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Approximation on slabs and uniqueness for Bernoulli percolation with a sublattice of defects

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Abstract. Let $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ be the d-dimensional hypercubic lattice. We consider a model of inhomogeneous Bernoulli percolation on $\mathbb{L}^d$ in which every edge inside the s-dimensional sublattice $\mathbb{Z}^s \times \{0\}^{d-s}$, $2 \leq s < d$, is open with probability q and every other edge is open with probability p. We prove the uniqueness of the infinite cluster in the supercritical regime whenever $p \neq p_c(d)$ and $2 \leq s < d-1$, full uniqueness when $s = d-1$ and that the critical point $(p, q_c(p))$ can be approximated on the phase space by the critical points of slabs, for any $p < p_c(d)$, where $p_c(d)$ denotes the threshold for homogeneous percolation.

1. Introduction

A percolation process on a graph $G = (V, E)$ is briefly defined as a probability measure on the set of the subgraphs of G. Among the many possible variants, this paper deals with bond percolation models, in which every edge of E can be ***retained (open)*** or ***removed (closed)***, states represented by 1 and 0, respectively. A typical percolation configuration is an element of $\Omega = \{0, 1\}^E$; this set can be regarded as the set of subgraphs of G induced by their open edges. That

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is, an element $\omega \in \Omega$ is associated with the subgraph $(\mathsf{V}(\omega), \mathsf{E}(\omega))$, where $\mathsf{E}(\omega) = \{e \in E : \omega(e) = 1\}$ and $\mathsf{V}(\omega) = \{x \in V : \exists e \in \mathsf{E}(\omega) \text{ such that } x \in e\}$, and conversely, a subgraph $(V', F) \subset G$ with no isolated vertices induces the configuration $\omega \in \Omega$, given by $\omega(e) = 1$ if $e \in F$ and $\omega(e) = 0$ otherwise. As usual, the underlying σ-algebra $\mathcal{F}$ of the process is the one generated by the finite-dimensional cylinder sets of Ω.

More specifically, we tackle the model studied by Iliev et al. (2015), which consists of Bernoulli percolation on the d-dimensional lattice, $d \geq 3$, with an s-dimensional sublattice of inhomogeneities, $2 \leq s < d$. Formally speaking, let $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, where $\mathbb{E}^d = \big\{\{x, y\} \subset \mathbb{Z}^d : \|x - y\|_1 = 1\big\}$ and $\|\cdot\|_1$ is the L_1-norm. Also, define $H := \mathbb{Z}^s \times \{0\}^{d-s}$ and $\mathsf{E}_H := \big\{e \in \mathbb{E}^d : e \subset H\big\}$. For $p, q \in [0, 1]$, the governing probability measure $P_{p,q}$ of the process is the product measure on $(\Omega, \mathcal{F})$ with densities q and p on E_H and $\mathbb{E}^d \setminus \mathsf{E}_H$, respectively. That is, each edge of E_H is open with probability q and each edge of $\mathbb{E}^d \setminus \mathsf{E}_H$ is open with probability p, independently of any other edge. In Iliev et al. (2015), the authors generalized several classical results of homogeneous bond percolation to this inhomogeneous setting. Besides, they presented the phase-diagram for percolation and showed that the critical curve $q_c(p)$ is strictly decreasing for $p \in [0, p_c(d)]$, where $p_c(d)$ is the threshold for homogeneous Bernoulli bond percolation on $\mathbb{L}^d$. This is particularly interesting since it guarantees the existence of a set of parameters (p, q) such that $p < p_c(d) < q < p_c(s)$ and there is an infinite cluster $P_{p,q}$-almost surely.

This model was also treated by Newman and Wu (1997), where the authors showed that, for large d, the critical point $q_c(p_c(d))$ is strictly between $p_c(s)$ and $p_c(d)$, when $2 \leq s \leq d - 3$. They have also proved that $q_c(p_c(d)) = 1$ if $s = 1$. The papers of Madras et al. (1994), Zhang (1994) and Friedli et al. (2013) deal with low-dimensional inhomogeneities as well. We also refer the reader to the book of Kesten (1982), which presents one of the earliest results on the study of the critical curve for inhomogeneous percolation: considering the square lattice $\mathbb{L}^2$ and assigning parameters p and q to the horizontal and vertical edges, respectively, the author proves that $q_c(p) = 1 - p$.

The present work addresses two fundamental problems in percolation theory which have not yet been considered for the model described above. The first one is to determine the number of infinite clusters in a percolation configuration. For invariant percolation on the d-dimensional lattice, major contributions to this topic are those of Aizenman et al. (1987) and Burton and Keane (1989). An extension of the latter's argument to more general graphs can be found in the book of Lyons and Peres (2016), where the authors make use of minimal spanning forests to establish the uniqueness of the infinite cluster under certain conditions. As we shall discuss further, the lack of invariance of the percolation measure $P_{p,q}$ under a transitive group of automorphisms of $\mathbb{L}^d$ plays against a direct application of the existing techniques. We will then explore some other properties of our model, so that we can overcome this issue and conveniently adapt the known arguments to prove uniqueness of the infinite cluster in the case where $p \neq p_c(d)$ and $1 \leq s < d - 1$, and full uniqueness when $s = d - 1$.

The second problem we address is whether for any $p \in [0, p_c(d))$, the critical point $(p, q_c(p)) \in [0, 1]^2$ can be approximated by the critical point of the restriction of the inhomogeneous process to a slab $\mathbb{Z}^2 \times \{-N, \ldots, N\}^{d-2}$, for large $N \in \mathbb{N}$. Here, the classical work of Grimmett and Marstrand (1990) serves as the standard reference for providing the building blocks that give an affirmative answer to this question. We undergo the construction of a suitable renormalization process, which possesses some particularities that arise with the introduction of inhomogeneities, in contrast with the usual approach of Grimmett and Marstrand (1990). As we shall see, in the supercritical regime of parameters (p, q) where $p < p_c(d)$, the exponential decay of the one arm event in $(\mathbb{Z}^d, \mathbb{E}^d \setminus \mathsf{E}_H)$ (Aizenman and Barsky (1987); Duminil-Copin and Tassion (2016); Menshikov (1986)) compels us to search for vertices connected to the origin lying near the sublattice H. Therefore, the finite-size criterion used in the construction of long-range connections must be modified accordingly.

In the following, we introduce the relevant notation and concepts that are necessary for the statement of the main results of this paper. Given a graph $G = (V, E)$ and a configuration $\omega \in \Omega$, an ***open path*** in G is a set of distinct vertices $v_0, v_1, \ldots, v_m \in V$, such that $\{v_i, v_{i+1}\} \in E$ and $\omega(\{v_i, v_{i+1}\}) = 1$ for every $i = 0, \ldots, m-1$. For $u, v \in V$, we say that u is ***connected*** to v in ω if either $u = v$ or there is an open path from u to v, this event being denoted by $\{u \leftrightarrow v\}$. The ***cluster*** $\mathcal{C}(u)$ of u in ω is the random set of vertices of V that are connected to u, that is,

$$\mathcal{C}(u) \coloneqq \{v \in V : u \leftrightarrow v\}.$$

If $|\mathcal{C}(u)| = \infty$, we say that the vertex u ***percolates*** and write $\{u \leftrightarrow \infty\}$ for the set of such configurations.

Since we are interested in investigating how many infinite clusters do exist, if any, in a configuration $\omega \in \Omega$, we define the ***number of infinite components*** of ω as the random variable $N_\infty : \Omega \to \mathbb{N} \cup \{+\infty\}$, given by

$$N_\infty \coloneqq |\{\mathcal{C}(u) : u \in V, |\mathcal{C}(u)| = \infty\}|.$$

We are now in a position to state the first theorem of this paper. As mentioned above, let $p_c(d) \coloneqq \sup\{p : P_{p,p}(o \leftrightarrow \infty \text{ in } \mathbb{L}^d) = 0\}$.

Theorem 1.1 (Uniqueness of the infinite cluster)**.** *Assume that $d \geq s \geq 1$. Then, for every $p, q \in [0, 1]$ such that $p \neq p_c(d)$, there is at most one infinite cluster almost surely. The conclusion also holds for $p = p_c(d)$ and $s = d - 1 \geq 1$.*

Before we move on to state the second result, let us briefly discuss the issues that appear in our model and are not covered by the existing literature regarding the determination of the number of infinite components. On amenable graphs such as $\mathbb{L}^d$, there is an important property used in Burton and Keane (1989) and Lyons and Peres (2016, Theorem 7.6) which plays a key role to determine the uniqueness of the infinite component in the supercritical phase, namely the invariance of the percolation measure under a transitive group of automorphisms of the graph. Under this condition, assuming $N_\infty = \infty$, one can find a positive lower bound for the probabilities of any vertex $x \in \mathbb{Z}^d$ to be a branching point. This fact together with the observation that the number of branching points lying inside any box of $\mathbb{Z}^d$ cannot exceed the size of its boundary implies the non-amenability of the graph, a contradiction. However, in our model, the group of automorphisms for which $P_{p,q}$ is invariant does not act transitively on $\mathbb{Z}^d$, hence the above argument cannot be applied. As a matter of fact, if $P_{p,q}(N_\infty = \infty) > 0$ and $p < p_c(d)$, the probability that a vertex x is a branching point decays exponentially fast with the distance between x and H, which leads us to the conclusion that the expected number of branching points in a box of length n is of order n^s, yielding no contradiction. On the other hand, when $p > p_c(d)$, we must ensure that setting the parameter q to any value other than p does not cause the appearance of any new infinite cluster around the sublattice H. We shall circumvent these difficulties by exploring additional properties of the percolation measure $P_{p,q}$.

For the statement of the second result, we introduce the ***critical parameter function***, $q_c : [0, 1] \to [0, 1]$, defined by

$$q_c(p) \coloneqq \sup\{q : P_{p,q}(o \leftrightarrow \infty) = 0\}.$$

We also denote by q_c^N the analogous function for the restriction of the Bernoulli percolation process on $\mathbb{Z}^d$ with sublattice of defects $H = \mathbb{Z}^s \times \{0\}^{d-s}$ to the slab $\mathbb{Z}^2 \times \{-N, \ldots, N\}^{d-2}$.

Theorem 1.2 (Approximation on slabs)**.** *Assume that $d \geq s \geq 2$. Let $p < p_c(d)$. Then, for every $\eta > 0$, there exists an $N \in \mathbb{N}$ such that*

$$q_c^N(p + \eta) < q_c(p) + \eta.$$

As we mentioned earlier, in Iliev et al. (2015) the authors showed that q_c is strictly decreasing in the interval $[0, p_c(d)]$. To complement the behavior of the critical curve, Theorem 1.2 arose as an effort to prove the (left)-continuity of q_c in the interval $[0, p_c(d))$. Although the idea of essential enhancements cannot be directly applied to determine the continuity of q_c, the work of Aizenman and Grimmett (1991) implies that q_c^N is continuous and strictly decreasing in the interval $[0, p_c(d))$, for every $N \in \mathbb{N}$. Therefore, the left-continuity of q_c would follow if we could replace $q_c^N(p+\eta)$ by $q_c^N(p)$ in the statement of Theorem 1.2. Since we were not able to make this change, continuity of q_c remains an open problem. Nevertheless, we have achieved some minor improvements, such as the openness of the set $\{(p,q) \in [0,1] : E_{p,q}|\mathcal{C}| < \infty\}$ and the continuity of q_c at $p = 0$.

In what follows, we devote Section 2 to prove these improvements. In Section 3, we deal with the uniqueness of the infinite cluster for our model. We also consider some uniqueness results on graphs of uniformly bounded degree (see Section 3.5). In Section 4, we study how the inhomogeneous percolation process on $\mathbb{Z}^d$ can be approximated by an analogous process on a slab $\mathbb{Z}^2 \times \{-N, \dots, N\}^{d-2}$, for large $N \in \mathbb{N}$.

2. Perturbative study of the subcritical regime

In this section we study what happens to the percolation behavior when we take parameters $p, q \in [0,1]$ such that $E_{p,q}|C| < \infty$ and increase both of them by some $\varepsilon > 0$. We conclude that if ε is small enough, then we still have $E_{p+\varepsilon,q+\varepsilon}|C| < \infty$.

Proposition 2.1. *The set* $\mathcal{A} = \big\{(p,q) \in [0,1]^2 : E_{p,q}|C| < \infty\big\}$ *is open.*

An immediate consequence of this proposition is the following result:

Corollary 2.2. *The function* $p \mapsto q_c(p)$ *is continuous at* 0.

Proof of Proposition 2.1: Let $(p,q) \in \mathcal{A}$ and define $B_m(v) := \{v + x : x \in B_m\}$, $v \in \mathbb{Z}^d$. We first claim that there exists $\lambda > 0$, such that, for every $v \in \mathbb{Z}^d$,

$$P_{p,q}(v \leftrightarrow \partial B_m(v)) \leq \exp(-\lambda m), \ \forall m \in \mathbb{N}. \tag{2.1}$$

To see this, note that if $v \leftrightarrow \partial B_m(v)$, then one of the following events occurs:

- $v \xleftrightarrow{\mathbb{E}^d \setminus \mathsf{E}_H} \partial B_m(v)$;
- for some $h \in H \cap B_m(v)$, there are two disjoint witnesses for $v \xleftrightarrow{\mathbb{E}^d \setminus \mathsf{E}_H} h$ and $h \leftrightarrow \partial B_m(v)$.

Thus, a simple union bound argument followed by the BK-inequality (see van den Berg and Kesten (1985)) yields

$$\begin{aligned} P_{p,q}\big(v \leftrightarrow \partial B_m(v)\big) &\leq P_{p,q}\big(v \xleftrightarrow{\mathbb{E}^d \setminus \mathsf{E}_H} \partial B_m(v)\big) + \textstyle\sum_{h \in H \cap B_m(v)} P_{p,q}\big(\{v \xleftrightarrow{\mathbb{E}^d \setminus \mathsf{E}_H} h\} \circ \{h \leftrightarrow \partial B_m(v)\}\big) \\ &\leq P_{p,q}\big(v \xleftrightarrow{\mathbb{E}^d \setminus \mathsf{E}_H} \partial B_m(v)\big) + \textstyle\sum_{h \in H \cap B_m(v)} P_{p,q}(v \xleftrightarrow{\mathbb{E}^d \setminus \mathsf{E}_H} h) P_{p,q}(h \leftrightarrow \partial B_m(v)). \end{aligned} \tag{2.2}$$

The events $\big\{v \xleftrightarrow{\mathbb{E}^d \setminus \mathsf{E}_H} \partial B_m(v)\big\}$ and $\big\{v \xleftrightarrow{\mathbb{E}^d \setminus \mathsf{E}_H} h\big\}$ do not depend on the states of the edges in E_H. Since $E_{p,q}|C| < \infty$, we have $p < p_c(d)$. Hence, by the exponential decay of the one-arm event in the subcritical regime for the homogeneous setting (see Aizenman and Barsky (1987); Duminil-Copin and Tassion (2016); Menshikov (1986)), there exists $\lambda_1 = \lambda_1(p) > 0$ such that

$$\begin{aligned} P_{p,q}(v \xleftrightarrow{\mathbb{E}^d \setminus \mathsf{E}_H} \partial B_m(v)) &\leq \exp(-\lambda_1 m), \\ P_{p,q}(v \xleftrightarrow{\mathbb{E}^d \setminus \mathsf{E}_H} h) &\leq \exp(-\lambda_1 \|h - v\|_\infty). \end{aligned} \tag{2.3}$$

Since $E_{p,q}|C| < \infty$, by Theorem 3 of Iliev et al. (2015), there exists $\lambda_2 = \lambda_2(p,q) > 0$ such that

$$P_{p,q}(o \leftrightarrow \partial B_m) \leq \exp(-\lambda_2 m), \ \forall m \in \mathbb{N}.$$

Hence, by the invariance of $P_{p,q}$ under the translations parallel to H, we have

$$P_{p,q}(h \leftrightarrow \partial B_m(v)) \leq P_{p,q}(h \leftrightarrow \partial B_{m-\|h-v\|_\infty}(h)) \\ \leq \exp(-\lambda_2(m - \|h-v\|_\infty)). \tag{2.4}$$

Combining estimates (2.3) and (2.4) in (2.2), we obtain

$$P_{p,q}\big(v \leftrightarrow \partial B_m(v)\big) \leq Km^s \exp(-\min(\lambda_1,\lambda_2)m) = \exp\{-(\log K + s\log m + \min(\lambda_1,\lambda_2)m)\},$$

for some constant $K > 0$, so that (2.1) follows for $\lambda = \min(\lambda_1, \lambda_2) + s + \log K$.

