ and $G(x) = \mathrm{e}^{\pm\mathrm{i}\varphi}\overline{x}\,|x|^{p-2}$ ($G(0) = 0$). This yields the inequality

$$\mathrm{Re}\Big(\mathrm{e}^{\pm\mathrm{i}\varphi} \begin{pmatrix} 1 & -\overline{\lambda} \\ -\lambda & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} \cdot_{\mathrm{Z}_2} \begin{pmatrix} z\,|z|^{p-2} \\ w\,|w|^{p-2} \end{pmatrix}\Big) \geq 0$$

for all choices of z, $w \in \mathbb{C}$ and $\lambda \in \mathbb{T}$. (Recall that $\cdot_{\mathrm{Z}_2}$ denotes the sesquilinear inner product on $\mathrm{L}^2(\mathrm{Z}_2)$.) If we replace w by λw in this inequality, we obtain the equivalent inequality

$$\mathrm{Re}\Big(\mathrm{e}^{\pm\mathrm{i}\varphi} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} \cdot_{\mathrm{Z}_2} \begin{pmatrix} z\,|z|^{p-2} \\ w\,|w|^{p-2} \end{pmatrix}\Big) \geq 0.$$

For $w = 0$ the inequality reduces to $|z|^p \cos\varphi \geq 0$, which poses no further restriction on φ. For $w \neq 0$ we can replace z by wz and find the equivalent inequality

$$\mathrm{Re}\Big(\mathrm{e}^{\pm\mathrm{i}\varphi} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} \cdot_{\mathrm{Z}_2} \begin{pmatrix} z\,|z|^{p-2} \\ 1 \end{pmatrix}\Big) \geq 0,$$

i.e.,

$$\mathrm{Re}\left(\mathrm{e}^{\pm\mathrm{i}\varphi}(z-1)(\overline{z}\,|z|^{p-2} - 1)\right) \geq 0.$$

Reformulating this as an inequality between real and imaginary part and letting $\varphi = \varphi_p$ as above reduces to the inequality (2.1) in [17] which is proven there. (Actually, our argument shows that the proof can be simplified since there is only one complex variable to deal with.)

Corollary 6.1 (Kriegler). *Let $-A$ be the generator of a symmetric contraction semigroup $S = (S_t)_{t\geq 0}$ over some measure space* X*, and let $1 < p < \infty$. Then S extends to an analytic semigroup of contractions on $\mathrm{L}^p(\mathrm{X})$ on the sector Σ_{φ_p}.*

Appendix: On homomorphisms of probability spaces

Suppose that $\mathrm{X} = (X, \Sigma, \mu)$ and $\mathrm{X}' = (X', \Sigma', \mu')$ are probability spaces and

$$\Phi : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{X}')$$

is a one-preserving isometric lattice homomorphism.[2] This means that Φ is an isometric embedding for the L^1-norms, $\Phi(\mathbf{1}) = \mathbf{1}$ and $|\Phi f| = \Phi\,|f|$ for all $f \in \mathrm{L}^1(\mathrm{X})$.

The positivity of Φ implies in particular that $\Phi(\overline{f}) = \overline{\Phi f}$ for all $f \in \mathrm{L}^1(\mathrm{X})$. Finally,

$$\int_{\mathrm{X}} f = \int_{\mathrm{X}'} \Phi f$$

for all $f \in \mathrm{L}^1(\mathrm{X})$, since this is true for all $f \geq 0$.

In this appendix we show how to (canonically) extend Φ to a homomorphic (as lattices and $*$-algebras) embedding

$$\Phi : \mathcal{M}(\mathrm{X}) \to \mathcal{M}(\mathrm{X}')$$

where $\mathcal{M}(\mathrm{X})$ and $\mathcal{M}(\mathrm{X}')$ denote the spaces of all measurable $\mathbb{C}$-valued functions modulo almost everywhere equality on X and X', respectively. Note that $\mathcal{M}(\mathrm{X})$ is a complete metric space with respect to the metric

$$\mathrm{d}_{\mathrm{X}}(f, g) := \int_{\mathrm{X}} \frac{|f - g|}{1 + |f - g|}.$$

The following lemma is the key property.

Lemma A.1. *In the situation from above, Φ restricts to an embedding of C^*-algebras $\Phi : \mathrm{L}^\infty(\mathrm{X}) \to \mathrm{L}^\infty(\mathrm{X}')$. Moreover, for any $f \in \mathrm{L}^1(\mathrm{X})$,*

$$\mu[\,|f| > 0\,] = \mu'[\,|\Phi f| > 0\,]$$

In particular, $[\,f = 0\,]$ is a μ-null set if and only if $[\,\Phi f = 0\,]$ is a μ'-null set.

Proof. It is clear that Φ restricts to a one-preserving isometric lattice homomorphism between the respective L^∞-spaces. So only the multiplicativity $\Phi(fg) = (\Phi f)(\Phi g)$ is to be shown. This is well known, see, e.g., [5, Chap. 7], but we repeat the argument for the convenience of the reader. By bilinearity, it suffices to consider $f, g \geq 0$. Then, by polarization, it suffices to consider $f = g$, which reduces the problem to establish that $\Phi(f^2) = (\Phi f)^2$. Now, for any $x \geq 0$, $x^2 = \sup_{t\geq 0} 2tx - t^2$.

[2] In [5, Chap. 12], this is called a *Markov embedding.* It is the functional-analytic analogue of a *factor map* (=homomorphism in the category of probability spaces) $\mathrm{X}' \to \mathrm{X}$.

Hence, $f^2 = \sup_{t\geq 0} 2tf - t^2\mathbf{1}$ in the Banach lattice sense. But Φ is a lattice homomorphism and $\Phi\mathbf{1} = \mathbf{1}$, therefore

$$\Phi(f^2) = \Phi\big(\sup_{t\geq 0} 2tf - t^2\mathbf{1}\big) = \sup_{t\geq 0} 2t(\Phi f) - t^2\mathbf{1} = (\Phi f)^2.$$

The remaining statement follows from:

$$\begin{aligned}\mu[\,|f| > 0\,] &= \lim_{n\to\infty} \int_{\mathrm{X}} (n\,|f| \wedge \mathbf{1}) = \lim_{n\to\infty} \int_{\mathrm{X}'} \Phi(n\,|f| \wedge \mathbf{1}) \\ &= \lim_{n\to\infty} \int_{\mathrm{X}'} n\,|\Phi f| \wedge \mathbf{1} = \mu'[\,|\Phi f| > 0\,]\,. \qquad \square\end{aligned}$$

Let $f \in \mathcal{M}(\mathrm{X})$. Then the function $e := \frac{1}{1+|f|}$ has the property that $e, ef \in \mathrm{L}^\infty(\mathrm{X})$. Moreover, by Lemma A.1, $[\,\Phi e = 0\,]$ is a μ'-null set. Hence, Φe is an invertible element in the algebra $\mathcal{M}(\mathrm{X}')$, and we can define

$$\widehat{\Phi} f := \frac{\Phi(ef)}{\Phi e} \in \mathcal{M}(\mathrm{X}').$$

Lemma A.2. *The so-defined mapping* $\widehat{\Phi} : \mathcal{M}(\mathrm{X}) \to \mathcal{M}(\mathrm{X}')$ *has the following properties:*

a) $\widehat{\Phi}$ *is an extension of* Φ.

b) $\widehat{\Phi}$ *is a unital* $*$*-algebra and lattice homomorphism.*

c) $\int_{\mathrm{X}'} \widehat{\Phi} f = \int_{\mathrm{X}} f$ *whenever* $0 \leq f \in \mathcal{M}(\mathrm{X})$.

d) $\widehat{\Phi}$ *is an isometry with respect to the canonical metrics* d_{X} *and* $\mathrm{d}_{\mathrm{X}'}$.

e) *If* Φ *is bijective then so is* $\widehat{\Phi}$.

f) *The mapping* $\widehat{\Phi} : \mathcal{M}(\mathrm{X}) \to \mathcal{M}(\mathrm{X}')$ *is uniquely determined by the property that it extends* Φ *and it is multiplicative, i.e., satisfies* $\widehat{\Phi}(fg) = \widehat{\Phi} f \cdot \widehat{\Phi} g$ *for all* $f,\, g \in \mathcal{M}(\mathrm{X})$.

Proof. a) and b) This is straightforward and left to the reader.
c) By the monotone convergence theorem,

$$\begin{aligned}\int_{\mathrm{X}} f &= \sup_{n\in\mathbb{N}} \int_{\mathrm{X}} (f \wedge n\mathbf{1}) = \sup_{n\in\mathbb{N}} \int_{\mathrm{X}} \Phi(f \wedge n\mathbf{1}) = \sup_{n\in\mathbb{N}} \int_{\mathrm{X}'} \widehat{\Phi}(f \wedge n\mathbf{1}) \\ &= \sup_{n\in\mathbb{N}} \int_{\mathrm{X}'} (\widehat{\Phi} f \wedge n\mathbf{1}) = \int_{\mathrm{X}'} \widehat{\Phi} f.\end{aligned}$$

d) Follows from b) and c).
e) Suppose that $\mathrm{L}^\infty(\mathrm{X}') \subseteq \operatorname{ran}(\Phi)$ and let $g \in \mathcal{M}(\mathrm{X}')$ be arbitrary. Then, by Lemma A.1, there are $e, h \in \mathrm{L}^\infty(\mathrm{X})$ such that

$$\Phi e = \frac{1}{1+|g|} \quad \text{and} \quad \Phi h = \frac{g}{1+|g|} = g\,\Phi e.$$

Again by Lemma A.1, $\mu[\,e = 0\,] = 0$, which is why we can define $f := \frac{h}{e} \in \mathcal{M}(\mathrm{X})$. It follows that $\Phi f = g$.
f) Suppose that $\Psi : \mathcal{M}(\mathrm{X}) \to \mathcal{M}(\mathrm{X}')$ is multiplicative and extends Φ. Let $f \in \mathcal{M}(\mathrm{X})$ and define $e := \frac{1}{1+|f|}$ as before. Then f, $ef \in \mathrm{L}^\infty(\mathrm{X})$ and hence

$$\Phi e \cdot \Psi f = \Psi e \cdot \Psi f = \Psi(ef) = \Phi(ef).$$

Since Φe is an invertible element in $\mathcal{M}(\mathrm{X}')$ (as seen above), it follows that

$$\Psi f = \frac{\Phi(ef)}{\Phi e} = \widehat{\Phi} f$$

as claimed. □

By abuse of notation, we write Φ again instead of $\widehat{\Phi}$. It is clear that Φ allows a further extension to $\mathbb{C}^d$-valued functions by

$$\Phi(\mathbf{f}) = \Phi(f_1, \ldots, f_d) := (\Phi f_1, \ldots, \Phi f_d) \quad \text{for } \mathbf{f} = (f_1, \ldots, f_d) \in \mathcal{M}(\mathrm{X}; \mathbb{C}^d).$$

Now we are well prepared for the final result of this appendix.

Theorem A.3. *Let* X *and* X′ *be probability spaces, and let* $\Phi : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{X}')$ *be a one-preserving isometric lattice isomorphism, with its canonical extension* $\Phi : \mathcal{M}(\mathrm{X}; \mathbb{C}^d) \to \mathcal{M}(\mathrm{X}'; \mathbb{C}^d)$, $d \in \mathbb{N}$. *Then*

$$\Phi\big(F(\mathbf{f})\big) = F(\Phi\mathbf{f}) \qquad \textit{almost everywhere} \tag{A.3}$$

for every Borel measurable function $F : \mathbb{C}^d \to \mathbb{C}$ *and every* $\mathbf{f} \in \mathcal{M}(\mathrm{X}; \mathbb{C}^d)$.

Proof. By linearity we may suppose that $F \geq 0$. Next, by approximating $F \wedge n\mathbf{1} \nearrow F$, we may suppose that F is bounded. Then F is a uniform limit of positive simple functions, hence we may suppose without loss of generality that $F = \mathbf{1}_B$, where B is a Borel set in $\mathbb{C}^d$. In this case, (A.3) becomes

$$\Phi\big(\mathbf{1}_{[\,(f_1,\ldots,f_d)\in B\,]}\big) = \mathbf{1}_{[\,(\Phi f_1,\ldots,\Phi f_d)\in B\,]} \qquad \text{almost everywhere.}$$

Let $\mathcal{B}$ be the set of all Borel subsets of $\mathbb{C}^d$ that satisfy this. Then $\mathcal{B}$ is a Dynkin system, so it suffices to show that each rectangle is contained in $\mathcal{B}$. Since Φ is multiplicative, this reduces the case to $d = 1$, f is real valued and $B = (a, b]$. Now $[\,a < f \leq b\,] = [\,a < f\,] \cap [\,b < f\,]^c$, which reduces the situation to $B = (a, \infty)$. Now

$$\mathbf{1}_{[\,a<f\,]} = \mathrm{L}^1\text{-}\lim_{n\to\infty} n(f - a\mathbf{1})^+ \wedge \mathbf{1},$$

and applying Φ concludes the proof. □

Remarks A.4.

1) As a consequence of Theorem A.3, $\Phi\,|f|^p = |\Phi f|^p$ for any $f \in \mathcal{M}(\mathrm{X})$ and $p > 0$, so Φ restricts to an isometric isomorphism of L^p-spaces for each $p > 0$.
2) The extension of the original L^1-isomorphism Φ to $\mathcal{M}(\mathrm{X})$ is uniquely determined by the requirement that Φ is continuous for the metrics d_X and $\mathrm{d}_{\mathrm{X}'}$.

3) One can extend Φ to a lattice homomorphism

$$\Phi : \mathcal{M}(\mathrm{X}; [0, \infty]) \to \mathcal{M}(\mathrm{X}'; [0, \infty])$$

by defining $\Phi f := \tau^{-1} \circ \Phi(\tau \circ f)$, where $\tau : [0, \infty] \to [0, 1]$ is any order-preserving bijection. Using this one can then show that Φ maps almost everywhere convergent sequences to almost everywhere convergent sequences.

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References

[1] Bakry, D. Sur l'interpolation complexe des semigroupes de diffusion. In *Séminaire de Probabilités, XXIII*, vol. 1372 of *Lecture Notes in Math.* Springer, Berlin, 1989, pp. 1–20.

[2] Carbonaro, A., and Dragicevic, O. Functional calculus for generators of symmetric contraction semigroups. arXiv:1308.1338v1.

[3] Chacon, R.V., and Krengel, U. Linear modulus of linear operator. *Proc. Amer. Math. Soc.* 15 (1964), 553–559.

[4] Cowling, M.G. Harmonic analysis on semigroups. *Ann. of Math.* (2) 117, 2 (1983), 267–283.

[5] Eisner, T., Farkas, B., Haase, M., and Nagel, R. *Operator Theoretic Aspects of Ergodic Theory*, vol. 272 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2015.

[6] Fukushima, M., Oshima, Y., and Takeda, M. *Dirichlet forms and symmetric Markov processes*, extended ed., vol. 19 of *de Gruyter Studies in Mathematics.* Walter de Gruyter & Co., Berlin, 2011.

[7] Gelfand, I. On normed rings. *C. R. (Doklady) Acad. Sci. URSS, n. Ser.* 23 (1939), 430–432.

[8] Gelfand, I., and Neumark, M. On the imbedding of normed rings into the ring of operators in Hilbert space. *Rec. Math. [Mat. Sbornik] N.S.* 12(54) (1943), 197–213.

[9] Grothendieck, A. *Produits tensoriels topologiques et espaces nucléaires*, vol. 16 of *Mem. Amer. Math. Soc.* American Mathematical Society, Providence, 1955.

[10] Haase, M. Convexity inequalities for positive operators. *Positivity* 11, 1 (2007), 57–68.

[11] Halmos, P.R. On a theorem of Dieudonné. *Proc. Nat. Acad. Sci. U.S.A.* 35 (1949), 38–42.

[12] Huang, S.-Z. Inequalities for submarkovian operators and submarkovian semigroups. *Math. Nachr.* 243 (2002), 75–91.

[13] Kakutani, S. Concrete representation of abstract (L)-spaces and the mean ergodic theorem. *Ann. of Math.* (2) 42 (1941), 523–537.

[14] Kakutani, S. Concrete representation of abstract (M)-spaces. (A characterization of the space of continuous functions.) *Ann. of Math.* (2) 42 (1941), 994–1024.

[15] Kantorovitch, L. Linear operations in semi-ordered spaces. I. *Rec. Math. Moscou, n. Ser.* 7 (1940), 209–284.

[16] Kriegler, C. Analyticity angle for non-commutative diffusion semigroups. *J. Lond. Math. Soc.* (2) 83, 1 (2011), 168–186.

[17] Liskevich, V.A., and Perel′muter, M.A. Analyticity of sub-Markovian semigroups. *Proc. Amer. Math. Soc.* 123, 4 (1995), 1097–1104.

[18] Luxemburg, W.A.J., and Zaanen, A.C. The linear modulus of an order bounded linear transformation. I & II. *Nederl. Akad. Wetensch. Proc. Ser. A* **74**=*Indag. Math.* 33 (1971), 422–447.

[19] Nagel, R., and Voigt, J. On inequalities for symmetric sub-Markovian operators. *Arch. Math.* (*Basel*) 67, 4 (1996), 308–311.

[20] Ouhabaz, E.M. *Analysis of heat equations on domains*, vol. 31 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2005.

[21] Schaefer, H.H. *Banach lattices and positive operators*. Springer-Verlag, New York-Heidelberg, 1974. Die Grundlehren der mathematischen Wissenschaften, Band 215.

[22] Segal, I.E. Equivalences of measure spaces. *Amer. J. Math.* 73 (1951), 275–313.

[23] Stein, E.M. *Topics in harmonic analysis related to the Littlewood–Paley theory.* Annals of Mathematics Studies, No. 63. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1970.

[24] Stone, M.H. Applications of the theory of Boolean rings to general topology. *Trans. Amer. Math. Soc.* 41, 3 (1937), 375–481.

[25] Voigt, J. One-parameter semigroups acting simultaneously on different L_p-spaces. *Bull. Soc. Roy. Sci. Liège* 61, 6 (1992), 465–470.

Markus Haase
Christian-Albrechts Universität zu Kiel
Mathematisches Seminar
Ludewig-Meyn-Straße 4
D-24098 Kiel, Germany
e-mail: haase@math.uni-kiel.de

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The Isomorphism Problem for Complete Pick Algebras: A Survey

Guy Salomon and Orr Moshe Shalit

Abstract. Complete Pick algebras – these are, roughly, the multiplier algebras in which Pick's interpolation theorem holds true – have been the focus of much research in the last twenty years or so. All (irreducible) complete Pick algebras may be realized concretely as the algebras obtained by restricting multipliers on Drury–Arveson space to a subvariety of the unit ball; to be precise: every irreducible complete Pick algebra has the form $\mathcal{M}_V = \{f\big|_V : f \in \mathcal{M}_d\}$, where $\mathcal{M}_d$ denotes the multiplier algebra of the Drury–Arveson space H^2_d, and V is the joint zero set of some functions in $\mathcal{M}_d$. In recent years several works were devoted to the classification of complete Pick algebras in terms of the complex geometry of the varieties with which they are associated. The purpose of this survey is to give an account of this research in a comprehensive and unified way. We describe the array of tools and methods that were developed for this program, and take the opportunity to clarify, improve, and correct some parts of the literature.

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1. Introduction

1.1. Motivation and background

Consider the following two classical theorems.

Theorem A (Gelfand, [18]). *Let X and Y be two compact Hausdorff spaces. The algebras of continuous functions $C(X)$ and $C(Y)$ are isomorphic if and only if X and Y are homeomorphic.*

Theorem B (Bers, [7]). *Let U and V be open subsets of $\mathbb{C}$. The algebras of holomorphic functions $\mathrm{Hol}(U)$ and $\mathrm{Hol}(V)$ are isomorphic if and only if U and V are biholomorphic.*

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The common theme of these two theorems is that an appropriate algebra of functions on a space encapsulates in its algebraic structure every aspect of the topological/complex-geometric structure of the space. The problem that we are concerned with in this paper has a very similar flavour. Let $\mathcal{M}_d$ denote the algebra of multipliers on Drury–Arveson space – precise definitions will be given in the next section, for now it suffices to say that $\mathcal{M}_d$ is a certain algebra of bounded analytic functions on the unit ball $\mathbb{B}_d \subseteq \mathbb{C}^d$. For every analytic variety $V \subseteq \mathbb{B}_d$ one may define the algebra

$$\mathcal{M}_V = \{f\big|_V : f \in \mathcal{M}_d\}.$$

The natural question to ask is: in what ways does the variety V determine the algebra $\mathcal{M}_V$, and vice versa? In other words, if $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic, can we conclude that V and W are "isomorphic" in some sense? Conversely, if V and W are, say, biholomorphic, can we conclude that the algebras are isomorphic?

As we shall explain below, $\mathcal{M}_V$ is also an operator algebra: it is the multiplier algebra of a certain reproducing kernel Hilbert space on V, and it is generated by the multiplication operators $[M_{z_i}h](z) = z_i h(z)$ (it will be convenient to denote henceforth $Z_i = M_{z_i}$). Thus one can ask: do the Banach algebraic or operator algebraic structures of $\mathcal{M}_V$ encode finer complex-geometric aspects of V?

These questions in themselves are interesting, natural, nontrivial, and studying them involves a collection of tools combining function theory, complex geometry and operator theory. However, it is worth noting that there are routes, other than analogy with Theorems A and B, that lead one to study the structure and classify the algebras $\mathcal{M}_V$ described above.

One path that leads to considering the algebras $\mathcal{M}_V$ comes from non-selfadjoint operator algebras: it is the study of operator algebras universal with respect to some polynomial relations. For simplicity consider the case in which $V = \mathcal{Z}_{\mathbb{B}_d}(\mathcal{I})$ is the zero set of a radical and homogeneous polynomial ideal $\mathcal{I} \triangleleft \mathbb{C}[z_1, \ldots, z_d]$, where

$$\mathcal{Z}_{\mathbb{B}_d}(\mathcal{I}) = \{\lambda \in \mathbb{B}_d \mid p(\lambda) = 0 \text{ for all } p \in \mathcal{I}\}.$$

Then $\mathcal{M}_V$ is the universal WOT-closed unital operator algebra, that is generated by a pure commuting row contraction $T = (T_1, \ldots, T_d)$ satisfying the relations in $\mathcal{I}$ (see [26, 30]). This means that

1. The d-tuple of operators $(Z_1, \ldots, Z_d)$, given by multiplication by the coordinate functions, is a pure, commuting row contraction satisfying the relations in $\mathcal{I}$, and it generates $\mathcal{M}_V$;
2. For any such tuple T, there is a unital, completely contractive and WOT-continuous homomorphism from $\mathcal{M}_V$ into $\overline{\mathrm{Alg}}^{\mathrm{WOT}}(1, T)$ determined by $Z_i \mapsto T_i$.

In general (when V is not necessarily the variety of a homogeneous polynomial ideal) it is a little more complicated to explain the universal property of $\mathcal{M}_V$.

Roughly, $\mathcal{M}_V$ is universal for tuples "satisfying the relations" in $J_V = \{f \in \mathcal{M}_d \mid f(\lambda) = 0 \text{ for all } \lambda \in V\}$.

Thus the algebras $\mathcal{M}_V$ are an operator algebraic version of the coordinate ring on an algebraic variety, and studying the relations between the structure of $\mathcal{M}_V$ and the geometry of V can be considered as rudimentary steps in developing "operator algebraic geometry".

A different road that leads one to consider the collection of algebras $\mathcal{M}_V$ runs from function theory, in particular from the theory of Pick interpolation. Let $\mathcal{H}$ be a reproducing kernel Hilbert space on a set X with kernel k. If $x_1, \ldots, x_n \in X$ and $A_1, \ldots, A_n \in M_k(\mathbb{C})$, then one may consider the problem of finding a matrix-valued multiplier $F : X \to M_k(\mathbb{C})$ which has multiplier norm 1 and satisfies

$$F(x_i) = A_i \ , \ i = 1, \ldots, d.$$

This is called the *Pick interpolation problem*. It is not hard to show that a necessary condition for the existence of such a multiplier is that the following matrix inequality hold:

$$[(1 - F(x_i)F(x_j)^*)K(x_i, x_j)]_{i,j=1}^n \geq 0. \tag{1.1}$$

G. Pick showed that for the Szegő kernel $k(z, w) = (1 - z\bar{w})^{-1}$ the condition (1.1) is also a sufficient condition for the existence of a solution to this problem [25]. Kernels for which condition (1.1) is a sufficient condition for the existence of a solution to the Pick interpolation problem have come to be called *complete Pick kernels*, and their multiplier algebras *complete Pick algebras*. We refer the reader to the monograph [2] for thorough introduction to Pick interpolation and complete Pick kernels. The connection to our problem is the following theorem, which states that under a harmless irreducibility assumption all complete Pick algebras are completely isometrically isomorphic to one of the algebras $\mathcal{M}_V$ described above.

Theorem C (Agler–McCarthy, [1]). *Let $\mathcal{H}$ be a reproducing kernel Hilbert space with an irreducible complete kernel k. Then there exists $d \in \mathbb{N} \cup \{\infty\}$ and there is an analytic subvariety $V \subseteq \mathbb{B}_d$ such that the multiplier algebra Mult($\mathcal{H}$) of $\mathcal{H}$ is unitarily equivalent to $\mathcal{M}_V$.*

In fact the theorem of Agler–McCarthy says much more: the Hilbert space $\mathcal{H}$ can (up to some rescaling) be considered as a Hilbert space of functions on V, which is a subspace of the Drury–Arveson space. Since we require this result only for motivation, we do not go into further detail.

Thus, by studying the algebras $\mathcal{M}_V$ in terms of the complex-geometric structure of V one may hope to obtain a structure theory of irreducible complete Pick algebras. In particular, we may hope to use the varieties as complete invariants of irreducible complete Pick algebras up to isomorphism – be it algebraic, isometric or spatial. This is why we call this study *The Isomorphism Problem for Complete Pick Algebras*.

1.2. About this survey

The goal of this survey is to present in a unified way the main results on the isomorphism problem for complete Pick algebras obtained in recent years. We do not provide all the proofs, but we do give proofs (or at least an outline) to most key results, in order to highlight the techniques involved. We give precise references so that all omitted details can be readily found by the interested reader. We also had to omit some results, but all results directly related to this survey may be found in the cited references.

Although one may treat the case where $V \subseteq \mathbb{B}_d$ and $W \subseteq \mathbb{B}_{d'}$ where d and d' might be different, we will only treat the case where $d = d'$. It is easy to see that this simplification results in no real loss.

This paper also contains some modest improvements to the results appearing in the literature. In some cases we unify, in others we simplify the proof somewhat, in one case we were able to extend a result from $d < \infty$ to $d = \infty$ (see Theorem 4.8). There is also one case where we correct a mistake that appeared in an earlier paper (see Remark 4.4).

Furthermore, we take this opportunity to call to attention a little mess that resides in the literature, and try to set it right. (The reader may skip the following paragraph and return to it after reading Section 2.5.) The results we review in this survey are based directly on results in the papers [4, 5, 10, 15, 16, 20, 23]. The papers [10, 16] relied in a significant way on many earlier results of Davidson and Pitts [12, 13, 14], and in particular on [12, Theorem 3.2]. The content of that theorem, phrased in the language of this survey, is that over every point of V there lies a unique character in the maximal ideal space $M(\mathcal{M}_V)$, and moreover that there are no characters over points of $\mathbb{B}_d \setminus V$. Unfortunately, at the time that the papers [10, 16] were in press it was observed by Michael Hartz that [12, Theorem 3.2] is true only under the assumption $d < \infty$, a counter example shows that it is false for $d = \infty$ (see the example on the first page of [11], or Example 2.4 in the arXiv version of [10]).

Luckily, the main results of [10, 16] survived this disaster, but significant changes in the arguments were required, and some of the results survived in a weaker form. The paper [10] has an erratum [11], and [16] contains some corrections made in proof. However, thorough revisions of the papers [10, 16] appeared on the arXiv, and when we refer to these papers we refer to the arXiv versions. We direct the interested reader to the arXiv versions.

1.3. Overview of main results

Sections 2 and 3 contain some basic results which are used in all of the classification schemes. The main results are presented in Sections 4, 5 and 6, which can be read independently after Sections 2 and 3. Some open problems are discussed in the final section.

The following table summarizes what is known and what is not known regarding the isomorphism problem of the algebras $\mathcal{M}_V$, where V is a variety in a finite-dimensional ball. (In some cases the result holds for $d = \infty$, see caption.)

Conditions on V, W	**Type of isomorphism** $\mathcal{M}_V \cong \mathcal{M}_W$	**Type of isomorphism** $V \cong W$	$\Rightarrow$	$\Leftarrow$	**Reference**
	Weak-$*$ continuous	Multiplier biholomorphic	$\surd$	$\times$	Corollary 3.4 Example 5.7
	Isometric	There is $F \in \mathrm{Aut}(\mathbb{B}_d)$ s.t. $F(W) = V$	$\surd$	$\surd$	Proposition 4.8 Theorem 4.6
	Completely isometric	There is $F \in \mathrm{Aut}(\mathbb{B}_d)$ s.t. $F(W) = V$	$\surd$	$\surd$	Theorem 4.6
	Unitary equivalence	There is $F \in \mathrm{Aut}(\mathbb{B}_d)$ s.t. $F(W) = V$	$\surd$	$\surd$	Theorem 4.6
Finite union of irreducible varieties and a discrete variety	Algebraic	Multiplier biholomorphic	$\surd$	$\times$	Theorem 5.5 Example 5.7
Irreducible	Algebraic	Multiplier biholomorphic	$\surd$	?	Theorem 5.5 Subsection 7.1
Homogeneous	Algebraic	There is $A \in \mathrm{GL}_d(\mathbb{C})$ s.t. $A(W) = V$	$\surd$	$\surd$	Theorem 5.14
Homogeneous	Algebraic	Biholomorphic	$\surd$	$\surd$	Theorem 5.14
Images of finite Riemann surfaces under a holomap that extends to be a 1-to-1 C^2-map on the boundary	Algebraic	Biholomorphism that extends to be a 1-to-1 C^2-map on the boundary	?	$\surd$	Corollary 5.18
Embedded discs	Algebraic	Biholomorphic	$\surd$	$\times$	Example 5.21

TABLE 1. Isomorphisms of varieties in $\mathbb{B}_d$ for $d < \infty$ corresponding to isomorphisms of the associated multiplier algebras. The first four lines also hold for $d = \infty$ with minor adjustments.

