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Abstract. Let X be a smooth projective curve of genus ≥ 2 over a number field. A natural variant of Grothendieck's Section Conjecture postulates that every section of the fundamental exact sequence for X which everywhere locally comes from a point of X in fact globally comes from a point of X. We show that $X/\mathbb{Q}$ satisfies this version of the Section Conjecture if it satisfies Kim's Conjecture for almost all choices of auxiliary prime p, and give the appropriate generalisation to S-integral points on hyperbolic curves. This gives a new "computational" strategy for proving instances of this variant of the Section Conjecture, which we carry out for the thrice-punctured line over $\mathbb{Z}[1/2]$.

Contents

1. Introduction

Let X be a smooth projective curve of genus ≥ 2 over a number field K with algebraic closure $\overline{K}$ and absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$. Via the *fundamental exact sequence*

$$1 \to \pi_1^{\text{ét}}(X_{\overline{K}}) \to \pi_1^{\text{ét}}(X) \to G_K \to 1 \tag{1.1}$$

one may study the K-rational points of X: each such point $x \in X(K)$ induces a section of this sequence, well-defined up to conjugation by $\pi_1^{\text{ét}}(X_{\overline{K}})$.[1] Grothendieck's famous *Section Conjecture* states that the set of rational points $X(K)$ should be in bijection with the set $\mathrm{Sec}(X/K)$ of $\pi_1^{\text{ét}}(X_{\overline{K}})$-conjugacy classes of splittings of (1.1). The injectivity is easy to show, but surjectivity is still a wide open problem, and there exist only a few examples where this is known to hold, all in the special case that $X(K) = \emptyset$ [HS09; LLSS22; Sti10; Sti11].

In this paper, we are interested in a variant of the Section Conjecture of local-to-global nature.

Definition 1.1. A section s of (1.1) is *Selmer* just when for every place v of K, the restriction $s|_{G_v}$ to a decomposition group at v is the section of the local fundamental exact sequence

$$1 \to \pi_1^{\text{ét}}(X_{\overline{K}_v}) \to \pi_1^{\text{ét}}(X_{K_v}) \to G_v \to 1 \tag{1.2}$$

coming from a K_v-rational point $x_v \in X(K_v)$. The set of $\pi_1^{\text{ét}}(X_{\overline{K}})$-conjugacy classes of Selmer sections is denoted by $\mathrm{Sec}(X/K)^{\mathrm{Sel}}$.

Sections which are induced globally by a K-rational point are clearly Selmer, so we have embeddings

$$X(K) \subseteq \mathrm{Sec}(X/K)^{\mathrm{Sel}} \subseteq \mathrm{Sec}(X/K). \tag{1.3}$$

It is natural to divide Grothendieck's Section Conjecture into two subproblems, stating that each of the two inclusions in (1.3) is a bijection:

1. Show that every Selmer section is induced by a K-rational point.
2. Show that every section is Selmer.

The second problem amounts to the p-adic Section Conjecture, i.e. the analogue of the Section Conjecture over finite extensions of $\mathbb{Q}_p$. Just like for number fields, the p-adic Section Conjecture is still open, although there are some partial results, most notably the proof of the Birational p-adic Section Conjecture [Koe05].

In this paper we are concerned with the first problem, which we call the Selmer Section Conjecture.[2]

Conjecture 1.2 (Selmer Section Conjecture, cf. [BS22])**.** *Let X/K be a smooth projective curve of genus ≥ 2 over a number field K. Then*

$$X(K) = \mathrm{Sec}(X/K)^{\mathrm{Sel}}.$$

[1] Strictly speaking, one should choose a geometric basepoint on $X_{\overline{K}}$ to make sense of these fundamental groups; we suppress this choice for the purposes of the introduction.

[2] As remarked above, this variant of the Section Conjecture is natural, both in being a possible route towards addressing the Section Conjecture itself, and also in its connection to the finite descent obstruction, as we explain in §2.1. Although this conjecture does not appear to be written down explicitly in the literature, a number of mathematicians have considered some version of it, including Mohamed Saïdi, who introduced the conjecture to the first author. Minhyong Kim informs us that Florian Pop and Akio Tamagawa were also interested in this question.