Having proved (2.1), let $m \in \mathbb{N}$, $v \in \mathbb{Z}^d$ and define the set $S_m(v) \coloneqq \{x \in \partial B_m(v) : x \overset{\mathsf{E}_{B_m(v)}}{\longleftrightarrow} v\}$. Then, there exists $K' > 0$ satisfying $|\partial B_m(v)| \leq K'm^d$ $\forall m \in \mathbb{N}$, so that

$$E_{p,q}|S_m(v)| \leq |\partial B_m(v)| P_{p,q}(v \leftrightarrow \partial B_m(v)) \leq K'm^d \exp(-\lambda m), \ \forall m \in \mathbb{N},$$

for every $v \in \mathbb{Z}^d$. Thus, choose $L \in \mathbb{N}$ such that,

$$\phi_v(p,q) \coloneqq K'L^d E_{p,q}|S_L(v)| \leq 1/2, \ \forall v \in \mathbb{Z}^d,$$

and note that $\phi_v(p,q)$ is an increasing polynomial in both p and q. Due to the symmetry of the percolation process and the invariance of $P_{p,q}$ under the translations parallel to H, there are at most $L+1$ different polynomials, each of which corresponding to a choice of v. Therefore, there exists $\varepsilon > 0$ such that

$$\phi(p+\varepsilon, q+\varepsilon) \coloneqq \sup_{v\in\mathbb{Z}^d} \phi_v(p+\varepsilon, q+\varepsilon) < 1.$$

Let $k \in \mathbb{N}$ and fix $v \in \mathbb{Z}^d$. If the event $\{v \leftrightarrow \partial B_{kL}(v)\}$ occurs, then $S_L(v) \neq \varnothing$ and there exists $y \in \partial B_L(v)$ such that $y \overset{\mathsf{E}_{B_{kL}} \setminus \mathsf{E}_{B_L(v)}}{\longleftrightarrow} \partial B_{kL}(v)$. These two events are independent, hence

$$\begin{aligned}
P_{p+\varepsilon,q+\varepsilon}(v \leftrightarrow \partial B_{kL}(v)) &\leq \sum_{\substack{S\subset\partial B_L(v)\\ S\neq\varnothing}} \sum_{y\in\partial B_L(v)} P_{p+\varepsilon,q+\varepsilon}(S_L(v) = S, y \overset{\mathsf{E}_{B_{kL}}\setminus\mathsf{E}_S}{\longleftrightarrow} \partial B_{kL}(v)) \\
&= \sum_{\substack{S\subset\partial B_L(v)\\ S\neq\varnothing}} \sum_{y\in\partial B_L(v)} P_{p+\varepsilon,q+\varepsilon}(S_L(v) = S) P_{p+\varepsilon,q+\varepsilon}(y \overset{\mathsf{E}_{B_{kL}}\setminus\mathsf{E}_S}{\longleftrightarrow} \partial B_{kL}(v)) \\
&\leq \sum_{\substack{S\subset\partial B_L(v)\\ S\neq\varnothing}} \sum_{y\in\partial B_L(v)} P_{p+\varepsilon,q+\varepsilon}(S_L(v) = S) P_{p+\varepsilon,q+\varepsilon}(y \leftrightarrow \partial B_{(k-1)L}(y)) \\
&= \sum_{y\in\partial B_L(v)} P_{p+\varepsilon,q+\varepsilon}(y \leftrightarrow \partial B_{(k-1)L}(y)) \sum_{\substack{S\subset\partial B_L(v)\\ S\neq\varnothing}} P_{p+\varepsilon,q+\varepsilon}(S_L(v) = S) \\
&\leq \sup_{y\in\mathbb{Z}^d} \big[P_{p+\varepsilon,q+\varepsilon}(y \leftrightarrow \partial B_{(k-1)L}(y))\big] K'L^d \sum_{\substack{S\subset\partial B_L(v)\\ S\neq\varnothing}} P_{p+\varepsilon,q+\varepsilon}(S_L(v) = S) \\
&\leq \sup_{y\in\mathbb{Z}^d} \big[P_{p+\varepsilon,q+\varepsilon}(y \leftrightarrow \partial B_{(k-1)L}(y))\big] \phi(p+\varepsilon, q+\varepsilon).
\end{aligned}$$

Since v is arbitrary, writing $b_k \coloneqq \sup_{y\in\mathbb{Z}^d}\big[P_{p+\varepsilon,q+\varepsilon}(y \leftrightarrow \partial B_{kL}(y))\big]$, $k \geq 1$, the above result implies

$$b_k = b_1[\phi(p+\varepsilon, q+\varepsilon)]^{k-1}.$$

Since $\phi(p+\varepsilon, q+\varepsilon) < 1$, we conclude that $P_{p+\varepsilon,q+\varepsilon}(o \leftrightarrow \partial B_m)$ decays exponentially with m, which implies $E_{p+\varepsilon,q+\varepsilon}|C| < \infty$. Therefore, $(p+\varepsilon, q+\varepsilon) \in \mathcal{A}$ and $\mathcal{A}$ is open. $\square$

3. Uniqueness for inhomogeneous percolation on $\mathbb{Z}^d$

We divide the proof of Theorem 1.1 in three cases: when $p < p_c(d)$, when $p > p_c(d)$ and when $p = p_c(d)$. Different techniques are used in each situation. To deal with the case $p < p_c(d)$, we develop a more general background regarding a less restrictive bond percolation measure $\mathbf{P}$ on a graph $G = (V, E)$, comprising the measure $P_{p,q}$ on the lattice $\mathbb{L}^d$ as a particular instance, and then show the impossibility of having more than one infinite cluster in the supercritical phase for the inhomogeneous percolation model. The argument used here is an adaptation of the use of minimal spanning forests as in Chapter 7 of Lyons and Peres (2016), together with the exponential decay of the probability of the one arm event for subcritical homogeneous percolation, derived by Menshikov (1986), Aizenman and Barsky (1987) and Duminil-Copin and Tassion (2016). When $p > p_c(d)$, we make use of the so-called mass transport principle as in Häggström and Peres (1999). Finally, we consider the case $p = p_c(d)$, which relies on ergodicity arguments together with the absence of the infinite cluster in the half space, proved by Barsky et al. (1991). As a future research project, we plan to provide a unified treatment of these cases and improve Theorem 1.1 by removing the $s = d - 1$ condition in the critical case.

3.1. *General background.* We begin with some definitions. A ***(vertex)-automorphism*** of a graph $G = (V, E)$ is a bijection $g : V \to V$ such that $\{g(u), g(v)\} \in E$ if and only if $\{u, v\} \in E$. We write $\mathrm{Aut}(G)$ for the group of automorphisms of G. Given a subgroup $\Gamma \subset \mathrm{Aut}(G)$, we say that Γ ***acts transitively*** on G if, for any $u, v \in V$, we have $g(u) = v$ for some $g \in \Gamma$. We say that G is ***transitive*** if $\mathrm{Aut}(G)$ itself acts transitively on G.

For any bond percolation process $(\Omega, \mathcal{F}, \mathbf{P})$ on $G = (V, E)$, note that every $g \in \mathrm{Aut}(G)$ induces a transformation $\hat{g} : \Omega \to \Omega$, given by

$$[\hat{g}(\omega)](\{u, v\}) = \omega\big(\{g^{-1}u, g^{-1}v\}\big), \quad \{u, v\} \in E.$$

We say that $\mathbf{P}$ is Γ***-invariant*** if $\mathbf{P}(\hat{g}A) = \mathbf{P}(A)$ for every $A \in \mathcal{F}$ and $g \in \Gamma$.

Now, let $\mathcal{I}_\Gamma \coloneqq \{A \in \mathcal{F} : \hat{g}A = A, \forall\, g \in \Gamma\}$. That is, $\mathcal{I}_\Gamma \subset \mathcal{F}$ is the σ-field of events of $\mathcal{F}$ that are invariant under the action of all elements of Γ. We call the measure $\mathbf{P}$ Γ***-ergodic*** if $\mathbf{P}(A) \in \{0, 1\}$ for every $A \in \mathcal{I}_\Gamma$.

Finally, given $\omega \in \Omega$ and $F \subset E$, let

$$\Pi_F \omega(e) \coloneqq \begin{cases} 1, & \text{if } e \in F, \\ \omega(e), & \text{if } e \notin F. \end{cases}$$

That is, $\Pi_F \omega \in \Omega$ is the configuration obtained by opening the edges of F in ω. We also denote by $\Pi_{\neg F}\omega$ the configuration obtained by closing the edges of F in ω (the same expression as above, but with 0 in place of 1). For any event $A \in \mathcal{F}$, we define $\Pi_F A \coloneqq \{\Pi_F \omega : \omega \in A\}$ and $\Pi_{\neg F} A \coloneqq \{\Pi_{\neg F}\omega : \omega \in A\}$.

A bond percolation process $\mathbf{P}$ on G is ***insertion tolerant*** (resp. ***deletion tolerant***) if $\mathbf{P}(\Pi_F A) > 0$ (resp. $\mathbf{P}(\Pi_{\neg F} A) > 0$) for any finite subset $F \subset E$ and any event $A \in \mathcal{F}$ satisfying $\mathbf{P}(A) > 0$. If a process is both insertion and deletion tolerant, it is said to have the ***finite energy property***.

Having defined all the relevant concepts, from now on we regard $\mathbf{P}$ as an insertion-tolerant bond percolation process on $G = (V, E)$, which is invariant and ergodic for some subgroup $\Gamma \subset \mathrm{Aut}(G)$. Moreover, for $S \subset V$, define $\mathsf{E}_S \coloneqq \{e \in E : e \subset S\}$, $\Gamma|_S \coloneqq \{g|_S : g \in \Gamma\}$ and $\mathcal{C}_S(u) \coloneqq \mathcal{C}(u) \cap S$. We shall also require that there exists $S \subset V$ such that $\Gamma|_S$ acts transitively on the subgraph (S, E_S) and

$$\mathbf{P}(|\mathcal{C}(u)| = \infty, |\mathcal{C}_S(u)| < \infty) = 0 \quad \text{for every } u \in V. \tag{3.1}$$

One can note that $P_{p,q}$ is a process of the above kind: as a matter of fact, $P_{p,q}$ is invariant under the translations of $\mathbb{Z}^d$ parallel to the sublattice $H = \mathbb{Z}^s \times \{0\}^{d-s}$, and insertion-tolerance comes from the fact that the states of the edges of $\mathbb{E}^d$ are independent of each other. For a proof of the ergodicity of $P_{p,q}$ under Γ, we refer to Proposition 7.3 of Lyons and Peres (2016). A proof of condition (3.1) with $S = H$ is postponed to the later sections.

For such percolation process $\mathbf{P}$, note that the action of any element of Γ on a configuration $\omega \in \Omega$ does not change the value of N_∞. Hence, N_∞ is measurable with respect to $\mathcal{I}_\Gamma$, and by ergodicity it is constant $\mathbf{P}$-a.s.. Under these conditions, we have the following result, due to Newman and Schulman (1981):

Theorem 3.1. *Let $G = (V, E)$ be a connected graph. Let $\mathbf{P}$ be an insertion-tolerant bond percolation process on G, which is invariant and ergodic under a subgroup $\Gamma \subset \mathrm{Aut}(G)$. Then $N_\infty \in \{0, 1, \infty\}$ $\mathbf{P}$-a.s..*

We refer the reader to Theorem 7.5 of Lyons and Peres (2016) for a proof of this result.

Thus, what comes next is intended to rule out the case $N_\infty = \infty$, using a similar approach to Theorem 7.9 of Lyons and Peres (2016). We emphasize that, unless $p = q$, this result cannot be applied directly in the present situation: if $p \neq q$, the only subgroup $\Gamma \subset \mathrm{Aut}(\mathbb{L}^d)$ for which $P_{p,q}$ is invariant is that of the translations parallel to the sublattice H, and Γ does not act transitively on $\mathbb{L}^d$, as required by the theorem.

First, we introduce some sets of vertices and edges of a graph $G = (V, E)$ that will be needed in our proof. For a subset $K \subset V$ and a subgraph $G' = (V', E') \subset G$, we define the ***interior vertex boundary*** of K in G', the ***exterior vertex boundary*** of K in G' and the ***exterior edge boundary*** of K in G' respectively as the sets

$$\begin{aligned} \partial_{G'} K &\coloneqq \{y \in K : \exists x \in V' \setminus K \text{ such that } \{x, y\} \in E'\}, \\ \Delta_v^{G'} K &\coloneqq \{y \in V' \setminus K : \exists x \in K \text{ such that } \{x, y\} \in E'\}, \\ \Delta_e^{G'} K &\coloneqq \{\{x, y\} \in E' : x \in K, y \in V' \setminus K\}. \end{aligned} \tag{3.2}$$

In particular, $\partial K = \partial_G K$, $\Delta_v K \coloneqq \Delta_v^G K$ and $\Delta_e K \coloneqq \Delta_e^G K$. For any vertex $u \in V$, we define the ***degree of the vertex*** u in K as the number $\deg_K(u) \coloneqq |\Delta_v\{u\} \cap K|$. We also write $\deg(u) \coloneqq \deg_V(u)$.

The relation between these sets we are going to use is expressed in the next result, which is Exercise 7.3 of Lyons and Peres (2016). The idea of the proof is highlighted in Lyons and Peres (2016) and therefore we shall omit it.

Lemma 3.2. *Let $T = (V_T, E_T)$ be a tree with $\deg(u) \geq 2$ for all $u \in V_T$ and consider the set $B \coloneqq \{u \in V_T : \deg(u) \geq 3\}$. Then, for every finite set $K \subset V_T$, we have*

$$|\Delta_v K| \geq |K \cap B| + 2. \tag{3.3}$$

Until the end of this section, it will be useful to keep in mind the correspondence between the space $\Omega = \{0, 1\}^E$ and the set of the subgraphs of $G = (V, E)$ induced by their open edges. We shall regard the configurations of Ω in both ways, referring to the most convenient manner when necessary.

We now state a version of Lemma 7.7 of Lyons and Peres (2016), specifically designed to deal with Bernoulli percolation on $\mathbb{Z}^d$ with a sublattice of defects and similar models. The proof of this version is carried out in the same way as that of its counterpart in Lyons and Peres (2016), with minor modifications, hence we shall omit it. For the statement of the lemma, we need the following definition: a vertex $u \in V$ is called a ***branching point*** of a configuration $\omega \in \Omega$ if u percolates in ω and removing all edges incident to u splits $\mathcal{C}(u)$ into at least three distinct infinite clusters. The ***set of branching points*** of a configuration ω will be denoted by $\Lambda(\omega)$. For $S \subset V$, recall that $\mathcal{C}_S(u) \coloneqq \mathcal{C}(u) \cap S$.

Lemma 3.3. *Let $G = (V, E)$ be a connected graph and $\mathbf{P}$ be an insertion-tolerant bond percolation process on G. Suppose there exist a subgroup $\Gamma \subset \mathrm{Aut}(G)$ and a connected set $S \subset V$ such that*

i. $\mathbf{P}$ *is invariant under* Γ*;*
ii. $\mathbf{P}(|\mathcal{C}(u)| = \infty, |\mathcal{C}_S(u)| < \infty) = 0$ *for every* $u \in V$*.*

If $\mathbf{P}(\omega : N_\infty(\omega) = \infty) > 0$*, then there exists, on a larger probability space* $\left(\widetilde{\Omega}, \widetilde{\mathbf{P}}\right)$*, a coupling* $(\mathfrak{F}, \omega)$ *with the following properties:*

a. $\mathfrak{F} \subset \omega$ *and* $\mathfrak{F}$ *is a random forest;*
b. *The distribution of the pair* $(\mathfrak{F}, \omega)$ *is* Γ*-invariant;*
c. $\widetilde{\mathbf{P}}(\Lambda(\mathfrak{F}) \cap S \neq \varnothing) > 0$*.*

When $\mathbf{P}$ is insertion-tolerant and invariant under $\mathrm{Aut}(G)$, the uniqueness of the infinite cluster is established for amenable graphs by proving that if $\mathbf{P}(\omega : N_\infty(\omega) = \infty) > 0$, then G is non-amenable, see for example Theorems 7.6 and 7.9 of Lyons and Peres (2016). What we shall exhibit in the next result is a simple and straightforward generalization of this fact. It will help us to make a proper argument regarding the uniqueness of the infinite cluster for Bernoulli percolation on $\mathbb{Z}^d$ with a sublattice of defects, which is not invariant under $\mathrm{Aut}(\mathbb{L}^d)$. Although the proof is carried out in the same way as the theorems mentioned above, we present it in the sequel to include the generalization step in the appropriate place.

For a graph $G = (V, E)$, let $S', S \subset V$ with $|S'| < \infty$, and define

$$\mathcal{C}(S'; S) := \{u \in S' : \exists v \in S \text{ such that } v \leftrightarrow u\}. \tag{3.4}$$

Lemma 3.4. *Let* $G = (V, E)$ *be a connected graph and* $\mathbf{P}$ *be an insertion-tolerant bond percolation process on* G*. Suppose there exist a subgroup* $\Gamma \subset \mathrm{Aut}(G)$ *and a connected set* $S \subset V$ *such that the following conditions hold:*

i. $\mathbf{P}$ *is invariant and ergodic under* Γ*;*
ii. $\mathbf{P}(|\mathcal{C}(u)| = \infty, |\mathcal{C}_S(u)| < \infty) = 0$ *for every* $u \in V$*;*
iii. $\Gamma|_S$ *acts transitively on the subgraph* (S, E_S)*, where* $\mathsf{E}_S := \{e \in E : e \subset S\}$*.*

If $\mathbf{P}(\omega : N_\infty(\omega) = \infty) > 0$*, then there exists a constant* $c > 0$ *such that, for every finite set* $R \subset V$ *satisfying* $R \cap S \neq \varnothing$*, we have*

$$\frac{\mathbf{E}|\mathcal{C}(\Delta_v R; S)|}{|R \cap S|} \geq c. \tag{3.5}$$

Before proving this result, note that if $\mathbf{P}$ is invariant and ergodic under Γ and Γ acts transitively on G, we can take $S = V$ and inequality (3.5) implies the non-amenability condition for $G = (V, E)$. Besides, we would like to stress the importance of the quantity $\mathbf{E}|\mathcal{C}(\Delta_v R; S)|$ in (3.5). If this term is replaced by a larger one, such as $|\Delta_v R|$, then it is not possible to extract any useful information from our percolation model. For instance, if $B_n = \{-n, \dots, n\}^d$ and we consider the inhomogeneous percolation process on $\mathbb{Z}^d$ defined in Section 1 for $d = 3$, $H = \mathbb{Z}^2 \times \{0\}$ and $p < p_c(3) < q < p_c(2)$, it follows that there is a constant $c > 0$ such that $|\Delta_v B_n| \geq c|B_n \cap H|$ for all $n \in \mathbb{N}$. Nevertheless, we shall see in the next section that inequality (3.5) does not hold for B_n on such model, therefore $P_{p,q}(\omega : N_\infty(\omega) = \infty) = 0$.