2. Notation and preliminaries

2.1. Basic notation

It this survey, d always stands for a positive integer or $\infty = \aleph_0$. The d-dimensional Hilbert space over $\mathbb{C}$ is denoted by $\mathbb{C}^d$ (when $d = \infty$, $\mathbb{C}^d$ stands for ℓ^2), and $\mathbb{B}_d$ denotes the open unit ball of $\mathbb{C}^d$. When $d = 1$, we usually write $\mathbb{D}$ instead of $\mathbb{B}_d$.

2.2. The Drury–Arveson space

Let H^2_d be the *Drury–Arveson space* (see [29]). H^2_d is the reproducing Hilbert space on $\mathbb{B}_d$, the unit ball of $\mathbb{C}^d$, with kernel functions

$$k_\lambda(z) = \frac{1}{1 - \langle z, \lambda \rangle} \quad \text{for } z, \lambda \in \mathbb{B}_d.$$

We denote by $\mathcal{M}_d$ the multiplier algebra $\mathrm{Mult}(H^2_d)$ of H^2_d.

2.3. Varieties and their reproducing kernel Hilbert spaces

We will use the term *analytic variety* (or just a *variety*) to refer to the common zero set of a family of H^2_d-functions. If $\mathcal{E}$ is a set of functions on $\mathbb{B}_d$ which is contained in H^2_d, let

$$V(\mathcal{E}) := \{\lambda \in \mathbb{B}_d \ : \ f(\lambda) = 0 \text{ for all } f \in \mathcal{E}\}.$$

On the ther hand, if S is a subset of $\mathbb{B}_d$ let

$$H_S := \{f \in H^2_d \ : \ f(\lambda) = 0 \text{ for all } \lambda \in S\},$$

and

$$J_S := \{f \in \mathcal{M}_d \ : \ f(\lambda) = 0 \text{ for all } \lambda \in S\}.$$

Proposition 2.1 ([16], Proposition 2.1). *Let $\mathcal{E}$ be a subset of H^2_d, and let $V = V(\mathcal{E})$. Then*

$$V = V(J_V).$$

Given an analytic variety V, we also define

$$\mathcal{F}_V := \overline{\mathrm{span}}\{k_\lambda \ : \ \lambda \in V\}.$$

This Hilbert space is naturally a reproducing kernel Hilbert space of functions living on the variety V.

Proposition 2.2 ([16], Proposition 2.3). *Let $S \subseteq \mathbb{B}_d$. Then*

$$\mathcal{F}_S := \overline{\mathrm{span}}\{k_\lambda \ : \ \lambda \in S\} = \mathcal{F}_{V(H_S)} = \mathcal{F}_{V(J_S)}.$$

2.4. The multiplier algebra of a variety

The reproducing kernel Hilbert space $\mathcal{F}_V$ comes with its multiplier algebra $\mathcal{M}_V = \mathrm{Mult}(\mathcal{F}_V)$. This is the algebra of all functions f on V such that $fh \in \mathcal{F}_V$ for all $h \in \mathcal{F}_V$. A standard argument shows that each multiplier determines a bounded linear operator $M_f \in B(\mathcal{F}_V)$ given by $M_f h := fh$. We will usually identify the function f with its multiplication operator M_f. We will also identify the subalgebra of $B(\mathcal{F}_V)$ consisting of the M_f's and the algebra of functions $\mathcal{M}_V$ (endowed with the same norm). We let Z_i denote both the multiplier corresponding to the ith coordinate function $z \mapsto z_i$, as well as the multiplication operator it gives rise to. In some cases, for emphasis, we write $Z_i\big|_V$ instead of Z_i.

Now consider the map from $\mathcal{M}_d$ into $B(\mathcal{F}_V)$ sending each multiplier f to $P_{\mathcal{F}_V} M_f|_{\mathcal{F}_V}$. One verifies that this map coincides with the map $f \mapsto f|_V$ and therefore its kernel is J_V. Thus, the multiplier norm of $f|_V$, for $f \in \mathcal{M}_d$, is $\|f + J_V\| = \|P_{\mathcal{F}_V} M_f|_{\mathcal{F}_V}\|$. The complete Nevanlinna–Pick property then implies that this map is completely isometric onto $\mathcal{M}_V$. This gives rise to the following proposition.

Proposition 2.3 ([16], Proposition 2.6). *Let V be an analytic variety in $\mathbb{B}_d$. Then*

$$\mathcal{M}_V = \{f|_V \ : \ f \in \mathcal{M}_d\}.$$

Moreover the mapping $\varphi : \mathcal{M}_d \to \mathcal{M}_V$ given by $\varphi(f) = f|_V$ induces a completely isometric isomorphism and weak-$*$ continuous homeomorphism of $\mathcal{M}_d/J_V$ onto $\mathcal{M}_V$. For any $g \in \mathcal{M}_V$ and any $f \in \mathcal{M}_d$ such that $f|_V = g$, we have $M_g = P_{\mathcal{F}_V} M_f|_{\mathcal{F}_V}$. Given any $F \in M_k(\mathcal{M}_V)$, one can choose $\widetilde{F} \in M_k(\mathcal{M}_d)$ so that $\widetilde{F}|_V = F$ and $\|\widetilde{F}\| = \|F\|$.

In the above proposition we referred to the weak-$*$ topology in $\mathcal{M}_V$; this is the weak-$*$ topology which $\mathcal{M}_V$ naturally inherits from $B(\mathcal{F}_V)$ by virtue of being a WOT-closed (hence weak-$*$ closed) subspace. The fact that $\mathcal{M}_V$ is a dual space has significant consequences for us. It is also useful to know the following.

Proposition 2.4 ([16], Lemma 3.1). *Let V be a variety in $\mathbb{B}_d$. Then the weak-$*$ and the weak-operator topologies on $\mathcal{M}_V$ coincide.*

2.5. The character space of $\mathcal{M}_V$

Let $\mathcal{A}$ be a unital Banach algebra. A *character* on $\mathcal{A}$ is a nonzero multiplicative linear functional. The set of all characters on $\mathcal{A}$, endowed with the weak-$*$ topology, is called the *character space* of $\mathcal{A}$, and will be denoted by $M(\mathcal{A})$. It is easy to check that a character is automatically unital and continuous with norm 1. If furthermore $\mathcal{A}$ is an operator algebra, then its characters are automatically completely contractive [24, Proposition 3.8].

The algebras we consider are semi-simple commutative Banach algebras, thus one might expect that the maximal ideal space will be a central part of the classification. However, these algebras are not uniform algebras; moreover, the topological space $M(\mathcal{M}_V)$ can be rather wild. Thus the classification does not use $M(\mathcal{M}_V)$ directly, but rather a subset of characters that can be identified with a subset of $\mathbb{B}_d$ and can be endowed with additional structure.

Let V be a variety in $\mathbb{B}_d$. Since $(Z_1, \dots, Z_d)$ is a row contraction, it holds that

$$\|(\rho(Z_1), \dots, \rho(Z_d))\| \le 1 \quad \text{for all } \rho \in M(\mathcal{M}_V).$$

The map $\pi : M(\mathcal{M}_V) \to \overline{\mathbb{B}_d}$, given by

$$\pi(\rho) = (\rho(Z_1), \dots, \rho(Z_d)),$$

is continuous as a map from $M(\mathcal{M}_V)$, with the weak-$*$ topology, into $\overline{\mathbb{B}_d}$ (endowed with the weak topology, in case $d = \infty$). Since π is continuous, $\pi(M(\mathcal{M}_V))$ is a compact subset of the closed unit ball. For every $\lambda \in \pi(M(\mathcal{M}_V))$, the set $\pi^{-1}\{\lambda\} \subseteq M(\mathcal{M}_V)$ is called the *fiber* over λ.

For every $\lambda \in V$, the fiber over λ contains the *evaluation functional* ρ_λ, which is given by

$$\rho_\lambda(f) = f(\lambda)\ ,\ f \in \mathcal{M}_V.$$

The following two results are crucial for much of the analysis of the algebras $\mathcal{M}_V$.

Proposition 2.5 ([16], Proposition 3.2). *V can be identified with the* WOT*-continuous characters of $\mathcal{M}_V$ via the correspondence $\lambda \leftrightarrow \rho_\lambda$.*

Proposition 2.6 ([16], Proposition 3.2). *If $d < \infty$, then*

$$\pi(M(\mathcal{M}_V)) \cap \mathbb{B}_d = V,$$

and for every $\lambda \in V$ the fiber over λ, that is $\pi^{-1}\{\lambda\}$, is a singleton.

2.6. Metric structure in $M(\mathcal{M}_V)$

Let $\nu \in \mathbb{B}_d$, and let Φ_ν be the automorphism of the ball that exchanges ν and 0 (see [28, p. 25]):

$$\Phi_\nu(z) := \frac{\nu - P_\nu z - s_\nu Q_\nu z}{1 - \langle z, \nu \rangle},$$

where

$$P_\nu = \begin{cases} \frac{\langle z,\nu\rangle}{\langle \nu,\nu\rangle}\nu & \text{if } \nu \neq 0, \\ 0 & \text{if } \nu = 0 \end{cases}, \quad Q_\nu = I - P_\nu, \quad \text{and} \quad s_\nu = (1 - \|\nu\|^2)^{\frac{1}{2}}.$$

If $\mu \in \mathbb{B}_d$ is another point, the *pseudohyperbolic distance* between μ and ν is defined to be

$$d_{\mathrm{ph}}(\mu, \nu) := \|\Phi_\nu(\mu)\| = \|\Phi_\mu(\nu)\|.$$

One can check that the pseudohyperbolic distance defines a metric on the open ball.

The following proposition will be useful in the sequel. Among other things it will imply that the metric structure induced on V by the pseudohyperbolic metric is an invariant of $\mathcal{M}_V$.

Proposition 2.7 ([16], Lemma 5.3). *Let V be a variety in $\mathbb{B}_d$.*

(a) *Let $\mu \in \partial\mathbb{B}_d$ and let $\varphi \in \pi^{-1}(\mu)$. Suppose that $\psi \in M(\mathcal{M}_V)$ satisfies $\|\psi - \varphi\| < 2$. Then $\psi \in \pi^{-1}(\mu)$.*

(b) *If $\mu, \nu \in V$, then*

$$d_{\mathrm{ph}}(\mu, \nu) = \frac{\|\rho_\mu - \rho_\nu\|}{\sup_{\|f\| \leq 1} \left|1 - f(\mu)\overline{f(\nu)}\right|}.$$

As a result,

$$d_{\mathrm{ph}}(\mu, \nu) \leq \|\rho_\mu - \rho_\nu\| \leq 2 d_{\mathrm{ph}}(\mu, \nu).$$

3. Weak-$*$ continuous isomorphisms

Let V and W be two varieties in $\mathbb{B}_d$. We say that V and W are *biholomorphic* if there exist holomorphic maps $F : \mathbb{B}_d \to \mathbb{C}^d$ and $G : \mathbb{B}_d \to \mathbb{C}^d$ such that $G \circ F|_V = \mathbf{id}_V$ and $F \circ G|_W = \mathbf{id}_W$. If furthermore the coordinate functions of F are multipliers, then we say that V and W are *multiplier biholomorphic*.

In this section we will see that in the finite-dimensional case, if there is a weak-$*$ continuous isomorphism between two multiplier algebras $\mathcal{M}_V$ and $\mathcal{M}_W$, then V and W are multiplier biholomorphic. We start with the following proposition, which is a basic tool in the theory.

Proposition 3.1 ([16], Proposition 3.4). *Let V and W be two varieties in $\mathbb{B}_d$, and let $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ be a unital homomorphism. Then φ gives rise to a function $F_\varphi : W \to \overline{\mathbb{B}_d}$ by*

$$F_\varphi = \pi \circ \varphi^*|_W.$$

Moreover, there exist multipliers $F_1, F_2, \ldots, F_d \in \mathcal{M}$ such that

$$F_\varphi = (F_1|_W, F_2|_W, \ldots, F_d|_W).$$

Furthermore, if φ is completely bounded or $d < \infty$, then F_φ extends to a holomorphic function defined on $\mathbb{B}_d$.

Here and below φ^* is the map from $M(\mathcal{M}_W)$ into $M(\mathcal{M}_V)$ given by $\varphi^*(\rho) = \rho \circ \varphi$ for all $\rho \in \mathcal{M}_W$.

Proof. Proposition 2.5 gives rise to the following commuting diagram

$$\begin{tikzcd}
\left\{\substack{\text{WOT-continuous} \\ \text{characters of } \mathcal{M}_W}\right\} \arrow[r, hook] \arrow[d, leftrightarrow, "\lambda \leftrightarrow \rho_\lambda"'] & M(\mathcal{M}_W) \arrow[r, "\varphi^*"] \arrow[d, "\pi"] & M(\mathcal{M}_V) \arrow[d, "\pi"] & \left\{\substack{\text{WOT-continuous} \\ \text{characters of } \mathcal{M}_V}\right\} \arrow[l, hook'] \arrow[d, leftrightarrow, "\lambda \leftrightarrow \rho_\lambda"] \\
W \arrow[r, hook] & \pi(M(\mathcal{M}_W)) & \pi(M(\mathcal{M}_V)) & V \arrow[l, hook']
\end{tikzcd}$$

and the composition of the thick arrows from W to $\overline{\pi}(M(\mathcal{M}_V)) \subseteq \overline{\mathbb{B}_d}$ yields the map F_φ. Now since $\varphi(Z_i) \in \mathcal{M}_W = \{f|_W \ : \ f \in \mathcal{M}\}$, there is an element $F_i \in \mathcal{M}$ such that $\varphi(Z_i) = F_i|_W$ and $\|F_i\| = \|\varphi(Z_i)\|$. Thus, for every $\lambda \in W$,

$$\begin{aligned} F_\varphi(\lambda) &= \pi(\varphi^*(\rho_\lambda)) \\ &= (\varphi^*(\rho_\lambda)(Z_1), \varphi^*(\rho_\lambda)(Z_2), \dots, \varphi^*(\rho_\lambda)(Z_d)) \\ &= (\varphi(Z_1)(\lambda), \varphi(Z_2)(\lambda), \dots, \varphi(Z_d)(\lambda)) \\ &= (F_1|_W(\lambda), F_2|_W(\lambda), \dots, F_d|_W(\lambda)) \,. \end{aligned}$$

It remains to show that if φ is completely bounded or $d < \infty$ then $(F_1, \dots, F_d)$ defines a function $\mathbb{B}_d \to \mathbb{C}^d$. If $d < \infty$ it is of course clear. If $d = \infty$ and φ is completely bounded then the norm of $(\varphi(Z_1), \varphi(Z_2), \dots)$ is finite, and the F_i's could have been chosen such that $\|(M_{F_1}, M_{F_2}, \dots)\| = \| (\varphi(Z_1), \varphi(Z_2), \dots) \|$. Hence, with this choice of the F_i's, $(F_1, F_2, \dots)$ defines a function $\mathbb{B}_\infty \to \ell^2$. $\square$

Remark 3.2. When $d = \infty$ and φ is not completely bounded, we cannot even say that the map $F_\varphi : W \to \overline{\mathbb{B}_d}$, in the above proposition, is a holomorphic map. The reason is that by definition a holomorphic function on a variety should be extendable to a holomorphic function on an open neighborhood of the variety. However, it is not clear whether there exists a choice of the F_i's and a neighborhood of W such that for any λ in this neighborhood $(F_1(\lambda), F_2(\lambda), \dots)$ belongs to ℓ^2.

Chasing the diagram in the proof of Proposition 3.1 shows that whenever φ^* takes weak-$*$ continuous characters of $\mathcal{M}_W$ to weak-$*$ continuous characters of $\mathcal{M}_V$, F_φ maps W into V. Therefore, if φ is a weak-$*$ continuous unital homomorphism, then $F_\varphi(W) \subseteq V$. This, together with the observation that the inverse of a weak-$*$ continuous isomorphism is weak-$*$ continuous, gives rise to the following corollary.

Corollary 3.3 ([16], Corollary 3.6). *Let V and W be varieties in $\mathbb{B}_d$. If $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ is a unital homomorphism that preserves weak-$*$ continuous characters, then $F_\varphi(W) \subseteq V$ and φ is given by*

$$\varphi(F) = f \circ F_\varphi, \quad f \in \mathcal{M}_V. \tag{3.2}$$

Moreover, if there exists a weak-$*$ continuous isomorphism $\varphi : \mathcal{M}_V \to \mathcal{M}_W$, then $F_\varphi(W) = V$, $F_{\varphi^{-1}}(V) = W$, and there are multipliers $F_1, \dots, F_d, G_1, \dots, G_d \in \mathcal{M}$ such that

$$F_\varphi = (F_1|_W, \dots, F_d|_W), \quad \text{and} \quad F_{\varphi^{-1}} = (G_1|_V, \dots, G_d|_V).$$

Proof. It remains only to verify (3.2), the rest follows from the discussion above. If $f \in \mathcal{M}_V$ and $\lambda \in W$, we find

$$\varphi(f)(\lambda) = \varphi^*(\rho_\lambda)(f) = \rho_{F_\varphi(\lambda)}(f) = f \circ F_\varphi(\lambda),$$

as required. $\square$

When $d < \infty$, we obtain the following result.

Corollary 3.4 ([16], Corollary 3.8). *Let V and W be varieties in $\mathbb{B}_d$ for $d < \infty$. If there exists a weak-$*$ continuous isomorphism $\varphi : \mathcal{M}_V \to \mathcal{M}_W$, then V and W are multiplier biholomorphic.*

The converse does not hold; see Example 5.7 (see also Corollary 6.9). We conclude this section with the following assertion which is a direct result of Proposition 2.7(b) together with the fact that isomorphisms are automatically bounded.

Corollary 3.5 ([10], Theorem 6.2). *Suppose $F : W \to V$ is a biholomorphism which induces (by composition) an isomorphism $\varphi : \mathcal{M}_V \to \mathcal{M}_W$. Then F must be bi-Lipschitz with respect to the pseudohyperbolic metric, i.e., there is a constant $c > 0$ such that*

$$c^{-1} d_{\mathrm{ph}}(\mu, \nu) \leq d_{\mathrm{ph}}(F(\mu), F(\nu)) \leq c d_{\mathrm{ph}}(\mu, \nu).$$

The converse does not hold; see [10, Example 6.6].

4. Isometric, completely isometric, and unitarily implemented isomorphisms

Let V and W be two varieties in $\mathbb{B}_d$. We say that V and W are *conformally equivalent* if there exists an automorphism of $\mathbb{B}_d$ (that is, a biholomorphism from $\mathbb{B}_d$ into itself) which maps V onto W. In this section we will see that if V and W are conformally equivalent then $\mathcal{M}_V$ and $\mathcal{M}_W$ are (completely) isometrically isomorphic (in fact, unitarily equivalent). When $d < \infty$ the converse also holds, and morally speaking it also holds for $d = \infty$. In fact, when $d = \infty$ it may happen that $\mathcal{M}_V$ and $\mathcal{M}_W$ are unitarily equivalent but V and W are not conformally equivalent. This, however, can only be the result of an unlucky embedding of V and W into $\mathbb{B}_\infty$, and is easily fixed.

4.1. Completely isometric and unitarily implemented isomorphisms

Proposition 4.1 ([16], Proposition 4.1). *Let V and W be varieties in $\mathbb{B}_d$. Let F be an automorphism of $\mathbb{B}_d$ that maps W onto V. Then $f \mapsto f \circ F$ is a unitarily implemented completely isometric isomorphism of $\mathcal{M}_V$ onto $\mathcal{M}_W$; i.e., $M_{f \circ F} = U M_f U^*$. The unitary U^* is the linear extension of the map*

$$U^* k_w = c_w k_{F(w)} \quad \text{for } w \in W,$$

where $c_w = (1 - \|F^{-1}(0)\|^2)^{\frac{1}{2}} \overline{k_{F^{-1}(0)}(w)}$.

The proof in [16] relies on Theorem 9.2 of [15], which uses Voiculescu's construction of automorphisms of the Cuntz algebra. For the convenience of the reader we give here a slightly different proof.

Proof. Let F be such an automorphism, and set $\alpha = F^{-1}(0)$. We first show that the linear transformation defined on reproducing kernels by $k_w \mapsto c_w k_{F(w)}$ extends to be a bounded operator of norm 1. First note that $\overline{c_w^{-1}} = (1 - \|\alpha\|^2)^{-\frac{1}{2}}(1 - \langle w, \alpha \rangle)$, so $\overline{c_w^{-1}}$ (as a function of w) is a multiplier. The transformation formula for ball automorphisms [28, Theorem 2.2.5], shows that

$$k_{F(w)}(F(z)) = c_w^{-1}\overline{c_z^{-1}}k_w(z) \quad \text{for } w, z \in \mathbb{B}_d.$$

Now,

$$\langle c_w k_{F(w)}, c_z k_{F(z)}\rangle = c_w \overline{c_z} k_{F(w)}(F(z)) = k_w(z) = \langle k_w, k_z\rangle.$$

Thus, the linear transformation $k_w \mapsto c_w k_{F(w)}$ extends to an isometry. We denote by U its adjoint. A short calculation shows that

$$Uh = (1 - \|\alpha\|^2)^{\frac{1}{2}} k_\alpha \cdot (h \circ F) \quad \text{for } h \in H_d^2.$$

We have already noted that U^* is an isometry, and since its range is evidently dense we conclude that U is a unitary.

Finally, we show that conjugation by U implements the isomorphism between $\mathcal{M}_V$ and $\mathcal{M}_W$ given by composition with F. For $f \in \mathcal{M}_V$ and $w \in W$,

$$UM_f^* U^* k_w = UM_f^* c_w k_{F(w)} = \overline{f(F(w))} U c_w k_{F(w)} = \overline{(f \circ F)(w)} k_w.$$

Therefore, $M_{f\circ F}$ is a multiplier on $\mathcal{F}_W$ and $M_{f\circ F} = UM_f U^*$. □

Before discussing the converse direction, we recall a few definitions on affine sets. The *affine span* (or *affine hull*) of a set $S \subseteq \mathbb{C}^d$ is the set $\mathrm{aff}(S) := \lambda + \mathrm{span}(S - \lambda)$ for $\lambda \in S$. This is independent of the choice of λ. An *affine set* is a set A with $A = \mathrm{aff}(A)$. The dimension $\dim(A)$ of an affine set A is the dimension of the subspace $\overline{A - \lambda}$ for $\lambda \in A$, and the codimension $\mathrm{codim}(A)$ is the dimension of the quotient space $\mathbb{C}^d / \overline{A - \lambda}$ for $\lambda \in A$. Both definitions, again, are independent of the choice of λ. By the *affine dimension* (resp. *codimension*) of a subset $S \subseteq \mathbb{C}^d$ we mean the dimension (resp. codimension) of $\mathrm{aff}(S)$. Furthermore, we use the term *affine subset of* $\mathbb{B}_d$ for any intersection $A \cap \mathbb{B}_d$, where A is affine in $\mathbb{C}^d$. By [28, Proposition 2.4.2], automorphisms of the ball map affine subsets of the ball to affine subsets of the ball. Therefore, we obtain the following lemma.

Lemma 4.2. *Let V and W be varieties in $\mathbb{B}_d$ and let F be an automorphism of $\mathbb{B}_d$ that maps W onto V. Then, $F(\overline{\mathrm{aff}}(V) \cap \mathbb{B}_d) = \overline{\mathrm{aff}}(W) \cap \mathbb{B}_d$. In particular, $\overline{\mathrm{aff}}(V)$ and $\overline{\mathrm{aff}}(W)$ have the same dimension and the same codimension.*

Proof. The first argument is clear, so it suffices to show that an automorphism of the ball preserves dimensions and codimensions of affine subsets. Indeed, as F is a diffeomorphism, its differential at any point of the ball is an invertible linear transformation. Let A be an affine subset of $\mathbb{B}_d$ and let $\lambda \in A$. Let $T_\lambda \mathbb{B}_d \cong \mathbb{C}^d$ be the tangent space of $\mathbb{B}_d$ at λ, and let $T_\lambda A \cong A - \lambda$ be the tangent space of A at λ. As A is a submanifold of $\mathbb{B}_d$, we may think of $T_\lambda A$ as a subspace of $T_\lambda \mathbb{B}_d$. Hence, the invertible linear transformation dF_λ maps the subspace $T_\lambda A$ onto $T_{F(\lambda)} F(A)$. We conclude that $T_\lambda A$ and $T_{F(\lambda)} F(A)$ must have the same dimension and the same codimension. □

Proposition 4.1 and Lemma 4.2 imply, in particular, that if there is an automorphism of the ball which sends W onto V, then V and W must have the same affine codimension, and this automorphism gives rise to a completely isometric isomorphism of $\mathcal{M}_V$ onto $\mathcal{M}_W$ (by precomposing this automorphism). The converse is also true: any completely isometric isomorphism of $\mathcal{M}_V$ onto $\mathcal{M}_W$, for V and W varieties in the ball having the same affine codimension, arises in this way.

Proposition 4.3. *Let V and W be varieties in $\mathbb{B}_d$, with the same affine codimension or with $d < \infty$. Then every completely isometric isomorphism $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ arises as composition $\varphi(f) = f \circ F$ where F is an automorphism of $\mathbb{B}_d$ mapping W onto V.*

Proof. Recall that Proposition 3.1 assures the existence of a holomorphic map $F : \mathbb{B}_d \to \overline{\mathbb{B}_d}$ representing $\varphi^*\big|_W$. A deep result of Kennedy and Yang [22, Corollary 6.4] asserts that $\mathcal{M}_V$ and $\mathcal{M}_W$ have strongly unique preduals. It then follows that every isometric isomorphism between these algebras, is also a weak-$*$ homeomorphism. Thus, by Corollary 3.3, $F(W) \subseteq V$ and $\varphi(f) = f \circ F$. (We note that if $d < \infty$, then we may argue differently: first one shows using the injectivity of φ that $F(\mathbb{B}_d) \subseteq \mathbb{B}_d$, and then one uses the assertion $V = \pi(M(\mathcal{M}_V)) \cap \mathbb{B}_d$ of Proposition 2.6 to obtain that φ preserves weak-$*$ continuous characters.) Similarly, $\varphi^{-1} : \mathcal{M}_W \to \mathcal{M}_V$ gives rise to a holomorphic map $G : \mathbb{B}_d \to \mathbb{B}_d$ such that $G(V) \subseteq W$ and $\varphi^{-1}(g) = g \circ G$. It is clear that $F \circ G|_V = \mathbf{id}|_V$ and $G \circ F|_W = \mathbf{id}|_W$, and so $F(W) = V$.

By Proposition 4.1 and Lemma 4.2, we may assume that V and W both contain 0, and that $F(0) = 0$. Some technical several-complex-variables arguments, which we will not present here, now show that $F|_{\overline{\mathrm{span}} W \cap \mathbb{B}_d}$ is an isometric linear transformation that maps $\overline{\mathrm{span}} W \cap \mathbb{B}_d$ onto $\overline{\mathrm{span}} V \cap \mathbb{B}_d$ (see [16, Lemma 4.4]). In particular, $\overline{\mathrm{span}} W$ and $\overline{\mathrm{span}} V$ have the same dimension. Since they also have the same codimension, we may extend the definition of $F|_{\overline{\mathrm{span}} W \cap \mathbb{B}_d}$ to a unitary map on $\mathbb{C}^d$. This yields the desired automorphism. $\square$

Remark 4.4. The original statement of Proposition 4.3 (which appears in [16, Theorem 4.5]) does not include the requirement that V and W have the same affine codimension. Example 4.5 below shows that this requirement is indeed necessary (for the case $d = \infty$). Nonetheless, it is clear that up to an isometric embedding of the original infinite ball into a "larger" one, the original statement does hold. For example, if we replace V and W with their images under the embedding $U : (z_1, z_2, \dots) \mapsto (z_1, 0, z_2, 0, \dots)$, then both V and W have an infinite affine codimension, and it is now true that $\mathcal{M}_V$ and $\mathcal{M}_W$ are completely isometrically isomorphic if and only if V and W are conformally equivalent.

Example 4.5. Let $V = \mathbb{B}_\infty$ and $W = \{(z_1, z_2, z_3, \dots) \in \mathbb{B}_\infty : z_1 = 0\}$. Let $F : W \to V$ be defined by

$$F(0, z_2, z_3, \dots) = (z_2, z_3, \dots).$$

Then F is a biholomorphism which cannot be extended to an automorphism of $\mathbb{B}_\infty$. Let $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ be defined by $\varphi(f) = f \circ F$. Then φ is a completely isometric isomorphism of $\mathcal{M}_V$ onto $\mathcal{M}_W$, which does not arise as a precomposition with an automorphism of the ball. The reason is of course that V has an affine codimension 0 while W has an affine codimension 1.

Combining Propositions 4.1 and 4.3 yields the following result.