In this paper, we propose a new strategy for proving instances of the Selmer Section Conjecture. The viewpoint we wish to promote, building on [BS22], is that p-adic methods for studying rational and S-integral points on curves can also be used to study Selmer sections. In [BS22], the first author and Jakob Stix applied the Lawrence–Venkatesh method to prove finiteness results for the Selmer section set of a general X. In this paper, we instead demonstrate that one can use the Chabauty–Kim method to *explicitly compute* the Selmer section set for particular curves.

To fix notation, suppose now that $K = \mathbb{Q}$ and $X/\mathbb{Q}$ is a smooth projective curve of genus ≥ 2. Suppose moreover that X has at least one rational point (for use as a basepoint for fundamental groups). The Chabauty–Kim method as described in [BDCKW18] defines for any prime p a *Chabauty–Kim locus*

$$X(\mathbb{Q}_p)_\infty \subseteq X(\mathbb{Q}_p)$$

containing the rational points $X(\mathbb{Q})$.[3] The Chabauty–Kim locus is by definition the common vanishing locus of some Coleman analytic functions on $X(\mathbb{Q}_p)$; in many cases, one can compute several of these Coleman functions explicitly, and this sometimes enables one to compute the set of rational points exactly. In fact, it is conjectured that in general these Coleman functions should always cut out exactly the set of rational points.

Conjecture 1.3 (Kim's Conjecture, [BDCKW18, Conjecture 3.1]). *Let $X/\mathbb{Q}$ be a smooth projective curve of genus ≥ 2 with $X(\mathbb{Q}) \neq \emptyset$. Let p be a prime. Then*

$$X(\mathbb{Q}) = X(\mathbb{Q}_p)_\infty .$$

The main theoretical result we will prove in this paper makes precise the relationship between Kim's Conjecture and the Selmer Section Conjecture.

Theorem 1.4 (= Theorem A, projective case). *Let $X/\mathbb{Q}$ be a smooth projective curve of genus ≥ 2 with $X(\mathbb{Q}) \neq \emptyset$. Suppose that Kim's Conjecture 1.3 holds for (X, p) for all primes p in a set of primes of Dirichlet density 1. Then the Selmer Section Conjecture 1.6 holds for X.*

This theorem thus gives a new strategy for proving instances of the Selmer Section Conjecture: if one can compute the Chabauty–Kim locus $X(\mathbb{Q}_p)_\infty$ for a density 1 set of primes p and show that it is equal to the set of rational points, then one obtains for free that the Selmer Section Conjecture holds for X. The difficulty here, of course, lies in carrying out Chabauty–Kim computations for infinitely many different primes p at once.

We will demonstrate the viability of this approach by working out one example of a hyperbolic curve where we can indeed verify Kim's Conjecture for infinitely many primes at once. The specific example we will study is that of the thrice-punctured line over $\mathbb{Z}[1/2]$, which is an *affine* hyperbolic curve. Therefore, we formulate a more general version of our main theorem which holds for S-integral points on arbitrary hyperbolic curves, projective or not.

[3] The Chabauty–Kim locus $X(\mathbb{Q}_p)_\infty$ is defined in [BDCKW18] under the assumption that p is a prime of good reduction. When we recall the definition in Definition 2.16, we will take care to formulate the definition in a way which does not require good reduction, though the general theory is not yet strong enough to study the Chabauty–Kim locus in the bad reduction case.

Definition 1.5. Let Y be a hyperbolic curve over a number field K, let S be a finite set of places of K containing all archimedean places, and let $\mathcal{Y}/\mathcal{O}_{K,S}$ be a regular model of Y, meaning that $\mathcal{Y}$ is the complement of a horizontal divisor in a flat proper regular $\mathcal{O}_{K,S}$-scheme and comes with an isomorphism $\mathcal{Y}_K \cong Y$ of its generic fibre with Y. We say that a section s of the global fundamental exact sequence (1.1) (with X replaced with Y) is *S-Selmer* just when the restriction $s|_{G_v}$ to a decomposition group at a place v is the section of the local fundamental exact sequence (1.2) coming from a

$$\begin{cases} K_v\text{-rational point } y_v \in Y(K_v) & \text{for all } v \in S, \\ \mathcal{O}_v\text{-integral point } y_v \in \mathcal{Y}(\mathcal{O}_v) & \text{for all } v \notin S. \end{cases}$$

Here $\mathcal{O}_v$ is the ring of integers of K_v. The set of $\pi_1^{\text{ét}}(Y_{\overline{K}})$-conjugacy classes of S-Selmer sections is denoted by $\mathrm{Sec}(\mathcal{Y}/\mathcal{O}_{K,S})^{\mathrm{Sel}}$.