Proof of Lemma 3.4: Let $\widetilde{\mathbf{P}}$ and $\mathfrak{F}$ be as in Lemma 3.3. Conditions ***i*** – ***iii*** imply that there is a constant $c > 0$ such that $\widetilde{\mathbf{P}}(u \in \Lambda(\mathfrak{F})) = c$ for every $u \in S$. Hence, the expected number of branching points of $\mathfrak{F}$ in $R \cap S$ is

$$\widetilde{\mathbf{E}}|\Lambda(\mathfrak{F}) \cap R \cap S| = \sum_{u \in R \cap S} \widetilde{\mathbf{P}}(u \in \Lambda(\mathfrak{F})) = c|R \cap S|. \tag{3.6}$$

Let $\mathscr{T}$ be the set of the infinite components (trees) of $\mathfrak{F}$. Also, consider the process of inductively removing the leaves of a tree. If we apply this process to any $T = (V_T, E_T) \in \mathscr{T}$, we are left, at the end of the procedure, with an infinite tree $T' = (V_{T'}, E_{T'}) \subset T$ that has no leaves and

$\Lambda(T') = \{u \in V_{T'} : \deg(u) \geq 3\} = \Lambda(T)$. Thus, an application of Lemma 3.2 with $K = R \cap V_{T'}$ yields

$$\begin{aligned}\left|\Delta_v^T(R \cap V_T)\right| &\geq \left|\Delta_v^{T'}(R \cap V_{T'})\right| \\ &\geq |R \cap V_{T'} \cap \Lambda(T')| = |R \cap \Lambda(T)|.\end{aligned}$$

Observing that $\left[\Delta_v^T(R \cap V_T)\right] \subset [\Delta_v R \cap V_T]$ and summing up the above inequality over all trees $T \in \mathscr{T}$, we arrive at

$$|\Delta_v R \cap V_{\mathfrak{F}_\infty}| \geq |R \cap \Lambda(\mathfrak{F}_\infty)| = |R \cap \Lambda(\mathfrak{F})|, \tag{3.7}$$

where $\mathfrak{F}_\infty := \bigcup_{T \in \mathscr{T}} T$.

Finally, by property **a.** of Lemma 3.3, we have $\mathfrak{F}_\infty \subset \omega_\infty$, where ω_∞ is the union of all the infinite components of ω. Since every vertex of ω_∞ is connected to S by condition ***ii*** and $\mathbf{P}(N_\infty = \infty) = 1$ by ergodicity, it follows that $\widetilde{\mathbf{P}}$-a.s.

$$|\Delta_v R \cap V_{\mathfrak{F}_\infty}| \leq |\Delta_v R \cap V_{\omega_\infty}| \leq |\mathcal{C}(\Delta_v R; S)|. \tag{3.8}$$

Combining equations (3.7) and (3.8), taking the expectation $\widetilde{\mathbf{E}}$ and using equality (3.6), we conclude that

$$\mathbf{E}|\mathcal{C}(\Delta_v R; S)| \geq c|R \cap S|.$$

□

3.2. *Proof of Theorem 1.1: the case $p < p_c(d)$.* Returning to the inhomogeneous percolation process on $\mathbb{Z}^d$ defined in Section 1, we recall that the conditions of Lemma 3.4 are satisfied for $\mathbf{P} = P_{p,q}$ and $S = H = \mathbb{Z}^s \times \{0\}^{d-s}$. In the case $p < p_c(d)$ and $P_{p,q}(N_\infty > 0) = 1$, condition (3.1) is trivially satisfied since there is no infinite cluster on $\mathbb{Z}^d \setminus H$ almost surely. By Theorem 3.1, we then have $N_\infty \in \{0, 1, \infty\}$ $P_{p,q}$-a.s.. However, going in the opposite direction of having infinitely many infinite clusters, we have the following result:

Proposition 3.5. *Let $B_n = \{-n, \ldots, n\}^d$, $n \in \mathbb{N}$. If $p < p_c(d)$ and $q \in [0, 1]$, then*

$$\frac{E_{p,q}|\mathcal{C}(\Delta_v B_n; H)|}{|B_n \cap H|} \xrightarrow[n \to \infty]{} 0.$$

Proof: By the exponential decay of the one arm event in the homogeneous model with parameter $p < p_c(d)$ (Aizenman and Barsky (1987); Duminil-Copin and Tassion (2016); Menshikov (1986)), there exists a positive constant $c_p > 0$ such that $P_{p,q}(u \leftrightarrow H) \leq \exp\{-c_p \operatorname{dist}(u, H)\}$ for any vertex $u \in \mathbb{Z}^d$, where $\operatorname{dist}(u, H)$ denotes the graph-theoretical distance between u and H. Therefore, taking $\alpha > (d - s - 1)/c_p$ and observing that $\Delta_v B_{n-1} \subset \partial B_n = B_n \setminus B_{n-1}$, we have

$$\begin{aligned}E_{p,q}|\mathcal{C}(\Delta_v B_{n-1}; H)| &\leq E_{p,q}|\mathcal{C}(\partial B_n; H)| \\ &= \sum_{\substack{u \in \partial B_n \\ \operatorname{dist}(u,H) < \alpha \log n}} P_{p,q}(u \leftrightarrow H) + \sum_{\substack{u \in \partial B_n \\ \operatorname{dist}(u,H) \geq \alpha \log n}} P_{p,q}(u \leftrightarrow H) \\ &\leq C\Big[n^{s-1}(\alpha \log n)^{d-s} + n^{d-1} \exp\{-c_p \alpha \log n\}\Big] \\ &\leq C'|B_{n-1} \cap H| \times \left[\frac{(\alpha \log n)^{d-s}}{n} + n^{d-s-1-c_p\alpha}\right],\end{aligned}$$

for positive constants $C = C(s, d)$ and $C' = C'(s, d)$. Observing that the last term in brackets goes to zero as $n \to \infty$, the result follows. □

As an immediate consequence of Lemma 3.4 and Proposition 3.5, we can rule out the case $N_\infty = \infty$ when $p < p_c(d)$.

Corollary 3.6. *If $p < p_c(d)$ and $q \in [0,1]$ then $N_\infty \in \{0,1\}$ $P_{p,q}$-a.s..*

3.3. *Proof of Theorem 1.1: the case $p > p_c(d)$.* In order to work with the case $p > p_c(d)$, recall that the set of edges whose vertices both belong to the sublattice $H = \mathbb{Z}^s \times \{0\}^{d-s}$ is denoted by $\mathsf{E}_H := \{e \in \mathbb{E}^d : e \subset H\}$ and let P be the probability measure associated with the family $\{U(e) : e \in \mathbb{E}^d\}$ of i.i.d. random variables having uniform distribution in $[0,1]$. Also, consider the decomposition $\mathbb{E}^d = E^+ \cup E^- \cup \mathsf{E}_H$, where $E^+ := \{\{x,y\} \in \mathbb{E}^d : (x_d \vee y_d) > 0\}$ and $E^- := \mathbb{E}^d \setminus (E^+ \cup \mathsf{E}_H)$, and for $p,q,t \in [0,1]$, let $\omega_{p,q,t} \in \{0,1\}^{\mathbb{E}^d}$ be the Bernoulli bond percolation process on $\mathbb{L}^d$ given by

$$\omega_{p,q,t}(e) := \begin{cases} \mathbf{1}_{\{U(e) \le p\}} & \text{if } e \in E^+, \\ \mathbf{1}_{\{U(e) \le q\}} & \text{if } e \in \mathsf{E}_H, \\ \mathbf{1}_{\{U(e) \le t\}} & \text{if } e \in E^-. \end{cases}$$

To establish the uniqueness of the infinite cluster when $p > p_c(d)$, we make use of the above coupling and the technique used in the proof of Proposition 3.1 of Häggström and Peres (1999) to derive the following result:

Proposition 3.7. *If $p > p_c(d)$ and $q \in (0,1)$, then $N_\infty = 1$ $P_{p,q}$-a.s..*

The proof of Proposition 3.7 relies on the so-called ***mass transport principle***. As pointed out in Häggström and Peres (1999), it was first used in the percolation setting by Häggström (1997) and fully developed by Benjamini et al. (1999). To our purposes, it suffices to state a particular case of this principle, based on Theorem 2.1 of Häggström and Peres (1999), to which we refer the reader for a proof.

Theorem 3.8 (The Mass-Transport Principle)**.** *Let $\Gamma \subset \mathrm{Aut}(\mathbb{L}^d)$ be the subgroup of translations parallel to the sublattice $H = \mathbb{Z}^s \times \{0\}^{d-s}$. If $(\Omega, \mathbf{P})$ is any Γ-invariant bond percolation process on $\mathbb{L}^d$ and $m(x,y,\omega)$ is a nonnegative function of $x,y \in H$, $\omega \in \Omega$ such that $m(x,y,\omega) = m(\gamma x, \gamma y, \gamma\omega)$ for all x, y and ω and $\gamma \in \Gamma$, then*

$$\sum_{y \in H} \int_\Omega m(x,y,\omega)\, \mathrm{d}\mathbf{P}(\omega) = \sum_{y \in H} \int_\Omega m(y,x,\omega)\, \mathrm{d}\mathbf{P}(\omega) \quad \forall x \in H. \tag{3.9}$$

This result can be viewed as the mass transport principle applied just on the sublattice H. To make proper use of this technique, we must establish condition (3.1), regarding the connected component $\mathcal{C}(v, \omega_{p,q,t})$ of $v \in \mathbb{Z}^d$ in the configuration $\omega_{p,q,t}$.

Lemma 3.9. *If $p > p_c(d)$ and $q \in [0,1]$, then for every $v \in H$ we have*

$$P(|\mathcal{C}(v, \omega_{p,0,0})| = \infty, |\mathcal{C}(v, \omega_{p,0,0}) \cap H| < \infty) = 0, \tag{3.10}$$

$$P(|\mathcal{C}(v, \omega_{p,q,p})| = \infty, |\mathcal{C}(v, \omega_{p,q,p}) \cap H| < \infty) = 0. \tag{3.11}$$

Proof: Since the critical point for homogeneous percolation in half-spaces is $p_c(d)$ Barsky et al. (1991), we have $P(|\mathcal{C}(o, \omega_{p,0,0})| = \infty) > 0$. Also, we know that P is Γ-invariant, hence ergodicity implies that there are P-a.s. infinitely many vertices in H belonging to an infinite cluster of $\omega_{p,0,0}$ when $p > p_c(d)$. Property (3.10) then follows from the uniqueness of the infinite cluster of $\omega_{p,0,0}$, as mentioned in Barsky et al. (1991).

To prove (3.11), suppose that for some $p > p_c(d)$ and $q \in [0,1]$, there is a finite set $F \subset H$ such that the event

$$B = \{U \in [0,1]^{\mathbb{E}^d} : |\mathcal{C}(o, \omega_{p,q,p}(U))| = \infty,\ \mathcal{C}(o, \omega_{p,q,p}(U)) \cap H = F\}$$

has positive probability. Since, for any $U \in B$, every edge within $\mathcal{C}(o, \omega_{p,q,p}(U))$ that is incident to H is contained in $\Delta_e F \cap \Delta_e H$, if we p-close every edge in $\Delta_e F \cap \Delta_e H$, we are mapped to a configuration U' such that, for some vertex $x \in \Delta_v F \setminus H$, we have $|\mathcal{C}(x, \omega_{p,t,p}(U'))| = \infty$ and

$|\mathcal{C}(x, \omega_{p,t,p}(U')) \cap H| < \infty$ not only for $t = q$, but for every $t \in [0, 1]$. In particular, this holds for $t = p$. Therefore, denoting by B' the event of such configurations, the finite energy property implies that $P(B') > 0$. But this is a contradiction, since ergodicity and uniqueness of the infinite cluster of $\omega_{p,p,p}$ imply that there are almost surely infinitely many vertices in H belonging to the infinite cluster of $\omega_{p,p,p}$ when $p > p_c(d)$. □

Proof of Proposition 3.7: We shall show that every infinite cluster of $\omega_{p,q,p}$ contains an infinite cluster of $\omega_{p,0,0}$. Uniqueness for $\omega_{p,q,p}$ follows from the fact that the cluster of $\omega_{p,0,0}$ is almost surely unique. This is the same proof as that of Proposition 3.1 of Häggström and Peres (1999). We present the reasoning again to indicate the places where Lemma 3.9 should be applied.

Let $\omega = (\omega_1, \omega_2)$ be the coupling of the processes $\omega_1 = \omega_{p,0,0}$ and $\omega_2 = \omega_{p,q,p}$, with $p > p_c(d)$ and $q \in [0, 1]$, and denote by P_i the marginal distribution of ω_i, $i = 1, 2$. Let $\mathcal{C}(u, \omega_i)$ be the connected component of $u \in \mathbb{Z}^d$ in the configuration ω_i and $\mathcal{C}(\infty, \omega_i)$ be the union of the infinite clusters in the configuration ω_i. Since P_i is invariant only by automorphisms $\gamma \in \mathrm{Aut}(\mathbb{L}^d)$ satisfying $\gamma(H) = H$, we shall use properties (3.10) and (3.11) to restrict our analysis to the sublattice H. Hence, we also consider the random sets $\mathcal{C}_H(u, \omega_i) \coloneqq \mathcal{C}(u, \omega_i) \cap H$ and $\mathcal{C}_H(\infty, \omega_i) \coloneqq \mathcal{C}(\infty, \omega_i) \cap H$.

For $u, v \in \mathbb{Z}^d$, recall that $\mathrm{dist}(u, v)$ denotes the graph-theoretic distance between u and v. Given $u \in H$, define

$$D_1(u) \coloneqq \inf\{\mathrm{dist}(u, v) : v \in \mathcal{C}_H(\infty, \omega_1)\};$$

$$A(u) \coloneqq \{D_1(u) > 0\} \cap \left\{D_1(u) = \min_{v \in \mathcal{C}_H(u,\omega_2)} D_1(v)\right\}.$$

That is, $A(u)$ is the event where $u \in H$ is one of the vertices of $\mathcal{C}_H(u, \omega_2)$ that are closest to $\mathcal{C}_H(\infty, \omega_1)$ in the configuration ω_1, this distance being positive.

By properties (3.10) and (3.11), every connected component of $\mathcal{C}(\infty, \omega_i)$, $i = 1, 2$, intersects H at infinitely many vertices almost surely. Hence, since $\omega_1 \subset \omega_2$, if $u \in \mathcal{C}_H(\infty, \omega_2)$, then one of the following events occur:

- $u \in \mathcal{C}_H(\infty, \omega_1)$;
- $u \notin \mathcal{C}_H(\infty, \omega_1)$, $\exists v \in \mathcal{C}(\infty, \omega_1)$ such that $u \in \mathcal{C}_H(v, \omega_2)$;
- $u \notin \mathcal{C}_H(\infty, \omega_1)$, $\forall v \in \mathcal{C}(\infty, \omega_1)$, $u \notin \mathcal{C}_H(v, \omega_2)$, $|\mathcal{C}(u, \omega_2)| = \infty$.

For any configuration in the first two events, it follows that $\mathcal{C}(u, \omega_2)$ contains an infinite cluster of $\mathcal{C}(\infty, \omega_1)$. For any $\omega = (\omega_1, \omega_2)$ in the last event, there exists a vertex $x \in \mathcal{C}_H(u, \omega_2)$ such that $D_1(x) = \min_{v \in \mathcal{C}_H(u,\omega_2)} D_1(v) > 0$. In other words, this configuration belongs to the event $\bigcup_{x \in H}[\{|\mathcal{C}(x, \omega_2)| = \infty\} \cap A(x)]$. Therefore, the proposition is proved if we show that

$$P(\{|\mathcal{C}(u, \omega_2)| = \infty\} \cap A(u)) = 0 \quad \forall u \in H.$$

We begin by analyzing the event $\{|\mathcal{C}(u, \omega_2)| = \infty\} \cap A(u) \cap \{D_1(u) > 1\}$. For $u, v \in H$, let

$$A_{u,v} \coloneqq \{v \in \mathcal{C}_H(u, \omega_2)\} \cap \left\{0 < D_1(v) < \min_{\substack{w \in \mathcal{C}_H(u,\omega_2) \\ w \neq v}} D_1(w)\right\},$$

that is, $A_{u,v}$ is the event in which $v \in \mathcal{C}_H(u, \omega_2)$ and is the only vertex of $\mathcal{C}_H(u, \omega_2)$ that is closest to $\mathcal{C}_H(\infty, \omega_1)$ in the configuration ω_1.