Theorem 4.6 ([16], Theorem 4.5). *Let V and W be varieties in $\mathbb{B}_d$, with the same affine codimension or with $d < \infty$. Then $\mathcal{M}_V$ is completely isometrically isomorphic to $\mathcal{M}_W$ if and only if there exists an automorphism F of $\mathbb{B}_d$ such that $F(W) = V$. In fact, under these assumptions, every completely isometric isomorphism $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ arises as composition $\varphi(f) = f \circ F$ where F is such an automorphism. In this case, φ is unitarily implemented by the unitary sending the kernel function $k_w \in \mathcal{F}_W$ to a scalar multiple of the kernel function $k_{F(w)} \in \mathcal{F}_V$.*

If V and W are not assumed to have the same affine codimension, then every completely isometric isomorphism $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ arises as composition with $U^* \circ F \circ U$, where $F \in Aut(\mathbb{B}_d)$ and U is the isometry from Remark 4.4, and is unitarily implemented.

4.2. Isometric isomorphisms

By Theorem 4.6 the conformal geometry of V is completely encoded by the operator algebraic structure $\mathcal{M}_V$ (and vice versa). It is natural to ask whether the Banach algebraic structure $\mathcal{M}_V$ also encodes some geometrical aspect of V. It turns out that within the family of irreducible complete Pick algebras, every isometric isomorphism of $\mathcal{M}_V$ and $\mathcal{M}_W$ is actually a completely isometric isomorphism, and the results of the previous section apply.

Lemma 4.7. *Let V and W be varieties in $\mathbb{B}_d$, and suppose that $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ is an isometric isomorphism. Then φ^* maps W onto V and preserves the pseudohyperbolic distance.*

Proof. The first assertion was obtained in the proof of Proposition 4.3. It then follows that φ is implemented by composition with $\varphi^*\big|_W$. Using this together with Proposition 2.7 (b), one obtains the second assertion. □

The following theorem appears in [16, Proposition 5.9] with the additional assumption that $d < \infty$. Here we remove this restriction.

Theorem 4.8 ([16], Proposition 5.9). *Let V and W be varieties in $\mathbb{B}_d$. Then every isometric isomorphism of $\mathcal{M}_V$ onto $\mathcal{M}_W$ is completely isometric, and thus is unitarily implemented.*

Proof. Without the loss of generality we may assume that V and W have the same affine codimension by embedding the original ball in a larger one, if needed (see Remark 4.4). Let φ be an isometric isomorphism of $\mathcal{M}_V$ onto $\mathcal{M}_W$. By Lemma 4.7, φ^* maps W onto V and preserves the pseudohyperbolic distance. Let $F = F_\varphi$.

As above, we may assume that 0 belongs to both V and W, and that $F(0) = 0$. Let $w_1, w_2, \ldots \in W$ be a sequence spanning a dense subset of $\overline{\operatorname{span}} W$. For every $p \geq 1$ let $v_p = F(w_p) = \varphi^*(w_p)$. Put $r_p := \|w_p\| = d_{\mathrm{ph}}(w_p, 0)$.

Then $\|v_p\| = d_{\mathrm{ph}}(v_p, 0) = r_p$. For every p let $h_p(z) := \langle z, \frac{v_p}{r_p}\rangle$. This is a continuous linear functional (restricted to V), and thus lies in $\mathcal{M}_V$. Furthermore, since $(Z_1, Z_2, \ldots, Z_d)$ is a row contraction it follows that $\|h_p\|_{\mathcal{M}_V} \leq 1$, and so $\|\varphi(h_p)\|_{\mathcal{M}_W} \leq 1$.

Now, let w be an arbitrary point in W, set $v = F(w) \in V$, and fix $p \geq 1$. Since, $\varphi(h_p)$ is a multiplier of norm at most 1 which satisfies $\varphi(h_p)(0) = 0$, $\varphi(h_p)(w_p) = h_p(v_p)$ and $\varphi(h_p)(w) = h_p(v)$, we have by a standard necessary condition for interpolation [2, Theorem 5.2] that

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & \frac{\overline{1-\langle v,v_p\rangle}}{1-\langle w_p,w\rangle} \\ 1 & \frac{1-\langle v,v_p\rangle}{1-\langle w,w_p\rangle} & \frac{1-|\langle v,v_p/r_p\rangle|^2}{1-\langle w,w\rangle} \end{bmatrix} \geq 0.$$

Examining the determinant we find that $\frac{1-\langle v,v_p\rangle}{1-\langle w,w_p\rangle} = 1$. Therefore,

$$\langle v, v_p\rangle = \langle w, w_p\rangle \quad \text{for all } p.$$

In particular, we obtain $\langle v_i, v_j\rangle = \langle w_i, w_j\rangle$ for all i, j. Therefore, there is a unitary operator $U : \overline{\mathrm{span}} W \to \overline{\mathrm{span}} V$ such that $Uw_i = v_i$ for all $1 \leq i \leq k$. Since $\mathrm{codim}(\overline{\mathrm{span}} W) = \mathrm{codim}(\overline{\mathrm{span}} V)$, it can be extended to a unitary operator U on $\mathbb{C}^d$. From here one shows that F agrees with the unitary U, and hence φ is implemented by an automorphism of the ball. Thus, by Proposition 4.1, φ is completely isometric and is unitarily implemented. $\square$

5. Algebraic isomorphisms

We now turn to study the algebraic isomorphism problem. It is remarkable that, under reasonable assumptions, purely algebraic isomorphism implies multiplier biholomorphism. Throughout this section we will assume that $d < \infty$.

5.1. Varieties which are unions of finitely many irreducible varieties and a discrete variety

Let V be a variety in the ball. We say that V is *irreducible* if for any regular point $\lambda \in V$, the intersection of zero sets of all multipliers vanishing on a small neighborhood $V \cap B_\epsilon(\lambda)$ is exactly V. We say that V is *discrete* if it has no accumulation points in $\mathbb{B}_d$. We will see that if V and W are two varieties in $\mathbb{B}_d$ $(d < \infty)$, which are the union of finitely many irreducible varieties and a discrete variety, then whenever $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic, V and W are multiplier biholomorphic.

Remark 5.1. The definition of irreducibility given in the previous paragraph is not to be confused with the classical notion of irreducibility (that is, that there is no non-trivial decomposition of the variety into subvarieties). Nonetheless, whenever a variety V is irreducible in the classical sense, it is also irreducible in our sense (see, e.g., [19, Theorem, H1]).

We open this section with two observations. The first is that every homomorphism between multiplier algebras is norm continuous. A general result in the theory of commutative Banach algebras, says that every homomorphism from a Banach algebra into a commutative semi-simple Banach algebra is norm continuous [9, Proposition 4.2]. As $\mathcal{M}_W$ is easily seen to be semi-simple, it holds that every homomorphism from $\mathcal{M}_V$ to $\mathcal{M}_W$ is norm continuous.

The second observation relates to isolated characters of a multiplier algebra. Suppose that ρ is an isolated point in $M(\mathcal{M}_V)$. By Shilov's idempotent theorem [8, Theorem 5], there is a function $0 \neq f \in \mathcal{M}_V$ such that every character except ρ annihilates f. As $f \neq 0$, there is $\lambda \in V$ such that $f(\lambda) \neq 0$. And so, $\rho \in \pi^{-1}(V)$. Thus, when $d < \infty$ any isolated character of a multiplier algebra is an evaluation. This gives rise to the following proposition.

Proposition 5.2 ([16], Lemma 5.2). *Let V and W be varieties in $\mathbb{B}_d$, with $d < \infty$. Let $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ be an algebra isomorphism. Suppose that λ is an isolated point in W. Then $\varphi^*(\rho_\lambda)$ is an evaluation functional at an isolated point in V.*

From the first observation above, together with Proposition 2.7, we obtain:

Proposition 5.3. *Let V and W be a varieties in $\mathbb{B}_d$, with $d < \infty$, and let $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ be a homomorphism. Let U be a connected subset of W. Then $\varphi^*(\pi^{-1}(U))$ is either a connected subset of $\pi^{-1}(V)$ (with respect to the norm topology induced by $\mathcal{M}_V^*$) or contained in a single fiber of the corona $M(\mathcal{M}_V) \setminus \pi^{-1}(V)$.*

Proposition 5.4 ([16], Corollary 5.4). *Let V and W be varieties in $\mathbb{B}_d$, $d < \infty$, and assume that each one is the union of a discrete variety and a finite union of irreducible varieties. Suppose that φ is an algebra isomorphism of $\mathcal{M}_V$ onto $\mathcal{M}_W$. Then φ^* must map W onto V.*

Proof. Let us write $V = D_V \cup V_1 \cup \cdots \cup V_m$ and $W = D_W \cup W_1 \cup \cdots \cup W_n$, where D_V and D_W are the discrete parts of V and W, and V_i, W_j are all irreducible varieties of dimension at least 1. By Proposition 5.2 φ^* maps D_W onto D_V.

First let us show that if W_1, say, is not mapped entirely into V then it is mapped into a single fiber of the corona $M(\mathcal{M}_V) \setminus \pi^{-1}(V)$. Suppose that λ is some regular point of W_1 mapped to a fiber of the corona. Without loss of generality, we may assume it is the fiber over $(1, 0, \ldots, 0)$. Then the connected component of λ in W_1 is mapped into the same fiber, by the previous proposition. If there exists another point $\mu \in W_1$ which is mapped into V or into another fiber in the corona, then by the previous proposition, the whole connected component of μ is mapped into V or into the other fiber. The function $h = \varphi(Z_1|_V) - 1|_W$ vanishes on the component of λ, but does not vanish on the component containing μ. This contradicts the fact that W_1 is irreducible.

Thus, to show that W_1 is mapped into V we must rule out the possibility that it is mapped into a single fiber of the corona. Fix $\lambda \in W_1 \setminus \bigcup_{i=2}^n W_i$. For each $2 \leq i \leq n$, there is a multiplier $h_i \in \mathcal{M}_d$ vanishing on W_i and satisfying $h_i(\lambda) \neq 0$. Moreover, since D_W is a variety, there is a multiplier k vanishing on

D_W and satisfying $k(\lambda) \neq 0$. Hence, $h := k \prod_{i=2}^n h_i$ belongs to $\mathcal{M}_W$ and vanishes on $D_W \cup \bigcup_{i=2}^n W_i$ but not on W_1. Therefore $\varphi^{-1}(h)$ is a non-zero element of $\mathcal{M}_V$.

Now suppose that $\varphi^*(W_1)$ is contained in a fiber over a point in $\partial\mathbb{B}_d$, say $(1, 0, \dots, 0)$. Since $(Z_1 - 1)|_V$ is never zero, we see that $(Z_1 - 1)|_V \varphi^{-1}(h)$ is not the zero function. However, $(Z_1 - 1)|_V \varphi^{-1}(h)$ vanishes on $\varphi^*(W_1)$. Therefore, $\varphi((Z_1 - 1)|_V \varphi^{-1}(h))$ vanishes on W_1 and on $D_W \cup \bigcup_{i=2}^n W_i$, contradicting the injectivity of φ. We deduced that W_1 is mapped into V. Replacing the roles of V and W shows that φ^* must map W onto V. □

From Proposition and 5.4 and Corollary 3.3 we obtain the following.

Theorem 5.5 ([16], Theorem 5.6). *Let V and W be varieties in $\mathbb{B}_d$, with $d < \infty$, which are each a union of finitely many irreducible varieties and a discrete variety. Let φ be an algebra isomorphism of $\mathcal{M}_V$ onto $\mathcal{M}_W$. Then there exist holomorphic maps F and G from $\mathbb{B}_d$ into $\mathbb{C}^d$ with coefficients in $\mathcal{M}_d$ such that*

(a) $F|_W = \varphi^*|_W$ *and* $G|_V = (\varphi^{-1})^*|_V$,
(b) $G \circ F|_W = \mathbf{id}_W$ *and* $F \circ G|_V = \mathbf{id}_V$,
(c) $\varphi(f) = f \circ F$ *for* $f \in \mathcal{M}_V$, *and*
(d) $\varphi^{-1}(g) = g \circ G$ *for* $g \in \mathcal{M}_W$.

Theorem 5.5 shows in particular that every automorphism of $\mathcal{M}_d = \mathcal{M}_{\mathbb{B}_d}$ is implemented as composition by a biholomorphic map of $\mathbb{B}_d$ onto itself, i.e., a conformal automorphism of $\mathbb{B}_d$. Proposition 4.1 shows that these automorphisms are unitarily implemented (hence, completely isometric). Thus, we obtain the following corollary.

Corollary 5.6 ([16], Corollary 5.8). *Every algebraic automorphism of $\mathcal{M}_d$ for d finite is completely isometric, and is unitarily implemented.*

The converse of Theorem 5.5 does not hold.

Example 5.7. Let

$$V = \left\{1 - \frac{1}{n^2} : n \in \mathbb{N}\right\} \quad \text{and} \quad W = \left\{1 - e^{-n^2} : n \in \mathbb{N}\right\}.$$

Since they both satisfy the Blaschke condition, they are analytic varieties in $\mathbb{D}$ (recall that $\{a_n \in \mathbb{C} : n \in \mathbb{N}\}$ satisfies the Blaschke condition if $\sum (1 - |a_n|) < \infty$). Let $B(z)$ be the Blaschke product with simple zeros at points in W. Define

$$h(z) = 1 - e^{\frac{1}{z-1}} \quad \text{and} \quad g(z) = \frac{\log(1-z) + 1}{\log(1-z)} \left(1 - \frac{B(z)}{B(0)}\right).$$

Then $g, h \in H^\infty = \mathcal{M}_{\mathbb{D}}$ and they satisfy

$$h \circ g|_W = \mathbf{id}_W \quad \text{and} \quad g \circ h|_V = \mathbf{id}_V.$$

However, by the corollary in [21, p. 204], W is an interpolating sequence and V is not. This implies that $\mathcal{M}_W$ is algebraically isomorphic to ℓ^∞ while V is not (see [16, Theorem 6.3]). Thus, $\mathcal{M}_V$ and $\mathcal{M}_W$ cannot be isomorphic.

5.2. Homogeneous varieties

Let V be a variety in the ball. We say that V is a *homogeneous variety* if it is the common vanishing locus of homogeneous polynomials.

We wish to apply Theorem 5.5 to homogeneous varieties in $\mathbb{B}_d$, $d < \infty$. It is well known that every algebraic variety can be decomposed into a finite union of irreducible varieties, but caution is required, since the well-known result is concerned with irreducibility in another sense than the one we used in Section 5.1. However, one may show that a homogeneous algebraic variety which is irreducible (in the sense of algebraic varieties) is also irreducible in our sense.

Proposition 5.8. *Every homogeneous variety in the ball is a union of finitely many irreducible varieties.*

Proof. Let V be a homogeneous variety and let $V = V_1 \cup \cdots \cup V_n$ be its decomposition into *algebraic* irreducible homogeneous varieties (in the sense of algebraic varieties). We will show that every V_i is irreducible in our sense. By [19, Theorem E19, Corollary E20], once we remove the set of singular points $S(V_i)$, the connected components of $V_i \setminus S(V_i)$ are such that their closures are varieties. Since $S(V_i)$ is a homogeneous variety, these connected components are invariant under nonzero scalar multiplication so their closures are homogeneous varieties. Thus, if there was more than one connected component we would obtain an algebraic decomposition of the variety V_i, so $V_i \setminus S(V_i)$ is connected. By the identity principle [19, Theorem, H1], the V_i's are irreducible in our sense. □

Thus we obtain the following theorem (the original proof of this theorem was somewhat different – see [15, Section 11]).

Theorem 5.9 ([15], Theorem 11.7(2)). *Let V and W be homogeneous varieties in $\mathbb{B}_d$, $d < \infty$. If $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic, then there is a multiplier biholomorphism mapping W onto V.*

The rest of this subsection is devoted towards the converse direction. Remarkably, a stronger result than the converse holds: it turns out that the existence of a biholomorphism from W onto V implies that the algebras are isomorphic.

We will start by showing that whenever a homogeneous variety $W \subseteq \mathbb{B}_d$ is the image of homogeneous variety $V \subseteq \mathbb{B}_d$ under a biholomorphism, then it is also the image of V under an invertible linear transformation. To see this, we first need to present the notion of the *singular nucleus* of a homogeneous variety. Lemma 4.5 of [15] and its proof say that a homogeneous variety V in $\mathbb{C}^d$ is either a linear subspace, or has singular points, and that whenever it is not a linear subspace, the set of singular points $S(V)$ (also known as the *singular locus*) of V is a homogeneous variety. Since the dimension of $S(V)$ must be strictly less than the dimension of V, there exists a smallest integer n such that $S(\ldots(S(S(V)))\ldots)$ (n times) is empty. The set

$$N(V) := \underbrace{S(\ldots(S(S(V)))\ldots)}_{n-1 \text{ times}}$$

is called the *singular nucleus* of V. By the above discussion, it is a subspace of $\mathbb{C}^d$. By basic complex differential geometry, a biholomorphism of V onto W must map $N(V)$ onto $N(W)$.

The following lemma – which seems to be of independent interest – was used implicitly in [15], but in fact does not appear anywhere in the literature. The proof follows closely the proof of [15, Proposition 4.7].

Lemma 5.10. *Let V and W be two biholomorphically equivalent homogeneous varieties in $\mathbb{B}_d$. Then there exists a biholomorphism F of V onto W that maps 0 to 0.*

Proof. Let G be a biholomorphism of V onto W. If $N(V) = N(W) = \{0\}$, then $G(0) = 0$, and we are done. Otherwise, $N(V) \cap \mathbb{B}_d$ and $N(W) \cap \mathbb{B}_d$ are both complex balls of the same dimension, say $d' \leq d$. As G takes $N(V) \cap \mathbb{B}_d$ onto $N(W) \cap \mathbb{B}_d$, we may think of G as an automorphism of $\mathbb{B}_{d'}$. We can find two discs $D_1 \subseteq N(V)$ and $D_2 \subseteq N(W)$ such that $G(D_1) = D_2$ (see [15, Lemma 4.6]). Define

$$\mathcal{O}(0; V) := \{z \in D_1 : z = F(0) \text{ for some automorphism } F \text{ of } V\}$$

and

$$\mathcal{O}(0; V, W) := \left\{ z \in D_2 \; : \; \begin{array}{l} z = F(0) \text{ for some biholomorphism} \\ F \text{ of } V \text{ onto } W \end{array} \right\}.$$

Since homogeneous varieties are invariant under multiplication by complex numbers, it is easy to check that these sets are circular, that is, for every $\mu \in \mathcal{O}(0; V)$ and $\nu \in \mathcal{O}(0; V, W)$, it holds that $C_{\mu, D_1} := \{z \in D_1 : |z| = |\mu|\} \subseteq \mathcal{O}(0; V)$ and $C_{\nu, D_2} := \{z \in D_2 : |z| = |\nu|\} \subseteq \mathcal{O}(0; V, W)$.

Now, as $G(0)$ belongs to $\mathcal{O}(0; V, W)$, we obtain that $C := C_{G(0), D_2} \subseteq \mathcal{O}(0; V, W)$. Therefore, the circle $G^{-1}(C)$ is a subset of $\mathcal{O}(0; V)$. As $\mathcal{O}(0; V)$ is circular, every point of the interior of the circle $G^{-1}(C)$ is a subset of $\mathcal{O}(0; V)$. Thus, the interior of the circle C must be a subset of $\mathcal{O}(0; V, W)$. We conclude that $0 \in \mathcal{O}(0; V, W)$. □

Proposition 5.11. *Let V and W be two biholomorphically equivalent homogeneous varieties in $\mathbb{B}_d$. Then there is a linear map on $\mathbb{C}^d$ which maps V onto W.*

Sketch of proof. By Lemma 5.10, V and W are biholomorphically equivalent via a 0 preserving biholomorphism; i.e., there exist two holomorphic maps F and G from $\mathbb{B}_d$ into $\mathbb{C}^d$ such that $G \circ F|_V = \mathbf{id}_V$ and $F \circ G|_W = \mathbf{id}_W$. Cartan's uniqueness theorem says that if there exists a 0 preserving biholomorphism between two bounded circular *regions*, then it must be a restriction of a linear transformation; see [28, Theorem 2.1.3]. Now, V and W are indeed circular (since they are homogeneous varieties) and bounded, but do not have to be "regions" (their interior might be empty). Nevertheless, it turns out that adapting the proof of Cartan's uniqueness theorem to the setting of varieties, rather than regions, does work (see [15, Theorem 7.4]). Thus, there exists a linear map $A : \mathbb{C}^d \to \mathbb{C}^d$ which agrees with F on V. □

Up to now we have seen that if $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic, then V and W are biholomorphically equivalent; and we have seen that if V and W are biholomorphically equivalent, then there is a linear map sending V onto W, and it is not hard to see that this map can be taken to be invertible. To close the circle, one needs to show that whenever there is an invertible linear transformation mapping a homogeneous variety $W \subseteq \mathbb{B}_d$ onto a homogeneous variety $V \subseteq \mathbb{B}_d$, we have that $\mathcal{M}_V$ and $\mathcal{M}_W$ are similar. In [15, Section 7], this statement was proved for a class of varieties which satisfy some extra assumptions (e.g., irreducible varieties, union of two irreducible components, hypersurfaces, and for the case $d \leq 3$). Later on, in [20] it was shown that these extra assumptions are superfluous, and that the statement holds for all homogeneous varieties. The main difficulty was in proving the following lemma.

Lemma 5.12 ([20]). *Let V and W be homogeneous varieties in $\mathbb{B}_d$, $d < \infty$, If there is a linear transformation $A : \mathbb{C}^d \to \mathbb{C}^d$ that maps W bijectively onto V, then the map $C_{A^*} : \mathcal{F}_W \to \mathcal{F}_V$, given by*

$$C_{A^*} k_\lambda = k_{A\lambda} \quad \text{for } \lambda \in W,$$

is a bounded linear transformation from $\mathcal{F}_W$ into $\mathcal{F}_V$.

We omit the proof of Lemma 5.12. The crucial step in its proof is to show that whenever $V_1, \ldots, V_n$ are subspaces of $\mathbb{C}^d$, the algebraic sum of the associated Fock spaces

$$\mathcal{F}(V_1) + \cdots + \mathcal{F}(V_n) \subseteq \mathcal{F}(\mathbb{C}^d)$$

is closed. In fact, most of [20] is devoted to proving this crucial step.

Theorem 5.13. *Let V and W be homogeneous varieties in $\mathbb{B}_d$, $d < \infty$. If there is an invertible linear transformation $A \in \mathrm{GL}_d(\mathbb{C})$ that maps W onto V, then the map $\varphi : \mathcal{M}_V \to \mathcal{M}_W$, given by*

$$\varphi(f) = f \circ A \quad \text{for } f \in \mathcal{M}_V,$$

is a completely bounded isomorphism, and when regarding $\mathcal{M}_V$ and $\mathcal{M}_W$ as operator algebras acting on $\mathcal{F}_V$ and $\mathcal{F}_W$, respectively, φ is given by

$$\varphi(M_f) = (C_{A^*})^* M_f (C_{A^*}^{-1})^* \quad \text{for } f \in \mathcal{M}_V.$$

Thus, $\mathcal{M}_V$ and $\mathcal{M}_W$ are similar.

Proof. By Lemma 5.12, both C_{A^*} and $C_{(A^{-1})^*}$ are bounded, and it is clear that $C_{(A^{-1})^*} = (C_{A^*})^{-1}$. A calculation shows that $M_{f\circ A} = (C_{A^*})^* M_f (C_{A^*}^{-1})^*$. □

We sum up the results of Theorems 5.11, 5.9 and 5.13 as follows.

Theorem 5.14 ([15, 20]). *Let V and W be homogeneous varieties in $\mathbb{B}_d$ with $d < \infty$. Then the following are equivalent:*

(a) *$\mathcal{M}_V$ and $\mathcal{M}_W$ are similar.*
(b) *$\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic.*
(c) *V and W are biholomorphically equivalent.*
(d) *There is an invertible linear map on $\mathbb{C}^d$ which maps W onto V.*

If a linear map A maps V onto W this means that A is length preserving on the homogeneous varieties $\tilde{V}$ and $\tilde{W}$, where $\tilde{V}$ is the homogeneous variety such that $V = \tilde{V} \cap \mathbb{B}_d$, and likewise $\tilde{W}$. This does not mean that A is isometric (as Example 5.16 shows), but it is true that A is isometric on the span of every irreducible component of W [15, Proposition 7.6]. Combining this fact with Proposition 4.1 we obtain the following result, which sharpens Corollary 5.6 substantially.

Theorem 5.15 ([15], Theorem 8.7). *Let V and W be homogeneous varieties in $\mathbb{B}_d$, $d < \infty$, such that W is either irreducible or a non-linear hypersurface. If $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic, then they are unitarily equivalent.*

Example 5.16. Suppose that V and W are each given as the union of two (complex) lines. There is always a linear map mapping W onto V that is length preserving on W, thus $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic. On the other hand, these algebras will be isometrically isomorphic if and only if the angle between the two lines is the same in each variety.

The case of three lines is also illuminating: it reveals how the algebra $\mathrm{Alg}(1, Z)$ and its WOT-closure, the algebra $\mathcal{M}_V$, each encodes different geometrical information. Indeed, suppose that $V = \mathrm{span}\{v_1\} \cup \mathrm{span}\{v_2\} \cup \mathrm{span}\{v_3\}$ and $W = \mathrm{span}\{w_1\} \cup \mathrm{span}\{w_2\} \cup \mathrm{span}\{w_3\}$, where v_i, w_j are all unit vectors in $\mathbb{C}^2$ spanning distinct lines. There always exists a bijective linear map from W onto V: indeed, define

$$A : w_1 \mapsto a_1 v_1 \, , \; w_2 \mapsto a_2 v_2 ,$$

and choose a_1, a_2 so that $w_3 = b_1 w_1 + b_2 w_2$ is mapped to v_3. One only has to choose a_1, a_2 such that $a_1 b_1 v_1 + a_2 b_2 v_2 = v_3$. It follows that the algebras $\mathrm{Alg}(1, Z\big|_V)$ and $\mathrm{Alg}(1, Z\big|_W)$ are isomorphic (the latter two algebras are easily seen to be isomorphic to the coordinate rings of the varieties).

On the other hand, if we require the linear map A to be length preserving on W, then $|a_1| = |a_2| = 1$. If $v_3 = c_1 v_1 + c_2 v_2$, then for such a map to exist we will need $a_1 b_1 = c_1$ and $a_2 b_2 = c_2$. This is possible if and only if $|b_1| = |c_1|$ and $|b_2| = |c_2|$. Thus the algebras $\mathcal{M}_V$ and $\mathcal{M}_W$ in this setup are rarely isomorphic.

5.3. Finite Riemann surfaces

In seeking a the converse of Theorem 5.5, it is natural to restrict attention to certain well-behaved classes of varieties. In the previous subsection it was shown that the converse of Theorem 5.5 holds within the class of homogeneous varieties. In this subsection we concentrate on generic one-dimensional subvarieties of $\mathbb{B}_d$, $d < \infty$.

A connected *finite Riemann surface* Σ is a connected open proper subset of some compact Riemann surface such that the boundary $\partial\Sigma$ is also the boundary of the closure and is the union of finitely many disjoint simple closed analytic curves. A general finite Riemann surface is a finite disjoint union of connected ones.

Let Σ be a connected finite Riemann surface and let $a \in \Sigma$ be some base-point. Let ω be the harmonic measure with respect to a, i.e., the measure on $\partial\Sigma$

with the property that

$$u(a) = \int_{\partial\Sigma} u(\zeta) d\omega(\zeta)$$

for every function u that is harmonic on Σ and continuous on $\overline{\Sigma}$. We denote by $H^2(\Sigma)$ the closure in $L^2(\omega)$ of the space $A(\Sigma) := \mathrm{Hol}(\Sigma) \cap C(\overline{\Sigma})$. In case that Σ is not connected we let $H^2(\Sigma)$ be the direct sum of the H^2 spaces of the connected components.

The multiplier algebra of $H^2(\Sigma)$ is $H^\infty(\Sigma)$, the bounded analytic functions on Σ. Note that the norm in $H^2(\Sigma)$ depends on the choice of base-point a, but the norm in $H^\infty(\Sigma)$ does not, as it is the supremum of the modulus on Σ; for more details see [3].

We say that a proper holomorphic map G from a finite Riemann surface Σ into a bounded open set $U \subseteq \mathbb{C}^d$ is a *holomap* if there is a finite subset Λ of Σ with the property that G is non-singular and injective on $\Sigma \setminus \Lambda$. We say that G is *transversal* at the boundary if

$$\langle DG(\zeta), G(\zeta) \rangle \neq 0 \quad \text{for all } \zeta \in \partial\Sigma.$$

The first result on this problem [4] showed that if $G : \mathbb{D} \to W$ is a biholomorphic unramified C^2-map that is transversal at the boundary, then there is an isomorphism of multiplier algebras from $\mathcal{M}_{\mathbb{D}} = H^\infty(\mathbb{D})$ to $\mathcal{M}_W$ (the assumptions appearing in [4] are slightly weaker – they only required C^1 and did not ask for the map to be unramified – but it seems that one needs a little more; see [5, p. 1132]). This was extended to planar domains in [5, Section 2.3.6], and to finite Riemann surfaces in [23]. Later, it was proved that a holomorphic C^1 embedding of a finite Riemann surface is automatically transversal at the boundary [10, Theorem 3.3]. Combining this automatic transversality result with [23, Theorem 4.2] we obtain:

Theorem 5.17 ([4, 5, 10, 23]). *Let Σ be a finite Riemann surface and W a variety in $\mathbb{B}_d$. Let $G : \Sigma \to \mathbb{B}_d$ be a holomap that maps Σ onto W, is C^2 up to $\partial\Sigma$, and is one-to-one on $\partial\Sigma$. Then the map*

$$\alpha : h \mapsto h \circ G \quad \textit{for } h \in \mathcal{F}_W$$

is an isomorphism from $\mathcal{F}_W$ onto $H^2_G(\Sigma) := H^2(\Sigma) \cap \{h \circ G : h \in \mathrm{Hol}(W)\}$. Consequently, the map $f \mapsto f \circ G$ implements an isomorphism of $\mathcal{M}_W$ onto $H^\infty_G(\Sigma) := H^\infty(\Sigma) \cap \{h \circ G : h \in \mathrm{Hol}(W)\}$.