The map sending an S-integral point to its corresponding section of (1.1) embeds $\mathcal{Y}(\mathcal{O}_{K,S})$ as a subset of $\mathrm{Sec}(\mathcal{Y}/\mathcal{O}_{K,S})^{\mathrm{Sel}}$, and the Section Conjecture would imply that this inclusion is an equality. So the Selmer Section Conjecture has the following natural analogue for S-integral points.

Conjecture 1.6 (S-Selmer Section Conjecture)**.** *Let Y/K be a hyperbolic curve over a number field K and let $\mathcal{Y}/\mathcal{O}_{K,S}$ be a regular S-integral model of Y. Then*

$$\mathcal{Y}(\mathcal{O}_{K,S}) = \mathrm{Sec}(\mathcal{Y}/\mathcal{O}_{K,S})^{\mathrm{Sel}}.$$

The Chabauty–Kim method also can be applied in the setting of affine hyperbolic curves. Suppose that $K = \mathbb{Q}$ and $\mathcal{Y}/\mathbb{Z}_S$ is a regular S-integral model of a hyperbolic curve Y, where S is a finite set of primes. Suppose moreover that the smooth compactification of Y has a rational point (so Y has either a rational point or a rational tangential point for use as a basepoint for fundamental groups). The Chabauty–Kim method, specifically the refined Chabauty–Kim method described in [BD20], defines for any prime $p \notin S$ a *refined Chabauty–Kim locus*

$$\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\infty} \subseteq \mathcal{Y}(\mathbb{Z}_p)$$

containing the S-integral points $\mathcal{Y}(\mathbb{Z}_S)$ and being defined as the common vanishing locus of some Coleman analytic functions on $\mathcal{Y}(\mathbb{Z}_p)$. The refined version of Kim's Conjecture predicts that this vanishing locus is precisely the set of S-integral points.

Conjecture 1.7 (Refined Kim's Conjecture)**.** *Let $Y/\mathbb{Q}$ be a hyperbolic curve and let $\mathcal{Y}/\mathbb{Z}_S$ be a regular S-integral model of Y for a finite set of primes S. Suppose that the smooth compactification of Y has a rational point. Let $p \notin S$ be prime. Then*

$$\mathcal{Y}(\mathbb{Z}_S) = \mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\infty}.$$

We can now state the general version of our main theorem in the setting of possibly affine hyperbolic curves.

Theorem A. *Let $\mathcal{Y}/\mathbb{Z}_S$ be a regular S-integral model of a hyperbolic curve $Y/\mathbb{Q}$ whose smooth compactification has a rational point. Suppose that the refined Kim's Conjecture 1.7 holds for $(\mathcal{Y}, S, p)$ for all primes p in a set of primes of Dirichlet density 1. Then the S-Selmer Section Conjecture 1.6 holds for $(\mathcal{Y}, S)$.*

Note that if the hyperbolic curve in question happens to be projective, everything in sight is independent of the set S and model $\mathcal{Y}$ and this theorem specialises to the one stated for projective curves above.

In order to show that Theorem A provides a viable strategy for proving instances of the S-Selmer Section Conjecture, we verify the refined Kim's Conjecture in the example of the thrice-punctured line over $\mathbb{Z}[1/2]$ for all odd primes p, which is the main computational result in this paper.