For every $\omega = (\omega_1, \omega_2) \in \{|\mathcal{C}(u, \omega_2)| = \infty\} \cap A(u) \cap \{D_1(u) > 1\}$, if we open (in ω_2 only) an edge $\{u, w\} \in \mathsf{E}_H$ with $D_1(w) = D_1(u) - 1$ and close every other edge incident to w, we are mapped to a configuration in $B_{w,w} \coloneqq \{|\mathcal{C}(w, \omega_2)| = \infty\} \cap A_{w,w}$. Since P_2 has the finite energy property, if we show that $P(B_{w,w}) = 0$, then we must have $P(\{|\mathcal{C}(u, \omega_2)| = \infty\} \cap A(u) \cap \{D_1(u) > 1\}) = 0$, and the first part of the proof is completed.

Define $m(u, v, \omega) \coloneqq \mathbf{1}_{A_{u,v}}(\omega)$ and, as in Theorem 3.8, let $\Gamma \subset \mathrm{Aut}(\mathbb{L}^d)$ be the subgroup of translations parallel to the sublattice $H = \mathbb{Z}^s \times \{0\}^{d-s}$. Since P is Γ-invariant, $m(x, y, \omega) =$

$m(\gamma x, \gamma y, \gamma\omega)$ for all $x, y \in H$, $\omega = (\omega_1, \omega_2)$ and $\gamma \in \Gamma$, and $A_{u,v} \cap A_{u,w} = \varnothing$ if $v \neq w$, the mass-transport principle (3.9) yields

$$\begin{aligned}\int_\Omega \sum_{v\in H} m(v,u,\omega)\,\mathrm{d}P(\omega) &= \sum_{v\in H}\int_\Omega m(u,v,\omega)\,\mathrm{d}P(\omega)\\ &= \sum_{v\in H} P(A_{u,v}) = P\Big(\bigcup_{v\in H} A_{u,v}\Big) < 1.\end{aligned} \tag{3.12}$$

By property (3.11), we have $|\mathcal{C}_H(u,\omega_2)| = \infty$ almost surely for every configuration $\omega \in B_{u,u} \coloneqq \{|\mathcal{C}(u,\omega_2)| = \infty\} \cap A_{u,u}$, and consequently $\sum_{v\in H} m(v,u,\omega) = \infty$ for every $\omega \in B_{u,u}$. This fact implies $P(B_{u,u}) = 0$ for all $u \in H$, since otherwise we would have

$$\int_\Omega \sum_{v\in H} m(v,u,\omega)\,\mathrm{d}P(\omega) \geq \int_{B_{u,u}} \sum_{v\in H} m(v,u,\omega)\,\mathrm{d}P(\omega) = \int_{B_{u,u}} \infty\,\mathrm{d}P(\omega) = \infty,$$

a contradiction with (3.12).

Now, it remains to show that $P(\{|\mathcal{C}(u,\omega_2)| = \infty\} \cap A(u) \cap \{D_1(u) = 1\}) = 0$. For a subset $V \subset \mathbb{Z}^d$ and $x, y \in V$, let $\mathrm{dist}_V(x,y)$ be the graph-theoretic distance between x and y in the subgraph of $\mathbb{Z}^d$ induced by V. For $w \in H$, define the random set

$$S(w) \coloneqq \begin{cases} \varnothing, & \text{if } w \notin \mathcal{C}_H(\infty,\omega_1),\\ \left\{\begin{array}{c} v \in \mathcal{C}_H(w,\omega_2) : \mathrm{dist}_{\mathcal{C}(w,\omega_2)}(v,w) \\ < \mathrm{dist}_{\mathcal{C}(w,\omega_2)}(v,x)\ \forall x \in \mathcal{C}(\infty,\omega_1)\setminus\{w\}\end{array}\right\}, & \text{if } w \in \mathcal{C}_H(\infty,\omega_1).\end{cases}$$

That is, $S(w)$ is the set of vertices $v \in \mathcal{C}_H(w,\omega_2)$ such that w is the only vertex of $\mathcal{C}(\infty,\omega_1)$ closest to v in the metric of $\mathcal{C}(w,\omega_2)$.

Note that, for any $\omega = (\omega_1,\omega_2) \in \{|\mathcal{C}(u,\omega_2)| = \infty\} \cap A(u) \cap \{D_1(u) = 1\}$, if we open (in ω_2 only) an edge $\{u,w\} \in \mathsf{E}_H$ with $w \in \mathcal{C}_H(\infty,\omega_1)$, we are mapped to a configuration in $\{|S(w)| = \infty\}$. Since P_2 is insertion-tolerant, we conclude that $P(\{|\mathcal{C}(u,\omega_2)| = \infty\} \cap A(u) \cap \{D_1(u) = 1\}) = 0$ if we show that $P(|S(w)| = \infty) = 0$.

Let $m(u,w,\omega) = \mathbf{1}_{\{u\in S(w)\}}$. Again by the mass-transport principle (3.9), we have

$$\begin{aligned}\int_\Omega \sum_{w\in H} m(w,u,\omega)\,\mathrm{d}P(\omega) &= \sum_{w\in H}\int_\Omega m(u,w,\omega)\,\mathrm{d}P(\omega)\\ &= \sum_{w\in H} P(u\in S(w)) = P\Big(\bigcup_{w\in H}\{u\in S(w)\}\Big) < 1.\end{aligned} \tag{3.13}$$

By property (3.11), we have $\sum_{w\in H} m(w,u,\omega) = \infty$ for any $\omega \in \{|S(u)| = \infty\}$, and this fact together with (3.13) implies $P(|S(u)| = \infty) = 0$, similarly to the previous case. Since P_2 is insertion-tolerant, we conclude that $P(\{|\mathcal{C}(u,\omega_2)| = \infty\} \cap A(u) \cap \{D_1(u) = 1\}) = 0$. □

Remark 3.10. By a similar reasoning, we can extend Proposition 3.7 to include the degenerate case $q = 0$. The case $q = 1$ is trivial.

3.4. *Proof of Theorem 1.1: the case $p = p_c(d)$, $s = d-1$.* We end the proof of Theorem 1.1 considering the case $p = p_c(d)$, $s = d-1$. The proof for this case also works for $p < p_c(d)$ and uses a more concrete approach than the one developed in Section 3.1. For simplicity, we assume that $q \notin \{0,1\}$: see Remark 3.12.

By Theorem 3.1, we know that $N_\infty \in \{0,1,\infty\}$ almost-surely for $P_{p,q}$. Then, let us proceed by contradiction and assume that $P_{p,q}(N_\infty = \infty) = 1$ for some $q \in [0,1]$ and $p \leq p_c(d)$. Let $\mathcal{G}$ be the set of all branching points that belong to H. Recall that Γ is the group of translations parallel to the hyperplane H. By the Γ-invariance of $P_{p,q}$ and the finite energy property, we have $P_{p,q}(x \in \mathcal{G}) = t > 0$ for every $x \in H$.

Let $B_n := \{0,1,\dots,n\}^s \times \{-\lfloor\log n\rfloor,\dots,\lfloor\log n\rfloor\}$, $n \in \mathbb{N}$. We shall study the consequences of having a "reasonable amount" of branching points inside $B_n \cap H$, for large values of n. First, let $\mathcal{G}_n := \mathcal{G} \cap B_n$. By the Γ-ergodicity of $P_{p,q}$ and the ergodic theorem, $|\mathcal{G}_n|/|B_n \cap H| \xrightarrow[n\to\infty]{} t$ in probability. Consequently,

$$P_{p,q}(|\mathcal{G}_n| \geq n^s t/2) \xrightarrow[n\to\infty]{} 1. \tag{3.14}$$

Next, given $n \in \mathbb{N}$, $\gamma \in \{-,+\}$, and writing $x = (x_1,\dots,x_d) \in \mathbb{Z}^d$, let $\partial B_n^\gamma := \{x \in \partial B_n : x_d = \gamma\lfloor\log n\rfloor\}$ and $\partial B_n^* := \partial B_n \setminus (\partial B_n^+ \cup \partial B_n^-)$.

As in the classical argument of Burton and Keane (1989), if $|\mathcal{G}_n| \geq k$ for some $k > 0$, then there are at least k vertices in ∂B_n connected to H within B_n. Let $\mathscr{K}_n$ be the set of such vertices and $\mathscr{K}_n^\gamma := \mathscr{K}_n \cap \partial B_n^\gamma$, for $\gamma \in \{-,+,*\}$. Note that if $|\mathscr{K}_n| \geq k$, then there exists $\gamma \in \{-,+,*\}$ such that $|\mathscr{K}_n^\gamma| \geq k/3$. Combining this observation with the limit (3.14) and the fact that $|\partial B_n^*|/n^s \xrightarrow[n\to\infty]{} 0$, we have

$$\begin{aligned} 1 = \liminf_{n\to\infty} P_{p,q}\Big(\big|\bigcup_\gamma \mathscr{K}_n^\gamma\big| \geq n^s t/2\Big) &\leq \liminf_{n\to\infty} P_{p,q}\Big(\bigcup_\gamma \Big\{\big|\mathscr{K}_n^\gamma\big| \geq n^s t/6\Big\}\Big) \\ &\leq 2\liminf_{n\to\infty} P_{p,q}\Big(\big|\mathscr{K}_n^-\big| \geq n^s t/6\Big), \end{aligned} \tag{3.15}$$

where the last inequality follows from the union bound and the symmetry of the events $\{|\mathscr{K}_n^\gamma| \geq k\}$, $\gamma \in \{+,-\}$.

Since $\operatorname{dist}(\partial B_n^-, H) = \lfloor\log n\rfloor$, any vertex of $\mathscr{K}_n^-$ is p-connected to distance $\lfloor\log n\rfloor$ within B_n. Hence, defining the half-space $\mathbb{H}_h = \{x \in \mathbb{Z} : x_d \geq h\}$, $h \in \mathbb{Z}$, we get

$$\liminf_{n\to\infty} P_{p,q}\Big(\big|\{v \in \partial B_n^- : v \text{ is } p\text{-connected to distance } \lfloor\log n\rfloor \text{ in } \mathbb{H}_{-\lfloor\log n\rfloor}\}\big| \geq n^s t/6\Big) > 0. \tag{3.16}$$

On the other hand, the result of Barsky et al. (1991) ensures that there is no infinite cluster in the half-space $\mathbb{Z}^{d-1} \times \mathbb{Z}_+$ when $p = q \leq p_c(d)$.

Then given $\varepsilon > 0$, there exists $r > 0$ such that $P_{p,p}(o \text{ is connected to distance } r \text{ in } \mathbb{H}_0) < \varepsilon$. By the Γ-ergodicity of $P_{p,q}$ and the ergodic theorem, we conclude that $P_{p,p}(|\{v \in B_n \cap H : v \text{ is connected to distance } r \text{ in } \mathbb{H}_0\}| \geq \varepsilon n^s) \xrightarrow[n\to\infty]{} 0$. In particular, we have

$$\lim_{n\to\infty} P_{p,p}\Big(\big|\{v \in B_n \cap H : v \text{ is connected to distance } \lfloor\log n\rfloor \text{ in } \mathbb{H}_0\}\big| \geq \varepsilon n^s\Big) = 0. \tag{3.17}$$

Take $\varepsilon = t/6$. Then, for every fixed n, the probabilities considered in Equations (3.16) and (3.17) are equal. Therefore, these two equations yield a contradiction and the proof is finished. □

Remark 3.11. Note that the above reasoning does not apply to the case $s < d-1$. As a matter of fact, the cardinality of any facet of $B_n := \{-n,\dots,n\}^s \times \{-\lfloor\log n\rfloor,\dots,\lfloor\log n\rfloor\}^{d-s}$ that does not intersect H consists of roughly $n^s(\log n)^{d-s-1}$ vertices. Therefore, (3.15) does not imply anymore that, for such a facet F, the proportion of vertices $v \in F$ such that $v \overset{B_n}{\longleftrightarrow} H$ stays away from zero with probability larger than some constant: it only gives that this proportion is larger than $c/(\log n)^{d-s-1}$ with controlled probability.

Remark 3.12. The argument as it is written uses the fact that p and q do not belong to $\{0,1\}$, because of finite energy. However, the argument readily adjusts to deal with degenerate values. The less trivial adjustment concerns degenerate values for q but not for p, and it is handled by working with $H + (0,\dots,0,1)$ instead of H: we then have at least four p-edges touching each vertex of the considered hyperplane, which suffices to craft branching points. Considering $q \in \{0,1\}$ makes what happens inside H trivial, but this does not make the result we obtain uninteresting: it is a statement about critical homogeneous Bernoulli percolation. Namely, in a critical homogeneous Bernoulli percolation, fully opening (resp. closing) a whole hyperplane cannot yield infinitely many infinite clusters.

3.5. *Uniqueness on graphs of uniformly bounded degree.* In this section, we make use of Lemma 3.4 and the mass-transport principle to discuss the uniqueness of inhomogeneous Bernoulli percolation in the case where $G = (V, E)$ is a connected graph and $\sup_{v \in V} \deg(v) = d < \infty$. We also suppose that there exist a subgroup $\Gamma \subset \mathrm{Aut}(G)$ and a connected set $S \subset V$ such that $\Gamma|_S$ acts transitively on (S, E_S).

For $p, q \in [0,1]$, let $P_{p,q}$ be the Bernoulli bond percolation measure on G, given by

$$P_{p,q}(e \text{ is open}) = \begin{cases} q, & \text{if } e \subset S, \\ p, & \text{otherwise.} \end{cases}$$

Given a vertex $x \in \mathbb{Z}^d$, a subset $S \ni x$ of $\mathbb{Z}^d$, and $p, q \in [0,1]$, define

$$\phi_{p,q}(x, S) := q \sum_{\{y,z\} \in \Delta S \cap \mathsf{E}_H} P_{p,q}(x \overset{S}{\leftrightarrow} y) \;+\; p \sum_{\{y,z\} \in \Delta S \cap \mathsf{E}_H^c} P_{p,q}(x \overset{S}{\leftrightarrow} y), \tag{3.18}$$

where $\Delta S := \{\{y,z\} \in \mathbb{E}^d : y \in S, z \notin S\}$ and $x \overset{S}{\leftrightarrow} y$ denotes the event that x is connected to y by an open path $\{x = x_1, x_2, \dots, x_k = y\} \subset S$.

Let $p < d^{-1}$. Then, for every $v \in V$, we have $\phi_{p,p}(v, \{v\}) \le dp < 1$. By a similar reasoning to the one used in the proof of the exponential decay presented in Duminil-Copin and Tassion (2016) (Theorem 1, Item 1), we can conclude that

$$P_{p,p}(v \leftrightarrow \partial B_n(v)) \le (dp)^n, \quad n \in \mathbb{N}, \tag{3.19}$$

where $B_n(v) = \{x \in V : \mathrm{dist}(x, v) \le n\}$.

Thus, in the setting described above, conditions **i-iii** of Lemma 3.4 are clearly satisfied for any $q \in [0,1]$ and $p < d^{-1}$.

Recall the notation for boundaries of sets defined in (3.2). Through the rest of this section, to avoid a cumbersome notation, given $A \subset V$, we shall write, $\Delta A = \Delta_v A$ and $\partial_S A = \partial_{(S, \mathsf{E}_S)} A$. We will also use $\mathrm{dist}_S(x, y)$ to denote the graph-theoretical distance between x and y in (S, E_S).

Proposition 3.13. *If (S, E_S) is amenable, then, for any $q \in [0,1]$ and $p < 1/d^2$, there exists a sequence $\{R_n\}_{n \in \mathbb{N}}$ of subsets of V, such that*

$$\frac{E_{p,q}|\mathcal{C}(\Delta R_n; S)|}{|R_n \cap S|} \xrightarrow[n \to \infty]{} 0.$$

Therefore, it follows that $N_\infty \in \{0,1\}$ almost surely.

Remark 3.14. Although we have exponential decay of connectivities for $p < d^{-1}$, it will be clear ahead that we must use the more restrictive condition $p < d^{-2}$, in order to compensate the growth of the vertices of a ball $B_n(v)$, as n increases.

Example 3.15. Let $T = (V_T, E_T)$ be an infinite tree whose vertices have uniformly bounded degree and $\mathbb{T} = (V_{\mathbb{T}}, E_{\mathbb{T}})$ be the cartesian product between T and $\mathbb{Z}$, i.e., the graph with vertex set $V_{\mathbb{T}} = V_T \times \mathbb{Z}$, and edge set

$$E_{\mathbb{T}} = \{\{(u,n), (u, n+1)\} : u \in V_T, n \in \mathbb{Z}\} \cup \{\{(u,n), (v,n)\} : \{u,v\} \in E_T, n \in \mathbb{Z}\}.$$

Let $\mathcal{P} = \{v_j : j \in \mathbb{Z}\} \subset V_T$ be a doubly-infinite path in T and $S = \mathcal{P} \times \mathbb{Z}$. In this case, (S, E_S) is isomorphic to the square lattice $\mathbb{L}^2$, and therefore is amenable. Given the percolation process $P_{p,q}$, defined above, Proposition 3.13 implies that, for any $q \in [0,1]$ and small values of p, there is a.s. at most one infinite cluster on $\mathbb{T}$, although it is a non-amenable graph.

Proof of Proposition 3.13: First, we construct an appropriate sequence $\{R_n\}_{n \in \mathbb{N}}$ of subsets of V. We proceed as follows:

For $v \in S$ and $n \in \mathbb{N}$, define

$$R_n(v) := \{x \in V : \mathrm{dist}(x, v) = \mathrm{dist}(x, S) \le n\}.$$

Note that, for every $w \in \Delta R_n(v)$, there exists a vertex $y \in S$ such that $\mathrm{dist}(w, y) = \mathrm{dist}(w, S) \leq \mathrm{dist}(w, v)$. Then, if $L_n(v, w)$ is the largest distance, in the metric of S, between v and the vertices $y \in S$ with the above property, one can define

$$L_n(v) \coloneqq \max\{L_n(v, w) : v \in S, w \in \Delta R_n(v)\}.$$

Since $\Gamma \subset \mathrm{Aut}(G)$ acts transitively on (S, E_S), we have $L_n(v) = L_n \in \mathbb{N}$ for every $v \in S$.