The main idea of the proof goes back to [4]. One first shows that α, given by the formula $h \mapsto h \circ G$, is a well-defined bounded and invertible map from $\mathcal{F}_W$ onto $H^2_G(\Sigma)$, by computing α^* and $\alpha\alpha^*$, and showing that $\alpha\alpha^*$ is an injective Fredholm operator. The key trick is to break up $\alpha\alpha^*$ as the sum of a Toeplitz operator and a Hilbert-Schmidt operator (see [23, Theorem 4.2] for details). Being positive and Fredholm, injectivity implies invertibility, and the first claim in the theorem follows. A straightforward computation then shows that the asserted isomorphism between $\mathcal{M}_W$ and $H^\infty_G(\Sigma)$ is the similarity induced by α.

Corollary 5.18. *Let Σ be a finite Riemann surface, and let V and W be varieties in $\mathbb{B}_d$ such that $W = G(\Sigma)$, where $G : \Sigma \to \mathbb{B}_d$ is a holomap which is C^2 on $\overline{\Sigma}$ and is one-to-one on $\partial\Sigma$. Let $F : W \to V$ be a biholomorphism that extends to be C^2 and one-to-one on $\overline{W}$. Then the map $\varphi : \mathcal{M}_V \to \mathcal{M}_W$, given by*

$$\varphi(f) = f \circ F \quad \text{for } f \in \mathcal{M}_V,$$

is an isomorphism.

As an application of the above results, we give the following theorem on extension of bounded holomorphic maps from a one-dimensional subvariety of the ball to the entire ball (under rather general assumptions). Such an extension theorem is difficult to prove using complex-analytic techniques, and it is pleasing to obtain it from operator theoretic considerations.

Corollary 5.19 ([4] and [23], Corollary 4.12). *Let W be as in Theorem 5.17. Then $\mathcal{M}_W = H^\infty(W)$, and the norms are equivalent. Consequently, every $h \in H^\infty(W)$ extends to a multiplier in $\mathcal{M}_d$, and in particular to a bounded holomorphic function on $\mathbb{B}_d$. Moreover, there exists a constant C such that for all $h \in H^\infty(W)$, there is an $\tilde{h} \in \mathcal{M}_d$ such that $\tilde{h}\big|_W = h$ and $\|\tilde{h}\|_\infty \leq \|\tilde{h}\|_{\mathcal{M}_d} \leq C\|h\|_\infty$.*

5.4. A class of counter-examples

In the last two subsections we saw classes of varieties, for which (well-behaved) biholomorphism of the varieties implies isomorphism of the multiplier algebras. We now turn to exhibiting a class of examples that show that, in general, biholomorphism of the varieties does not imply that the multiplier algebras are isomorphic. In particular, these examples show that biholomorphic varieties need not be multiplier biholomorphic.

Proposition 5.20. *Suppose that $G : \mathbb{D} \to \mathbb{B}_d$ is a proper injective holomorphic map which extends to a differentiable map on $\mathbb{D} \cup \{-1, 1\}$ such that the extension, also denoted by G, satisfies $G(1) = G(-1)$. If $V = G(\mathbb{D})$ is a variety, then $G^{-1} \notin \mathcal{M}_V$. In particular, the embedding*

$$\mathcal{M}_V \to \mathcal{M}_\mathbb{D} = H^\infty, \quad f \mapsto f \circ G$$

is not surjective.

One way to prove this proposition is to observe that such a map G can not be bi-Lipschitz with respect to the pseudohyperbolic metric, and then invoke Corollary 3.5 (see [10, Remark 6.3] for details). For an alternative proof, we refer the reader to [10, Theorem 5.1].

Example 5.21. Fix $r \in (0, 1)$, and let

$$b(z) = \frac{z - r}{1 - rz}.$$

Note that $b(1) = 1$ and $b(-1) = -1$. Define

$$G(z) = \frac{1}{\sqrt{2}} \left(z^2, b(z)^2\right).$$

It is not hard to verify that this map is a biholomorphism satisfying the hypotheses of Proposition 5.20. Therefore, $\mathcal{M}_V \subsetneq H^\infty(V)$, and G^{-1} is not a multiplier. By Corollary 6.4 below we obtain that $\mathcal{M}_V$ is not isomorphic to $\mathcal{M}_{\mathbb{D}} = H^\infty$.

6. Embedded discs in $\mathbb{B}_\infty$

6.1. Some general observations

In this section we will examine multiplier algebras $\mathcal{M}_V$ where $V = G(\mathbb{D}) \subseteq \mathbb{B}_d$ is a biholomorphic image of a disc via a biholomorphism $G : \mathbb{D} \to \mathbb{B}_d$. The case that interests us most is $d = \infty$.

Theorem 6.1 ([10], Theorem 2.5). *Let V and W be two varieties in $\mathbb{B}_d$, biholomorphic to a disc via the maps G_V and G_W, respectively. Furthermore, assume that*

(a) *for every $\lambda \in V$, the fiber $\pi^{-1}\{\lambda\}$ is the singleton $\{\rho_\lambda\}$, and*
(b) *$\pi(M(\mathcal{M}_V)) \cap \mathbb{B}_d = V$.*

If $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ is an algebra isomorphism, then $F = F_\varphi|_W$ is a multiplier biholomorphism $F : W \to V$, such that $\varphi(f) = f \circ F$ for all $f \in \mathcal{M}_V$.

Here $F = F_\varphi$ is the function provided by Proposition 3.1. By saying that F is a *multiplier biholomorphism* we mean that (i) $F = (F_1, F_2, \ldots)$ where every $F_i \in \mathcal{M}_W$, i.e., is a multiplier, and (ii) F is holomorphic on W, in the sense that for every $\lambda \in W$ there is a ball $B_\epsilon(\lambda)$ and a holomorphic function $\tilde{F} : B_\epsilon(\lambda) \to \mathbb{C}^d$ such that $F\big|_{B_\epsilon(\lambda)\cap W} = \tilde{F}\big|_{B_\epsilon(\lambda)\cap W}$. We require slightly different terminology (compared to Section 3) because we are dealing with $d = \infty$, and we are not making any complete boundedness assumptions (see Remark 3.2). For more details about holomorphic maps in this setting of discs embedded in $\mathbb{B}_\infty$ see [10, Section 2].

Proof. We assume that $d = \infty$. There are two issues here: we need to prove that F is a biholomorphism, and that $F(W) = V$ in the isomorphic case. For the first issue, let $\alpha = (\alpha_i)_{i=1}^\infty \in \ell^2$. Then

$$\langle F \circ G_W(z), \alpha \rangle = \sum_{i=1}^\infty \overline{\alpha_i} h_i(z),$$

where $h_i(z) := F_i \circ G_W(z)$. As characters are completely contractive, we have

$$\sum_{i=1}^\infty |h_i(z)|^2 = \|F(G_W(z))\|^2 = \|\rho_{G_W(z)}(\varphi(Z|_W))\|^2 \le \|Z|_W\|^2 = 1.$$

Thus, $\sum_{i=1}^\infty \overline{\alpha_i} h_i$ converges uniformly on W since by the Cauchy–Schwartz inequality,

$$\sum_{n=N}^\infty |\overline{\alpha_n} h_n(z)| \le \left(\sum_{n=N}^\infty |\alpha_n|^2 \right)^{\frac{1}{2}} \xrightarrow{N\to\infty} 0.$$

Therefore, $\langle F \circ G_W(\cdot), \alpha\rangle$ is holomorphic for all α, and it follows that F is holomorphic (see [10, Section 2]).

We now show that the injectivity of φ implies that F is not constant, and that this implies $F(W) \subseteq \mathbb{B}_\infty$. Suppose that F is the constant function λ ($\lambda \in \overline{\mathbb{B}_d}$). Then for every i we have $\varphi(\lambda_i - Z_i|_V) = \lambda_i - F_i = 0$. By the injectivity of φ, $Z_i|_V = \lambda_i$, which is impossible as V is not a singleton. Thus, F is not constant. If $\mu = F(\lambda)$ lies in $\partial\mathbb{B}_\infty$ for some $\lambda \in W$, then $\langle F \circ G_W(\cdot), \mu\rangle$ is a holomorphic function into $\overline{\mathbb{D}}$, which is equal to 1 at λ. The maximum modulus principle would then imply that this function is constant, so this cannot happen.

In view of the previous paragraph, $F(W) \subseteq \mathbb{B}_\infty$. Since for every $\lambda \in W$, $\varphi^*(\rho_\lambda) \in \pi^{-1}\{F(\lambda)\} \subseteq \pi^{-1}\mathbb{B}_\infty$, by the assumptions (a) and (b), we conclude that F maps W into V, and therefore (by Corollary 3.3) that $\varphi(f) = f \circ F$. In particular, φ is weak-$*$ continuous, and so (as φ is an isomorphism) φ^{-1} is weak-$*$ continuous too. Thus, both φ^* and $(\varphi^{-1})^*$ map point evaluations to point evaluations. We conclude that F is a biholomorphism, mapping W onto V. $\square$

Remark 6.2. We do not know when precisely conditions (a) and (b) in the above theorem hold. We do not have an example in which they fail. We do know that if a variety V in $\mathbb{B}_\infty$ is the intersection of zero sets of a family of polynomials (or more generally, elements in $\mathcal{M}_\infty$ that are norm limits of polynomials) then (b) holds (see [10, Proposition 2.8]).

By a familiar result [21, p. 143] the automorphisms of H^∞ are the maps $C_\theta(h) := h \circ \theta$ for some Möbius map θ (i.e., $\theta(z) = \lambda\left(\frac{z-a}{1-\overline{a}z}\right)$ for $a \in \mathbb{D}$, and $\lambda \in \partial\mathbb{D}$). If G is a biholomorphic map of the disc onto a variety V in $\mathbb{B}_d$, then one can transfer the Möbius maps to conformal automorphisms of V by sending θ to $G \circ \theta \circ G^{-1}$. Since this can be reversed, these are precisely the conformal automorphisms of V. We say that $\mathcal{M}_V$ is *automorphism invariant* if composition with all these conformal maps yields automorphisms of $\mathcal{M}_V$.

Proposition 6.3. *Let V and W be two varieties in $\mathbb{B}_d$, biholomorphic to a disc via the maps G_V and G_W, respectively. Assume that V satisfies the conditions* (a) *and* (b) *of Theorem* 6.1*. Let $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ be an algebra isomorphism. Then there is a Möbius map θ such that the diagram*

$$\begin{tikzcd} \mathcal{M}_V \arrow[r, "\varphi"] \arrow[d, "C_{G_V}"'] & \mathcal{M}_W \arrow[d, "C_{G_W}"] \\ H^\infty \arrow[r, "C_\theta"'] & H^\infty \end{tikzcd}$$

commutes.

The proof follows by Theorem 6.1 and the above discussion. We omit the details.

Suppose that the automorphism θ can be chosen to be the identity, or equivalently, that C_F, where $F = G_V \circ G_W^{-1}$, is an isomorphism of $\mathcal{M}_V$ onto $\mathcal{M}_W$. Then we will say that $\mathcal{M}_V$ and $\mathcal{M}_W$ are *isomorphic via the natural map.*

Corollary 6.4. *Let V and W be two varieties in $\mathbb{B}_d$, biholomorphic to a disc via the maps G_V and G_W, respectively. Assume that V satisfies the conditions* (a) *and* (b) *of Theorem* 6.1. *If $\mathcal{M}_V$ or $\mathcal{M}_W$ is automorphism invariant, then $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic if and only if they are isomorphic via the natural map C_F, where $F = G_V \circ G_W^{-1}$. In particular, if $\mathcal{M}_V$ is isomorphic to H^∞, then C_{G_V} implements the isomorphism.*

6.2. A special class of embeddings

We now consider a class of embedded discs in $\mathbb{B}_\infty$. The principal goal is to exhibit a large class of multiplier biholomorphic discs in $\mathbb{B}_\infty$ for which we may classify the obtained multiplier algebras. Though this goal is not obtained fully, we are able to tell when one of these multiplier algebras is isomorphic to $H^\infty := H^\infty(\mathbb{D})$. Moreover, we obtain an uncountable family of embeddings of the disc into $\mathbb{B}_\infty$ such that all obtained multiplier algebras are mutually non-isomorphic, while the one-dimensional varieties associated with them are all multiplier biholomorphic to each other, via a biholomorphism that extends continuously and one-to-one up to the boundary.

Let $(b_n)_{n=1}^\infty$ be an ℓ^2-sequence of norm 1 and $b_1 \neq 0$. Define $G : \mathbb{D} \to \mathbb{B}_\infty$ by

$$G(z) = (b_1 z, b_2 z^2, b_3 z^3, \dots) \quad \text{for } z \in \mathbb{D}.$$

Then $G : \mathbb{D} \to G(\mathbb{D}) \subseteq \mathbb{B}_\infty$ is a biholomorphism with inverse $b_1^{-1} Z_1|_{G(\mathbb{D})}$ and these maps are multipliers. Moreover, $G(\mathbb{D})$ is a variety because the conditions on the sequence (b_n) (namely, that it has norm 1 and that $b_1 \neq 0$) imply that

$$V := V(\{b_n z_1^n - b_1^n z_n \ : \ n \geq 2\}) = G(\mathbb{D}).$$

It is easy to see that any two varieties arising this way are multiplier biholomorphic.

Remark 6.5. One may also consider embeddings similar to the above but with the difference that $\sum |b_n|^2 < 1$, and the results obtained are in some sense analogous to what we describe here, but also contain some surprises. Since the varieties involved are technically different from those on which we concentrate in this survey, we do not elaborate; the reader is referred to [10, Section 8].

Define a kernel on $\mathbb{D}$ by

$$k_G(z, w) = \frac{1}{1 - \langle G(z), G(w) \rangle} \quad \text{for } z, w \in \mathbb{D},$$

and let $\mathcal{H}_G$ be the Hilbert function space on $\mathbb{D}$ with reproducing kernel k_G. Then we can define a linear map $U : \mathcal{F}_V \to \mathcal{H}_G$ by $Uh = h \circ G$. Since

$$\langle k_{G(z)}, k_{G(w)} \rangle = \frac{1}{1 - \langle G(z), G(w) \rangle} = \langle (k_G)_z, (k_G)_w \rangle \quad \text{for all } z, w \in \mathbb{D},$$

it follows that $Uk_{G(z)} = (k_G)_z$ extends to a unitary map of $\mathcal{F}_V$ onto $\mathcal{H}_G$. Hence composition with G determines a unitarily implemented completely isometric isomorphism $C_G : \mathcal{M}_V \to \mathrm{Mult}(\mathcal{H}_G)$. Therefore, we can work with multiplier algebras of Hilbert function spaces on the disc rather than the algebras $\mathcal{M}_V$ itself.

Now write

$$k_G(z,w) = \frac{1}{1-\sum_{n=1}^\infty |b_n|^2 (z\overline{w})^n} =: \sum_{n=0}^\infty a_n (z\overline{w})^n$$

for a suitable sequence $(a_n)_{n=0}^\infty$. A direct computation shows that the sequence (a_n) satisfies the recursion

$$a_0 = 1 \quad \text{and} \quad a_n = \sum_{k=1}^n |b_k|^2 a_{n-k} \quad \text{for } n \geq 1.$$

Moreover, $0 < a_n \leq 1$ for all $n \in \mathbb{N}$.

Due to the special form of the kernel k_G, we may compute the multiplier norm of monomials in $\mathcal{H}_G$.

Lemma 6.6 ([10], Lemma 7.2). *For every $n \in \mathbb{N}$, it holds that*

$$\|z^n\|^2_{\mathrm{Mult}(\mathcal{H}_G)} = \|z^n\|^2_{\mathcal{H}_G} = \frac{1}{a_n}.$$

We now compare between two varieties embedded discs V and W as above. We let $(b_n^V)_{n=1}^\infty$ and $(b_n^W)_{n=1}^\infty$ be two ℓ^2-sequence of norm 1 and $b_1^V \neq 0 \neq b_1^W$, and define $G_V, G_W : \mathbb{D} \to \mathbb{B}_\infty$ by

$$G_V(z) = (b_1^V z, b_2^V z^2, b_3^V z^3, \dots) \quad \text{and} \quad G_W(z) = (b_1^W z, b_2^W z^2, b_3^W z^3, \dots).$$

As before, we consider also the sequences $(a_n^V)_{n=0}^\infty$ and $(a_n^W)_{n=0}^\infty$ which satisfy

$$k_{G_V}(z,w) = \sum_{n=0}^\infty a_n^V (z\overline{w})^n \quad \text{and} \quad k_{G_W}(z,w) = \sum_{n=0}^\infty a_n^W (z\overline{w})^n.$$

Theorem 6.7 ([10], Proposition 7.5). *The algebras $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic via the natural map of composition with $G_V \circ G_W^{-1}$ if and only if the sequences (a_n^V) and (a_n^W) are comparable, i.e., if and only if there is some $c > 0$ such that $c^{-1}|a_n^V| \leq |a_n^W| \leq c|a_n^V|$ for all n.*

Furthermore, if $\pi^{-1}\{\lambda\} = \{\rho_\lambda\}$ for every $\lambda \in W$ and $\mathcal{M}_W$ is automorphism invariant, then $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic if and only if they are isomorphic via the natural map.

Proof. If (a_n^V) and (a_n^W) are comparable, then by Lemma 6.6 the norms in $\mathcal{H}_{G_V}$ and $\mathcal{H}_{G_W}$ of the orthogonal base $\{z^n : n \in \mathbb{N}\}$ are comparable. Thus, the identity map is an invertible bounded operator between $\mathcal{H}_{G_V}$ and $\mathcal{H}_{G_W}$. Therefore, $\mathrm{Mult}(\mathcal{H}_{G_V}) = \mathrm{Mult}(\mathcal{H}_{G_W})$, so that $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic via the natural map.

Conversely, if $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic via the natural map then $\mathrm{Mult}(\mathcal{H}_{G_V}) = \mathrm{Mult}(\mathcal{H}_{G_W})$. Therefore the identity map is an isomorphism between these two semisimple Banach algebras, so the isomorphism is topological. By Lemma 6.6, the sequences (a_n^V) and (a_n^W) are comparable.

If if $\pi^{-1}\{\lambda\} = \{\rho_\lambda\}$ for every $\lambda \in W$ and $\mathcal{M}_W$ is automorphism invariant, then by Corollary 6.4, this is equivalent to $\mathcal{M}_V$ being isomorphic to $\mathcal{M}_W$ via any isomorphism. □

Corollary 7.4 of [10] states that if $\mathcal{M}_W$ is automorphism invariant and $\sup_{n\geq 1}(a_n^W/a_{n-1}^W) < \infty$, then $\pi^{-1}\{\lambda\} = \{\rho_\lambda\}$ for every $\lambda \in W$. This gives rise to examples in which the second part of Theorem 6.7 is meaningful. For example, the following corollary follows by the above by setting $(b_1^W, b_2^W, b_3^W, \dots) = (1, 0, 0, \dots)$, and noting that $a_n^W = 1$ for all $n \in \mathbb{N}$.

Corollary 6.8. *$\mathcal{M}_V$ is isomorphic to H^∞ if and only if the sequence (a_n^V) is bounded below.*

In terms of the sequence (b_n) the result reads as follows.

Corollary 6.9. *Let $V = G(\mathbb{D})$ where $G(z) = (b_1 z, b_2 z^2, b_3 z^3, \dots)$, where $\|(b_n)\|_{\ell^2} = 1$ and $b_1 \neq 0$. Then $\mathcal{M}_V$ is isomorphic to H^∞ if and only if*

$$\sum_{n=1}^{\infty} n|b_n|^2 < \infty.$$

Proof. By the Erdős–Feller–Pollard theorem (see [17, Chapter XIII, Section 11]) we know that

$$\lim_{n\to\infty} a_n = \frac{1}{\sum_{n=1}^{\infty} n|b_n|^2},$$

where $1/\infty = 0$. Hence, (a_n) is bounded below if and only is the series converges. □

Example 6.10 ([10], Example 7.9). For every $s \in [-1, 0]$, consider the reproducing kernel Hilbert spaces $\mathcal{H}_s$ with kernel

$$k^s(z, w) = \sum_{n=0}^{\infty} (n+1)^s (z\overline{w})^n \quad \text{for } z, w \in \mathbb{D}.$$

It is shown in [10] that these kernels arise from embeddings as above, and also that these embeddings satisfy all the conditions of Theorem 6.7. We have that $a_n^s = (n+1)^s$ in this case, and obviously the sequences $\big((n+1)^s\big)_{n=0}^\infty$ and $\big((n+1)^{s'}\big)_{n=0}^\infty$ are not comparable for $s \neq s'$. Thus the family of algebras $\mathrm{Mult}(\mathcal{H}_s)$ is an uncountable family of multiplier algebras of the type we consider which are pairwise non-isomorphic. Note that all these algebras live on varieties that are multiplier biholomorphic via a biholomorphism that extends continuously to the boundary.

7. Open problems

Though we have accumulated a body of satisfactory results, and although we have a rich array of examples and counter examples, the isomorphism problem for irreducible Pick algebras is far from being solved. We close this survey by reviewing some open problems.

7.1. Finite unions of irreducible varieties

Theorem 5.5 implies that in the case where V and W are finite unions of irreducible varieties in $\mathbb{B}_d$ (for $d < \infty$), we have that if $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic then V and W are multiplier biholomorphic. It is not known whether the converse holds. We did see an example of multiplier biholomorphic varieties which are *infinite* unions of irreducible varieties but with non-isomorphic multiplier algebras; see Example 5.7. We also saw an example (Example 5.21) of biholomorphic irreducible varieties, with non-isomorphic multiplier algebras; this, however, was not a *multiplier* biholomorphism. And so the question, whether a multiplier biholomorphism of varieties which are a finite union of irreducible ones implies that the multiplier algebras are isomorphic, remains unsolved for $d < \infty$ (for $d = \infty$ the answer is *no*, see Example 6.10).

7.2. Maximal ideal spaces of multiplier algebras

As we remarked in the introduction, in the case $d = \infty$ there are multiplier algebras $\mathcal{M}_V$ for which there are points in $\pi^{-1}\mathbb{B}_\infty \subseteq M(\mathcal{M}_V)$ which are not point evaluations; similarly, there are also multiplier algebras $\mathcal{M}_V$ with characters in fibers over points in $\mathbb{B}_\infty \setminus V$ [10, Example 2.4]. Nevertheless, when we restrict attention to "sufficiently nice" varieties, it might be the case that the characters over the varieties do behave appropriately, in the sense that for every $\lambda \in V$ the fiber $\pi^{-1}\{\lambda\}$ is the singleton $\{\rho_\lambda\}$, and $\pi(M(\mathcal{M}_V)) \cap \mathbb{B}_\infty = V$. In particular, it will be interesting to obtain such a result for the family of discs embedded in $\mathbb{B}_\infty$ by $G(z) = (b_1 z, b_2 z^2, \ldots)$ as in Section 6.2. This will amount to obtaining a better understanding of the maximal ideal space of the algebras $\mathrm{Mult}(\mathcal{H}_G)$.

7.3. The correct equivalence relation

Theorem 5.5 says (under some assumptions) that if $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic then V and W are multiplier biholomorphic. We have seen a couple of counter examples showing that the converse is not true, but to clarify the nature of the obstruction let us point out the following: *multiplier biholomorphism is not an equivalence relation*, while, on the other hand, isomorphism is an equivalence relation; see [10, Remark 6.7]. This leads to the problem: describe the equivalence relation $\cong$ on varieties given by "$V \cong W$ iff $\mathcal{M}_V$ is isomorphic to $\mathcal{M}_W$" in complex geometric terms.

7.4. Structure theory

The central problem dealt with up to now was the isomorphism problem: when are $\mathcal{M}_V$ and $\mathcal{M}_W$ isomorphic (or isometrically isomorphic)? For isometric isomorphisms the problem is completely resolved: the structure of the Banach algebra $\mathcal{M}_V$ is completely determined by the conformal structure of V. As for algebraic isomorphisms, we know that the biholomorphic structure of V is an invariant of the algebra $\mathcal{M}_V$. This opens the door for a profusion of delicate questions on how to read the (operator) algebraic information from the variety, and vice versa. For example, how is the dimension of V reflected in $\mathcal{M}_V$? If V is a finite Riemann surface with m handles and n boundary components, what in the algebraic structure of $\mathcal{M}_V$ reflects the m handles and the n boundary components? What about algebraic-geometric invariants, such as number of irreducible components or degree?

7.5. Embedding dimension

A particular question in the flavour of the above broad question, is this: given an irreducible complete Pick algebra $\mathcal{A}$, what is the minimal $d \in \{1, 2, \ldots, \infty\}$ such that $\mathcal{A}$ is isomorphic to $\mathcal{M}_V$, with $V \subseteq \mathbb{B}_d$? This question is interesting – and the answer is unknown – even for the case of the multiplier algebra of the well-studied *Dirichlet space* $\mathcal{D}$ (see [6]).

7.6. Other algebras. Norm closed algebras of multipliers

The isomorphism problem makes sense on many natural algebras, for examples, one may wonder whether, given two varieties $V, W \subseteq \mathbb{B}_d$, is it true that the algebra $H^\infty(V)$ is (isometrically) isomorphic to $H^\infty(W)$ precisely when V is biholomorphic to W? Answering this question will require an understanding of the maximal ideal spaces of the bounded analytic functions of a variety.

Another natural class of algebras is given by the norm closures of the polynomials in $\mathcal{M}_V$,

$$\mathcal{A}_V = \overline{\mathbb{C}[z]}^{\|\cdot\|_{\mathcal{M}_V}}.$$

(These algebras are sometimes referred to as the *continuous multipliers on* $\mathcal{F}_V$, but this terminology is misleading since in general $\mathcal{A}_V \subsetneq C(\overline{V}) \cap \mathcal{M}_V$; see [29, Section 5.2].) In fact, the isomorphism problem was studied in [15] first for the algebras $\mathcal{A}_V$. It was later realized that the norm closed algebras present some delicate difficulties; see [16, Section 7]. In fact, subtleties arise already in the case $d = 1$; see [16, Section 8].

7.7. Approximation and Nullstellensatz

One of the problems in studying the isomorphism problem for the norm closed algebras $\mathcal{A}_V$ is the following (see [16, Section 7] for an explanation of how these issues relate). Denote by $\mathcal{A}_d$ the norm closed algebra generated by the polynomials in $\mathcal{M}_d$. Let $V \subseteq \mathbb{B}_d$ be a variety, and assume that $d < \infty$, and that V is determined by polynomials. Consider the following ideals $K_V = \{p \in \mathbb{C}[z] : p\big|_V = 0\}$, $I_V = \{f \in \mathcal{A}_d : f\big|_V = 0\}$, and $J_V = \{f \in \mathcal{M}_d : f\big|_V = 0\}$. A natural question is whether I_V is the norm closure of K_V, and whether J_V is the WOT-closure of I_V. In other words, we know that every $f \in I_V$ is the norm limit of polynomials, but does the fact that f vanishes on V imply that it can be approximated in norm using only polynomials from K_V? Likewise, is every function in J_V the limit of a bounded and pointwise convergent sequence of polynomials in K_V (or functions in I_V)?

It is very natural to conjecture that the answer is *yes*, and this was indeed proved for homogeneous ideals; see [16, Corollary 6.13] (see also [27, Corollary 2.1.31] for the WOT case). As may be expected, this approximation result is closely related to an analytic Nullstellensatz: $\sqrt{\mathcal{I}} = I(V(\mathcal{I}))$ (here $\mathcal{I}$ is some norm closed ideal in $\mathcal{A}_d$, $V(\mathcal{I})$ is the zero locus of the ideal $\mathcal{I}$, $I(V(\mathcal{I}))$ is the ideal of all functions in $\mathcal{A}_d$ vanishing on $V(\mathcal{I})$, and $\sqrt{\mathcal{I}}$ is an appropriately defined radical; see [16, Theorem 6.12] and [27, 2.1.30]). However, we understand very little about these issues in the non-homogeneous case.

Note added in proof. In the time that passed since this survey was written, a few papers appeared on the isomorphism problem for complete Pick algebras. We mention the papers:

1. M. Hartz, On the isomorphism problem for multiplier algebras of Nevanlinna–Pick spaces, *Canad. J. Math.* (to appear), arXiv:1505.05108 (2015).
2. M. Hartz and M. Lupini, The classification problem for operator algebraic varieties and their multiplier algebras, arXiv:1508.07044 (2015).
3. J.E. McCarthy and O.M. Shalit, Spaces of Dirichlet series with the Complete Pick property, *Israel J. Math.* (to appear), arXiv:1507.04162 (2015).

References

[1] J. Agler and J.E. McCarthy. Complete Nevanlinna–Pick Kernels. *J. Funct. Anal.*, 175:11–124, 2000.

[2] J. Agler and J.E. McCarthy. *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics, 44, Providence, RI: American Mathematical Society, 2002.

[3] P.R. Ahern and D. Sarason. The H^p spaces of a class of function algebras. *Acta Math.* 117, 123–163, 1967.

[4] D. Alpay, M. Putinar and V. Vinnikov. A Hilbert space approach to bounded analytic extension in the ball. *Commun. Pure Appl. Anal.*, 2(2): 139–145, 2003.