Theorem B. *The refined Kim's Conjecture 1.7 holds for $S = \{2\}$, $\mathcal{Y} = \mathbb{P}^1_{\mathbb{Z}[1/2]} \smallsetminus \{0, 1, \infty\}$ and all odd primes p.*

Consequently, we obtain

Theorem C. *Conjecture 1.6 holds for $K = \mathbb{Q}$, $S = \{2\}$ and $\mathcal{Y} = \mathbb{P}^1_{\mathbb{Z}[1/2]} \smallsetminus \{0, 1, \infty\}$.*

Remark 1.8. Theorem B is, to the best of our knowledge, the first example of a hyperbolic curve $\mathcal{Y}/\mathbb{Z}_S$ with $\mathcal{Y}(\mathbb{A}^f_{\mathbb{Q},S}) \neq \emptyset$ for which the refined Kim's Conjecture can be verified for infinitely many values of p. (For curves with $\mathcal{Y}(\mathbb{A}^f_{\mathbb{Q},S}) = \emptyset$, the refined Kim's Conjecture holds for more or less trivial reasons, see [BBK+23, Remark 2.8].)

On the other hand, the particular case of Conjecture 1.6 proved in Theorem C is already known due to work of Jakob Stix [Sti15, Corollary 6] (for $K = \mathbb{Q}$, $\mathcal{Y} = \mathbb{P}^1_{\mathbb{Z}_S} \smallsetminus \{0, 1, \infty\}$, and any S). However, our strategy is completely different to Stix's, and demonstrates the applicability of the Chabauty–Kim method to questions related to the Section Conjecture.

There are a couple of other cases where one can show a Kim-like conjecture for infinitely many primes p. The first involves the unrefined Chabauty–Kim loci $\mathcal{Y}(\mathbb{Z}_p)_{S,\infty}$ for an affine hyperbolic curve, as originally defined in [Kim05; Kim09]. These are a priori larger than the refined loci $\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\infty}$ but still conjectured to be equal to the set of S-integral points $\mathcal{Y}(\mathbb{Z}_S)$.

Theorem D. *The original (i.e. unrefined) Kim's Conjecture holds for $S = \emptyset$, $\mathcal{Y} = \mathbb{P}^1_{\mathbb{Z}} \smallsetminus \{0, 1, \infty\}$ and all odd primes p.*

We can also show a Kim-like conjecture for certain projective hyperbolic curves.

Theorem E. *Let $X/\mathbb{Q}$ be a smooth projective curve of genus ≥ 2 with a rational point, and suppose that the Jacobian of X has a factor of dimension ≥ 2 with Mordell–Weil rank 0 and finite Tate–Shafarevich group. Then there is a finite closed subscheme $Z \subset X_{\mathrm{cl}}$ such that*

$$X(\mathbb{Q}_p)_\infty \subseteq Z(\mathbb{Q}_p) \tag{1.4}$$

for all primes p of good reduction for X. In particular, if Z can be chosen to be the set of rational points of X, then the refined Kim's Conjecture 1.7 holds for X.

In particular, both of these give examples where we prove the S-Selmer Section Conjecture using our strategy. (In fact, the containment (1.4) is already enough to conclude the Selmer Section Conjecture.) Again, neither of these cases of the S-Selmer Section Conjecture are new: for $\mathcal{Y} = \mathbb{P}^1_{\mathbb{Z}} \smallsetminus \{0, 1, \infty\}$, the S-Selmer section set is empty since $\mathcal{Y}(\mathbb{Z}_2) = \emptyset$, while in the setting of Theorem E, the Selmer Section Conjecture is essentially a theorem of Stoll (combine [Sto07, Theorem 8.6] and Proposition 2.13).

Let us now say a little about the proof of Theorem B. The original unrefined Chabauty–Kim method for the thrice-punctured line has been studied extensively in work of Ishai Dan-Cohen, Stefan Wewers and David Corwin [DCW15; DCW16; DC20; CDC20a; CDC20b]. They give some explicit Coleman functions vanishing on $\mathcal{Y}(\mathbb{Z}[1/2])$, but it is far from clear whether these should cut out exactly $\mathcal{Y}(\mathbb{Z}[1/2])$. What we do in this paper is to carry out the refined versions of the computations of [CDC20a], obtaining the following description of the refined Chabauty–Kim locus.