By the amenability of (S, E_S), there exists a sequence $\{S_n\}_{n\in\mathbb{N}}$ of finite subsets of S, such that $|\partial_S S_n|/|S_n| \to 0$ as $n \to \infty$. In particular, given $n \in \mathbb{N}$, let $M_n \in \mathbb{N}$ be such that

$$|\partial_S S_{M_n+j}|/|S_{M_n+j}| < d^{-L_n-2n-2} \tag{3.20}$$

for every $j \geq 1$. Finally, let

$$\begin{aligned} U_{n,j} &\coloneqq \{x \in S : \mathrm{dist}_S(x, S_{M_n}) = j\}, \quad j \in \mathbb{N}, \\ S_n^* &\coloneqq S_{M_n} \cup \Big[\bigcup_{1\leq j\leq L_n} U_{n,j}\Big], \\ R_n &\coloneqq \bigcup_{v\in S_n^*} R_n(v). \end{aligned}$$

Having defined the sets R_n, $n \in \mathbb{N}$, we claim that for every $v \in S_{M_n}$, if $x \in R_n(v)$ and $\mathrm{dist}(x, S) < n$, then $\Delta\{x\} \subset R_n$. To see this, suppose $v \in S_{M_n}$ and consider a vertex $x \in R_n(v)$, such that $\mathrm{dist}(x, S) = m < n$. Then, if $w \in \Delta\{x\}$, one of the following alternatives hold:

- $w \in R_{m+1}(v) \subset R_n$;
- $w \in \Delta R_m(v) \cap \Delta R_n(v)$. In this case, there is a vertex $y \in S$, $y \neq v$, and a path from w to y, such that

$$\mathrm{dist}(w, y) = \mathrm{dist}(w, S) < \mathrm{dist}(w, v) \leq \mathrm{dist}(w, x) + \mathrm{dist}(x, v) = 1 + m \leq n,$$

  therefore $w \in R_n(y)$. If $y \in S_{M_n} \subset S_n^*$, then we trivially have $w \in R_n$. If $y \in S \setminus S_{M_n}$, note that $\mathrm{dist}_S(y, S_{M_n}) \leq \mathrm{dist}_S(y, v) \leq L_n$, which implies that $y \in U_{n,j}$ for some $j = 1, \ldots, L_n$, and consequently $y \in S_n^*$ and $w \in R_n$.

Then, the claim is true and we can conclude that, for every $v \in S_{M_n}$, if $x \in \Delta R_n \cap \Delta R_n(v)$, then $\mathrm{dist}(x, S) \geq n$.

Now, let $q \in [0, 1]$ and $p < 1/d^2$. To find a suitable upper bound for $E_{p,q}\mathcal{C}(\Delta R_n; S)$, we rely on two estimates for $\sum_{x\in\Delta R_n(v)} P_{p,q}(x \leftrightarrow S)$; both use the fact, that since $\sup_{x\in V} \deg(x) = d < \infty$, we have $|\{x \in V : \mathrm{dist}(x, v) \leq m\}| \leq d^{m+1}$ for every $v \in V$.

First, for every $n \in \mathbb{N}$ and $v \in S$,

$$\sum_{x\in\Delta R_n(v)} P_{p,q}(x \leftrightarrow S) \leq d^{n+1}. \tag{3.21}$$

Second, for any $n \in \mathbb{N}$ and $v \in S$, the exponential decay (3.19) implies

$$\sum_{\substack{x\in\Delta R_n(v)\\ \mathrm{dist}(x,S)\geq n}} P_{p,q}(x \leftrightarrow S) \leq \sum_{\substack{x\in\Delta R_n(v)\\ \mathrm{dist}(x,S)\in\{n,n+1\}}} (dp)^n \leq d^{n+2}(dp)^n = d^2(d^2p)^n. \tag{3.22}$$

Thus, using the facts that

$$\Delta R_n \cap \Delta R_n(v) \subset \Delta R_n(v), \tag{3.23}$$

$$|U_{n,j}| \leq |\partial S_{M_n}| d^j, \tag{3.24}$$

along with estimates (3.21) and (3.22), we arrive at

$$\begin{aligned}
E_{p,q}|\mathcal{C}(\Delta R_n;S)| &= \sum_{x\in\Delta R_n} P_{p,q}(x\leftrightarrow S)\\
&\le \sum_{\substack{v\in S_n^*\\ x\in\Delta R_n\cap\Delta R_n(v)}} P_{p,q}(x\leftrightarrow S)\\
&\le \sum_{\substack{v\in S_{M_n}\\ x\in\Delta R_n(v)}} P_{p,q}(x\leftrightarrow S) + \sum_{\substack{v\in S_n^*\setminus S_{M_n}\\ x\in\Delta R_n(v)}} P_{p,q}(x\leftrightarrow S)\\
&\le \sum_{v\in S_{M_n}} \sum_{\substack{x\in\Delta R_n(v)\\ \mathrm{dist}(x,S)\ge n}} P_{p,q}(x\leftrightarrow S) + |S_n^*\setminus S_{M_n}|d^{n+1}\\
&\le |S_{M_n}|d^2(d^2p)^n + \sum_{j=1}^{L_n}|U_{n,j}|d^{n+1}\\
&\le |S_{M_n}|d^2(d^2p)^n + \sum_{j=1}^{L_n}|\partial_S S_{M_n}|d^j d^{n+1}\\
&\le |S_{M_n}|d^2(d^2p)^n + |\partial_S S_{M_n}|d^{L_n+n+2}\\
&\le |S_{M_n}|d^2(d^2p)^n + |S_{M_n}|d^{-n},
\end{aligned}$$

where the last inequality is a consequence of the choice of M_n (3.20). Since $p<d^{-2}$, dividing both sides by $|R_n\cap S| = |S_n^*| \ge |S_{M_n}|$ and letting $n\to\infty$ yields the desired result. □

From Proposition 3.13, we conclude that uniqueness holds for every pair of parameters in $\{(p,q) : p<d^{-2}, q>q_c(p)\}$. We now claim that this set can be extended with the aid of the mass-transport principle, described in Section 3.3. As a matter of fact, one can note that Theorem 3.8 can be reformulated in a more general setting, where

- $\mathbb{L}^d$ is replaced by an infinite and connected graph, $G=(V,E)$, of uniformly bounded degree;
- the sublattice H and the translations parallel to it are replaced by a subgraph (S,E_S) and a subgroup $\Gamma\subset \mathrm{Aut}(G)$, such that $\Gamma|_S$ acts transitively on (S,E_S).

Also, one should observe that, in the proof of Proposition 3.7, the key fact that allowed us to show that every infinite cluster of $\omega_{p,q,p}$ contains an infinite cluster of $\omega_{p,0,0}$ is that, whenever an infinite cluster exists, it intersects H at an infinite number of vertices, in both percolation configurations. Thus, letting $p_c(G)$ be the threshold for homogeneous Bernoulli percolation on G, we can establish the following result:

Proposition 3.16. *Let (S,E_S) be amenable, $0<p_c(S,\mathsf{E}_S)<1$, and define $q^* := \lim_{p\uparrow d^{-2}} q_c(p)$. If there exist $p\in(0,1)$ and $q>q^*$ such that*

$$P_{p,q}(v\leftrightarrow\infty)>0,$$
$$P_{p,q}(|\mathcal{C}(v)|=\infty, |\mathcal{C}(v)\cap S|<\infty)=0 \quad \forall v\in V,$$

then $P_{p,q}(N_\infty=1)=1$.

Proof: Since $\sup_{x\in V}\deg(x)=d<\infty$ and q_c is non-increasing, if $q>q^*$, there exists $p'<d^{-2}$ such that

$$P_{p',q}(v\leftrightarrow\infty)>0, \tag{3.25}$$
$$P_{p',p'}(v\leftrightarrow\infty)=0. \tag{3.26}$$

By the amenability of (S, E_S), Proposition 3.13 and (3.25) imply that $P_{p',q}(N_\infty = 1) = 1$. Additionally, the finite energy property along with (3.26) imply that

$$P_{p',q}(|\mathcal{C}(v)| = \infty, |\mathcal{C}(v) \cap S| < \infty) = 0 \quad \forall v \in V.$$

By hypothesis, we also have $P_{p,q}(|\mathcal{C}(v)| = \infty, |\mathcal{C}(v) \cap S| < \infty) = 0$ for every $v \in V$.

Now, let P be the probability associated with the family $\{U(e) : e \in E\}$ of i.i.d. random variables having uniform distribution in $[0,1]$ and, for $p, q \in [0,1]$, let $\omega_{p,q} \in \{0,1\}^E$ be the bond percolation process on G, given by

$$\omega_{p,q}(e) := \begin{cases} \mathbf{1}_{\{U(e) \le p\}} & \text{if } e \in E \setminus \mathsf{E}_S, \\ \mathbf{1}_{\{U(e) \le q\}} & \text{if } e \in \mathsf{E}_S. \end{cases}$$

By the same reasoning used in the proof of Proposition 3.7, we conclude that every infinite cluster of $\omega_{p,q}$ contains an infinite cluster of $\omega_{p',q}$ almost surely. Since $P_{p',q}(N_\infty = 1) = 1$, it follows that $P_{p,q}(N_\infty = 1) = 1$. □

Corollary 3.17. *Let (S, E_S) and q^* be as in Proposition 3.16. Then $N_\infty = 1$ almost surely for every $p \in (0, p_c(G))$ and $q > q^* \vee q_c(p)$.*

Proof: In this case, we have $P_{p,q}(v \leftrightarrow \infty) > 0$ and $P_{p,q}(|\mathcal{C}(v)| = \infty, |\mathcal{C}(v) \cap S| < \infty) = 0$ for every $v \in V$, therefore Proposition 3.16 holds. □

4. Approximation on slabs

We accomplish the proof of Theorem 1.2 using the ideas developed by Grimmett and Marstrand (1990). Nevertheless, they must be adapted to the inhomogeneous setting, and we do so in the sequel. Every result stated in the next section has an analogous counterpart in Grimmett and Marstrand (1990), and this correspondence will be indicated. Proofs that do not differ from their original counterpart are omitted. We shall also highlight the relevant aspects that are particular to our case. From now on, we denote $\theta(p,q) := P_{p,q}(o \leftrightarrow \infty)$.

4.1. *Technical lemmas.* Recall that $H = \mathbb{Z}^s \times \{0\}^{d-s}$ and that $\Delta_v S$ and $\Delta_e S$ denote the external vertex and edge boundaries of a set $S \subset \mathbb{Z}^d$, respectively, and ∂S denotes internal vertex boundary of S. For $m \in \mathbb{N}$, let $B_m := \{-m, \dots, m\}^d$ and $B_m^H := B_m \cap H$.

Given $\alpha, \beta > 0$ and $n \in \mathbb{N}$, let

$$S_n^{\alpha,\beta} := \{x \in H : \beta n + 1 \le \|x\|_\infty \le \beta n + \alpha n\},$$

and, for $m \in \mathbb{N}$ with $\beta n > m$, consider the random set

$$U_n^{\alpha,\beta} := \left\{ x \in \Delta_v S_n^{\alpha,\beta} : x \xleftrightarrow{B_{\beta n + \alpha n} \setminus S_n^{\alpha,\beta}} B_m^H \right\}. \tag{4.1}$$

Our first task is to show that, in the regime $p < p_c(d) < q < p_c(s)$, if the cluster of B_m^H is infinite for some $m \in \mathbb{N}$, then it is unlikely that $U_n^{\alpha,\beta}$ consists of just a few vertices, as $n \to \infty$. We work with definition (4.1) because, unlike the homogeneous percolation process, since we are considering $\theta(p,q) > 0$ and $p < p_c(d)$, when we search for vertices that are connected to B_m^H and distant from the origin, we are compelled to look for candidates near the sublattice H. The following result is the equivalent of Lemma 3 of Grimmett and Marstrand (1990). Its proof is carried out anew due to the definition of $U_n^{\alpha,\beta}$.

Lemma 4.1. *For any $k, m \in \mathbb{N}$, $\alpha, \beta > 0$ and $p < p_c(d) < q < p_c(s)$, we have*

$$P_{p,q}\big(|U_n^{\alpha,\beta}| \le k, B_m^H \leftrightarrow \infty\big) \underset{n}{\to} 0.$$

Proof: Under the conditions of the lemma we have

$$P_{p,q}\big(|U_n^{\alpha,\beta}| \le k, B_m^H \leftrightarrow \infty\big) \le P_{p,q}\big(|U_n^{\alpha,\beta}| = 0, B_m^H \leftrightarrow \infty\big) + P_{p,q}\big(1 \le |U_n^{\alpha,\beta}| \le k\big).$$

Hence, the result is proved if we show that the two probabilities on the right-hand side of the inequality above go to zero as $n \to \infty$. To see this, note that since $p < p_c(d)$, the exponential decay of the radius of the open cluster (Aizenman and Barsky (1987); Duminil-Copin and Tassion (2016); Menshikov (1986)) implies that there is a constant $c_p > 0$ such that

$$P_{p,q}\big(|U_n^{\alpha,\beta}| = 0, B_m^H \leftrightarrow \infty\big) \le P_{p,q}\Big(B_{\beta n} \xleftrightarrow{B_{\beta n+\alpha n}\setminus S_n^{\alpha,\beta}} \partial B_{\beta n+\alpha n}\Big) \le |\partial B_{\beta n}| e^{-c_p \alpha n}. \tag{4.2}$$

Also, since the random variable $|U_n^{\alpha,\beta}|$ does not depend on the states of the edges in $\Delta_e S_n^{\alpha,\beta}$, given $j \in \{1, \dots, k\}$, we have

$$\begin{aligned}
P_{p,q}\big(|U_n^{\alpha,\beta}| = j\big)(1-q)^k &\le P_{p,q}\big(|U_n^{\alpha,\beta}| = j\big)(1-q)^j \\
&\le P_{p,q}\big(|U_n^{\alpha,\beta}| = j, \Delta_e U_n^{\alpha,\beta} \cap \Delta_e S_n^{\alpha,\beta} \text{ closed}\big) \\
&\le P_{p,q}\Big(\big\{B_m^H \leftrightarrow \partial B_{\beta n},\ B_m^H \not\leftrightarrow \partial B_{\beta n+\alpha n}\big\} \cup \Big\{B_m^H \xleftrightarrow{B_{\beta n+\alpha n}\setminus S_n^{\alpha,\beta}} \partial B_{\beta n+\alpha n}\Big\}\Big) \\
&\le P_{p,q}\big(B_m^H \leftrightarrow \partial B_{\beta n}, |\mathcal{C}(B_m^H)| < \infty\big) + |\partial B_{\beta n}| e^{-c_p \alpha n},
\end{aligned}$$

where $\mathcal{C}(B_m^H)$ denotes the open cluster of B_m^H. Consequently, it follows that

$$\begin{aligned}
P_{p,q}\big(1 \le |U_n^{\alpha,\beta}| \le k\big) &= (1-q)^{-k} \sum_{j=1}^{k} (1-q)^k P_{p,q}\big(|U_n^{\alpha,\beta}| = j\big) \\
&\le (1-q)^{-k} k \Big[P_{p,q}\big(B_m^H \leftrightarrow \partial B_{\beta n}, |\mathcal{C}(B_m^H)| < \infty\big) + |\partial B_{\beta n}| e^{-c_p \alpha n}\Big].
\end{aligned} \tag{4.3}$$

Thus, the proof is completed by observing that the right-hand sides of (4.2) and (4.3) go to zero as $n \to \infty$. □

The next result we state is the equivalent of Lemma 4 of Grimmett and Marstrand (1990). It says that if B_m^H percolates, then for sufficiently large n, there is always a portion of $\Delta_v S_n^{\alpha,\beta}$ where we can find as many sites connected to B_m^H as we like with positive probability, which goes to one as $m \to \infty$.

Define the sets

$$\begin{aligned}
F_n^{\alpha,\beta} &:= [\beta n + 1, \beta n + \alpha n] \times [0, \beta n + \alpha n]^{s-1} \times \{0\}^{d-s}, \\
T_n^{\alpha,\beta} &:= \Delta_v F_n^{\alpha,\beta} \cap B_{\beta n+\alpha n}, \\
V_n^{\alpha,\beta} &:= \Big\{x \in T_n^{\alpha,\beta} : x \xleftrightarrow{B_{\beta n+\alpha n}\setminus F_n^{\alpha,\beta}} B_m^H\Big\}.
\end{aligned}$$

Lemma 4.2. *For any $k, m \in \mathbb{N}$, $\alpha, \beta > 0$ and $p < p_c(d) < q < p_c(s)$, we have*

$$\liminf_n P_{p,q}\big(|V_n^{\alpha,\beta}| \ge k\big) \ge 1 - P_{p,q}\big(B_m^H \not\leftrightarrow \infty\big)^{1/s2^s}.$$

The proof of this result consists in an application of the FKG-inequality Fortuin et al. (1971) together with Lemma 4.1. Since it is analogous to its counterpart in Grimmett and Marstrand (1990), we shall omit it.