[5] N. Arcozzi, R. Rochberg and E.T. Sawyer. Carleson measures for the Drury–Arveson Hardy space and other Besov–Sobolev spaces on complex balls. *Adv. Math.*, 218(4): 1107–1180, 2008.

[6] N. Arcozzi, R. Rochberg, E.T. Sawyer, and B. Wick. The Dirichlet space: a survey. *New York J. Math*, 17:45–86, 2011.

[7] L. Bers. On rings of analytic functions. *Bull. Amer. Math. Soc.*, 54:311–315, 1948.

[8] F.F. Bonsall and J. Duncan. *Complete normed algebras.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 80. Springer-Verlag, New York-Heidelberg, 1973.

[9] H.G. Dales. Automatic continuity: a survey. *Bull. London Math. Soc.*, 10(2):192–183, 1978.

[10] K.R. Davidson, M. Hartz and O.M. Shalit. Multipliers of embedded discs. *arXiv:* 1307.3204 [*math.OA*], 2014. Also in *Complex Anal. Oper. Theory*, 9(2):287–321, 2015.

[11] K.R. Davidson, M. Hartz and O.M. Shalit. Erratum to: Multpliers of embedded discs. *Complex Anal. Oper. Theory*, 9(2):323–327, 2015.

[12] K.R. Davidson and D.R. Pitts. The algebraic structure of non-commutative analytic Toeplitz algebras. *Math. Ann.*, 311(2):275–303, 1998.

[13] K.R. Davidson and D.R. Pitts. Nevanlinna–Pick interpolation for non-commutative analytic Toeplitz algebras. *Integral Equations Operator Theory*, 31:321–337, 1998.

[14] K.R. Davidson and D.R. Pitts. Invariant subspaces and hyper-reflexivity for free semigroup algebras. *Proc. Lond. Math. Soc.*, 78:401–430, 1999.

[15] K.R. Davidson, C. Ramsey and O.M. Shalit. The isomorphism problem for some universal operator algebras. *Adv. Math.*, 228(1):167–218, 2011.

[16] K.R. Davidson, C. Ramsey and O.M. Shalit. Operator algebras for analytic varieties. *arXiv:*1201.4072 [*math.OA*], 2014. Also in *Trans. Amer. Math. Soc.*, 367:1121–1150, 2015.

[17] W. Feller. *An introduction to probability theory and its applications*, Vol. I, 3rd ed., John Wiley & sons Inc., 1968.

[18] I.M. Gelfand, D.A. Raikov and G.E. Shilov. Commutative normed rings. *Uspehi Matem. Nauk* (*N.S.*) 1(2):48–146, 1946.

[19] R.C. Gunning. *Introduction to holomorphic functions of several variables*, Vol. II, Local theory. Monterey, CA, 1990.

[20] M. Hartz. Topological isomorphisms for some universal operator algebras. *J. Funct. Anal.*, 263(11):3564–3587, 2012.

[21] K. Hoffman. *Banach spaces of analytic functions*. Prentice Hall Series in Modern Analysis Prentice Hall, Inc., Englewood Cliffs, N. J. 1962.

[22] M. Kennedy and D. Yang. A non-self-adjoint Lebesgue decomposition. *Anal. PDE*, 7(2):497–512, 2014.

[23] M. Kerr, J.E. McCarthy and O.M. Shalit. On the isomorphism question for complete pick multiplier algebras. *Integral Equations Operator Theory*, 76(1):39–53, 2013.

[24] V.I. Paulsen. *Completely Bounded Maps and Operator Algebras*. Cambridge University Press, Cambridge, 2002.

[25] G. Pick. Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden. *Math Ann.*, 77(1):7–23, 1915.

[26] G. Popescu. Operator theory on noncommutative varieties. *Indiana Univ. Math. J.*, 5(2):389–442, 2006.

[27] C. Ramsey. *Maximal ideal space techniques in non-selfadjoint operator algebras*. PhD. thesis, University of Waterloo, 2013. http://hdl.handle.net/10012/7464.

[28] W. Rudin. *Function theory in the unit ball of* $\mathbb{C}^n$. Springer-Verlag, 1980.

[29] O.M. Shalit. *Operator theory and function theory in Drury-.Arveson space and its quotients*. In *Handbook of Operator Theory*, editor Daniel Alpay, Springer, 2015, pp. 1125–1180.

[30] O.M. Shalit and B. Solel. Subproduct Systems. *Doc. Math.*, 14:801–868, 2009.

Guy Salomon and Orr Moshe Shalit
Department of Mathematics
Technion – Israel Institute of Technology
Haifa 3200003, Israel
e-mail: guy.salomon@tx.technion.ac.il
oshalit@tx.technion.ac.il

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The Stationary State/Signal Systems Story

Olof J. Staffans

Abstract. We give an introduction to the theory of linear stationary s/s (state/signal) systems in continuous time. A s/s system has a state space which plays the same role as the state space of an ordinary i/s/o (input/state/output) system, but it differs from an i/s/o systems in the sense that the interaction signal which connects the system to the outside world has not been divided a priori into one part which is called the "input" and another part which is called the "output". The class of s/s systems can be used to model, e.g., linear time-invariant circuits which may contain both lumped and distributed components. To each s/s system corresponds in general an infinite number of i/s/o systems which differ from each other by the choice of how the interaction signal has been divided into an input part and output part. Each such i/s/o system is called an i/s/o representation of the given s/s system.

We begin by giving an introduction to the time domain theory for i/s/o and s/s systems, then continue by taking a brief look at the frequency domain theory for i/s/o and s/s systems, and end with a short overview of the notions of passivity and conservativity of i/s/o and s/s systems. In all cases the s/s results that we present can be formulated in such a way that they do not depend on any particular i/s/o representation of the s/s system, but it is still true that there is a strong connection between the central properties of a s/s system and the corresponding properties of its i/s/o representations.

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1. Introduction to state/signal systems

1.1. Input/state/output systems in the time domain

A "well-posed" linear stationary discrete time i/s/o (input/state/output) system is of the form

$$\Sigma_{\mathrm{i/s/o}}: \begin{cases} x(n+1) = Ax(n) + Bu(n), \\ \quad y(n) = Cx(n) + Du(n), \end{cases} \qquad n \in \mathbb{Z}^+. \tag{1.1}$$

Here the *input* u, the *state* x, and the *output* y take their values in three Hilbert spaces, the input space $\mathcal{U}$, the state space $\mathcal{X}$, and the output space $\mathcal{Y}$, respectively, $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$, and A, B, C, and D, are bounded linear operators with the appropriate domain and range spaces. These operators are called as follows: A is the main operator, B is the control operator, C is the observation operator, and D is the feed-through operator. By a *future trajectory* of $\Sigma_{\mathrm{i/s/o}}$ we mean a sequence $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ defined on $\mathbb{Z}^+$ with values in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$ which satisfies (1.1) for all $n \in \mathbb{Z}^+$.

If we here replace the discrete time axis $\mathbb{Z}^+$ by the continuous time axis $\mathbb{R}^+ = [0, \infty)$ and at the same time replace the first equation in (1.1) by the corresponding differential equation, then we get a bounded linear stationary continuous time i/s/o system of the form

$$\Sigma_{\mathrm{i/s/o}}: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \qquad t \in \mathbb{R}^+. \tag{1.2}$$

The input u, the state x, and the output y still take their values in the Hilbert spaces $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$, respectively, and the main operator A, the control operator B, the observation operator C, and the feed-through operator D are still bounded linear operators. By a *classical future trajectory* of $\Sigma_{\mathrm{i/s/o}}$ we mean a triple of functions $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ which satisfies (1.2) for all $t \in \mathbb{R}^+$, with x continuously differentiable with values in $\mathcal{X}$ and $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ continuous with values in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$.

Unfortunately, typical stationary i/s/o systems modelled by partial differential equations are *not bounded* in the sense that even if it might be possible to describe the dynamics of the system with an equation of the type (1.2), the operators A, B, C, and D need not be bounded. For this reason a more general version of (1.2) is needed. Clearly, equation (1.2) can be rewritten in the form

$$\Sigma_{\mathrm{i/s/o}}: \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \tag{1.3}$$

where S is the bounded block matrix operator $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. We get a much more general class of linear stationary continuous time i/s/o systems by simply allowing the operator S in (1.3) bo be unbounded (but still closed) and rewriting (1.3) in the form

$$\Sigma_{\mathrm{i/s/o}}\colon \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \operatorname{dom}(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+. \tag{1.4}$$

This class of systems covers "all" the standard models from mathematical physics. We call S *the generator* of $\Sigma_{\mathrm{i/s/o}}$. Usually the domain $\operatorname{dom}(S)$ of S is assumed to be dense in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$.

Definition 1.1.

(i) By a regular (continuous time stationary) *i/s/o* (*input/state/output*) node we mean a colligation $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, where $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ are Hilbert spaces, and $S\colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ is a closed linear operator with dense domain.

(ii) The *main operator* A of $\Sigma_{\mathrm{i/s/o}}$ (or of S) is defined by

$$\begin{aligned} \operatorname{dom}(A) &:= \left\{ x \in \mathcal{X} \,\middle|\, \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \operatorname{dom}(S) \right\}, \\ Ax &:= \begin{bmatrix} 1_{\mathcal{X}} & 0 \end{bmatrix} S \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right], \qquad x \in \operatorname{dom}(A). \end{aligned} \tag{1.5}$$

Here $\begin{bmatrix} 1_{\mathcal{X}} & 0 \end{bmatrix}$ stands for the operator which maps $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ into x.

(iii) By a *classical future trajectory* of $\Sigma_{\mathrm{i/s/o}}$ we mean a triple of functions $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ which satisfies (1.4) for all $t \in \mathbb{R}^+$, with x continuously differentiable with values in $\mathcal{X}$ and $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ continuous with values in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$.

(iv) By a *generalized future trajectory* of $\Sigma_{\mathrm{i/s/o}}$ we mean a triple of functions $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ which is the limit of a sequence $\left[\begin{smallmatrix} x_n \\ u_n \\ y_n \end{smallmatrix}\right]$ of classical future trajectories of $\Sigma_{\mathrm{i/s/o}}$ in the sense that $x_n \to x$ in $C(\mathbb{R}^+; \mathcal{X})$ and $\left[\begin{smallmatrix} u_n \\ y_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ in $L^2_{\mathrm{loc}}(\mathbb{R}^+; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right])$.

(v) By a regular (time domain) *i/s/o system* system we mean an i/s/o node $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ together with the sets of all classical and generalized future trajectories of Σ.

Above $C(\mathbb{R}^+; \mathcal{X})$ stands for the space of continuous function on $\mathbb{R}^+$ with values in $\mathcal{X}$, and convergence in $C(\mathbb{R}^+; \mathcal{X})$ means uniform convergence on each finite subinterval of $\mathbb{R}^+$. The space $L^2_{\mathrm{loc}}(\mathbb{R}^+; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right])$ consists of functions which belong locally to L^2 over $\mathbb{R}^+$ with values in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, and convergence in $L^2_{\mathrm{loc}}(\mathbb{R}^+; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right])$ means convergence in L^2 on each finite subinterval of $\mathbb{R}^+$.

Note that if S is bounded, then S has a block matrix decomposition $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, and (1.4) is equivalent to (1.2).

1.2. State/signal systems in the time domain

The idea behind the definition of a s/s (state/signal) system is to remove the distinction between the "input" and the "output" of an i/s/o system. This can be done in several ways. One way is to define the *signal space* to be the product $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$ of $\mathcal{X}$ and $\mathcal{Y}$, and to replace the input u and the output y by the combined i/o (input/output) *signal* $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$. After that one absorbs the "output" equation in (1.4) into the domain of a new operator F (whose domain will no longer be dense in $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$), and rewrites (1.4) in the form

$$\Sigma\colon\ \begin{cases}\begin{bmatrix}x(t)\\ w(t)\end{bmatrix}\in\operatorname{dom}(F),\\[4pt] \dot x(t)=F\left(\begin{bmatrix}x(t)\\ w(t)\end{bmatrix}\right),\end{cases}\qquad t\in\mathbb{R}^+,\tag{1.6}$$

$$\begin{aligned}\operatorname{dom}(F)&=\left\{\begin{bmatrix}x_0\\ \left[\begin{smallmatrix}u_0\\ y_0\end{smallmatrix}\right]\end{bmatrix}\in\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]\ \middle|\ \left[\begin{smallmatrix}x_0\\ u_0\end{smallmatrix}\right]\in\operatorname{dom}(S),\ y_0=\begin{bmatrix}0 & 1_{\mathcal{Y}}\end{bmatrix}S\left[\begin{smallmatrix}x_0\\ u_0\end{smallmatrix}\right]\right\},\\ F\begin{bmatrix}x_0\\ \left[\begin{smallmatrix}u_0\\ y_0\end{smallmatrix}\right]\end{bmatrix}&=\begin{bmatrix}1_{\mathcal{X}} & 0\end{bmatrix}S\left[\begin{smallmatrix}x_0\\ u_0\end{smallmatrix}\right],\end{aligned}$$

where $\begin{bmatrix}0 & 1_{\mathcal{Y}}\end{bmatrix}$ stands for the operator which maps $\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]\in\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ into y. Note that (1.6) can be regarded as a special case of (1.4) with $\mathcal{U}=\mathcal{W}$ and $\mathcal{Y}=\{0\}$, apart from the fact that $\operatorname{dom}(F)$ need no longer be dense in $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$.

We can also go one step further and replace the operator F in (1.6) by its graph $V=\operatorname{gph}(F)$. More precisely, we still take $\mathcal{W}=\left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{Y}\end{smallmatrix}\right]$, define the *node space* $\mathfrak{K}$ to be $\mathfrak{K}=\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$, and rewrite (1.6) in the form

$$\Sigma\colon\ \begin{bmatrix}\dot x(t)\\ x(t)\\ w(t)\end{bmatrix}\in V,\qquad t\in\mathbb{R}^+.\tag{1.7}$$

The *generating subspace* $V=\operatorname{gph}(F)$ of Σ can alternatively be interpreted as a reordered version of the graph of the original generator S in (1.4):

$$\begin{aligned}V&=\left\{\begin{bmatrix}z_0\\ x_0\\ \left[\begin{smallmatrix}u_0\\ y_0\end{smallmatrix}\right]\end{bmatrix}\in\mathfrak{K}\ \middle|\ \left[\begin{smallmatrix}x_0\\ u_0\end{smallmatrix}\right]\in\operatorname{dom}(S),\ \left[\begin{smallmatrix}z_0\\ y_0\end{smallmatrix}\right]=S\left[\begin{smallmatrix}x_0\\ u_0\end{smallmatrix}\right]\right\}\\ &=\left\{\begin{bmatrix}z_0\\ x_0\\ \left[\begin{smallmatrix}u_0\\ y_0\end{smallmatrix}\right]\end{bmatrix}\in\mathfrak{K}\ \middle|\ \begin{bmatrix}x_0\\ \left[\begin{smallmatrix}u_0\\ y_0\end{smallmatrix}\right]\end{bmatrix}\in\operatorname{dom}(F),\ z_0=F\begin{bmatrix}x_0\\ \left[\begin{smallmatrix}u_0\\ y_0\end{smallmatrix}\right]\end{bmatrix}\right\}.\end{aligned}\tag{1.8}$$

Definition 1.2.

(i) By a *s/s* (*state/signal*) *node* we mean a colligation $\Sigma=(V;\mathcal{X},\mathcal{W})$, where $\mathcal{X}$ and $\mathcal{W}$ are Hilbert spaces and V is a closed subspace of the product space space $\mathfrak{K}=\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$.

(ii) By a *classical future trajectory* of Σ we mean a pair of functions $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right]$ which satisfies (1.7) for all $t\in\mathbb{R}^+$, with x continuously differentiable with values in $\mathcal{X}$ and w continuous with values in $\mathcal{W}$.

(iii) By a *generalized future trajectory* of Σ we mean a pair of functions $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right]$ which is the limit of a sequence $\left[\begin{smallmatrix}x_n\\ w_n\end{smallmatrix}\right]$ of classical future trajectories of Σ in the sense that $x_n\to x$ in $C(\mathbb{R}^+;\mathcal{X})$ and $w_n\to w$ in $L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W})$.

(iv) By a (time domain) *s/s system* system we mean an s/s node $\Sigma=(V;\mathcal{X},\mathcal{W})$ together with the sets of all classical and generalized future trajectories of Σ.

It is also possible to go in the opposite direction, i.e., to start with a state/signal system of the type (1.7), and to rewrite it into an i/s/o system of the type (1.4) under some additional "regularity" assumptions on the generating subspace V. In this case we start by decomposing the signal space $\mathcal{W}$ (which now is supposed to be an arbitrary Hilbert space with no particular structure) into a direct sum $\mathcal{W} = \mathcal{U} \dot{+} \mathcal{Y}$, and try to rewrite (1.7) into the form (1.4) with $\mathcal{U}$ as input space and $\mathcal{Y}$ as output space, for some closed operator S with dense domain. This will *not be possible for every possible decomposition* $\mathcal{W} = \mathcal{U} \dot{+} \mathcal{Y}$. The closedness of S is not a problem (since the graph of S can be "identified" with V after the permutation of some of the components of V), but the *existence* of a (single-valued) *operator S with dense domain* is more problematic. This is equivalent to the following two conditions on V and on the decomposition $\mathcal{W} = \mathcal{U} \dot{+} \mathcal{Y}$:

(i) if $\left[\begin{smallmatrix} z \\ 0 \\ y \end{smallmatrix}\right] \in V$ and $y \in \mathcal{Y}$, then $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right] = 0$,

(ii) the projection onto the second component of V and $\mathcal{U}$ along the first component of V and $\mathcal{Y}$ is dense in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$.

The first of these conditions means that the z-component and the y-component of a vector $\left[\begin{smallmatrix} z \\ x \\ u+y \end{smallmatrix}\right] \in V$ is determined uniquely by x and u, and the second conditions says that the map from $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$ should have dense domain. If these two conditions hold, and if we denote the linear map from $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$ by S, then S is the generator of a regular i/s/o node $\Sigma_{\mathrm{i/s/o}}$, and V has the graph representation

$$V := \left\{ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \,\middle|\, \begin{bmatrix} x \\ P_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} \in \operatorname{dom}(S) \text{ and } \begin{bmatrix} z \\ P_{\mathcal{Y}}^{\mathcal{U}} w \end{bmatrix} = S \begin{bmatrix} x \\ P_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} \right\}. \tag{1.9}$$

Here $P_{\mathcal{U}}^{\mathcal{Y}}$ is the projection onto $\mathcal{U}$ along $\mathcal{Y}$, and $P_{\mathcal{Y}}^{\mathcal{U}}$ is the complementary projection.

Definition 1.3. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node. By a *regular i/s/o representation* of Σ we mean a regular i/s/o node $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, where $\mathcal{U} \dot{+} \mathcal{Y}$ is a direct sum decomposition of $\mathcal{W}$ and V and S are connected to each other by (1.9).

Not every s/s node has a regular i/s/o representation. It is not difficult to see that if $\Sigma = (V; \mathcal{X}, \mathcal{W})$ has a regular i/s/o representation, then Σ must be "regular" in the following sense:

Definition 1.4. A s/s node $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is *regular* if it satisfies the following two conditions:

(i) $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V \Rightarrow z = 0$;

(ii) The projection of V onto its middle component is dense in $\mathcal{X}$.

The two conditions (i) and (ii) above have the following interpretations: (i) means that $\dot{x}(t)$ in (1.7) is determined uniquely by $x(t)$ and $w(t)$, and (ii) permits the set of all initial states $x(0)$ of a classical future trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ to be dense in the state space $\mathcal{X}$

Theorem 1.5. *Every regular i/s/o node has at least one* (*and usually infinitely many*) *regular i/s/o representations.*

The proof of this theorem is found in [AS16, Chapter 2].

If a s/s node $\Sigma = (V; \mathcal{X}, \mathcal{W})$ has a *bounded* i/s/o representation, then V must satisfy the stronger conditions (i)–(iii) listed below:

Definition 1.6. A s/s node $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is *bounded* if it satisfies the following conditions:

(i) $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \Rightarrow z = 0$;

(ii) For every $x_0 \in \mathcal{X}$ there exists some $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ such that $\begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V$.

(iii) The projection of V onto its second and third components is closed in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$.

The interpretation of condition (i) in Definition 1.6 is the same as in Definition 1.4. This condition is equivalent to the condition that V has a graph representation

$$V = \left\{ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \middle| \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \operatorname{dom}(F) \text{ and } z = F \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \right\} \tag{1.10}$$

for some closed operator $F \colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \to \mathcal{X}$. Condition (iii) says that $\operatorname{dom}(F)$ is closed in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$, and hence by the closed graph theorem, F is continuous. In other words, $\dot{x}(t)$ in (1.7) depends continuously on $\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$. Finally, condition (ii) permits every $x_0 \in \mathcal{X}$ to be the initial state $x(0)$ of some classical future trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ.

Theorem 1.7. *Every bounded s/s node has at least one* (*and usually infinitely many*) *bounded i/s/o representations.*

Also the proof of this theorem is found in [AS16, Chapter 2].

As we noticed above, a s/s node Σ cannot have a regular i/s/o representation unless Σ is regular. From time to time it is useful to also study s/s nodes which are not regular. In that case it is still possible to obtain i/s/o representations, but these will no longer be regular. Instead they will be i/s/o nodes of the following type:

Definition 1.8.

(i) By a (continuous time stationary) *i/s/o* (*input/state/output*) node we mean a colligation $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, where $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ are Hilbert spaces, and $S \colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ is a closed multi-valued linear operator.

(ii) The (multi-valued) *main operator* A of $\Sigma_{\mathrm{i/s/o}}$ (or of S) is defined by

$$\begin{aligned} \operatorname{dom}(A) &:= \{x \in \mathcal{X} \,|\, \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \operatorname{dom}(S)\}, \\ z \in Ax \Leftrightarrow z &\in \begin{bmatrix} 1_{\mathcal{X}} & 0 \end{bmatrix} S \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right], \qquad x \in \operatorname{dom}(A). \end{aligned} \tag{1.11}$$

(iii) By a *classical future trajectory* of $\Sigma_{\mathrm{i/s/o}}$ we mean a triple of functions $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ which satisfies

$$\Sigma_{\mathrm{i/s/o}} \colon \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \operatorname{dom}(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} \in S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+, \tag{1.12}$$

with x continuously differentiable with values in $\mathcal{X}$ and $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ continuous with values in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$.

(iv) By a *generalized future trajectory* of $\Sigma_{\mathrm{i/s/o}}$ we mean a triple of functions $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ which is the limit of a sequence $\left[\begin{smallmatrix} x_n \\ u_n \\ y_n \end{smallmatrix}\right]$ of classical future trajectories of $\Sigma_{\mathrm{i/s/o}}$ in the sense that $x_n \to x$ in $C(\mathbb{R}^+;\mathcal{X})$ and $\left[\begin{smallmatrix} u_n \\ y_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ in $L^2_{\mathrm{loc}}(\mathbb{R}^+;\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right])$.

(v) By a (time domain) *i/s/o system* system we mean an i/s/o node $\Sigma_{\mathrm{i/s/o}} = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ together with the sets of classical and generalized future trajectories of $\Sigma_{\mathrm{i/s/o}}$.

See, e.g., [AS16] for a short introduction to the notion of a multi-valued linear operator.

Definition 1.9. Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a s/s node. By an *i/s/o representation* of Σ we mean an i/s/o node $\Sigma_{\mathrm{i/s/o}} = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$, where $\mathcal{U} \dotplus \mathcal{Y}$ is a direct sum decomposition of $\mathcal{W}$ and V and S are connected to each other by

$$V := \left\{ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \,\middle|\, \begin{bmatrix} x \\ P^{\mathcal{Y}}_{\mathcal{U}} w \end{bmatrix} \in \operatorname{dom}(S) \text{ and } \begin{bmatrix} z \\ P^{\mathcal{U}}_{\mathcal{Y}} w \end{bmatrix} \in S \begin{bmatrix} x \\ P^{\mathcal{Y}}_{\mathcal{U}} w \end{bmatrix} \right\}. \tag{1.13}$$

See [AS16, Chapter 2] for a more detailed description of this class of non-regular i/s/o nodes and i/s/o representations.

1.3. Various notions for state/signal systems

The definition of a (regular or non-regular) i/s/o representation of a (regular or non-regular) s/s node immediately implies the following results:

Lemma 1.10. *Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a s/s node, and let $\Sigma_{\mathrm{i/s/o}} = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ be an i/s/o representation of Σ. Then $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical or generalized future trajectory of Σ if and only if $\left[\begin{smallmatrix} x \\ P^{\mathcal{Y}}_{\mathcal{U}} w \\ P^{\mathcal{U}}_{\mathcal{Y}} w \end{smallmatrix}\right]$ is a classical or generalized future trajectory of $\Sigma_{\mathrm{i/s/o}}$.*

Thanks to Lemma 1.10, it is possible to extend all those notions for i/s/o systems that can be expressed in terms of properties of classical or generalized future trajectories of i/s/o systems to the s/s case. In this way it is possible to introduce and study, e.g., the following notions for s/s systems:

- driving variable and output nulling representations of s/s systems,
- existence and uniqueness of classical and generalized trajectories of s/s systems,
- well-posedness of s/s systems,
- s/s systems of boundary control type,
- controllability and observability of s/s systems,
- stability, stabilizability, and detectability of s/s systems,
- past, future, and two-sided time domain behaviors of s/s systems,
- frequency domain analysis of s/s systems,
- external equivalence of s/s systems,
- intertwinements of s/s systems.
- similarities and pseudo-similarities of s/s systems,

- restrictions, projections, compressions, and dilations of s/s systems,
- minimal s/s systems,
- the dual and the adjoint of a s/s system,
- passive past, future, and two-sided time domain behaviors,
- passive frequency domain behaviors,
- optimal and $*$-optimal s/s systems (available storage and required supply),
- passive balanced s/s systems,
- energy and co-energy preserving s/s systems,
- controllable energy-preserving and observable co-energy preserving realizations of passive signal bundles,
- quadratic optimal control and KYP-theory for s/s systems,
- s/s systems with extra symmetries (reality, reciprocity, real-reciprocity),
- relationships between the symmetries of a s/s system and the symmetries of its i/s/o representations,
- s/s versions of the de Branges complementary spaces of type $\mathcal{H}$ and $\mathcal{D}$.

Some of these notions are discussed in [AS16], some of them are discussed in the other articles listed in the reference list, and some of them still remain to be properly developed.

In this article we shall still take a closer look at

- i/s/o and s/s systems in the frequency domain,
- The characteristic node and signal bundles of a s/s system,
- $\mathcal{J}$-passive and $\mathcal{J}$-conservative i/s/o systems,
- passive and conservative s/s systems,
- passive signal bundles,
- conservative realizations of passive signal bundles.

2. State/signal systems in the frequency domain

2.1. Input/state/output systems in the frequency domain

Let $\Sigma_{\mathrm{i/s/o}} = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ be an i/s/o node, and let $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ be a classical future trajectory of $\Sigma_{\mathrm{i/s/o}}$. If x, $\dot{x}$, u, and y in (1.4) are Laplace transformable, then it follows from (1.4) (since we assume S to be closed) that the Laplace transforms $\hat{x}$, $\hat{u}$, and $\hat{y}$ of x, u, and y satisfy the *i/s/o resolvent equation* (with $x^0 := x(0)$)

$$\widehat{\Sigma}_{\mathrm{i/s/o}}\colon \left\{ \begin{aligned} \left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{smallmatrix}\right] &\in \mathrm{dom}\,(S)\,, \\ \begin{bmatrix} \lambda\hat{x}(\lambda) - x^0 \\ \hat{y}(\lambda) \end{bmatrix} &\in S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \end{aligned} \right. \tag{2.1}$$

for all those $\lambda \in \mathbb{C}$ for which the Laplace transforms converge (to see this it suffices to multiply by (1.4) by $\mathrm{e}^{-\lambda t}$ and integrate by parts in the $\dot{x}$-component). If $\Sigma_{\mathrm{i/s/o}}$ is regular, or more generally, if S is single-valued, then we may replace the second inclusion "$\in$" in (2.1) by the equality "$=$"

Definition 2.1. Let $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o node.

(i) $\lambda \in \mathbb{C}$ belongs to the *resolvent set* $\rho(\Sigma_{\mathrm{i/s/o}})$ *of* $\Sigma_{\mathrm{i/s/o}}$ if for every $x^0 \in \mathcal{X}$ and for every $\hat{u}(\lambda) \in \mathcal{U}$ there is a unique pair of vectors $\left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ satisfying the i/s/o resolvent equation (2.1). This set is alternatively called the *i/s/o resolvent set of* S and denoted by $\rho_{\mathrm{i/s/o}}(S)$.

(ii) For each $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$ we define the *i/s/o resolvent matrix* $\widehat{\mathfrak{S}}(\lambda)$ *of* $\Sigma_{\mathrm{i/s/o}}$ *at* λ to be the linear operator $\left[\begin{smallmatrix} x^0 \\ \hat{u}(\lambda) \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{smallmatrix}\right]$.