Proposition 1.9. *For $S = \{2\}$, $\mathcal{Y} = \mathbb{P}^1_{\mathbb{Z}[1/2]} \smallsetminus \{0, 1, \infty\}$ and p any odd prime, the refined Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)^{\min}_{\{2\},\infty}$ is contained in the union of the S_3-translates of the locus in $\mathcal{Y}(\mathbb{Z}_p)$ defined by the equations*

$$\log(z) = 0 \quad \textit{and} \quad \mathrm{Li}_k(z) = 0 \quad \textit{for } k \geq 2 \textit{ even,}$$

where $\log$ and Li_k are the p-adic (poly)logarithms and the action of S_3 on $\mathcal{Y}$ is the usual one permuting the three cusps $\{0, 1, \infty\}$.

So Theorem B follows once we verify that log and Li_{p-3} have no common zeroes in $\mathcal{Y}(\mathbb{Z}_p)$ other than $z = -1$ for $p \geq 5$, and that the only zero of log in $\mathcal{Y}(\mathbb{Z}_3)$ is $z = -1$. The same argument will also imply Theorem D.

Remark 1.10. When the refined Kim's Conjecture holds, a natural question is what depth in the fundamental group one needs to go to in order to cut out exactly the S-integral points. Our proof of Theorem B establishes that the refined Kim's Conjecture already holds in depth $n = p - 3$ for $p \geq 5$, which notably depends on the prime p. We strongly believe this to be an artefact of the proof: in fact, in [BBK+23] we conjectured that $n = 2$ is enough in this case, and verified this computationally for $3 \leq p \leq 10^5$.

Remark 1.11. The computations of Corwin and Dan-Cohen [CDC20a] which we use in our proof of Theorem B use a rather different foundation of Chabauty–Kim theory than the original papers of Kim, using the $\mathbb{Q}$-pro-unipotent motivic fundamental groupoid of $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ as constructed by Deligne–Goncharov [DG05] in place of the $\mathbb{Q}_p$-pro-unipotent étale fundamental groupoid. So in order to apply their results, we need to show that the two approaches are equivalent. This ultimately amounts to what seems to be a folklore result, that the étale realisation functor from mixed Tate motives to mixed Tate Galois representations is an equivalence "after tensoring with $\mathbb{Q}_p$". Here is a precise statement, which we will prove in the Appendix A.

Theorem 1.12. *Let K be a number field and S a finite set of primes of $\mathcal{O}_K$. Let $\mathrm{MT}(\mathcal{O}_{K,S}, \mathbb{Q})$ denote the Tannakian category of mixed Tate motives over $\mathcal{O}_{K,S}$ with coefficients in $\mathbb{Q}$ [DG05, (1.7)], and for an odd rational prime p not dividing any element of S, let $\mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_K)$ denote the Tannakian category of mixed Tate Galois representations which are unramified outside $S \cup \{\mathfrak{p} \mid p\}$ and crystalline at all $\mathfrak{p} \mid p$. Let $G^{\mathrm{MT}}_{K,S}$ and $G^{\mathrm{MTR}}_{K,S}$ denote the Tannaka groups of $\mathrm{MT}(\mathcal{O}_{K,S}, \mathbb{Q})$ and $\mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_K)$ at their canonical fibre functors, respectively. Then the étale realisation functor*

$$\rho_{\text{ét}} \colon \mathrm{MT}(\mathcal{O}_{K,S}, \mathbb{Q}) \to \mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_K)$$

induces an isomorphism

$$G^{\mathrm{MTR}}_{K,S} \xrightarrow{\sim} G^{\mathrm{MT}}_{K,S,\mathbb{Q}_p}$$

of affine group schemes over $\mathbb{Q}_p$.[4]

1.1. **Comparing the conjectures.** Let us say a little about our proof of Theorem A, about the relationship between the S-Selmer Section Conjecture and the refined Kim's Conjecture. We will relate the two conjectures by relating them both to a third conjecture in obstruction theory. If $\mathcal{Y}/\mathcal{O}_{K,S}$ is a regular S-integral model of a hyperbolic curve Y/K, then the finite descent obstruction cuts out a *finite descent locus*

$$\mathcal{Y}(\mathbb{A}_{K,S})_\bullet^{\text{f-cov}} \subseteq \mathcal{Y}(\mathbb{A}_{K,S})_\bullet$$

inside the modified S-adelic points $\mathcal{Y}(\mathbb{A}_{K,S})_\bullet$, containing the S-integral points $\mathcal{Y}(\mathcal{O}_{K,S})$ (see §2.1 for precise definitions). According to a conjecture of Stoll, this inclusion should be an equality.