Now, we go one step further and show that, if the origin percolates for some $p < p_c(d) < q < p_c(s)$, then for sufficiently large n and m, it is very likely to have B_m^H connected to some translate $x + B_m^H$ which is contained in $F_n^{\alpha,\beta}$ and whose edges are all open. That is, we shall establish the equivalent of Lemma 5 of Grimmett and Marstrand (1990). Although the proof of our result is carried out similarly as its counterpart, one of its steps uses a more general argument. This is done to avoid the verification, at a certain point of the proof, that $2m+1$ divides both $\alpha n+1$ and $\alpha n+\beta n+1$, for some $\alpha,\beta>0$ and $m,n\in\mathbb{N}$. For the sake of clarity, we will present the full proof.

For $m\in\mathbb{N}$ and $x\in H$, we say that $x+B_m^H$ is an m***-seed*** if every edge in $x+B_m^H$ is open. Thus, we define, for $\alpha n>2m+1$,

$$K_{m,n}^{\alpha,\beta}:=\Big\{x\in T_n^{\alpha,\beta}:\exists y\in F_n^{\alpha,\beta},\{x,y\}\in\mathbb{E}^d,\omega(\{x,y\})=1,y\text{ is in an }m\text{-seed in }F_n^{\alpha,\beta}\Big\}.$$

The strategy here is the following: provided that we can find any large number of vertices in $|V_n^{\alpha,\beta}|$ with probability as high as we need, we additionally require that some fixed number of these vertices are connected to a seed in $F_n^{\alpha,\beta}$. Using the structure of $\mathbb{Z}^d$ we can ensure that these candidates are far away from each other in such a way that all the possible seeds are mutually disjoint. Hence, if we have many such candidates, we can conclude that B_m^H is connected to $K_{m,n}^{\alpha,\beta}$ with high probability.

The following assertion describes the structural property of $\mathbb{Z}^d$ we will make use of:

Claim 4.3. *For every $M,k\in\mathbb{N}$, $M\geq 2$, there exists $T(M,k)\in\mathbb{N}$ such that if $A\subset\mathbb{Z}^d$ and $|A|>T(M,k)$, then there is a subset $\{x_1,\dots,x_M\}\subset A$ satisfying $\|x_i-x_j\|_\infty>k$ for every $i\neq j$, where $1\leq i,j\leq M$.*

Lemma 4.4. *If $\theta(p,q)>0$ and $p<p_c(d)<q<p_c(s)$, then for every $\alpha,\beta,\eta\in(0,\infty)$, there exist $m,n\in\mathbb{N}$ such that*

$$P_{p,q}\Big(B_m^H\xleftrightarrow{B_{\beta n'+\alpha n'}}K_{m,n'}^{\alpha,\beta}\Big)>1-\eta\quad\textit{for all } n'\geq n.$$

Proof: If $\theta(p,q)>0$, then there exists $m\in\mathbb{N}$ such that

$$P_{p,q}(B_m^H\leftrightarrow\infty)>1-\Big(\frac{\eta}{2}\Big)^{s2^s}. \tag{4.4}$$

Let $M\in\mathbb{N}$ be such that

$$pP_{p,q}(B_m^H\text{ is an }m\text{-seed})>1-\Big(\frac{\eta}{2}\Big)^{1/M} \tag{4.5}$$

and fix $l=T(M,2(2m+1)+2)$ as in Claim 4.3. By Lemma 4.2 and (4.4), it follows that there exists an $n\in\mathbb{N}$ such that

$$P_{p,q}\big(|V_{n'}^{\alpha,\beta}|\geq l\big)>1-\frac{\eta}{2}\quad\text{for all } n'\geq n. \tag{4.6}$$

Now, let $n'\geq n$ and note that Claim 4.3 ensures that, for every configuration in the event $\big\{\big|V_{n'}^{\alpha,\beta}\big|\geq l\big\}$, there is a subset $\{x_1,\dots,x_M\}\subset V_{n'}^{\alpha,\beta}$ satisfying $\|x_i-x_j\|_\infty>2(2m+1)+2$ for every $i\neq j$, where $1\leq i,j\leq M$. Hence, if y_i is the unique neighbor of x_i that belongs to $F_{n'}^{\alpha,\beta}$ and $B_{m,i}^H\subset H$ is a box of side length $2m$ containing y_i, then $B_{m,i}^H\cap B_{m,j}^H=\varnothing$ for every $i\neq j$, $1\leq i,j\leq M$. Since the event $\big\{|V_{n'}^{\alpha,\beta}|\geq l\big\}$ does not depend on the states of the edges in $S_{n'}^{\alpha,\beta}$ and of $\Delta_e S_{n'}^{\alpha,\beta}$, inequalities (4.5) and (4.6) imply

$$\begin{aligned}P_{p,q}\Big(B_m^H\xleftrightarrow{B_{\beta n'+\alpha n'}}K_{m,n'}^{\alpha,\beta}\Big)&\geq P_{p,q}\Big(\big\{\big|V_{n'}^{\alpha,\beta}\big|\geq l\big\}\cap\Big[\bigcup_{i=1}^{M}\{x_i\in K_{m,n'}^{\alpha,\beta}\}\Big]\Big)\\&\geq 1-\eta.\end{aligned}$$

□

The previous result illustrates what kind of long-range connections we intend to use in the proof of Theorem 1.2. To properly use them, we consider the following improvement of Lemma 4.4, which is the equivalent of Lemma 6 of Grimmett and Marstrand (1990).

Recall that, for $S\subset\mathbb{Z}^d$, we have $\mathsf{E}_S:=\big\{e\in\mathbb{E}^d:e\subset S\big\}$, and let P be the probability measure associated with the family $\big\{U(e):e\in\mathbb{E}^d\big\}$ of i.i.d. random variables having uniform distribution in $[0,1]$. In this context, for $p \in [0,1]$, we say that $e \in \mathbb{E}^d$ is p-***open*** if $U(e) \leq p$ and p-***closed*** otherwise. We also say that a subset $F \subset \mathbb{E}^d$ is (p,q)-***open*** if every edge of $F \cap (\mathbb{E}^d \setminus \mathsf{E}_H)$ is p-open and every edge of $F \cap \mathsf{E}_H$ is q-open.

Lemma 4.5 (Finite-size criterion). *Assume that $\theta(p,q) > 0$ for some $p < p_c(d) < q < p_c(s)$. Then, for every $\epsilon, \delta > 0$ and $\alpha, \beta > 0$, there exist $m, n \in \mathbb{N}$ with the following property:*

Suppose $n' \in \mathbb{N}$ and $R \subset \mathbb{Z}^d$ satisfy $B_m^H \subset R \subset B_{\beta n' + \alpha n'}$ and $(R \cup \Delta_v R) \cap T_{n'}^{\alpha,\beta} = \varnothing$. Also, let $\gamma : \Delta_e R \cap \mathsf{E}_{B_{\beta n' + \alpha n'}} \to [0, 1-\delta]$ be any function and define the events

$$E_{n'} := \left\{ \begin{array}{c} \textit{there is a path joining } R \textit{ to } K_{m,n'}^{\alpha,\beta} \textit{ which is } (p,q)\textit{-open} \\ \textit{outside } \Delta_e R \textit{ and } (\gamma(f)+\delta)\textit{-open in its only edge } f \in \Delta_e R \end{array} \right\},$$

$$F_{n'} := \left\{ f \textit{ is } \gamma(f)\textit{-closed for every } f \in \Delta_e R \cap \mathsf{E}_{B_{\beta n' + \alpha n'}} \right\}.$$

Then $P(E_{n'} | F_{n'}) > 1 - \epsilon$ for every $n' \geq n$.

The proof is analogous to its counterpart, therefore we refer the reader to Lemma 6 of Grimmett and Marstrand (1990).

The idea for proving Theorem 1.2 is to recursively grow the cluster of the origin of $\mathbb{Z}^d$ to more distant regions, jumping from a recently obtained seed to a farther one, and keep this process going indefinitely with positive probability. Similarly to Grimmett and Marstrand (1990), due to the geometrical nature of our connections, it is not possible to perform such exploration independently. As a matter of fact, any attempt to reach a new open seed from a recently obtained one always involves an already explored region of $\mathbb{Z}^d$ that contains closed edges in its external boundary, creating a problem to the direct application of Lemma 4.4. Lemma 4.5 solves this issue by stating that if we give these explored closed edges a small extra chance to be open, then the desired long-range connections can be attained with high probability P.

Remark 4.6. It is important to emphasize the condition "for every $n' \geq n$" in the statement of Lemma 4.5. Further on, we will need to choose a finite number of pairs $(\alpha_1, \beta_1), \ldots, (\alpha_l, \beta_l)$, and check that there exists $n_0 \in \mathbb{N}$ such that $P(E_{n_0}^{\alpha_i,\beta_i} | F_{n_0}^{\alpha_i,\beta_i})$ is sufficiently large, for every $i = 1, \ldots, l$. Since for each pair (α_i, β_i), there exists $n(\alpha_i, \beta_i) \in \mathbb{N}$ such that $P(E_{n'}^{\alpha_i,\beta_i} | F_{n'}^{\alpha_i,\beta_i})$ is sufficiently large for every $n' \geq n(\alpha_i, \beta_i)$, the desired result is achieved if we consider $n_0 = \max_{1 \leq i \leq l} n(\alpha_i, \beta_i)$. The necessity of working with boxes of multiple sizes is particular to our setting. This technicality differs from Grimmett and Marstrand (1990), where the authors needed to use just one size of box in their renormalization process.

The last technical result we need is Lemma 1 of Grimmett and Marstrand (1990), stated in the following.

Let $G = (V, E)$ be an infinite and connected graph. Suppose we have a collection of random variables $\{Z(x) \in \{0,1\} : x \in V\}$ defined in some probability space $(\Omega, \mathcal{F}, \mu)$, let $f_1, f_2, \ldots$ be an ordering of the edges in E and fix $x_1 \in V$. Consider the following random sequence $\mathcal{S} = \{S_t = (A_t, B_t)\}_{t \in \mathbb{N}}$ of ordered pairs of subsets of V: let

$$S_1 = \begin{cases} (\{x_1\}, \varnothing), & \text{if } Z(x_1) = 1, \\ (\varnothing, \{x_1\}), & \text{if } Z(x_1) = 0. \end{cases}$$

Having obtained $S_1, \ldots, S_t$ for $t \geq 1$, we define S_{t+1} in the following manner: denote $f_i = \{u_i, v_i\}$ and let $j_{t+1} = \inf\{i : u_i \in A_t, v_i \in V \setminus (A_t \cup B_t)\}$, with the convention that $\inf \varnothing = \infty$. If $j_{t+1} < \infty$, let $x_{t+1} = v_{j_{t+1}}$ and declare

$$S_{t+1} = \begin{cases} (A_t \cup \{x_{t+1}\}, B_t), & \text{if } Z(x_{t+1}) = 1 \\ (A_t, \{x_{t+1}\} \cup B_t), & \text{if } Z(x_{t+1}) = 0. \end{cases}$$

Otherwise, declare $S_{t+1} = S_t$. We call $\mathcal{S}$ the ***cluster-growth process*** of the vertex x_1 with respect to $(Z(x))_{x\in V}$. Note that the, in the context of site percolation, the open cluster $\mathcal{C}(x_1)$ of x_1 with respect to $(Z(x))_{x\in V}$ is the set $A_\infty = \bigcup_{t\geq 1} A_t$ and its external vertex boundary is the set $B_\infty = \bigcup_{t\geq 1} B_t$.

Now, let $p_c^{\text{site}}(G) \in (0,1)$ be the Bernoulli site percolation threshold for G and define

$$\rho(\mathcal{S},t) := \begin{cases} \mu(Z(x_{t+1}) = 1|S_1,\dots,S_t), & \text{if } j_{t+1} < \infty, \\ 1, & \text{otherwise.} \end{cases}$$

The next result states that the cluster of x_1 with respect to $(Z(x))_{x\in V}$ is infinite with positive probability μ provided that, when performing the cluster-growth process of x_1, the conditional probability of augmenting the set A_t at any step $t \in \mathbb{N}$ exceeds the parameter of a supercritical Bernoulli site percolation process on G.

Lemma 4.7 (Renormalization condition)**.** *If there exists* $\lambda \in (p_c^{site}(G), 1)$ *such that*

$$\rho(\mathcal{S},t) \geq \lambda \quad \textit{for all } t \in \mathbb{N}, \tag{4.7}$$

then $\mu(|A_\infty| = \infty) > 0$.

We refer the reader to Lemma 1 of Grimmett and Marstrand (1990) for a proof. We also stress that an analogous result also holds if we introduce an orientation to the edges of G. This is particularly important in our case, since the renormalized graph we shall consider in the sequel is an oriented one.

4.2. *The renormalization process.* To prove Theorem 1.2, it suffices to show that, for any $p < p_c(d)$, $\eta > 0$ and $q = q_c(p) + \eta/2$, there exists $N \in \mathbb{N}$ such that, with positive probability, the origin lies in an infinite $(p+\eta/2, q+\eta/2)$-open cluster within $\mathbb{Z}^2 \times \{-N,\dots,N\}^{d-2}$. As already mentioned, we rely on the classical approach of Grimmett and Marstrand (1990) to show that the restriction of the inhomogeneous process with parameters $p+\eta/2$ and $q+\eta/2$ to the slab $\mathbb{Z}^2 \times \{-N,\dots,N\}^{d-2}$ stochastically dominates a supercritical percolation process on the graph $G = (V,E)$, with vertex set $V = \{x \in \mathbb{Z}^+ \times \mathbb{Z} : x_1 + x_2 \text{ is even}\}$ and edge set $E = \{\{x, x+(1,\pm 1)\} : x \in V\}$. The orientation of the edges is to be taken from x to $x+(1,\pm 1)$, for every $x \in V$. The stochastic domination occurs in the sense that if the cluster of the origin of the latter is infinite, then the cluster of the origin of the former is infinite as well.

The above idea is carried out with the aid of the following renormalization scheme: we construct a (dependent) oriented site percolation process on G, defined in terms of some special events lying on the space $([0,1]^{\mathbb{E}^d}, P)$, where P denotes the probability measure associated with the family $\{U(e) : e \in \mathbb{E}^d\}$ of i.i.d. random variables having uniform distribution in $[0,1]$. We do this by specifying a collection of random variables $\{Z(x) \in \{0,1\} : x \in V\}$, which encode information about the existence of large $(p+\eta/2, q+\eta/2)$-open paths in $\mathbb{Z}^2 \times \{-N,\dots,N\}^{d-2}$. In particular, when considering the cluster-growth process of the origin with respect to $(Z(x))_{x\in V}$, we will require that

- **i.** property (4.7) holds for some $\lambda \in (\vec{p}_c^{\text{site}}(G), 1)$, so that $|A_\infty|$ is infinite with positive probability by Lemma 4.7;
- **ii.** if $|A_\infty| = \infty$, then the origin percolates in $\mathbb{Z}^2 \times \{-N,\dots,N\}^{d-2}$ by a $(p+\eta/2, q+\eta/2)$-open path.

It is clear that these two conditions combined immediately imply the desired conclusion. Thus, we proceed to the construction of the process $(Z(x))_{x\in V}$.

Having fixed $p < p_c(d)$, let $\eta > 0$ be small and define

$$q = q_c(p) + \eta/2 \qquad \delta = \frac{1}{16}\eta, \qquad \epsilon = \frac{1}{150}\big(1 - \vec{p}_c^{\text{site}}(G)\big). \tag{4.8}$$

Also, consider $\alpha_1 = \alpha_2 = \alpha_3 = \alpha = 1/100$ and $\beta_1 = \beta_2/2 = \beta_3/(2+\alpha+\alpha^2) = 1$. Since $\theta(p,q) > 0$, Lemma 4.5 guarantees the existence of $m, n \in \mathbb{N}$ such that $P(E_n|F_n) > 1-\epsilon$ for each given pair (α_i, β_i).

For a vertex $x \in V$ and a subset $A \subset V$, let $x + A \coloneqq \{x + a : a \in A\}$. Also, let $\vec{u}_1, \dots, \vec{u}_d$ be the canonical basis of $\mathbb{R}^d$ and, for $N = 6n$, let $\Lambda(N) = B_N \cup (2N\vec{u}_2 + B_N)$. The fundamental blocks of the renormalized lattice are the ***site-blocks***

$$\Lambda_x = \Lambda_x(N) \coloneqq 4Nx + \Lambda(N), \quad x \in V,$$

which can be written as the union of a "lower" and an "upper" translate of B_N, namely

$$\Lambda_x^l = \Lambda_x^l(N) \coloneqq 4Nx + B_N,$$
$$\Lambda_x^u = \Lambda_x^u(N) \coloneqq 2N\vec{u}_2 + \Lambda_x^l(N).$$

The adjacency relation between site-blocks is the one inherited from $G = (V, E)$. That is, for $x, y \in V$, the boxes Λ_x and Λ_y are adjacent if and only if $\{x, y\} \in E$. The long-range connections in $\mathbb{Z}^2 \times \{-N, \dots, N\}^{d-2}$ we are going to build will occur between adjacent site-blocks, using its edges and the edges within the ***passage-blocks***

$$\Pi_x = \Pi_x(N) \coloneqq [\Lambda_x + 2N(\vec{u}_1 + \vec{u}_2)] \cup [\Lambda_x + 2N(\vec{u}_1 - \vec{u}_2)], \quad x \in V.$$

Having set up the renormalization structure, we are now in a position to define the random variables $Z(x)$, $x \in V$. We will specify them recursively, considering the first coordinate of each $x = (x_1, x_2) \in V$. The idea is to make $Z(x)$ encode information about connections between seeds inside the site-blocks Λ_x, $\Lambda_{x+(1,1)}$ and $\Lambda_{x+(1,-1)}$. These open paths will be contained in $\Lambda_x \cup \Pi_x \cup \Lambda^l_{x+(1,1)} \cup \Lambda^u_{x+(1,-1)}$ and possess connectivity features such that requirements **i.** and **ii.** are fullfilled for $\lambda = \left[1 + \vec{p}_c^{\,\text{site}}(G)\right]/2$.