Since S is assumed to be closed, also $\widehat{\mathfrak{S}}(\lambda)$ is closed (see [AS16, Chapter 5] for details). Therefore by the closed graph theorem, for each $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$ the i/s/o resolvent matrix $\widehat{\mathfrak{S}}(\lambda)$ is a bounded linear operator. In particular, this implies that $\widehat{\mathfrak{S}}(\lambda)$ has a block matrix representation

$$\widehat{\mathfrak{S}}(\lambda) = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix}, \quad \lambda \in \rho(\Sigma_{\mathrm{i/s/o}}), \tag{2.2}$$

where each of the components $\widehat{\mathfrak{A}}(\lambda)$, $\widehat{\mathfrak{B}}(\lambda)$, $\widehat{\mathfrak{C}}(\lambda)$, and $\widehat{\mathfrak{D}}(\lambda)$ is a bounded linear operator with the appropriate domain and range space. Thus, if $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$, then (2.1) holds if and only if

$$\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix}. \tag{2.3}$$

Conversely, if there exist four bounded linear operators $\widehat{\mathfrak{A}}(\lambda)$, $\widehat{\mathfrak{B}}(\lambda)$, $\widehat{\mathfrak{C}}(\lambda)$, and $\widehat{\mathfrak{D}}(\lambda)$ with the appropriate domain and ranges spaces such that (2.1) is equivalent to (2.3), then $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$, and the operator $\widehat{\mathfrak{S}}(\lambda)$ defined by (2.2) is the i/s/o resolvent matrix of $\Sigma_{\mathrm{i/s/o}}$ at the point λ.

Definition 2.2. The components $\widehat{\mathfrak{A}}$, $\widehat{\mathfrak{B}}$, $\widehat{\mathfrak{C}}$, and $\widehat{\mathfrak{D}}$ of the i/s/o resolvent matrix $\widehat{\mathfrak{S}}$ are called as follows:

(i) $\widehat{\mathfrak{A}}$ is the *s/s* (*state/state*) *resolvent function* of $\Sigma_{\mathrm{i/s/o}}$,
(ii) $\widehat{\mathfrak{B}}$ is the *i/s* (*input/state*) *resolvent function* of $\Sigma_{\mathrm{i/s/o}}$,
(iii) $\widehat{\mathfrak{C}}$ is the *s/o* (*state/output*) *resolvent function* of $\Sigma_{\mathrm{i/s/o}}$,
(iv) $\widehat{\mathfrak{D}}$ is the *i/o* (*input/output*) *resolvent function* of $\Sigma_{\mathrm{i/s/o}}$.

The state/state resolvent function $\widehat{\mathfrak{A}}$ is the usual resolvent of the main operator A of $\Sigma_{\mathrm{i/s/o}}$. Here the resolvent set of A and the resolvent of A is defined in the same way as in Definition 2.1 with $\mathcal{U} = \mathcal{Y} = \{0\}$, i.e., λ belongs to the resolvent set $\rho(A)$ of A if it is true for every $x^0 \in \mathcal{X}$ that there exists a unique $z_\lambda \in \mathcal{X}$ such that $\lambda z_\lambda - x^0 \in A z_\lambda$, in which case the bounded linear operator which maps x^0 into z_λ is called the resolvent of A (evaluated at λ). This operator is usually denoted by $(\lambda - A)^{-1}$ since it is the (single-valued) inverse of the (possibly multi-valued) operator $\lambda - A$.

The i/o resolvent function $\widehat{\mathfrak{D}}$ is known in the literature under different names, such as "the transfer function", or "the characteristic function", or "the Weyl function". In operator theory the i/s resolvent function $\widehat{\mathfrak{B}}$ is sometimes called the Γ-field.

The fact that (2.1) and (2.3) are equivalent to each other leads to the following graph representations of S and $S - \left[\begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix}\right]$ which will be needed later:

Lemma 2.3. *Let $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o node with $\rho(\Sigma_{\mathrm{i/s/o}}) \neq \emptyset$. Then for each $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$ the graph of $(S - \left[\begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix}\right])$ has the representation*

$$\operatorname{gph}\left(S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}\right) = \operatorname{rng}\left(\left[\begin{array}{cc} -1_{\mathcal{X}} & 0 \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \\ \hline \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ 0 & 1_{\mathcal{U}} \end{array}\right]\right), \tag{2.4}$$

where $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ is the i/s/o resolvent matrix of $\Sigma_{\mathrm{i/s/o}}$, and the graph of S has the representation

$$\operatorname{gph}(S) = \operatorname{rng}\left(\left[\begin{array}{cc} \lambda\widehat{\mathfrak{A}}(\lambda) - 1_{\mathcal{X}} & \lambda\widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \\ \hline \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ 0 & 1_{\mathcal{U}} \end{array}\right]\right). \tag{2.5}$$

Definition 2.1 above is both natural and simple, and it may be surprising that in the case where S is single-valued and densely defined the above definition is equivalent to the condition that S is a so-called "operator node" in the sense of [Sta05].

Definition 2.4 ([Sta05, Definition 4.7.2]). By an *operator node* (in the sense of [Sta05]) on a triple of Hilbert spaces $(\mathcal{X}, \mathcal{U}, \mathcal{Y})$ we mean a linear operator $S\colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ with the following properties:

- (i) S is closed.
- (ii) The main operator A of S has dense domain and nonempty resolvent set.
- (iii) $\begin{bmatrix} 1_{\mathcal{X}} & 0 \end{bmatrix} S$ can be extended to a bounded linear operator $\begin{bmatrix} A_{-1} & B \end{bmatrix} : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \mathcal{X}_{-1}$, where $\mathcal{X}_{-1}$ is the so-called *extrapolation space* induced by A (i.e., the completion of $\mathcal{X}$ with respect to the norm $\|x\|_{\mathcal{X}_{-1}} = \|(\alpha - A)^{-1}x\|_{\mathcal{X}}$ where α is some fixed point in $\rho(A)$).
- (iv) $\operatorname{dom}(S) = \left\{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right] \,\middle|\, A_{-1}x + Bu \in \mathcal{X}\right\}$.

Theorem 2.5. *An operator $S\colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ is an operator node in the sense of Definition* 2.4 *if and only if* $\operatorname{dom}(S)$ *is dense in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ and $\rho_{\mathrm{i/s/o}}(S) \neq \emptyset$. Moreover, if $\rho_{\mathrm{i/s/o}}(S) \neq \emptyset$, then $\rho_{\mathrm{i/s/o}}(S) = \rho(A)$ where A is the main operator of S.*

The proof of this theorem is given in [AS16, Chapter 5].

As the following lemma shows, it is possible to use the s/s resolvent function $\widehat{\mathfrak{A}}$ to check the regularity of an i/s/o system $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ with nonempty resolvent set.

Lemma 2.6. *Let $\Sigma_{\rm i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a i/s/o node with $\rho(\Sigma_{\rm i/s/o}) \neq \emptyset$, with main operator A, and with and s/s resolvent function $\widehat{\mathfrak{A}}$. Then*

(i) *The following conditions are equivalent:*
   (a) *S is single-valued;*
   (b) *A is single-valued;*
   (c) *$\widehat{\mathfrak{A}}(\lambda)$ is injective for some $\lambda \in \rho(\Sigma_{\rm i/s/o})$ (or equivalently, for all $\lambda \in \rho(\Sigma_{\rm i/s/o})$).*

(ii) *Also the following conditions are equivalent:*
   (a) $\operatorname{dom}(S)$ *is dense in* $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$*;*
   (b) $\operatorname{dom}(A)$ *is dense in $\mathcal{X}$;*
   (c) *$\widehat{\mathfrak{A}}(\lambda)$ has dense range for some $\lambda \in \rho(\Sigma_{\rm i/s/o})$ (or equivalently, for all $\lambda \in \rho(\Sigma_{\rm i/s/o})$).*

In particular, $\Sigma_{\rm i/s/o}$ is a regular i/s/o system if and only if A is single-valued and has dense domain, or equivalently, if and only if $\widehat{\mathfrak{A}}(\lambda)$ is injective and has dense range for some $\lambda \in \rho(\Sigma_{\rm i/s/o})$ (or equivalently, for all $\lambda \in \rho(\Sigma_{\rm i/s/o})$).

The proof of this lemma is given in [AS16, Chapter 5].

The i/s/o resolvent matrix has the following properties:

Lemma 2.7. *Let $\Sigma_{\rm i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o node with $\rho(\Sigma_{\rm i/s/o}) \neq \emptyset$. Then the resolvent set $\rho(\Sigma_{\rm i/s/o})$ of $\Sigma_{\rm i/s/o}$ is open, the i/s/o resolvent matrix $\widehat{\mathfrak{S}}$ of $\Sigma_{\rm i/s/o}$ is analytic on $\rho(\Sigma_{\rm i/s/o})$, and it satisfies the* i/s/o resolvent identity

$$\widehat{\mathfrak{S}}(\lambda) - \widehat{\mathfrak{S}}(\mu) = \widehat{\mathfrak{S}}(\mu) \begin{bmatrix} (\mu - \lambda) & 0 \\ 0 & 0 \end{bmatrix} \widehat{\mathfrak{S}}(\lambda) = \widehat{\mathfrak{S}}(\lambda) \begin{bmatrix} (\mu - \lambda) & 0 \\ 0 & 0 \end{bmatrix} \widehat{\mathfrak{S}}(\mu) \tag{2.6}$$

for all μ, $\lambda \in \rho(\Sigma_{\rm i/s/o})$. In terms of the components of the i/s/o resolvent matrix $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ the above identity can be rewritten into the equivalent form

$$\begin{aligned}
\widehat{\mathfrak{A}}(\lambda) - \widehat{\mathfrak{A}}(\mu) &= (\mu - \lambda)\widehat{\mathfrak{A}}(\mu)\widehat{\mathfrak{A}}(\lambda) = (\mu - \lambda)\widehat{\mathfrak{A}}(\lambda)\widehat{\mathfrak{A}}(\mu), \\
\widehat{\mathfrak{B}}(\lambda) - \widehat{\mathfrak{B}}(\mu) &= (\mu - \lambda)\widehat{\mathfrak{A}}(\mu)\widehat{\mathfrak{B}}(\lambda) = (\mu - \lambda)\widehat{\mathfrak{A}}(\lambda)\widehat{\mathfrak{B}}(\mu), \\
\widehat{\mathfrak{C}}(\lambda) - \widehat{\mathfrak{C}}(\mu) &= (\mu - \lambda)\widehat{\mathfrak{C}}(\mu)\widehat{\mathfrak{A}}(\lambda) = (\mu - \lambda)\widehat{\mathfrak{C}}(\lambda)\widehat{\mathfrak{A}}(\mu), \\
\widehat{\mathfrak{D}}(\lambda) - \widehat{\mathfrak{D}}(\mu) &= (\mu - \lambda)\widehat{\mathfrak{C}}(\mu)\widehat{\mathfrak{B}}(\lambda) = (\mu - \lambda)\widehat{\mathfrak{C}}(\lambda)\widehat{\mathfrak{B}}(\mu).
\end{aligned} \tag{2.7}$$

The proof of this lemma is given in [AS16, Chapter 5].

Motivated by Lemma 2.7 we make the following definition.

Definition 2.8. Let Ω be an open subset of the complex plane $\mathbb{C}$. An analytic $\mathcal{B}(\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right])$-valued function $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ defined in Ω is called an *i/s/o pseudo-resolvent in* $(\mathcal{X}, \mathcal{U}, \mathcal{Y}; \Omega)$ if it satisfies the identity (2.6) for all μ, $\lambda \in \Omega$.

Thus, the i/s/o resolvent matrix $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ of an i/s/o node $\Sigma_{\rm i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ with $\rho(\Sigma_{\rm i/s/o}) \neq \emptyset$ is an i/s/o pseudo-resolvent in $\rho(\Sigma_{\rm i/s/o})$.

In [Opm05] Mark Opmeer makes systematic use of the notion of an i/s/o pseudo-resolvent, but instead of calling $\widehat{\mathfrak{S}}$ an i/s/o pseudo-resolvent he calls $\widehat{\mathfrak{S}}$ a "resolvent linear system", and calls $\widehat{\mathfrak{A}}$ the "pseudo-resolvent", $\widehat{\mathfrak{B}}$ the "incoming wave function", $\widehat{\mathfrak{C}}$ the "outgoing wave function", and $\mathfrak{D}$ the "characteristic function" of the resolvent linear system $\widehat{\mathfrak{S}}$. In the same article he also investigates what can be said about time domain trajectories (in the distribution sense) of resolvent linear systems satisfying some additional conditions. One of these additional set of conditions is that Ω should contain some right half-plane and that $\widehat{\mathfrak{S}}$ should satisfy a polynomial growth bound in this right half-plane.

The converse of Lemma 2.7 is also true in the following form.

Theorem 2.9. *Let Ω be an open subset of the complex plane $\mathbb{C}$. Then every i/s/o pseudo-resolvent $\widehat{\mathfrak{S}}$ in $(\mathcal{X},\mathcal{U},\mathcal{Y};\Omega)$ is the restriction to Ω of the i/s/o resolvent of some i/s/o node $\Sigma_{\mathrm{i/s/o}} = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ satisfying $\rho(\Sigma_{\mathrm{i/s/o}}) \supset \Omega$. The i/s/o node $\Sigma_{\mathrm{i/s/o}}$ is determined uniquely by $\widehat{\mathfrak{S}}(\lambda)$ where λ is some arbitrary point in Ω, and $\widehat{\mathfrak{S}}$ has a unique extension to $\rho(\Sigma_{\mathrm{i/s/o}})$. This extension is maximal in the sense that $\widehat{\mathfrak{S}}$ cannot be extended to an i/s/o pseudo-resolvent on any larger open subset of $\mathbb{C}$.*

See [AS16, Chapter 5] for the proof.

Theorem 2.9 is well known in the case where the system has no input and no output (so that S is equal to its main operator A), and where $\widehat{\mathfrak{A}}(\lambda)$ is injective and has dense range for some $\lambda \in \Omega$; see, e.g., [Paz83, Theorem 9.3, p. 36]. A multi-valued version of this theorem, still with no input and output, is found in [DdS87, Remark, pp. 148–149].

2.2. State/signal systems in the frequency domain

Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a s/s node, and let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be a classical future trajectory of Σ. If x, $\dot{x}$, and w in (1.7) are Laplace transformable, then it follows from (1.7) (since we assume V to be closed) that the Laplace transforms $\hat{x}$, and $\hat{w}$ x and w satisfy (with $x^0 := x(0)$)

$$\widehat{\Sigma}\colon \begin{bmatrix} \lambda\hat{x}(\lambda) - x^0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in V \tag{2.8}$$

for all those $\lambda \in \mathbb{C}$ for which the Laplace transforms converge (to see this it suffices to multiply by (1.4) by $\mathrm{e}^{-\lambda t}$ and integrate by parts in the $\dot{x}$-component). This formula can be rewritten in the form

$$\begin{bmatrix} x^0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) := \begin{bmatrix} -1_{\mathcal{X}} & \lambda & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V. \tag{2.9}$$

Definition 2.10. The family of subspaces $\widehat{\mathfrak{E}} : \{\widehat{\mathfrak{E}}(\lambda) \mid \lambda \in \mathbb{C}\}$ of $\mathfrak{K} = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ defined in (2.9) is called the *characteristic node bundle* of the s/s node $\Sigma = (V;\mathcal{X},\mathcal{W})$.

The characteristic node bundle is a special case of a vector bundle:

Definition 2.11. Let $\mathcal{Z}$ be a Hilbert vector space.

(i) By a *vector bundle in* $\mathcal{Z}$ we mean a family of subspaces $\mathfrak{G} = \{\mathfrak{G}(\lambda)\}_{\lambda \in \mathrm{dom}(\mathfrak{G})}$ of $\mathcal{Z}$ parameterized by a complex parameter $\lambda \in \mathrm{dom}\,(\mathfrak{G}) \subset \mathbb{C}$.

(ii) For each $\lambda \in \mathrm{dom}\,(\mathfrak{G})$, the subspace $\mathfrak{G}(\lambda)$ of $\mathcal{Z}$ is called the *fiber* of $\mathfrak{G}$ at λ.

(iii) The vector bundle $\mathfrak{G}$ is *analytic* at a point $\lambda_0 \in \mathrm{dom}\,(\mathfrak{G})$ if there exists a neighborhood $\mathcal{O}(\lambda_0)$ of λ_0 and some direct sum decomposition $\mathcal{Z} = \mathcal{U} \dotplus \mathcal{Y}$ of $\mathcal{Z}$ such that the restriction of $\mathfrak{G}$ to $\mathcal{O}(\lambda_0)$ is the graph of an analytic $\mathcal{B}(\mathcal{U};\mathcal{Y})$-valued function in $\mathcal{O}(\lambda_0)$.

(iv) The vector bundle $\mathfrak{G}$ is *analytic* if $\mathrm{dom}\,(\mathfrak{G})$ is open and $\mathfrak{G}$ is analytic at every point in $\mathrm{dom}\,(\mathfrak{G})$.

(v) The vector bundle $\mathfrak{G}$ is *entire* if $\mathfrak{G}$ is analytic in the full complex plane $\mathbb{C}$.

Lemma 2.12. *The characteristic node bundle $\widehat{\mathfrak{E}}$ of a s/s node $\Sigma = (V;\mathcal{X},\mathcal{W})$ is an entire vector bundle in the node space $\mathfrak{K} = \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$.*

This is easy to see (and proved in [AS16, Chapter 1]).

Lemma 2.13. *Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a s/s node with the i/s/o representation $\Sigma_{\mathrm{i/s/o}} = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$, suppose that $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$. Denote the characteristic node bundle of Σ by $\widehat{\mathfrak{E}}$ and the i/s/o resolvent matrix of $\Sigma_{\mathrm{i/s/o}}$ by $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix}\widehat{\mathfrak{A}} & \widehat{\mathfrak{B}}\\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}}\end{smallmatrix}\right]$. Then V and $\widehat{\mathfrak{E}}(\lambda)$ have the representations*

$$V = \mathrm{rng}\left(\begin{bmatrix} 1_{\mathcal{X}} - \lambda\widehat{\mathfrak{A}}(\lambda) & -\lambda\widehat{\mathfrak{B}}(\lambda)\\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda)\\ \mathcal{I}_{\mathcal{Y}}\widehat{\mathfrak{C}}(\lambda) & \mathcal{I}_{\mathcal{U}} + \mathcal{I}_{\mathcal{Y}}\widehat{\mathfrak{D}}(\lambda)\end{bmatrix}\right), \tag{2.10}$$

$$\widehat{\mathfrak{E}}(\lambda) = \mathrm{rng}\left(\begin{bmatrix} 1_{\mathcal{X}} & 0\\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda)\\ \mathcal{I}_{\mathcal{Y}}\widehat{\mathfrak{C}}(\lambda) & \mathcal{I}_{\mathcal{U}} + \mathcal{I}_{\mathcal{Y}}\widehat{\mathfrak{D}}(\lambda)\end{bmatrix}\right), \tag{2.11}$$

where $\mathcal{I}_{\mathcal{U}}$ and $\mathcal{I}_{\mathcal{Y}}$ are the injection operators $\mathcal{I}_{\mathcal{U}}\colon \mathcal{U} \hookrightarrow \mathcal{W}$ and $\mathcal{I}_{\mathcal{Y}}\colon \mathcal{Y} \hookrightarrow \mathcal{W}$.

This follows from (1.13), Lemma 2.3, and (2.9) (see [AS16, Chapter 5] for details).

Note that (2.11) can be interpreted as a graph representation of $\widehat{\mathfrak{E}}(\lambda)$ over the first copy of $\mathcal{X}$ and the input space $\mathcal{U}$. It follows from Lemma 2.13 that V (and $\widehat{\mathfrak{E}}(\lambda)$) are determined uniquely by the decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ and the i/s/o resolvent matrix $\widehat{\mathfrak{S}}$ of $\Sigma_{\mathrm{i/s/o}}$ evaluated at some arbitrary point $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$.

In i/s/o systems theory one is often interested in the "pure i/o behavior", which one gets by "ignoring the state". More precisely, one takes the initial state $x^0 = 0$, and looks at the relationship between the input u and the output y, ignoring the state x. If we in the frequency domain setting take $x^0 = 0$ and ignore $\hat{x}$, then the full frequency domain identity (2.3) simplifies into $\hat{y}(\lambda) = \widehat{\mathfrak{D}}(\lambda)\hat{u}(\lambda)$, where $\widehat{\mathfrak{D}}(\lambda)$ is the i/o resolvent function of $\Sigma_{\mathrm{i/s/o}}$.

The same procedure can be carried out in the case of a s/s system: We take $x^0 = 0$ and ignore the values of $\hat{x}(\lambda)$ in (2.8). Then it follows from (2.9) that

$\hat{w}(\lambda) \in \widehat{\mathfrak{F}}(\lambda)$, where

$$\widehat{\mathfrak{F}}(\lambda) = \left\{ w \in \mathcal{W} \,\middle|\, \begin{bmatrix} 0 \\ z \\ w \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) \text{ for some } z \in \mathcal{X} \right\}. \tag{2.12}$$

Definition 2.14. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node. The family of subspaces $\widehat{\mathfrak{F}}$: $\{\widehat{\mathfrak{F}}(\lambda) \mid \lambda \in \mathbb{C}\}$ of $\mathcal{W}$ defined by (2.12) is called the *characteristic signal bundle* of Σ.

Whereas the characteristic node bundle $\widehat{\mathfrak{E}}$ of Σ is an entire vector bundle, the same is not true for the signal bundle $\widehat{\mathfrak{F}}$ of Σ. Even the dimension of the fibers $\widehat{\mathfrak{F}}(\lambda)$ may change from one point to another. However, the following result is true:

Lemma 2.15. *If $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is an i/s/o representation of the s/s node $\Sigma = (V; \mathcal{X}, \mathcal{W})$ with $\rho(\Sigma_{\mathrm{i/s/o}}) \neq \emptyset$, then for each $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$ the fibers of the characteristic signal bundle $\widehat{\mathfrak{F}}$ of Σ have the graph representation*

$$\widehat{\mathfrak{F}}(\lambda) = \left\{ w \in \mathcal{W} \,\middle|\, P_{\mathcal{Y}}^{\mathcal{U}} w = \widehat{\mathfrak{D}}(\lambda) P_{\mathcal{U}}^{\mathcal{Y}} \right\}, \qquad \lambda \in \rho(\Sigma_{\mathrm{i/s/o}}). \tag{2.13}$$

This follows from Lemma 2.13.

Lemma 2.16. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node with the i/s/o representation $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, suppose that $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$. Denote the characteristic node bundle of Σ by $\widehat{\mathfrak{E}}$. Then, for each $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$ the fiber $\widehat{\mathfrak{E}}(\lambda)$ of $\widehat{\mathfrak{E}}$ is a closed subspace of $\mathfrak{K} = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$, and it has the following properties:*

(i) $\left[\begin{smallmatrix} 0 \\ x \\ 0 \end{smallmatrix}\right] \in \widehat{\mathfrak{E}}(\lambda) \Rightarrow x = 0$;

(ii) *For every $z \in \mathcal{X}$ there exists some $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ such that $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in \widehat{\mathfrak{E}}(\lambda)$.*

(iii) *The projection of $\widehat{\mathfrak{E}}(\lambda)$ onto its first and third components is closed in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$.*

This follows from Lemma 2.13.

Another equivalent way of formulating Lemma 2.16 is to say that for each $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$ the fiber $\widehat{\mathfrak{E}}(\lambda)$ becomes a bounded s/s node after we interchange the first and the second component of $\widehat{\mathfrak{E}}(\lambda)$.

Definition 2.17. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node with node bundle $\widehat{\mathfrak{E}}$. Then the *resolvent set* $\rho(\Sigma)$ of Σ consists of all those points $\lambda \in \mathbb{C}$ for which conditions (i)–(iii) in Lemma 2.16 hold.

Theorem 2.18. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node. Then $\rho(\Sigma)$ is the union of the resolvent sets of all i/s/o representations of Σ.*

See [AS16, Chapter 5] for the proof.

Lemma 2.19. *The characteristic signal bundle $\widehat{\mathfrak{F}}$ of a s/s node Σ is analytic in $\rho(\Sigma)$.*

This follows from Definition 2.11, Lemma 2.15, and Theorem 2.18.

3. Passive and conservative i/s/o and s/s systems

In this section we have, for simplicity, restricted the discussion to the regular case, i.e., the case where both the s/s system and its i/s/o representations are regular. As shown in [AS16], the extension to the non-regular case is straightforward.

3.1. $\mathcal{J}$-passive and $\mathcal{J}$-conservative i/s/o systems

Definition 3.1. Let $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a regular i/s/o node.

(i) $\Sigma_{\mathrm{i/s/o}}$ is (forward) *solvable* if it is true that for every $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathrm{dom}\,(S)$ there exists at least one classical future trajectory $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ of $\Sigma_{\mathrm{i/s/o}}$ with $\left[\begin{smallmatrix} x(0) \\ u(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]$.

(ii) The *adjoint* of $\Sigma_{\mathrm{i/s/o}}$ is the i/s/o node $\Sigma^*_{\mathrm{i/s/o}} = (S^*; \mathcal{X}, \mathcal{Y}, \mathcal{U})$, where S^* is the adjoint of S.

Definition 3.2. Let $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a regular i/s/o node with adjoint $\Sigma^*_{\mathrm{i/s/o}} = (S^*; \mathcal{X}, \mathcal{Y}, \mathcal{U})$, and suppose that both $\Sigma_{\mathrm{i/s/o}}$ and $\Sigma^*_{\mathrm{i/s/o}}$ are solvable.

(i) $\Sigma_{\mathrm{i/s/o}}$ is *scattering conservative* if all its classical future trajectories $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ satisfy the balance equation

$$\|x(t)\|^2_{\mathcal{X}} + \int_0^t \|y(s)\|^2_{\mathcal{Y}}\,\mathrm{d}s = \|x(0)\|^2_{\mathcal{X}} + \int_0^t \|u(s)\|^2_{\mathcal{U}}\,\mathrm{d}s, \quad t \in \mathbb{R}^+, \tag{3.1}$$

and the adjoint system $\Sigma^*_{\mathrm{i/s/o}}$ has the same property. If the above conditions hold with the equality sign in (3.1) by "$\le$" then $\Sigma_{\mathrm{i/s/o}}$ is *scattering passive*.

(ii) Let $\Psi\colon \mathcal{Y} \to \mathcal{U}$ be a unitary operator. Then $\Sigma_{\mathrm{i/s/o}}$ is Ψ*-impedance conservative* if all its classical future trajectories (u, x, y) satisfy the balance equation

$$\|x(t)\|^2_{\mathcal{X}} = \|x(0)\|^2_{\mathcal{X}} + 2\Re \int_0^t \langle u(s), \Psi y(s)\rangle_{\mathcal{U}}\,\mathrm{d}s, \quad t \in \mathbb{R}^+, \tag{3.2}$$

and the adjoint system $\Sigma^*_{\mathrm{i/s/o}}$ has the same property with Ψ replaced by Ψ^*. If the above conditions hold with the equality sign in (3.1) by "$\le$" then $\Sigma_{\mathrm{i/s/o}}$ is Ψ*-impedance passive.*

(iii) Let $\mathcal{J}_{\mathcal{U}}$ and $\mathcal{J}_{\mathcal{Y}}$ be signature operators in $\mathcal{U}$ respectively $\mathcal{Y}$ (i.e., $\mathcal{J}_{\mathcal{U}} = \mathcal{J}^*_{\mathcal{U}} = \mathcal{J}^{-1}_{\mathcal{U}}$ and $\mathcal{J}_{\mathcal{Y}} = \mathcal{J}^*_{\mathcal{Y}} = \mathcal{J}^{-1}_{\mathcal{Y}}$). Then $\Sigma_{\mathrm{i/s/o}}$ is $(J_{\mathcal{U}}, J_{\mathcal{Y}})$*-transmission conservative* if all its classical future trajectories (u, x, y) satisfy the balance equation

$$\begin{aligned} \|x(t)\|^2_{\mathcal{X}} &+ \int_0^t \langle y(s), J_{\mathcal{Y}} y(s)\rangle_{\mathcal{Y}}\,\mathrm{d}s \\ &= \|x(0)\|^2_{\mathcal{X}} + \int_0^t \langle u(s), J_{\mathcal{U}} u(s)\rangle_{\mathcal{U}}\,\mathrm{d}s, \quad t \in \mathbb{R}^+, \end{aligned} \tag{3.3}$$

and the adjoint system $\Sigma^*_{\mathrm{i/s/o}}$ has the same property with $(J_{\mathcal{U}}, J_{\mathcal{Y}})$ replaced by $(J_{\mathcal{Y}}, J_{\mathcal{U}})$. If the above conditions hold with the equality sign in (3.1) by "$\le$" then $\Sigma_{\mathrm{i/s/o}}$ is $(J_{\mathcal{U}}, J_{\mathcal{Y}})$*-transmission passive.*

The three different balance equations in Lemma 3.3 can all be written in the common form

$$\|x(t)\|^2_{\mathcal{X}} = \|x(0)\|^2_{\mathcal{X}} + \int_0^t \left\langle \begin{bmatrix} u(s) \\ y(s) \end{bmatrix}, \mathcal{J} \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} \right\rangle_{\mathcal{U}\oplus\mathcal{Y}} \mathrm{d}s, \quad t \in \mathbb{R}^+, \tag{3.4}$$

where $\mathcal{J}$ *is a signature operator* in the product space $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$:

(i) $\mathcal{J} = \mathcal{J}_{\mathrm{scat}} = \left[\begin{smallmatrix} 1_{\mathcal{U}} & 0 \\ 0 & -1_{\mathcal{Y}} \end{smallmatrix}\right]$ in the *scattering* case,
(ii) $\mathcal{J} = \mathcal{J}_{\mathrm{imp}} = \left[\begin{smallmatrix} 0 & \Psi \\ \Psi^* & 0 \end{smallmatrix}\right]$ in the Ψ-*impedance* case,
(iii) $\mathcal{J} = \mathcal{J}_{\mathrm{tra}} = \left[\begin{smallmatrix} J_{\mathcal{U}} & 0 \\ 0 & -J_{\mathcal{Y}} \end{smallmatrix}\right]$ in the $(\mathcal{J}_{\mathcal{U}}, \mathcal{J}_{\mathcal{Y}})$-*transmission* case.