Conjecture 1.13 (Strong sufficiency of finite descent)**.** *Let Y/K be a hyperbolic curve over a number field K and let $\mathcal{Y}/\mathcal{O}_{K,S}$ be a regular S-integral model of Y. Then*

$$\mathcal{Y}(\mathcal{O}_{K,S}) = \mathcal{Y}(\mathbb{A}_{K,S})_\bullet^{\text{f-cov}}\,.$$

Remark 1.14. Conjecture 1.13 implies as a special case the equivalence

$$\mathcal{Y}(\mathbb{A}_{K,S})_\bullet^{\text{f-cov}} = \emptyset \quad \Leftrightarrow \quad \mathcal{Y}(\mathcal{O}_{K,S}) = \emptyset,$$

i.e. that the finite descent obstruction is sufficient to explain the failure of the Hasse Principle for S-integral points on models of hyperbolic curves. This is why we call Conjecture 1.13 the strong sufficiency of finite descent.

Our proof of Theorem A then consists of two halves.

(1) The S-Selmer Section Conjecture 1.6 holds for some $(\mathcal{Y}, S)$ if and only if Conjecture 1.13 holds for $(\mathcal{Y}, S)$. (Proposition 2.13.)
(2) If the refined Kim's Conjecture 1.7 holds for $(\mathcal{Y}, S, p)$ for all primes p in a set of Dirichlet density 1, then Conjecture 1.13 holds for $(\mathcal{Y}, S)$. (Corollary 2.20.)

The second of these points is rather easier than the first. It follows more-or-less directly from the definitions that the projection of the finite descent locus on $\mathcal{Y}(\mathbb{Z}_p)$ is contained in the refined Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)_{S,\infty}^{\min}$ for all $p \notin S$. So if Conjecture 1.7 held for all primes p in a set $\mathfrak{P}$ of Dirichlet density 1, then we would deduce that the projection of the finite descent locus on $\prod_{p\in\mathfrak{P}} \mathcal{Y}(\mathbb{Z}_p)$ is contained in

$$\prod_{p\in\mathfrak{P}} \mathcal{Y}(\mathbb{Z}_S)\,.$$

There are then two problems to be overcome to deduce Conjecture 1.13 from this: one needs to know that the projection of $\mathcal{Y}(\mathbb{A}_{\mathbb{Q},S})_\bullet^{\text{f-cov}}$ on $\prod_{p\in\mathfrak{P}} \mathcal{Y}(\mathbb{Z}_p)$ is injective, and that its image lands in $\mathcal{Y}(\mathbb{Z}_S)$ embedded diagonally in the product. Both of these problems are resolved by a theorem due to Stoll (in the projective case; we extend Stoll's argument to the general case in Theorem 2.6) which we dub the *theorem of the diagonal*.

The proof of the first point, which holds over any number field K, revolves around a theorem of Harari and Stix [HS12] which shows that the finite descent locus

[4] We strongly believe that this result should still be true when $p = 2$, but our proof does not show this.

$\mathcal{Y}(\mathbb{A}_{K,S})_\bullet^{\text{f-cov}}$ is the image of the S-Selmer section set under a certain localisation map

$$\mathrm{loc}\colon \mathrm{Sec}(\mathcal{Y}/\mathcal{O}_{K,S})^{\mathrm{Sel}} \to \mathcal{Y}(\mathbb{A}_{K,S})_\bullet\,.$$

The surjectivity of the map $\mathrm{Sec}(\mathcal{Y}/\mathcal{O}_{K,S})^{\mathrm{Sel}} \twoheadrightarrow \mathcal{Y}(\mathbb{A}_{K,S})_\bullet^{\text{f-cov}}$ immediately gives the "only if" direction of (1). The "if" direction is harder as we do not know the localisation map to be injective; instead we give a rather indirect argument using the calculus of sections and a second application of the theorem of the diagonal.