We begin by determining the event $\{Z(o) = 1\}$. This will be achieved through the application of a sequential algorithm, which constructs an increasing sequence $E_1, E_2, \dots$ of edge-sets by making repeated use of Lemma 4.5. At each step k of the algorithm, we acquire information about the values of $U(e)$ for certain $e \in \mathbb{E}^d$, and record this information into suitable functions $\gamma_k, \zeta_k : \mathbb{E}^d \to [0,1]$, in such a way that every $e \in \mathbb{E}^d$ is $\gamma_k(e)$-closed and $\zeta_k(e)$-open and

$$\gamma_k(e) \le \gamma_{k+1}(e), \qquad \zeta_k(e) \ge \zeta_{k+1}(e).$$

In this context, we respectively regard γ_k and ζ_k as the acquired "negative" and "positive" information about the states of the edges of $\mathbb{E}^d$ up to step k. At the end of each step, the ζ_k-open cluster of the origin within $\mathbb{Z}^2 \times \{-N, \dots, N\}^{d-2}$ will have grown larger and closer to the site-blocks $\Lambda_{(1,1)}$ and $\Lambda_{(1,-1)}$, as we use Lemma 4.5 to reach new open seeds from the previously open ones in a coordinated manner.

In our process, a single attempt of growing the cluster of the origin in the setting of Lemma 4.5 will be called a ***step*** of the exploration. The determination of $Z(o) = 1$ is constituted by a (finite) sequence of successful steps, specified in the sequel. To make the construction clear, we gather some particular subsequences of steps together, according to the "direction of growth" of the cluster, and call them ***phases*** of the exploration. A picture of a configuration such that $Z(o) = 1$ is illustrated in Figure 4.5. This event occurs if we succeed in each of the following phases:

Phase 1: Let $E_1 = \mathsf{E}_{B_m^H}$. This phase is successful if every edge in E_1 is q-open. In this case, we set

$$\gamma_1(e) = 0, \qquad \text{for all } e \in \mathbb{E}^d,$$

$$\zeta_1(e) = \begin{cases} q, & \text{if } e \in E_1, \\ 1, & \text{otherwise,} \end{cases}$$

so that every edge $e \in \mathbb{E}^d$ is $\gamma_1(e)$-closed and $\zeta_1(e)$-open.

Phase 2: Provided that Phase 1 is successful, we attempt to connect the open seed B_m^H to another q-open m-seed lying in the passage-block Π_o by using Lemma 4.5 in the first series of steps in the same direction.

Let $\mathcal{P}$ be the collection of all paths in $\mathbb{Z}^d$ and denote the ***edge-boundary*** of a subset $E' \subset \mathbb{E}^d$ by $\Delta E' \coloneqq \{f \in \mathbb{E}^d \backslash E' : \exists e \in E' \text{ such that } |f \cap e| = 1\}$. Given $V' \subset \mathbb{Z}^d$, $E' \subset \mathbb{E}^d$ with $(E' \cup \Delta E') \subset \mathsf{E}_{V'}$, and $\gamma : \mathbb{E}^d \to [0,1]$, define

$$\begin{aligned}\mathcal{P}(V', E', \gamma) \coloneqq \big\{\pi = \{x_1, \dots, x_k\} \in \mathcal{P} : \pi \subset V', \{x_1, x_2\} \in \Delta E' \text{ and is } \gamma(\{x_1, x_2\})\text{-open}, \\ \{x_i, x_{i+1}\} \in (E' \cup \Delta E')^c \text{ and is } (p,q)\text{-open } \forall i = 2, \dots, k-1\big\}, \\ \mathcal{V}(V', E', \gamma) \coloneqq \bigcup_{\pi \in \mathcal{P}(V', E', \gamma)} \pi.\end{aligned}$$

Now, set $D_1 = B_{n+\alpha n}$ and let $E_2 = E_1 \cup \widetilde{E}_2$, where $\widetilde{E}_2$ is the set of all edges with both vertices in $\mathcal{V}(D_1, E_1, \gamma_1 + \delta)$. This step is successful if there exists an edge in E_2 having an endvertex in

$$\begin{aligned}K_{m,n}^{\alpha,1} = \big\{x \in T_n^{\alpha,1} : \exists y \in F_n^{\alpha,1} \text{ such that } \{x, y\} \in E \text{ and is } (p,q)\text{-open}, \\ y \text{ is in a } q\text{-open } m\text{-seed in } F_n^{\alpha,1}\big\}.\end{aligned}$$

Conditioned that Phase 1 is successful, Lemma 4.5 implies that this step is successful with probability at least $1 - \epsilon$. In this case, let

$$\gamma_2(e) = \begin{cases} \gamma_1(e), & \text{if } e \notin \mathsf{E}_{D_1}, \\ \gamma_1(e) + \delta, & \text{if } e \in \Delta E_1 \setminus E_2, \\ q, & \text{if } e \in (\Delta E_2 \setminus \Delta E_1) \cap \mathsf{E}_{D_1} \cap \mathsf{E}_H, \\ p, & \text{if } e \in (\Delta E_2 \setminus \Delta E_1) \cap \mathsf{E}_{D_1} \cap \mathsf{E}_H^c, \\ 0, & \text{otherwise,} \end{cases}$$

$$\zeta_2(e) = \begin{cases} \zeta_1(e), & \text{if } e \in E_1, \\ \gamma_1(e) + \delta, & \text{if } e \in \Delta E_1 \cap E_2, \\ q, & \text{if } e \in E_2 \setminus (E_1 \cup \Delta E_1) \cap \mathsf{E}_{D_1} \cap \mathsf{E}_H, \\ p, & \text{if } e \in E_2 \setminus (E_1 \cup \Delta E_1) \cap \mathsf{E}_{D_1} \cap \mathsf{E}_H^c, \\ 1, & \text{otherwise.} \end{cases}$$

Figure 4.1 illustrates a successful realization of the first step.

Having succeeded with the first step, let $b_2 \in \mathbb{Z}^s \times \{0\}^{d-s}$ be the center of the earliest seed in $F_n^{\alpha,1}$ (in some ordering of all centers) connected to B_m^H and let

$$D_2 = b_2 + B_{n+\alpha n}.$$

In this second step, we proceed to link the seed $b_2 + B_m^H$ to a new seed $b_3 + B_m^H$ inside D_2, in such a way that if we denote $b_k = (b_{k,1}, \dots, b_{k,d})$, we have

$$\begin{aligned} b_{3,1} - b_{2,1} \in [n, n + \alpha n], \\ |b_{3,i}| \le n + \alpha n, \ \forall i = 2, \dots, s, \\ b_{3,i} = 0, \ \forall i = s+1, \dots, d. \end{aligned}$$

Observe that the first condition imposes a direction for the cluster of the origin to grow and the second condition constrains it to some adequate boundaries. The third condition is the requirement $b_3 + B_m^H \subset H$. They can be achieved through a steering argument analogous to the one in Grimmett and Marstrand (1990): for a vertex $v = (v_1, \dots, v_d) \in \mathbb{Z}^d$, let $\sigma_v : \mathbb{Z}^d \to \mathbb{Z}^d$ be the application given

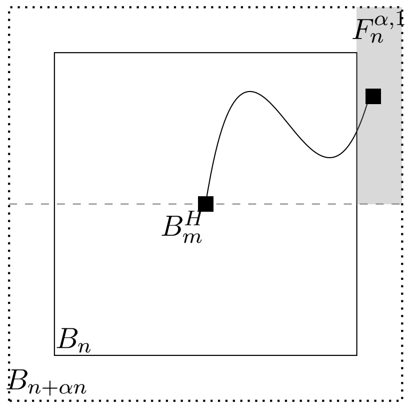

FIGURE 4.1. A successful realization of the first step, projected onto $\mathbb{Z}^2 \times \{0\}^{d-2}$. The black squares represent the q-open m-seeds, connected by a ζ_2-open path indicated by the black curve, whose edges are contained in the dotted box $B_{n+\alpha n}$. The gray region represents the set $F_n^{\alpha,1}$, where the new seed is found.

by

$$[\sigma_v(x)]_i = \begin{cases} -\operatorname{sgn}(v_i)x_i, & \text{if } i = 2, \dots, s, \\ x_i, & \text{if } i = 1 \text{ or } i = s+1, \dots, d. \end{cases} \tag{4.9}$$

We regard σ_v as the ***steering function***, which, in the present Phase, is given by (4.9). Its definition will be modified for the subsequent Phases, whenever necessary.

Let $E_3 = E_2 \cup \widetilde{E}_3$, where $\widetilde{E}_3$ is the set of all edges with both vertices in $\mathcal{V}(D_2, E_2, \gamma_2 + \delta)$. This step is successful if there exists an edge in E_3 having an endvertex in

$$\begin{aligned} b_2 + \sigma_{b_2} K_{m,n}^{\alpha,1} := \big\{ & x \in b_2 + \sigma_{b_2} T_n^{\alpha,1} : \exists y \in b_2 + \sigma_{b_2} F_n^{\alpha,1} \text{ such that } \{x,y\} \in E, \\ & \{x,y\} \text{ is } (p,q)\text{-open and } y \text{ is in a } q\text{-open } m\text{-seed in } b_2 + \sigma_{b_2} F_n^{\alpha,1} \big\}. \end{aligned}$$

Just as before, in case of success, we update the values of the random variables $U(e)$, $e \in \mathbb{E}^d$, recording them into the functions $\gamma_3, \zeta_3 : \mathbb{E}^d \to [0,1]$. Note that, by Lemma 4.5, conditioned that Phase 1 and the previous step are successful, this step is successful with probability greater than $1 - \epsilon$.

The above procedure illustrates how we should proceed with the sequential algorithm in order to find our suitable seed in Π_o: from the ζ_k-open cluster of $b_k + B_m^H$ inside the box $D_k = b_k + B_{n+\alpha n}$, we give a small increase $\delta > 0$ on the parameter of the edges in its external boudary in order to open some of them. In turn, from the endpoints of these newly open edges, we try to find a (p,q)-open path to a new q-open m-seed $b_{k+1} + B_m^H$, satisfying

$$\begin{aligned} b_{k+1,1} - b_{k,1} &\in [n, n+\alpha n], \\ |b_{k+1,i}| &\leq n + \alpha n, \ \forall i = 2, \dots, s \\ b_{k+1,i} &= 0, \ \forall i = s+1, \dots, d. \end{aligned} \tag{4.10}$$

Given that the previous steps are successful, this happens with probability at least $1 - \epsilon$, since in each application of Lemma 4.5, the already explored region R together with its external vertex boundary, $\Delta_v R$, never intersects $b_k + \sigma_{b_k} T_n^{\alpha,1}$. In this case, the updated values of the random variables $U(e)$, $e \in \mathbb{E}^d$, are recorded into functions $\gamma_{k+1}, \zeta_{k+1} : \mathbb{E}^d \to [0,1]$ accordingly.

The exploration process stops when we finally find a q-open m-seed $\left(c_2 + B_m^H\right) \subset \Pi_o$, such that

$$c_{2,1} \in [9n, 10n + \alpha n],$$
$$|c_{2,i}| \leq n + \alpha n,\ \forall i = 2, \dots, s$$
$$c_{2,i} = 0,\ \forall i = s + 1, \dots, d,$$

and we say that Phase 2 is successful if such seed is reached. Since (4.10) implies that $b_{k+1,1} \geq b_{k,1}+n$ and our initial seed is $o + B_m^H$, this is possible after the application of at most nine of the described steps. Therefore, conditioned that Phase 1 is successful, we have

$$P(\text{Phase 2 successful}|\text{Phase 1 successful}) \geq (1 - \epsilon)^9,$$

and every edge $e \in \mathbb{E}^d$ is $\gamma_{10}(e)$-closed and $\zeta_{10}(e)$-open at the end of the procedure. Figure 4.2 represents a successful connection between B_m^H and $c_2 + B_m^H$.

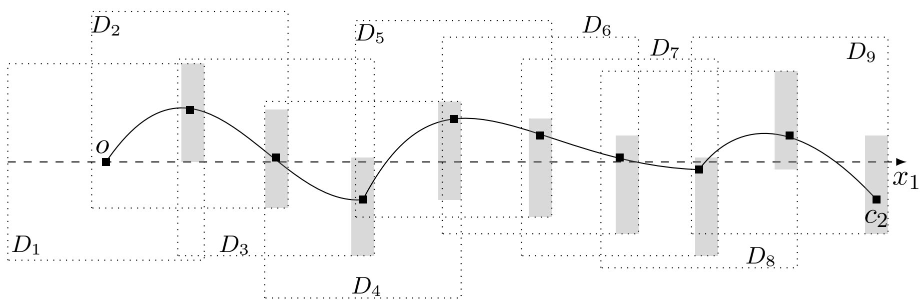

FIGURE 4.2. A successful realization of Phase 1, linking B_m^H to $c_2 + B_m^H$. Each black square represents the open seed obtained at the end of each step. They are linked by paths indicated by the black curves, obtained through successive applications of Lemma 4.5. Each application of the lemma considers a box $D_k = b_k + B_{n+\alpha n}$, $k = 1, \dots, 9$, depicted by the dotted boxes. The gray regions represent the sets $b_k + \sigma_{b_k} F_n^{\alpha,1}$, where seed $b_{k+1} + B_m^H$ is found at the end of the k-th step. The dashed line is the reference by which the steering occurs, relative to the x_1-axis.

Phase 3: So far, the sequential algorithm has been applied following the restrictions imposed by (4.10), which can be interpreted as requiring the cluster of the origin to "grow along the x_1-axis in the positive direction, keeping its coordinates bounded in the other directions". Having reached seed $c_2 + B_m^H \subset \Pi_o$, we continue the exploration process in order to find a path in $\Pi_o \cup \Lambda^l_{(1,1)} \cup \Lambda^u_{(1,-1)}$ to open seeds in the site-blocks $\Lambda^l_{(1,1)}$ and $\Lambda^u_{(1,-1)}$, which means that a change of direction is necessary. As a condition for applying Lemma 4.5, this needs to be done in such a way that we do not analyze previously explored edges in the region where we intend to place the next seeds. Hence, we branch out the cluster of $c_2 + B_m^H$ into an upper and a lower component by inspecting, in two steps, the edges inside boxes of sizes $2n + \alpha n$ and $2n + 2\alpha n$, both centered in c_2.

To put it rigorously, let $\mathcal{L} : \mathbb{R}^d \to \mathbb{R}^d$ be the linear mapping given by

$$\mathcal{L}(x_1, x_2, x_3, \dots, x_d) = (x_2, -x_1, x_3, \dots, x_d),$$

and define the steering function $\sigma_v : \mathbb{Z}^d \to \mathbb{Z}^d$, $v \in \mathbb{Z}^d$, by

$$[\sigma_v(x)]_i = \begin{cases} -\operatorname{sgn}(v_i)x_i, & \text{if } i = 3, \dots, s, \\ x_i, & \text{if } i = 1, 2 \text{ or } i = s + 1, \dots, d. \end{cases}$$

The application $\mathcal{L}$ is a rotation of the x_1x_2-plane by $-\pi/2$ and introduces the change of direction of the exploration process from being parallel to the x_1-axis to being parallel to the x_2-axis. As before, σ_v will act to keep the other $d-2$ coordinates bounded. Let

$$D_{10} = c_2 + B_{2n+\alpha n}$$

and $E_{11} = E_{10} \cup \widetilde{E}_{11}$, where $\widetilde{E}_{11}$ is the set of all edges with both vertices in $\mathcal{V}(D_{10}, E_{10}, \gamma_{10} + \delta)$. This step is successful if there exists an edge in E_{11} having an endvertex in

$$c_2 + \mathcal{L}\sigma_{c_2} K^{\alpha,2}_{m,n} := \big\{x \in c_2 + \mathcal{L}\sigma_{c_2} T^{\alpha,2}_n : \exists y \in c_2 + \mathcal{L}\sigma_{c_2} F^{\alpha,2}_n \text{ such that } \{x,y\} \in E, \\ \{x,y\} \text{ is } (p,q)\text{-open and } y \text{ is in a } q\text{-open } m\text{-seed in } c_2 + \mathcal{L}\sigma_{c_2} F^{\alpha,2}_n\big\}.$$

After succeeding, we record the updated values of the random variables $U(e)$ into the functions $\gamma_{11}, \zeta_{11} : \mathbb{E}^d \to [0,1]$ and repeat the same step using a slightly bigger box than D_{10},

$$D_{11} = c_2 + B_{2n+2\alpha n+\alpha^2 n},$$

this time to find an edge in E_{12} having an endvertex in $c_2 - \mathcal{L}\sigma_{c_2} K^{\alpha,2+\alpha+\alpha^2}_{m,n}$. The size D_{11} is bigger to ensure that the edges incident to $c_2 - \mathcal{L}\sigma_{c_2} T^{\alpha,2+\alpha+\alpha^2}_n$ have not been explored before. If we succeed, we call the "lower" and the "upper" seeds $c^l_3 + B^H_m$ and $c^u_3 + B^H_m$, respectively. Thus,

$$P(\text{Phase 3 successful}|\text{Phases 1 and 2 successful}) \geq (1-\epsilon)^2,$$

and every edge $e \in \mathbb{E}^d$ is $\gamma_{12}(e)$-closed and $\zeta_{12}(e)$-open in this case. Figure 4.3 illustrates a successful connection at Phase 3.