It is also possible of combine the three different parts of Definition 3.2 into one general definition. In that definition we need two different signature operators, one in the space $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, and the other in the space $\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$. The connection between these two operators is the following: If $\mathcal{J}$ is a signature operator in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, then we define the operator $\mathcal{J}_*$ by

$$\mathcal{J}_* = \begin{bmatrix} 0 & -1_{\mathcal{Y}} \\ 1_{\mathcal{U}} & 0 \end{bmatrix} \mathcal{J} \begin{bmatrix} 0 & 1_{\mathcal{U}} \\ -1_{\mathcal{Y}} & 0 \end{bmatrix}. \tag{3.5}$$

It is easy to see that $\mathcal{J}_*$ is a signature operator in $\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$ whenever $\mathcal{J}$ is a signature operator in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$ and that $(\mathcal{J}_*)_* = \mathcal{J}$.

Definition 3.3. Let $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a regular i/s/o node with adjoint $\Sigma^*_{\mathrm{i/s/o}} = (S^*; \mathcal{X}, \mathcal{Y}, \mathcal{U})$, and suppose that both $\Sigma_{\mathrm{i/s/o}}$ and $\Sigma^*_{\mathrm{i/s/o}}$ are solvable. Let $\mathcal{J}$ be a signature operator in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, and define $\mathcal{J}_*$ by (3.5). Then $\Sigma_{\mathrm{i/s/o}}$ is $\mathcal{J}$-conservative if all its classical future trajectories $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ satisfy the balance equation (3.4), and the adjoint system $\Sigma^*_{\mathrm{i/s/o}}$ has the same property with $\mathcal{J}$ replaced by $\mathcal{J}_*$. If the above conditions hold with the equality sign in (3.4) by "$\le$" then $\Sigma_{\mathrm{i/s/o}}$ is *$\mathcal{J}$-passive.*

The reader is invited to check that Definition 3.2 can indeed be interpreted as a special case of Definition 3.3 (with the appropriate choice of $\mathcal{J} = \mathcal{J}_{\mathrm{scat}}$, $\mathcal{J} = \mathcal{J}_{\mathrm{imp}}$, or $\mathcal{J} = \mathcal{J}_{\mathrm{tra}}$).

Formula (3.4) treats the *input u and the output y in an equal way*: the operator $\mathcal{J}$ *is simply a signature operator in the signal space* $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, and it defines a Kreĭn space inner product in $\mathcal{W}$. From the point of view of (3.4) it does not matter if u is the input and y the output, or the other way around, or if neither u nor y is the input or output.

It is well known that one can pass from a Ψ-impedance or $(J_{\mathcal{U}}, J_{\mathcal{Y}})$-transmission passive or conservative i/s/o system to a scattering passive or conservative i/s/o system by simply *reinterpreting which part of the combined i/o signal* $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ *is the input, and which part is the input.*

(i) If $\Sigma_{\mathrm{i/s/o}}$ is Ψ-impedance conservative, and if we take the new input and output to be

$$\begin{bmatrix} u_{\mathrm{scat}} \\ y_{\mathrm{scat}} \end{bmatrix} = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1_{\mathcal{U}} & \Psi \\ \Psi^* & -1_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} u_{\mathrm{imp}} \\ y_{\mathrm{imp}} \end{bmatrix},$$

then the resulting i/s/o system is scattering conservative.

(ii) If $\Sigma_{\mathrm{i/s/o}}$ is $(J_{\mathcal{U}}, J_{\mathcal{Y}})$-transmission conservative, and if we take the new input and output to be

$$\begin{bmatrix} u_{\mathrm{scat}} \\ y_{\mathrm{scat}} \end{bmatrix} = \begin{bmatrix} P_{\mathcal{U}^+} & P_{\mathcal{Y}^-} \\ P_{\mathcal{U}^-} & P_{\mathcal{Y}^+} \end{bmatrix} \begin{bmatrix} u_{\mathrm{tra}} \\ y_{\mathrm{tra}} \end{bmatrix},$$

where $(P_{\mathcal{U}^+}, P_{\mathcal{U}^-})$ and $(P_{\mathcal{Y}^+}, P_{\mathcal{Y}^-})$ are complementary projections onto the positive and negative subspaces of $J_{\mathcal{U}}$ and $J_{\mathcal{Y}}$, respectively, then the resulting i/s/o system is again scattering conservative.

The two transforms described above have the following common interpretation: We decompose the Kreĭn space $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$ with the $\mathcal{J}$-inner product into a positive part and an orthogonal negative part (= a fundamental decomposition), and choose the input to be the positive part of $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ and the output to be the negative part of w. Of course, these transformations lead to new dynamic equations with new generators S_{scat}, which can be explicitly derived from the original generators S_{imp} and S_{tra}, but the formulas for S_{scat} tend to be complicated, especially when S_{imp} and S_{tra} are unbounded. For this reason it makes sense to reformulate the $\mathcal{J}$-passivity and $\mathcal{J}$-conservativity conditions described above into a state/signal setting.[1]

3.2. Passive and conservative state/signal systems

Let $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a regular i/s/o system, and let $\Sigma = (V; \mathcal{X}, \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right])$ be the s/s system induced by $\Sigma_{\mathrm{i/s/o}}$, i.e., the generating subspace V of Σ is given by (1.8). If $\Sigma_{\mathrm{i/s/o}}$ is $\mathcal{J}$-passive or $\mathcal{J}$-conservative for some signature operator $\mathcal{J}$ in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, then what does this tell us about the s/s system Σ?

First of all, the solvability condition of $\Sigma_{\mathrm{i/s/o}}$ implies an analogous condition for Σ:

Definition 3.4. A s/s node $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is (forward) *solvable* if it is true that for every $\left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V$ there exists at least one classical future trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ satisfying $\left[\begin{smallmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right]$.

It follows from Definitions 3.1 and 3.4 and Lemma 1.10 that if $\Sigma_{\mathrm{i/s/o}}$ is a regular i/s/o representation of a s/s node Σ, then Σ is solvable if and only if $\Sigma_{\mathrm{i/s/o}}$ is solvable.

By Lemma 1.10, $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ is a classical future trajectory of $\Sigma_{\mathrm{i/s/o}}$ if and only if $\left[\begin{smallmatrix} x \\ x \end{smallmatrix}\right]$ is a classical future trajectory Σ, where $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$. Thus, if $\Sigma_{\mathrm{i/s/o}}$ is $\mathcal{J}$-conservative,

[1] This was the primary motivation for the development of the s/s systems theory in the first place.

then every classical future trajectory $\left[\begin{smallmatrix} x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{smallmatrix}\right]$ of Σ satisfies (3.4). If instead $\Sigma_{\mathrm{i/s/o}}$ is $\mathcal{J}$-passive, then every classical future trajectory $\left[\begin{smallmatrix} x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{smallmatrix}\right]$ of Σ satisfies (3.4) with "=" replace by "$\le$".

Up to this point we have throughout assumed that the signal space $\mathcal{W}$ of a s/s node is a *Hilbert space*, but it follows from (3.4) that in the study of passive and conservative systems it more natural to allow $\mathcal{W}$ to be a *Kreĭn space*, i.e., to allow the inner product in $\mathcal{W}$ to be indefinite. More precisely, we let $\mathcal{W}$ be the product space $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$ equipped with the Kreĭn space inner product

$$\left[\begin{bmatrix} u_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ y_2 \end{bmatrix}\right]_{\mathcal{W}} = \left\langle \begin{bmatrix} u_1 \\ y_1 \end{bmatrix}, \mathcal{J} \begin{bmatrix} u_2 \\ y_2 \end{bmatrix} \right\rangle_{\mathcal{U}\oplus\mathcal{Y}}, \qquad \begin{bmatrix} u_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ y_2 \end{bmatrix} \in \begin{bmatrix} \mathcal{U} \\ \mathcal{Y} \end{bmatrix}. \tag{3.6}$$

With this notation (3.4) becomes

$$\|x(t)\|^2_{\mathcal{X}} = \|x(0)\|^2_{\mathcal{X}} + \int_0^t [w(s), w(s)]_{\mathcal{W}}\, \mathrm{d}s, \quad t \in \mathbb{R}^+. \tag{3.7}$$

Differentiating (3.7) with respect to t we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} = [w(t), w(t)]_{\mathcal{W}}, \qquad t \in \mathbb{R}^+,$$

or equivalently,

$$-\langle \dot{x}(t), x(t)\rangle_{\mathcal{X}} - \langle x(t), \dot{x}(t)\rangle_{\mathcal{X}} + [w(t), w(t)]_{\mathcal{W}} = 0, \qquad t \in \mathbb{R}^+. \tag{3.8}$$

In particular, this equation is true for $t = 0$. If we assume that Σ is solvable (or equivalently, $\Sigma_{\mathrm{i/s/o}}$ is solvable), then it follows from (3.8) that

$$-\langle z_0, x_0\rangle_{\mathcal{X}} - \langle x_0, z_0\rangle_{\mathcal{X}} + [w_0, w_0]_{\mathcal{W}} = 0, \qquad \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V. \tag{3.9}$$

We can make also the *node space* $\mathfrak{K} = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ into a Kreĭn space by introducing the following *node inner product* in $\mathfrak{K}$:

$$\left[\left[\begin{smallmatrix} z_1 \\ x_1 \\ w_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} z_2 \\ x_2 \\ w_2 \end{smallmatrix}\right]\right]_{\mathfrak{K}} = -(z_1, x_2)_{\mathcal{X}} - (x_1, z_2)_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}}, \quad \left[\begin{smallmatrix} z_1 \\ x_1 \\ w_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} z_2 \\ x_2 \\ w_2 \end{smallmatrix}\right] \in \mathfrak{K}. \tag{3.10}$$

Clearly (3.9) says that $V \subset V^{[\perp]}$, where

$$V^{[\perp]} := \left\{ \left[\begin{smallmatrix} z_* \\ x_* \\ w_* \end{smallmatrix}\right] \in \mathfrak{K} \,\middle|\, \left[\left[\begin{smallmatrix} z_* \\ x_* \\ w_* \end{smallmatrix}\right], \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right]\right]_{\mathfrak{K}} = 0 \text{ for all } \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V \right\}. \tag{3.11}$$

In other words, V is a *neutral subspace of* $\mathfrak{K}$. If $\Sigma_{\mathrm{i/s/o}}$ is $\mathcal{J}$-passive instead of $\mathcal{J}$-conservative, then the same argument shows that

$$\left[\left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right], \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right]\right]_{\mathfrak{K}} \ge 0 \text{ for all } \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V,$$

i.e., V is a *nonnegative subspace of* $\mathfrak{K}$.

Above we have used only one half of Definition 3.3, namely the half with refers to the i/s/o representation $\Sigma_{\mathrm{i/s/o}}$ itself, and not the half which refers to the adjoint i/s/o node $\Sigma^*_{\mathrm{i/s/o}}$. By adding the conditions imposed on $\Sigma^*_{\mathrm{i/s/o}}$ to the above argument it is possible to show that

(i) $\Sigma_{\mathrm{i/s/o}}$ is $\mathcal{J}$-conservative if and only if Σ satisfies $V = V^{[\perp]}$ (i.e., V is a *Lagrangian* subspace of $\mathfrak{K}$), and
(ii) $\Sigma_{\mathrm{i/s/o}}$ is $\mathcal{J}$-passive if and only if V is a maximal nonnegative subspace of $\mathfrak{K}$ (i.e., V is nonnegative, and it is not strictly contained in any other nonnegative subspace of $\mathfrak{K}$).

This motivates the following definition:

Definition 3.5.

(i) By a *conservative s/s system* Σ we mean a regular s/s system whose signal space $\mathcal{W}$ is a Kreĭn space, and whose generating subspace V *is a Lagrangian subspace* of the node space $\mathfrak{K}$ (with respect to the inner product (3.10)).
(ii) By a *passive s/s system* Σ we mean a regular s/s system whose signal space $\mathcal{W}$ is a Kreĭn space, and whose generating subspace V *is a maximal nonnegative subspace* of the node space $\mathfrak{K}$ (with respect to the inner product (3.10)).

Thus, in particular, every conservative s/s system is also passive.

Note that Definition 3.5 does *not explicitly require that* Σ *must be solvable*, which was assumed in the derivation of (3.9). However, it turns out that this condition is redundant in Definitions 3.5, i.e., the regularity of Σ combined with either the condition $V = V^{[\perp]}$ or the assumption that V is maximal nonnegative implies that Σ is solvable.

3.3. Passive and conservative realizations

In i/s/o systems theory one is often interested in the "converse problem" of finding a "realization" of a given analytic "transfer function" φ with some "additional properties". By a realization we mean an i/s/o system whose i/o resolvent function coincides with φ is some specified open subset Ω of $\mathbb{C}$. For example,

(i) φ is a *"Schur function"* over $\mathbb{C}^+$, and one wants to construct a *scattering conservative realization* $\Sigma_{\mathrm{i/s/o}}$ of φ,
(ii) φ is a *"positive real function"* over $\mathbb{C}^+$, and one wants to construct an *impedance conservative realization* $\Sigma_{\mathrm{i/s/o}}$ of φ.
(iii) φ is a *"Potapov function"* over $\mathbb{C}^+$, and one wants to construct a *transmission conservative realization* $\Sigma_{\mathrm{i/s/o}}$ of φ.

In the state/signal setting all *these three problems collapse into one and the same problem*: Given a *passive signal bundle* over $\mathbb{C}^+$ (this notion will be defined in Definition 3.7 below), we want to construct a *conservative s/s realization* of this signal bundle, i.e., a conservative s/s system Σ with $\mathbb{C}^+ \subset \rho(\Sigma)$ such that the given passive signal bundle coincides with the characteristic signal bundle $\widehat{\mathfrak{F}}$ of Σ in $\mathbb{C}^+$.

Theorem 3.6. *Let Σ be a passive s/s system with signal space space $\mathcal{W}$ and characteristic signal bundle $\widehat{\mathfrak{F}}$. Then*

(i) $\mathbb{C}^+ \subset \rho(\Sigma)$ *(and hence $\widehat{\mathfrak{F}}$ is analytic in $\mathbb{C}^+$),*
(ii) *for each $\lambda \in \mathbb{C}^+$ the fiber $\widehat{\mathfrak{F}}(\lambda)$ of $\widehat{\mathfrak{F}}$ is a maximal nonnegative subspace of $\mathcal{W}$.*

See [AS16] for the proof of this theorem.

Definition 3.7. By a *passive signal bundle* in a Kreĭn (signal) space $\mathcal{W}$ we mean an analytic signal bundle Ψ in $\mathbb{C}^+$ with the property that for each $\lambda \in \mathbb{C}^+$ the fiber $\Psi(\lambda)$ is a maximal nonnegative subspace of $\mathcal{W}$.

This leads us to the following problem:

Problem 3.8 (Conservative state/signal realization problem). *Given a passive signal bundle Ψ, find a conservative s/s system Σ such that the characteristic signal bundle of Σ coincides with Ψ in $\mathbb{C}^+$.*

One such construction is carried out in [AKS11]. The setting in [AKS11] is different from the one described here, but it follows from [AKS11], e.g., that every passive signal bundle Ψ has a "simple" conservative s/s realization, and that such a realization is unique up to a unitary similarity transformation in the state space. Here "simplicity" means that the system is minimal within the class of conservative s/s systems, i.e., a conservative s/s system is simple if and only if it does not have any nontrivial conservative compression.

4. A short history

I first met Dima (Prof. Damir Arov) at the MTNS conference 1998 in Padova where he gave a plenary talk on "Passive Linear Systems and Scattering Theory". Five years later, in the fall of 2003, Dima came to work with me in Åbo for one month, and that was the beginning of our joint stationary state/signal systems story. We decided to "join forces" to study the relationship between the (external) reciprocal symmetry of a conservative linear system and the (internal) symmetry structure of the system in three different settings, namely the scattering, the impedance, and the transmission setting. Instead of writing three separate papers with three separate sets of results and proofs we wanted to rationalize and to find some "general setting" that would cover the "common part" of the theory. The basic plan was to first develop the theory in such a "general setting" as far as far as possible, before discussing the three related symmetry problems mentioned above in detail.

After a couple of days we realized that the "behavioral approach" of [BS06] seemed to provide a suitable "general setting". This setting gave us a natural mathematical model for a "linear time-invariant circuit" which may contain both lumped and distributed components.

To make the work more tractable from a technical point of view we decided to begin by studying the discrete time case. As time went by the borderline between the "general theory" and the application to the original symmetry problem kept moving forward. Our first paper had to be split in two because it became too long. Then the second part had to be split in two because it became too long, then the third part had to be split in to, and so on. Every time the paper was split into two the original symmetry problem was postponed to the second unfinished half, and our "general solution" to the symmetry problem was not submitted until 2011. By that time we had published more than 500 pages on the s/s systems theory in

13 papers (in addition to numerous conference papers). The specific applications of our symmetry paper to the scattering, impedance, and transmission settings is still "work in progress".

In 2006 Mikael Kurula joined the s/s team, and together with him we begun to also study the continuous time problem. See the reference list for details.

Since 2009 Dima and I have spent most of our common research time on writing a book on linear stationary systems in continuous time. It started out as a manuscript about s/s systems in discrete time. In 2012 we shifted the focus to s/s systems in continuous time. After one more year the manuscript was becoming too long to be published as a single volume, so we decided to split the book into two volumes. A partial preliminary draft of the first volume of this book is available as [AS16].

References

[AKS11] Damir Z. Arov, Mikael Kurula, and Olof J. Staffans, *Canonical state/signal shift realizations of passive continuous time behaviors*, Complex Anal. Oper. Theory **5** (2011), 331–402.

[AKS12a] ______, *Boundary control state/signal systems and boundary triplets*, Operator Methods for Boundary Value Problems, Cambridge University Press, 2012.

[AKS12b] ______, *Passive state/signal systems and conservative boundary relations*, Operator Methods for Boundary Value Problems, Cambridge University Press, 2012.

[AS04a] Damir Z. Arov and Olof J. Staffans, *Passive and conservative infinite-dimensional linear state/signal systems*, Proceedings of MTNS2004, 2004.

[AS04b] ______, *Reciprocal passive linear time-invariant systems*, Proceedings of MTNS2004, 2004.

[AS05] ______, *State/signal linear time-invariant systems theory. Part I: Discrete time systems*, The State Space Method, Generalizations and Applications (Basel Boston Berlin), Operator Theory: Advances and Applications, vol. 161, Birkhäuser Verlag, 2005, pp. 115–177.

[AS06] ______, *Affine input/state/output representations of state/signal systems*, Proceedings of MTNS2006, 2006.

[AS07a] ______, *State/signal linear time-invariant systems theory: passive discrete time systems*, Internat. J. Robust Nonlinear Control **17** (2007), 497–548.

[AS07b] ______, *State/signal linear time-invariant systems theory. Part III: Transmission and impedance representations of discrete time systems*, Operator Theory, Structured Matrices, and Dilations, Tiberiu Constantinescu Memorial Volume, Theta Foundation, 2007, available from American Mathematical Society, pp. 101–140.

[AS07c] ______, *State/signal linear time-invariant systems theory. Part IV: Affine representations of discrete time systems*, Complex Anal. Oper. Theory **1** (2007), 457–521.

[AS09a] ______, *A Kreĭn space coordinate free version of the de Branges complementary space*, J. Funct. Anal. **256** (2009), 3892–3915.

[AS09b] ______, *Two canonical passive state/signal shift realizations of passive discrete time behaviors*, J. Funct. Anal. **257** (2009), 2573–2634.

[AS10] ______, *Canonical conservative state/signal shift realizations of passive discrete time behaviors*, J. Funct. Anal. **259** (2010), 3265–3327.

[AS12] ______, *Symmetries in special classes of passive state/signal systems*, J. Funct. Anal. **262** (2012), 5021–5097.

[AS14] ______, *The i/s/o resolvent set and the i/s/o resolvent matrix of an i/s/o system in continuous time*, Proceedings of MTNS2014, 2014.

[AS16] ______, *Linear Stationary Input/State/Output and State/Signal Systems*, 2015, Book manuscript, available at
http://users.abo.fi/staffans/publ.html.

[BS06] Joseph A. Ball and Olof J. Staffans, *Conservative state-space realizations of dissipative system behaviors*, Integral Equations Operator Theory **54** (2006), 151–213.

[DdS87] A. Dijksma and H. S. V. de Snoo, *Symmetric and selfadjoint relations in Kreĭn spaces. I*, Operators in indefinite metric spaces, scattering theory and other topics (Bucharest, 1985), Oper. Theory Adv. Appl., vol. 24, Birkhäuser, Basel, 1987, pp. 145–166.

[KS07] Mikael Kurula and Olof J. Staffans, *A complete model of a finite-dimensional impedance-passive system*, Math. Control Signals Systems **19** (2007), 23–63.

[KS10] ______, *Well-posed state/signal systems in continuous time*, Complex Anal. Oper. Theory **4** (2010), 319–390.

[KS11] ______, *Connections between smooth and generalized trajectories of a state/signal system*, Complex Anal. Oper. Theory **5** (2011), 403–422.

[Kur10] Mikael Kurula, *On passive and conservative state/signal systems in continuous time*, Integral Equations Operator Theory **67** (2010), 377–424, 449.

[Opm05] Mark R. Opmeer, *Infinite-dimensional linear systems: A distributional approach*, Proc. London Math. Soc. **91** (2005), 738–760.

[Paz83] Amnon Pazy, *Semi-groups of linear operators and applications to partial differential equations*, Springer-Verlag, Berlin, 1983.

[Sta05] Olof J. Staffans, *Well-posed linear systems*, Cambridge University Press, Cambridge and New York, 2005.

[Sta06] ______, *Passive linear discrete time-invariant systems*, Proceedings of the International Congress of Mathematicians, Madrid, 2006, 2006, pp. 1367–1388.

Olof J. Staffans
Åbo Akademi University
Department of Mathematics
FIN-20500 Åbo, Finland
`http://users.abo.fi/staffans/`
e-mail: `olof.staffans@abo.fi`

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Dichotomy, Spectral Subspaces and Unbounded Projections

Christian Wyss

Abstract. The existence of spectral subspaces corresponding to the spectrum in the right and left half-plane is studied for operators on a Banach space where the spectrum is separated by the imaginary axis and both parts of the spectrum are unbounded. This is done under different assumptions on the decay of the resolvent along the imaginary axis, including the case of bisectorial operators. Moreover, perturbation results and an application are presented.

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1. Introduction

Let S be a linear operator on a Banach space X such that a strip around the imaginary axis belongs to the resolvent set of S, i.e.,

$$\big\{\lambda \in \mathbb{C} \,\big|\, |\operatorname{Re}\lambda| \le h\big\} \subset \varrho(S)$$

for some $h > 0$. The problem we want to address is the separation of the spectrum of S along the imaginary axis: Do there exist closed invariant subspaces X_+ and X_- which correspond to the part of the spectrum in the open right and left half-plane, respectively:

$$\sigma(S|_{X_+}) = \sigma(S) \cap \mathbb{C}_+, \qquad \sigma(S|_{X_-}) = \sigma(S) \cap \mathbb{C}_-.$$

There are two simple cases where such a separation is possible:

(i) If X is a Hilbert space and S is a selfadjoint or normal operator, then the spectral calculus yields projections onto the spectral subspaces corresponding to $\mathbb{C}_+$ and $\mathbb{C}_-$.

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(ii) In the Banach space setting, if one of the parts $\sigma(S) \cap \mathbb{C}_+$ or $\sigma(S) \cap \mathbb{C}_-$ of the spectrum is bounded, then there is the associated Riesz projection

$$P = \frac{-1}{2\pi i} \int_\Gamma (S-\lambda)^{-1}\, d\lambda.$$

The integration contour Γ is positively oriented and such that it contains the bounded part of the spectrum in its interior. P then projects onto the spectral subspace corresponding to the bounded part.

Here we will consider the case that X is a Banach space and both parts of the spectrum are unbounded. There have been several articles devoted to this problem, in particular [1, 2, 7, 11, 16]. We present some results from these publications, with a focus on the recently published [16].

In Section 2 we start with general theorems on the existence of the invariant subspaces $X_\pm$. If the projections associated with $X_\pm$ are bounded, then the operator S is called dichotomous, but we will see that the case of unbounded projections is possible and can be handled too; in fact, this will later allow for a more general perturbation result. In Section 3 the spectral separation problem is studied for bisectorial and almost bisectorial operators. Such operators have a certain decay of the resolvent along the imaginary axis, which leads to simplifications in the existence results for $X_\pm$. The final Section 4 contains perturbation results for dichotomy and an application involving a Hamiltonian block operator matrix from control theory.

2. Dichotomy and unbounded spectral projections

We will consider the following general setting: X is a Banach space, S is a densely defined, closed operator on X, and there exists $h > 0$ such that

$$\big\{\lambda \in \mathbb{C} \,\big|\, |\operatorname{Re}\lambda| \le h\big\} \subset \varrho(S). \tag{1}$$

Moreover, we denote by $\mathbb{C}_+$ and $\mathbb{C}_-$ the open right and left half-plane, respectively.

Definition 2.1. (i) The operator S is called *dichotomous* if there exists a decomposition $X = X_+ \oplus X_-$ into closed, S-invariant subspaces $X_\pm$ such that

$$\sigma(S|_{X_+}) \subset \mathbb{C}_+, \qquad \sigma(S|_{X_-}) \subset \mathbb{C}_-.$$

(ii) S is called *strictly dichotomous* if in addition

$$\sup_{\lambda \in \mathbb{C}_\mp} \|(S|_{X_\pm} - \lambda)^{-1}\| < \infty.$$

(iii) Finally, S is *exponentially dichotomous* if it is dichotomous and $-S|_{X_+}$ and $S|_{X_-}$ generate exponentially stable semigroups.

From the definition it is immediate that exponential dichotomy implies strict dichotomy which in turn implies dichotomy. Moreover, exponential dichotomy is equivalent to S being the generator of an exponentially stable bisemigroup [2].

If S is dichotomous, then S *decomposes* with respect to $X = X_+ \oplus X_-$, i.e., the domain of S decomposes as

$$\mathcal{D}(S) = (\mathcal{D}(S) \cap X_+) \oplus (\mathcal{D}(S) \cap X_-), \tag{2}$$

see [11, Lemma 2.4]. As a consequence, S admits the block operator representation

$$S = \begin{pmatrix} S|_{X_+} & 0 \\ 0 & S|_{X_-} \end{pmatrix},$$

the spectrum satisfies

$$\sigma(S) = \sigma(S|_{X_+}) \cup \sigma(S|_{X_-}), \tag{3}$$

and the subspaces $X_\pm$ are also $(S - \lambda)^{-1}$-invariant. Note that in both (2) and (3) the non-trivial inclusion is "$\subset$". Finally there are the bounded complementary projections $P_\pm$ associated with $X = X_+ \oplus X_-$ which project onto $X_\pm$ and satisfy $I = P_+ + P_-$.

The concept of an exponentially dichotomous operator was introduced in 1986 by Bart, Gohberg and Kaashoek [2]. In this and subsequent papers they applied it, e.g., to canonical factorisation of matrix functions analytic on a strip and to Wiener–Hopf integral operators. Perturbation results for exponential dichotomy and applications to Riccati equations were studied by Ran and van der Mee [10, 14]. For a comprehensive account on exponential dichotomy and its applications, see the monographs [3, 13].

Building up on the central spectral separation result from [2] (Theorem 2.5 in the present article), plane dichotomy was studied in 2001 by Langer and Tretter [7] for the special class of bisectorial operators. There, and in the following works [6, 11], perturbation results were derived and applied to Dirac and Hamiltonian block operator matrices and associated Riccati equations.

A different approach to the problem of spectral separation can be found in [5]: here complex powers of bisectorial operators are used to obtain equivalent conditions for dichotomy.

We consider now the question of uniqueness of the decomposition $X = X_+ \oplus X_-$ of a dichotomous operator. It is easy to see that the eigenvector part of such a decomposition is always unique:

Lemma 2.2. *Let S be dichotomous with respect to $X = X_+ \oplus X_-$. Then:*

(i) *If x is a (generalized) eigenvector of S with eigenvalue $\lambda \in \mathbb{C}_\pm$, then $x \in X_\pm$.*
(ii) *Suppose that S has a complete system of generalized eigenvectors. Then the spaces $X_\pm$ are uniquely determined as*

$$X_\pm = \overline{\operatorname{span}\{x \in X \mid x \ (\textit{gen.})\ \textit{eigenvector corresp. to}\ \lambda \in \mathbb{C}_\pm\}}.$$

On the other hand, there are simple examples of dichotomous operators whose decomposition is not unique:

Example 2.3. Let S be a linear operator with $\sigma(S) = \varnothing$, e.g., the generator of a nilpotent semigroup. Then S is trivially dichotomous with respect to the two choices

$$X_+ = X, \quad X_- = \{0\},$$

and

$$X_+ = \{0\}, \quad X_- = X.$$

From this, an example with non-empty spectrum is readily obtained by taking the direct sum $S = S_0 \oplus S_+ \oplus S_-$ where $\sigma(S_0) = \varnothing$ and $\sigma(\pm S_\pm) \subset \{\operatorname{Re}\lambda \geq h\}$, $h > 0$.

The notion of strict dichotomy (Definition 2.1(ii)) has been introduced in [16] in order to ensure the uniqueness of the decomposition $X = X_+ \oplus X_-$. For the stronger condition of exponential dichotomy, this uniqueness was already obtained in [2].

Lemma 2.4 ([16, Lemma 3.7]). *Let S be strictly dichotomous with respect to the decomposition $X = X_+ \oplus X_-$. Then $X_\pm$ are uniquely determined as $X_\pm = G_\pm$ where*

$$G_\pm = \left\{x \in X \,\middle|\, (S-\lambda)^{-1}x \text{ has a bounded analytic extension to } \overline{\mathbb{C}_\mp}\right\}. \tag{4}$$

We remark that the subspaces $G_\pm$ are well defined for any operator satisfying $i\mathbb{R} \subset \varrho(S)$, and that $G_+ \cap G_- = \{0\}$ always.

Having dealt with the uniqueness of the subspaces $X_\pm$, we turn now to the existence of dichotomous decompositions. In their paper from 1986, Bart, Gohberg and Kaashoek obtained the following fundamental result:

Theorem 2.5 ([2, Theorem 3.1]). *Suppose that S satisfies the condition*

$$\sup_{|\operatorname{Re}\lambda| \leq h} \|(S-\lambda)^{-1}\| < \infty. \tag{5}$$

If the expression

$$Px = \frac{1}{2\pi i}\int_{h-i\infty}^{h+i\infty} \frac{1}{\lambda^2}(S-\lambda)^{-1}S^2x\,d\lambda, \quad x \in \mathcal{D}(S^2), \tag{6}$$

defines a bounded linear operator on X, then S is dichotomous with $P_+ = P$.

Remark 2.6.

(i) The integral is well defined since $(S-\lambda)^{-1}$ is uniformly bounded on the strip $\{|\operatorname{Re}\lambda| \leq h\}$.

(ii) As we assumed S to be densely defined and $0 \in \varrho(S)$, the subspace $\mathcal{D}(S^2)$ is dense in X, and so P has a unique bounded extension to X as soon as it is bounded on $\mathcal{D}(S^2)$.

(iii) If S is bounded then a simple calculation shows that the above expression for P reduces to the formula for the Riesz projection for the spectrum in $\mathbb{C}_+$.

(iv) There is an analogous formula for the projection P_-:

$$P_-x = \frac{-1}{2\pi i}\int_{-h-i\infty}^{-h+i\infty} \frac{1}{\lambda^2}(S-\lambda)^{-1}S^2x\,d\lambda, \quad x \in \mathcal{D}(S^2).$$

(v) The proofs in [6, 7, 11] which show that certain operators remain dichotomous after a perturbation are all based on Theorem 2.5.

There are simple examples where the operator P from the previous theorem will be unbounded.

Example 2.7. On the sequence space $X = \ell^2$ we consider the block diagonal operator

$$S = \begin{pmatrix} S_1 & & \\ & S_2 & \\ & & \ddots \end{pmatrix}, \qquad S_n = \begin{pmatrix} n & 2n^2 \\ 0 & -n \end{pmatrix}.$$

Eigenvectors of the block S_n for the eigenvalues $\lambda = n$ and $\lambda = -n$, respectively, are

$$v_{n+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad v_{n-} = \begin{pmatrix} -n \\ 1 \end{pmatrix};$$

the corresponding spectral projections are

$$P_{n+} = \begin{pmatrix} 1 & n \\ 0 & 0 \end{pmatrix}, \qquad P_{n-} = \begin{pmatrix} 0 & -n \\ 0 & 1 \end{pmatrix}.$$

Moreover, straightforward calculations show that $\sigma(S) = \mathbb{Z} \setminus \{0\}$ and

$$\sup_{|\operatorname{Re}\lambda| \le \frac{1}{2}} \|(S-\lambda)^{-1}\| < \infty,$$

i.e., S satisfies condition (5) of Theorem 2.5. However, the projections of the blocks P_{n+} and P_{n-} are unbounded in n and consequently the projections P_+ and P_- for the whole operator S will be unbounded, too. In particular, S is not dichotomous and the integral expression (6) in Theorem 2.5 will yield an unbounded operator P. Note here that the precise reasoning uses Lemma 2.2: If S were dichotomous, then the eigenvectors of S corresponding to $\lambda = \pm n$ would belong to $X_\pm$, and hence $P_\pm$ would contain $P_{n\pm}$ and had to be unbounded. So S is not dichotomous and thus P from (6) must be unbounded.

Motivated by the last example, we look at properties of unbounded projections. The following definition and basic facts can be found in [1].

Definition 2.8. A linear operator $P : \mathcal{D}(P) \subset X \to X$ is called a (possibly unbounded) *projection* if

$$\mathcal{R}(P) \subset \mathcal{D}(P) \quad \text{and} \quad P^2 = P,$$

i.e., P is a linear projection in the algebraic sense on the vector space $\mathcal{D}(P)$.

A projection P yields a decomposition of its domain,

$$\mathcal{D}(P) = \mathcal{R}(P) \oplus \ker P,$$

and the complementary projection is given by

$$Q = I - P, \qquad \mathcal{D}(Q) = \mathcal{D}(P).$$

On the other hand, for every pair of linear subspaces $X_1, X_2 \subset X$ such that $X_1 \cap X_2 = \{0\}$, i.e., $X_1 \oplus X_2 \subset X$, there is a corresponding projection P with $\mathcal{D}(P) = X_1 \oplus X_2$, $\mathcal{R}(P) = X_1$, and $\ker P = X_2$.

A projection P is closed if and only if $\mathcal{R}(P)$ and $\ker P$ are closed subspaces. In this case, P is bounded if and only if $\mathcal{R}(P) \oplus \ker P$ is closed.

It turns out that under condition (5) the integral formula (6) from the theorem of Bart, Gohberg and Kaashoek always defines a closed projection and that the associated subspaces are invariant and correspond to the parts of the spectrum in $\mathbb{C}_+$ and $\mathbb{C}_-$, even if S is not dichotomous:

Theorem 2.9 ([16, Theorem 4.1]). *Let* $\sup_{|\operatorname{Re}\lambda|\le h} \|(S-\lambda)^{-1}\| < \infty$. *Then:*

(i) *There exist closed complementary projections* $P_\pm = S^2 A_\pm$ *where* $A_\pm \in L(X)$ *are given by*

$$A_\pm = \frac{\pm 1}{2\pi i}\int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda^2}(S-\lambda)^{-1}\,d\lambda. \tag{7}$$

(ii) $\mathcal{D}(S^2) \subset \mathcal{D}(P_\pm)$ *and*

$$P_\pm = \frac{\pm 1}{2\pi i}\int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda^2}(S-\lambda)^{-1} S^2 x\,d\lambda, \qquad x \in \mathcal{D}(S^2).$$

(iii) *The subspaces* $X_\pm = \mathcal{R}(P_\pm)$ *are closed,* S*- and* $(S-\lambda)^{-1}$*-invariant,*

$$\sigma(S|_{X_\pm}) \subset \mathbb{C}_\pm, \quad \sigma(S) = \sigma(S|_{X_+}) \cup \sigma(S|_{X_-}),$$
$$\sup_{\lambda\in\mathbb{C}_\mp} \|(S|_{X_\pm} - \lambda)^{-1}\| < \infty.$$

(iv) S *is strictly dichotomous if and only if* P_+ *is bounded.*

Remark 2.10.

(i) One can also show that always $\mathcal{R}(P_\pm) = G_\pm$, with $G_\pm$ defined in (4).

(ii) The theorem implies that all dichotomous operators obtained via the Bart–Gohberg–Kaashoek theorem, in particular those in [6, 7, 11], are in fact strictly dichotomous.

(iii) The fact that the projections $P_\pm$ are always closed will play an important role in the proof of the perturbation results in Section 4, see Remark 4.2.

The proof of Theorem 2.9 is based on the following construction of closed projections which commute with an operator:

Lemma 2.11 ([16, Lemma 2.3]). *Let* S *be a closed operator such that* $0 \in \varrho(S)$ *and let* $A_1, A_2 \in L(X)$ *with*

$$A_1 + A_2 = S^{-2}, \qquad A_1A_2 = A_2A_1 = 0,$$
$$A_j S^{-1} = S^{-1} A_j, \quad j = 1, 2.$$

Then the operators $P_j = S^2 A_j$ *are closed, complementary projections, their ranges* $X_j = \mathcal{R}(P_j)$ *are* S*- and* $(S-\lambda)^{-1}$*-invariant,*

$$\sigma(S) = \sigma(S|_{X_1}) \cup \sigma(S|_{X_2}),$$

$\mathcal{D}(S^2) \subset \mathcal{D}(P_j)$ *and* $P_j x = A_j S^2 x$, $x \in \mathcal{D}(S^2)$.

Proof. It is clear that P_j is closed. Since A_j commutes with S^{-1} we have

$$SA_jx = A_jSx \quad \text{for all} \quad x \in \mathcal{D}(S). \tag{8}$$

If $x \in \mathcal{D}(P_1)$, i.e., $A_1x \in \mathcal{D}(S^2)$, then $A_2P_1x = S^2A_2A_1x = 0$. Hence $A_1P_1x = S^{-2}P_1x \in \mathcal{D}(S^2)$ and so $P_1x \in \mathcal{D}(P_1)$ with $P_1^2x = P_1x$, i.e., P_1 is a projection. The identity $A_1 + A_2 = S^{-2}$ implies that P_2 is the projection complementary to P_1. From (8) it follows that $(S-\lambda)^{-1}A_j = A_j(S-\lambda)^{-1}$ for all $\lambda \in \varrho(S)$. Hence $X_1 = \ker P_2 = \ker A_2$ is invariant under S and $(S-\lambda)^{-1}$, similarly for X_2. Moreover (8) yields $\mathcal{D}(S^2) \subset \mathcal{D}(P_j)$ and $P_jx = A_jS^2x$ for $x \in \mathcal{D}(S^2)$. Finally we show

$$\varrho(S) = \varrho(S|_{X_1}) \cap \varrho(S|_{X_2}).$$

The inclusion "$\subset$" is trivial, so let $\lambda \in \varrho(S|_{X_1}) \cap \varrho(S|_{X_2})$. If $(S-\lambda)x = 0$, then $Sx \in \mathcal{D}(S^2) \subset \mathcal{D}(P_j)$ and $P_jSx = S^3A_jx = SP_jx$. Therefore $(S|_{X_j} - \lambda)P_jx = P_j(S-\lambda)x = 0$ and thus $P_jx = 0$. We obtain $x = 0$, so $S - \lambda$ is injective. To show that it is also surjective, set $T = (S|_{X_1} - \lambda)^{-1}A_1 + (S|_{X_2} - \lambda)^{-1}A_2$. Then $(S-\lambda)T = A_1 + A_2 = S^{-2}$ from which we conclude that $(S-\lambda)S^2T = I$. □

The proof of Theorem 2.9 now proceeds as follows: The operators $A_\pm$ defined by (7) satisfy $A_+ + A_- = S^{-2}$, $A_+A_- = A_-A_+ = 0$ and $A_\pm S^{-1} = S^{-1}A_\pm$. The previous lemma thus yields the closed projections $P_\pm$ and the invariance properties of $X_\pm$. An explicit integral formula for $(S|_{X_\pm} - \lambda)^{-1}$ on $\mathbb{C}_\mp$ then implies $\sigma(S|_{X_\pm}) \subset \mathbb{C}_\pm$ and, in conjunction with an application of the Phragmén-Lindelöf theorem, the boundedness of $\|(S|_{X_\pm} - \lambda)^{-1}\|$ on $\mathbb{C}_\mp$. This finally yields the strict dichotomy of S (when P_+ is bounded).

Remark 2.12. Theorem 2.9 and its proof use and combine existing results and ideas from the papers by Bart, Gohberg and Kaashoek [2] and Arendt and Zamboni [1]:

(i) The definition of $A_\pm$ along with the identities $A_+ + A_- = S^{-2}$ and $A_+A_- = A_-A_+ = 0$ can be found in [2]. Also the integral representation of $(S|_{X_\pm} - \lambda)^{-1}$ on $\mathbb{C}_\mp$ and the spaces $G_\pm$ from (4) are taken from this paper.

(ii) In [1] unbounded projections of the form $P_\pm = SB_\pm$ were constructed for the case of bisectorial S, where the bounded operators $B_\pm$ satisfy $B_+ + B_- = S^{-1}$ and $B_+B_- = B_-B_+ = 0$.

What is genuinely new here compared to [1, 2], is the invariance of $X_\pm$ under S and the fact that the decomposition of the spectrum $\sigma(S) = \sigma(S|_{X_+}) \cup \sigma(S|_{X_-})$ also holds in the absence of dichotomy. Again we remark here that the inclusion "$\subset$" is non-trivial.

3. Bisectorial and almost bisectorial operators

In this section we look at the spectral separation problem for the special classes of bisectorial and almost bisectorial operators. This will lead to certain simplifications as well as additional results compared with the general setting of Section 2.

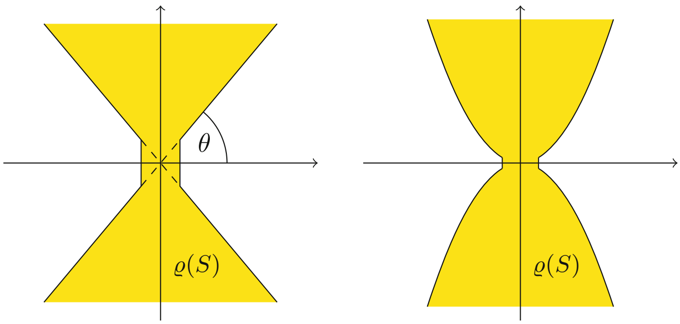

FIGURE 1. Resolvent sets of bisectorial and almost bisectorial operators

Definition 3.1. Let $i\mathbb{R} \subset \varrho(S)$. Then S is called *bisectorial* if

$$\|(S-\lambda)^{-1}\| \leq \frac{M}{|\lambda|}, \qquad \lambda \in i\mathbb{R} \setminus \{0\}, \tag{9}$$

with some constant $M > 0$. The operator S is called *almost bisectorial* if there exist $M > 0$, $0 < \beta < 1$ such that

$$\|(S-\lambda)^{-1}\| \leq \frac{M}{|\lambda|^{\beta}}, \qquad \lambda \in i\mathbb{R} \setminus \{0\}. \tag{10}$$

Note that here we only consider bisectorial operators satisfying $0 \in \varrho(S)$. If S is bisectorial with $0 \in \varrho(S)$, then S is also almost bisectorial for any $0 < \beta < 1$. On the other hand, an estimate (10) with $\beta < 1$ already implies that $0 \in \varrho(S)$.

If S is bisectorial, then a bisector $\theta \leq |\arg \lambda| \leq \pi - \theta$ belongs to the resolvent set and estimate (9) actually holds on this bisector. Similarly, if S is almost bisectorial, then (10) holds on a parabola shaped region, see Figure 1. For more details about bisectorial and almost bisectorial operators, see [1, 13, 16].

If $i\mathbb{R} \subset \varrho(S)$ and S is (almost) bisectorial, then estimate (5) holds and hence Theorem 2.9 applies. Moreover, its assertions may be simplified and strengthened:

Theorem 3.2 ([16, Theorem 5.6]). *Let S be (almost) bisectorial with $i\mathbb{R} \subset \varrho(S)$. Then the closed projections $P_\pm$ satisfy $P_\pm = SB_\pm$ with $B_\pm \in L(X)$,*

$$B_\pm = \frac{\pm 1}{2\pi i} \int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda}(S-\lambda)^{-1}\, d\lambda.$$

The inclusion $\mathcal{D}(S) \subset \mathcal{D}(P_\pm)$ holds and $P_\pm x = B_\pm S x$ for $x \in \mathcal{D}(S)$. Moreover, the restrictions $\pm S|_{X_\pm}$ are (almost) sectorial, i.e., an estimate

$$\|(S|_{X_\pm} - \lambda)^{-1}\| \leq \frac{M}{|\lambda|^{\beta}}, \quad \lambda \in \mathbb{C}_\mp,$$

holds. Here the constants M and β are the same as for S in Definition 3.1. ($\beta = 1$ *if S is bisectorial.*)

We note that the integral defining $B_\pm$ is well defined due to the resolvent estimates (9) and (10), respectively.

One may ask whether the resolvent decay of a bisectorial or almost bisectorial operator already implies that it is dichotomous, i.e., that $P_\pm$ are bounded. This is not the case:

Example 3.3. Let us modify Example 2.7 by taking for S_n the matrix

$$S_n = \begin{pmatrix} n & 2n^{1+p} \\ 0 & -n \end{pmatrix}.$$

For $0 < p < 1$ we then obtain that S is almost bisectorial with

$$\|(S-\lambda)^{-1}\| \leq \frac{M}{|\lambda|^{1-p}}, \qquad \lambda \in i\mathbb{R} \setminus \{0\}.$$

The eigenvectors of S_n are now

$$v_{n+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad v_{n-} = \begin{pmatrix} -n^p \\ 1 \end{pmatrix},$$

and the corresponding spectral projections are

$$P_{n+} = \begin{pmatrix} 1 & n^p \\ 0 & 0 \end{pmatrix}, \qquad P_{n-} = \begin{pmatrix} 0 & -n^p \\ 0 & 1 \end{pmatrix}.$$

Again, $P_\pm$ are unbounded and S is not dichotomous.

If in the last example we take $p = 0$, then S is bisectorial, the projections $P_\pm$ are bounded, and S is strictly dichotomous. But even in the bisectorial case, S may fail to be dichotomous. An example was given by McIntosh and Yagi [9], [16, Example 8.2].

There is yet another integral representation for the projections $P_\pm$ in the (almost) bisectorial setting:

Corollary 3.4 ([16, Corollary 5.9]). *If S is (almost) bisectorial, then*

$$P_+x - P_-x = \frac{1}{\pi i}\int_{-i\infty}^{i\infty\,\prime} (S-\lambda)^{-1}x\,d\lambda, \qquad x \in \mathcal{D}(S).$$

Here the prime denotes the Cauchy principal value at infinity. In particular, the integral exists for all $x \in \mathcal{D}(S)$.

In a Krein space setting with J-accretive, bisectorial S, such an integral representation has been used in [7, 11] to derive that the subspaces X_+ and X_- are J-nonnegative and J-nonpositive, respectively.

4. Perturbation results

We present two perturbation results for dichotomy: one in the general setting of Section 2 and one for (almost) bisectorial operators.

Theorem 4.1 ([16, Theorem 7.1]). *Let S, T be densely defined operators such that S is strictly dichotomous and T is closed. Suppose there exist $h, M, \varepsilon > 0$ such that*

(i) $\{\lambda \in \mathbb{C} \,|\, |\operatorname{Re}\lambda| \le h\} \subset \varrho(S) \cap \varrho(T)$,

(ii) $\|(S-\lambda)^{-1} - (T-\lambda)^{-1}\| \le \dfrac{M}{|\lambda|^{1+\varepsilon}}$ *for* $|\operatorname{Re}\lambda| \le h$,

(iii) $\mathcal{D}(S^2) \cap \mathcal{D}(T^2) \subset X$ *is dense.*

Then T is strictly dichotomous.

Sketch of the proof. The strict dichotomy of S implies that the corresponding projection P_+^S is bounded and, moreover, that $\|(S-\lambda)^{-1}\|$ is bounded for $|\operatorname{Re}\lambda| \le h$ (with a possibly smaller constant $h > 0$). From (ii) it follows that $\|(T-\lambda)^{-1}\|$ is also bounded for $|\operatorname{Re}\lambda| \le h$. Hence Theorem 2.9 applies to T and yields a closed projection P_+^T. For $x \in \mathcal{D}(S^2) \cap \mathcal{D}(T^2)$ one gets

$$\begin{aligned} P_+^S x - P_+^T x &= \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \frac{1}{\lambda^2} \big((S-\lambda)^{-1} S^2 x - (T-\lambda)^{-1} T^2 x\big)\, d\lambda \\ &= \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \big((S-\lambda)^{-1} x - (T-\lambda)^{-1} x\big)\, d\lambda. \end{aligned}$$

By (ii) this last integral converges in the uniform operator topology, and thus $P_+^S - P_+^T$ is bounded on $\mathcal{D}(S^2) \cap \mathcal{D}(T^2)$. Since P_+^S is bounded, we obtain that P_+^T is bounded on $\mathcal{D}(S^2) \cap \mathcal{D}(T^2)$. Now this is a dense subset of X and P_+^T is a closed operator, so we conclude that $P_+^T \in L(X)$ and hence T is strictly dichotomous. □

Remark 4.2.

(i) Assumption (iii) allows for situations where $\mathcal{D}(S) \ne \mathcal{D}(T)$, i.e., it is not required that $T = S + R$ with $R : \mathcal{D}(S) \to X$.

(ii) Theorem 4.1 generalizes a similar result for exponentially dichotomous operators [2, Theorem 5.1], where $\varepsilon = 1$ and $\mathcal{D}(T^2) \subset \mathcal{D}(S^2)$ were assumed.

(iii) The proof shows that since we know that P_+^T is closed, it suffices to show the boundedness of P_+^T on any dense subspace of $\mathcal{D}(T^2) \subset \mathcal{D}(P_+^T)$. This allows us to use assumption (iii) instead of the much more restrictive $\mathcal{D}(T^2) \subset \mathcal{D}(S^2)$ from [2].

As before, when considering (almost) bisectorial operators, some conditions can be simplified.

Theorem 4.3 ([16, Theorem 7.3]). *Let S, T be densely defined operators such that S is (almost) bisectorial and strictly dichotomous and T is closed. Suppose there exist $M, \varepsilon > 0$ such that*

(i) $i\mathbb{R} \subset \varrho(T)$,

(ii) $\|(S-\lambda)^{-1} - (T-\lambda)^{-1}\| \le \dfrac{M}{|\lambda|^{1+\varepsilon}}$ *for* $\lambda \in i\mathbb{R} \setminus \{0\}$,

(iii) $\mathcal{D}(S) \cap \mathcal{D}(T) \subset X$ *is dense.*

Then T is (almost) bisectorial (with the same β as for S) and strictly dichotomous.

A special case of this theorem was proved in [11]. There S was assumed to be bisectorial, $\mathcal{D}(T) = \mathcal{D}(S)$, $T = S + R$, and the perturbation $R : \mathcal{D}(S) \to X$ was *p-subordinate* to S. This means that there exist $0 \le p < 1$ and $c > 0$ such that

$$\|Rx\| \le c\|x\|^{1-p}\|Sx\|^p, \quad x \in \mathcal{D}(S).$$

For such a perturbation, assumption (ii) of Theorem 4.3 holds with $\varepsilon = 1 - p$.

Finally, we look at an application from systems theory [16, Example 8.8]. We consider the so-called Hamiltonian operator matrix

$$T = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}$$

with unbounded control and observation. The Hamiltonian is connected to the control algebraic Riccati equation

$$A^*\Pi + \Pi A - \Pi BB^*\Pi + C^*C = 0.$$

An operator Π is a solution of the Riccati equation, at least formally, if and only if the graph subspace of Π is invariant under the Hamiltonian. For more information on the optimal control problem see, e.g., [4, 15]. The aim here is to derive conditions for the dichotomy of T and then use it to construct invariant graph subspaces.

The setting is as follows: A is a sectorial operator on the Hilbert space X and $0 \in \varrho(A)$. We consider the interpolation spaces $X_s \subset X \subset X_{-s}$, $0 < s \le 1$, associated with A: Take $X_1 = \mathcal{D}(A)$ equipped with the graph norm and let X_{-1} be the completion of X with respect to the norm $\|A^{-1}x\|$. For $s < 1$, X_s and X_{-s} are obtained by complex interpolation between X_1, X and X_{-1}, see, e.g., [8, Chapter 1]. For A^* the corresponding spaces are $X_s^d \subset X \subset X_{-s}^d$. The spaces X_s and X_{-s}^d are dual with respect to the pivot space X: The inner product on X extends to a sesquilinear form on $X_s \times X_{-s}^d$ by which the dual space X_s' can be identified with X_{-s}^d. Similarly, X_s^d is dual to X_{-s}. More details on this construction can be found in [12, §§2.9, 2.10, 3.4]. For selfadjoint A, the spaces X_s and X_s^d coincide with the domains of the fractional powers of A, see [17, §3].

Now the control and observation operators B and C are assumed to be bounded linear operators $B : U \to X_{-s}$, $C : X_s \to Y$ where $s < 1/2$ and U, Y are additional Hilbert spaces. The aim is to make sense of T as an operator on $V = X \times X$ and then to show that it is strictly dichotomous. The difficulty is that by the above duality relations, $C^*C : X_s \to X_{-s}^d$ and $BB^* : X_s^d \to X_{-s}$, i.e., for $s > 0$, BB^* and C^*C map out of the space X. (This is what is meant here by unbounded control and observation.) We decompose T as

$$T = S + R, \qquad S = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -BB^* \\ -C^*C & 0 \end{pmatrix}.$$

Then S is bisectorial and strictly dichotomous on $V = X \times X$. The perturbation R is a bounded operator $R : V_s \to V_{-s}$ where $V_s = X_s \times X_s^d$, $V_{-s} = X_{-s} \times X_{-s}^d$. The operator S can be extended to an operator $S : V_{1-s} \to V_{-s}$, so that $T = S + R$ is well defined as an operator on V_{-s}. To consider T as an operator on V, we set $\mathcal{D}(T) = \{x \in V_{1-s} \,|\, Tx \in V\}$.

One can now check that the conditions of Theorem 4.3 are satisfied: Using perturbation and interpolation arguments, one can derive that $\lambda \in \varrho(T)$ and $\|(S-\lambda)^{-1} - (T-\lambda)^{-1}\| \le M/|\lambda|^{1+\varepsilon}$ for $\lambda \in i\mathbb{R}$ with $|\lambda|$ large enough and $\varepsilon = 1-2s$. The structure of T then implies that $i\mathbb{R} \subset \varrho(T)$. For more details on this see [16]. In a typical setting from systems theory, B and C are boundary operators. In this case $\mathcal{D}(T) \neq \mathcal{D}(S)$, but $\mathcal{D}(S) \cap \mathcal{D}(T)$ is in fact dense in V. Therefore Theorem 4.3 implies that T is bisectorial and strictly dichotomous.

In the next step, one now wants to show that the invariant subspaces V_+ and V_- of T are graph subspaces. The idea is to use the same approach as in [11, 17]: The symmetry of the Hamiltonian with respect to an indefinite inner product implies that V_+ and V_- are neutral subspaces for this inner product. Neutrality together with an approximate controllability condition then yields the graph subspace property. The details will be presented in a forthcoming paper.

References

[1] W. Arendt, A. Zamboni. *Decomposing and twisting bisectorial operators.* Studia Math., **197(3)** (2010), 205–227.

[2] H. Bart, I. Gohberg, M.A. Kaashoek. *Wiener–Hopf factorization, inverse Fourier transforms and exponentially dichotomous operators.* J. Funct. Anal., **68(1)** (1986), 1–42.

[3] H. Bart, I. Gohberg, M.A. Kaashoek, A.C. Ran. *A state space approach to canonical factorization with applications.* Basel: Birkhäuser, 2010.

[4] R.F. Curtain, H.J. Zwart. *An Introduction to Infinite Dimensional Linear Systems Theory.* Springer, New York, 1995.

[5] G. Dore, A. Venni. *Separation of two (possibly unbounded) components of the spectrum of a linear operator.* Integral Equations Operator Theory, **12(4)** (1989), 470–485.

[6] H. Langer, A.C.M. Ran, B.A. van de Rotten. *Invariant subspaces of infinite dimensional Hamiltonians and solutions of the corresponding Riccati equations.* In *Linear Operators and Matrices*, volume 130 of Oper. Theory Adv. Appl., pages 235–254. Birkhäuser, Basel, 2002.

[7] H. Langer, C. Tretter. *Diagonalization of certain block operator matrices and applications to Dirac operators.* In *Operator theory and analysis (Amsterdam,* 1997), volume 122 of Oper. Theory Adv. Appl., pages 331–358. Birkhäuser, Basel, 2001.

[8] J.-L. Lions, E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I.* Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[9] A. McIntosh, A. Yagi. *Operators of type ω without a bounded H_∞ functional calculus.* In *Miniconference on Operators in Analysis (Sydney,* 1989), volume 24 of Proc. Centre Math. Anal. Austral. Nat. Univ., pages 159–172. Austral. Nat. Univ., Canberra, 1990.

[10] A.C.M. Ran, C. van der Mee. *Perturbation results for exponentially dichotomous operators on general Banach spaces.* J. Funct. Anal., **210(1)** (2004), 193–213.

[11] C. Tretter, C. Wyss. *Dichotomous Hamiltonians with unbounded entries and solutions of Riccati equations.* J. Evol. Equ., **14(1)** (2014), 121–153.

[12] M. Tucsnak, G. Weiss. *Observation and control for operator semigroups.* Birkhäuser Advanced Texts. Birkhäuser Verlag, Basel, 2009.

[13] C. van der Mee. *Exponentially dichotomous operators and applications*, volume 182 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, 2008. Linear Operators and Linear Systems.

[14] C.V. van der Mee, A. C. Ran. *Additive and multiplicative perturbations of exponentially dichotomous operators on general Banach spaces.* In *Recent advances in operator theory and its applications*, volume 160 of Oper. Theory Adv. Appl., pages 413–424. Basel: Birkhäuser, 2005.

[15] M. Weiss, G. Weiss. *Optimal control of stable weakly regular linear systems.* Math. Control Signals Systems, **10(4)** (1997), 287–330.

[16] M. Winklmeier, C. Wyss. *On the Spectral Decomposition of Dichotomous and Bisectorial Operators.* Integral Equations Operator Theory, (2015), 1–32.

[17] C. Wyss, B. Jacob, H.J. Zwart. *Hamiltonians and Riccati equations for linear systems with unbounded control and observation operators.* SIAM J. Control Optim., **50** (2012), 1518–1547.

Christian Wyss
Fachgruppe Mathematik und Informatik
Bergische Universität Wuppertal
Gaußstr. 20
D-42097 Wuppertal, Germany
e-mail: wyss@math.uni-wuppertal.de