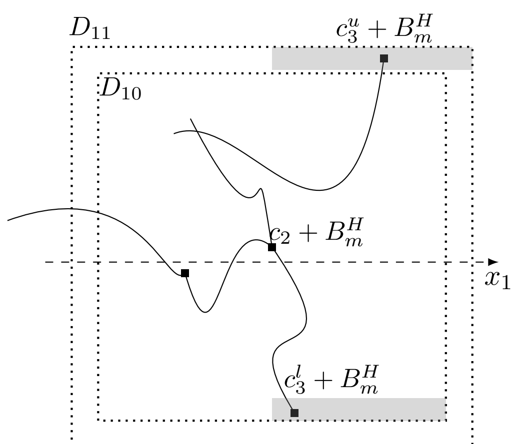

FIGURE 4.3. A successful connection at Phase 3, projected onto $\mathbb{Z}^2 \times \{0\}^{d-2}$. The connections between seeds occur in the same way as described in Figure 4.2.

One can notice that Lemma 4.5 is not applicable if, instead of using the box $c_2 + B_{2n+\alpha n}$, we had considered $D_{10} = c_2 + B_{n+\alpha n}$. In this situation, as shown in Figure 4.4, we have $D_9 \cap \big(c_2 + \mathcal{L}\sigma_{c_2} T^{\alpha,1}_n\big) \neq \varnothing$, which implies that the vertices of this region may have been revealed in the previous step. Therefore, the requiremets for the subset R in the statement of Lemma 4.5 are not satisfied under this setting. This fact also explains why the renormalization scheme of Grimmett and Marstrand (1990) cannot be adapted in a straightforward manner, using only one size of box, as mentioned in Remark 4.6.

Phase 4: From now on, all the subsequent phases will consist in explorations analogous to the ones in Phases 2 and 3, hence we will only give a brief explanation on how the cluster grows and mention the number of steps necessary for the accomplishment of each phase.

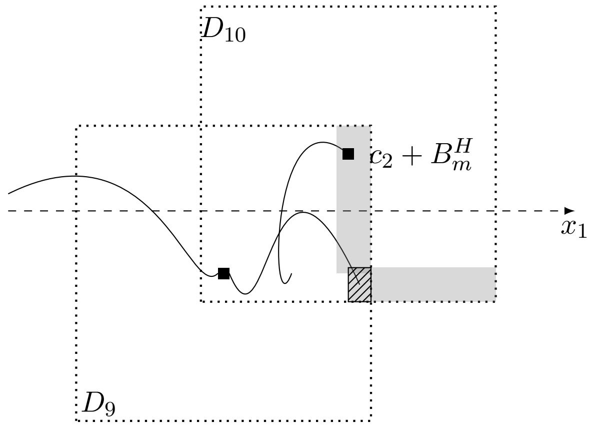

FIGURE 4.4. An illustration of the issue that appears if we consider $D_{10} = c_2 + B_{n+\alpha n}$. Once seed $c_2 + B^H_m$ is reached from the open paths obtained at previous steps (indicated by the black curves), we should make a change of direction, as explained in Phase 3. However, if we attempt to make such change using D_{10} as a translate of $B_{n+\alpha n}$, then a portion of the region where seed $c^l_3 + B^H_m$ (or $c^r_3 + B^H_m$, depending on the position of D_9) is supposed to be found may have already been explored. As the hatched region indicates, this might be the case when we applied Lemma 4.5 using the box D_9.

At Phase 4, we attempt to link $c^l_3 + B^H_m$ to a q-open m-seed $(c_4 + B^H_m) \subset \Pi_o$, such that

$$\begin{aligned} c_{4,1} &\in [9n, 15n], \\ c_{4,2} &\in [-9n, -10n - \alpha n] \\ |c_{4,i}| &\leq 3n, \ \forall i = 3, \dots, s \\ c_{4,i} &= 0, \ \forall i = s+1, \dots, d. \end{aligned}$$

This phase is analogous to Phase 2, with the difference that, in the present case, we grow the cluster along the x_2-axis in the negative direction and use the plane $x_1 = 12n$ as the reference for steering the first coordinate. The steering reference for the other $s-2$ coordinates do not change. Since $c^l_{3,2} \leq n$ and $c_{4,2} \in [-9n, -10n - \alpha n]$, it takes at most 12 applications of Lemma 4.5 to reach a seed as mentioned above. Therefore, Phase 4 is successful with probability at least $(1-\epsilon)^{12}$, conditioned that we succeed at the previous phases.

Phase 5: Here we prepare another change of direction in the explored open cluster, analogous to the step used in Phase 3. We attempt to link $c_4 + B^H_m$ to a q-open m-seed $(c_5 + B^H_m) \subset \sigma_{c_4} F^{\alpha,2}_n$, where $\sigma_v : \mathbb{Z}^d \to \mathbb{Z}^d$, $v \in \mathbb{Z}^d$ is the steering function

$$[\sigma_v(x)]_i = \begin{cases} -x_2, & \text{if } i = 2, \\ -\operatorname{sgn}(v_i) x_i, & \text{if } i = 3, \dots, s, \\ x_i, & \text{if } i = 1 \text{ or } i = s+1, \dots, d. \end{cases}$$

This phase is successful with probability at least $1 - \epsilon$, conditioned that the previous phases are successful as well.

Phase 6: Here we complete the exploration of the lower branch of the cluster of the origin. We attempt to link $c_5 + B_m^H$ to a seed $({}_o + B_m^H) \subset \Lambda^u_{(1,-1)}$, with ${}_o = ({}_{o,1}, \ldots, {}_{o,d}) \in \mathbb{Z}^d$ satisfying

$$
\begin{aligned}
{}_{o,1} &\in [24n, 25n + \alpha n],\\
{}_{o,2} &\in [-9n, -15n]\\
|{}_{o,i}| &\le 3n,\ \forall i = 3, \ldots, s\\
{}_{o,i} &= 0,\ \forall i = s+1, \ldots, d.
\end{aligned}
$$

We perform a process similar to that of Phases 2 and 4, growing the cluster of $c_5 + B_m^H$ along the x_1-axis in the positive direction, using the plane $x_2 = -12n$ as the reference for steering the second coordinate and keeping the steering rule for the remaining coordinates the same as before. As usual, we use a translate of $B_{n+\alpha n}$ in each application of Lemma 4.5. If such seed is reached, we declare Phase 6 successful. Since $c_{5,1} \ge 11n$ and ${}_{o,1} \in [24n, 25n + \alpha n]$, this is achieved within at most 13 applications of Lemma 4.5, hence the probability of success is at least $(1-\epsilon)^{13}$.

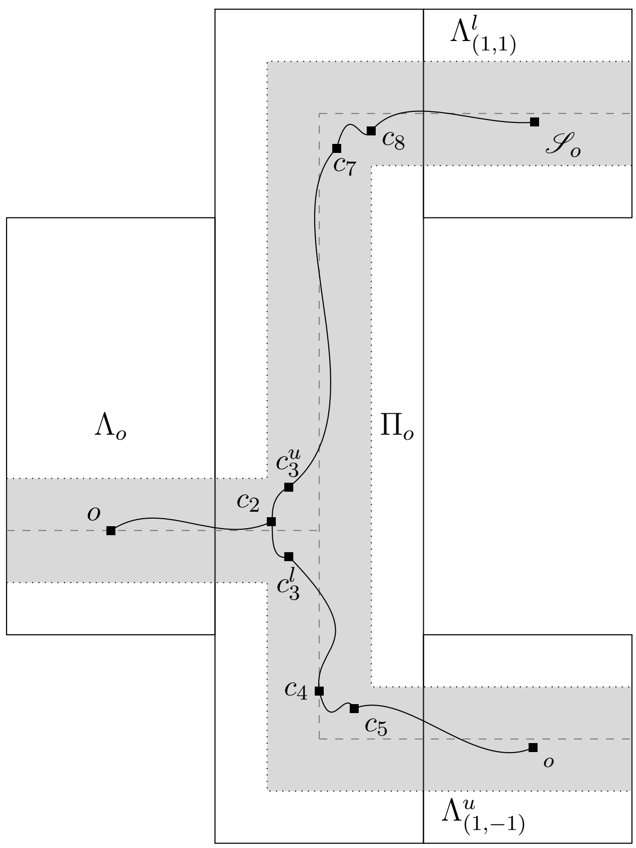

FIGURE 4.5. A configuration in the event $\{Z(o) = 1\}$, projected onto $\mathbb{Z}^2 \times \{0\}^{d-2}$. Each tiny black square represents the open seed obtained at the end of each phase. They are linked by paths represented by the black curves, obtained through successive applications of Lemma 4.5. The dashed lines represent the reference by which the steering occurs, relative to the x_1x_2-plane. As a consequence of adopting this reference and the parameters (α_i, β_i), $i = 1, 2, 3$, every open seed found in the exploration process lies inside the gray region, within a distance of $3n$ from the dashed lines.

Phases 7, 8 and 9: These are essentially reproductions of Phases 4, 5 and 6, respectively. This time, we apply the sequential algorithm to the "upper" branch of the cluster of the origin, attempting to link $(c_3^u + B_m^H) \subset \Pi_o$ to an open seed $(\mathscr{S}_o + B_m^H) \subset \Lambda^l_{(1,1)}$. The only relevant difference occurs at Phase 7, where Lemma 4.5 must be applied at most 24 times, instead of 12 times as in Phase 4. This is so because the box $2N\vec{u}_1 + \Lambda_o^u \subset \Pi_o$ necessarily needs to be entirely crossed during the exploration process along the x_2-axis in the positive direction.

If we succeed at all these phases, we declare $Z(o) = 1$. A configuration of this kind is illustrated in Figure 4.5. During this process, we have used Lemma 4.5 at most 75 times, therefore (4.8) implies that

$$P\big(Z(o) = 1 \big| B_m^H \text{ is a seed}\big) \geq (1-\epsilon)^{75} \geq 1 - 75\epsilon \geq \frac{1}{2}\big(1 + \vec{p}_c^{\,\text{site}}(G)\big). \tag{4.11}$$

We should also have updated the functions γ_k and ζ_k to the same extent. Thus, if $k_{\max} \in \mathbb{N}$ is the maximum number of steps used in the determination of $Z(o)$, it follows that $k_{\max} \leq 75$. Moreover, we claim that

$$\gamma_{k_{\max}}(e) \leq \zeta_{k_{\max}}(e) \leq q\mathbf{1}_{\mathsf{E}_H}(e) + p\mathbf{1}_{\mathsf{E}_H^c}(e) + 8\delta \quad \forall e \in E_{k_{\max}}, \tag{4.12}$$

which implies that every edge of $E_{k_{\max}}$ is $(p+\eta/2, q+\eta/2)$-open, since $8\delta \leq \eta/2$ by (4.8).

As a matter of fact, note that the general rule for updating the edges of $\mathbb{Z}^d$ is

$$\gamma_{k+1}(e) = \begin{cases} \gamma_k(e), & \text{if } e \notin \mathsf{E}_{D_k}, \\ \gamma_k(e) + \delta, & \text{if } e \in \Delta E_k \setminus E_{k+1}, \\ q, & \text{if } e \in (\Delta E_{k+1} \setminus \Delta E_k) \cap \mathsf{E}_{D_k} \cap \mathsf{E}_H, \\ p, & \text{if } e \in (\Delta E_{k+1} \setminus \Delta E_k) \cap \mathsf{E}_{D_k} \cap \mathsf{E}_H^c, \\ 0, & \text{otherwise,} \end{cases}$$

$$\zeta_{k+1}(e) = \begin{cases} \zeta_k(e), & \text{if } e \in E_k, \\ \gamma_k(e) + \delta, & \text{if } e \in \Delta E_k \cap E_{k+1}, \\ q, & \text{if } e \in E_{k+1} \setminus (E_k \cup \Delta E_k) \cap \mathsf{E}_{D_k} \cap \mathsf{E}_H, \\ p, & \text{if } e \in E_{k+1} \setminus (E_k \cup \Delta E_k) \cap \mathsf{E}_{D_k} \cap \mathsf{E}_H^c, \\ 1, & \text{otherwise.} \end{cases}$$

This means that any edge $e \in \mathbb{Z}^d$ such that $\zeta_{k+1}(e) = \gamma_k(e) + \delta$ or $\gamma_{k+1}(e) = \gamma_k(e) + \delta$ belong to ΔE_k. By definition of the exploration process, this inspected edge must be contained in the box D_k. Since a box D_k, $k = 1, \dots, k_{\max}$, intersects at most 8 other boxes (this is the case of boxes D_{10} and D_{11} used at Phase 3), such an edge is inspected at most 8 times. Therefore, $\zeta_{k_{\max}}(e) \leq q\mathbf{1}_{\mathsf{E}_H}(e) + p\mathbf{1}_{\mathsf{E}_H^c}(e) + 8\delta$ for every $e \in E_{k_{\max}}$.

If $Z(o) = 1$, we continue to apply the exploration process described, in order to determine the states of the random variables $Z(1,-1)$ and $Z(1,1)$. For each random variable, the process goes the same way as for $Z(o)$: we start with $({}_o + B_m^H) \subset \Lambda_{(1,-1)}$ and $(\mathscr{S}_o + B_m^H) \subset \Lambda_{(1,1)}$ as the initial q-open m-seeds, respectively, and apply Lemma 4.5 at most 75 times, reproducing Phases 2-9 in the relevant site and passage blocks. This involves augmenting the set of explored edges $E_{k_{\max}}$ by successive applications of Lemma 4.5. By the observations made in the previous paragraph, it follows that every edge in the augmented set is $(p+\eta/2, q+\eta/2)$-open.

In general, for $x \in V \subset \mathbb{Z}^2$, we say that $Z(x) = 1$ if Phases 2-9 can be successfully performed in the region $\Lambda_x \cup \Pi_x \cup \Lambda^l_{x+(1,1)} \cup \Lambda^u_{x+(1,-1)}$, using $\big({}_{x-(1,-1)} + B_m^H\big) \subset \Lambda_x^l$ as the initial q-open m-seed, if it exists, or $(\mathscr{S}_{x-(1,1)} + B_m^H) \subset \Lambda_x^u$, if such seed exists and the former do not. Otherwise, we say that $Z(x) = 0$.

The definition of $Z(x)$ together with the choice of $N = 6n$ imply that, for any $l \in \mathbb{N}$, given that the variables $Z((x_1, x_2))$ with $x_1 < l$ have been determined, the states of the variables $Z((x_1, x_2))$ with $x_1 = l$ are independent of each other, since the set of edges used in the exploration of the corresponding boxes are all disjoint. We use this fact to conclude the proof of Theorem 1.2 in the following manner: for $x, y \in V$, we say that $x \le y$ if $x_1 \le y_1$ or $x_1 = y_1$ and $x_2 \le y_2$. This naturally defines an ordering of the sites of V. If we consider the cluster-growth of $o \in V$ with respect to $(Z(x))_{x\in V}$ according to this ordering, it follows that, at each stage, conditioned on the past exploration, the chance of augmenting the open cluster by one vertex is at least $\big(1 + \vec{p}_c^{\,\mathrm{site}}(G)\big)/2$ by (4.11), so that (4.7) is satisfied. By Lemma 4.7, it follows that there is a positive probability of the cluster of the origin on $G = (V, E)$ induced by $(Z(x))_{x\in V}$ to be infinite. On this event, there exists an infinite $(p + \eta/2, q + \eta/2)$-open path of $\mathbb{Z}^d$ within the slab $\mathbb{Z}^2 \times \{-N, \dots, N\}^{d-2}$. □

Figure 4.6 shows a cluster-growth process with all possible types of open and closed site-blocks.

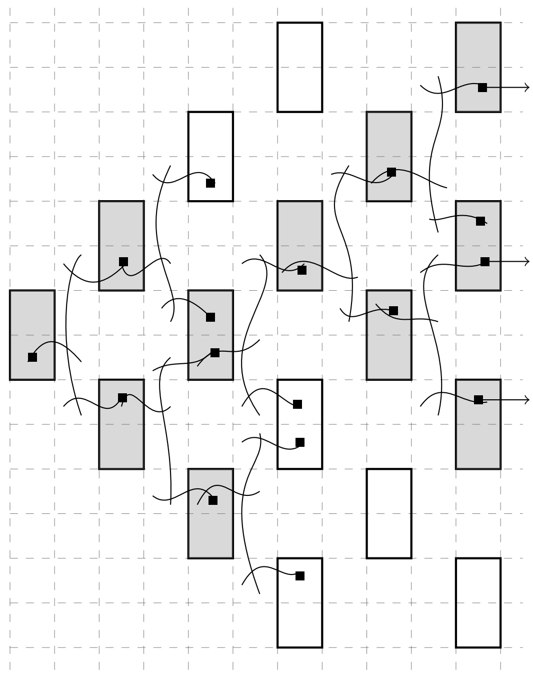

FIGURE 4.6. A cluster-growth process of $o \in V$ with respect to $(Z(x))_{x\in V}$. The gray site-blocks indicate $Z(x) = 1$ and the white ones indicate $Z(x) = 0$. Successful paths between adjacent site-blocks are indicated by the black curves and unsuccessful paths are omitted. Every possible combination between the placement of seeds and the value of $Z(x)$ is represented above.

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