1.2. **Overview of sections.** We begin in Section 2 by proving Theorem A: that the refined Kim's Conjecture implies the S-Selmer Section Conjecture. We then take a detour in Section 3 to explain how the Chabauty–Kim method as formulated in [Kim05; Kim09; BDCKW18; BD20] is related to the motivic Chabauty–Kim method used in [DCW15; DCW16; DC20; CDC20a; CDC20b]. This is essentially a very careful checking that the two methods agree. The most technical motivic arguments in this section are hived off into Appendix A.

Section 4 recalls the setup of the refined Chabauty–Kim method, and lays the groundwork for the computation of the refined Chabauty–Kim locus for the thrice-punctured line over $\mathbb{Z}[1/2]$, which we carry out in Section 5. Section 5 contains the proofs of Theorems B, D and E.

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2. Selmer sections, finite descent and Chabauty–Kim

We begin by discussing the relation between the S-Selmer Section Conjecture 1.6, strong sufficiency of finite descent (Conjecture 1.13) and the refined Kim's Conjecture 1.7.

We will adopt the following notation: for a geometrically connected variety V over a field F, we write $\mathrm{Sec}(V/F)$ for the set of splittings of the fundamental exact sequence

$$1 \to \pi_1^{\text{ét}}(V_{\overline{F}}) \to \pi_1^{\text{ét}}(V) \to G_F \to 1$$

up to $\pi_1^{\text{ét}}(V_{\overline{F}})$-conjugacy. We also write $\mathrm{Sec}(V/F)_{\mathrm{ab}}$ for the set of sections of the geometrically abelian fundamental exact sequence

$$1 \to \pi_1^{\mathrm{ab}}(V_{\overline{F}}) \to \pi_1^{\mathrm{gab}}(V) \to G_F \to 1$$

given by pushing out the fundamental exact sequence along the abelianisation map $\pi_1^{\text{ét}}(V_{\overline{F}}) \to \pi_1^{\mathrm{ab}}(V_{\overline{F}})$. If $x \in V(F)$ is an F-rational point, we write $\kappa(x) \in \mathrm{Sec}(V/F)$ for the induced section, and $\kappa_{\mathrm{ab}}(x) \in \mathrm{Sec}(V/F)_{\mathrm{ab}}$ for the abelian section obtained via pushout.

Throughout this section, K will be a number field and Y/K a hyperbolic curve (or sometimes a curve of non-positive Euler characteristic).[5] Our principal object of study will be the set of *Selmer sections* of Y/K, defined as follows.

Definition 2.1. Let Y be a hyperbolic curve over a number field K. A section $s \in \mathrm{Sec}(Y/K)$ (resp. abelian section $s_{\mathrm{ab}} \in \mathrm{Sec}(Y/K)_{\mathrm{ab}}$) is called *Selmer* just when there exists an adelic point $(y_v)_v \in Y(\mathbb{A}_K)$ such that

$$s_{(\mathrm{ab})}|_{G_v} = \kappa_{(\mathrm{ab})}(y_v)$$

is the (abelian) section coming from y_v for all places v. We write $\mathrm{Sec}(Y/K)^{\mathrm{Sel}} \subseteq \mathrm{Sec}(Y/K)$ and $\mathrm{Sec}(Y/K)^{\mathrm{Sel}}_{\mathrm{ab}} \subseteq \mathrm{Sec}(Y/K)_{\mathrm{ab}}$ for the set of Selmer sections and abelian Selmer sections, respectively.

Note that the S-Selmer section set $\mathrm{Sec}(\mathcal{Y}/\mathcal{O}_K)^{\mathrm{Sel}}$ of Definition 1.5 is exactly the subset of $\mathrm{Sec}(Y/K)^{\mathrm{Sel}}$ consisting of those Selmer sections for which the adelic point $(y_v)_v$ can be chosen to be an S-adelic point on the model $\mathcal{Y}$.

It turns out that the adelic point $(y_v)_v$ associated to a Selmer section is unique, up to a certain ambiguity at infinite places. To explain this, we need the following injectivity result for the local section maps.

Lemma 2.2. *Let K_v be a local field of characteristic 0 and let Y/K_v be a hyperbolic curve.*

(1) If K_v is non-archimedean, then the section map (resp. abelian section map)

$$Y(K_v) \hookrightarrow \mathrm{Sec}(Y/K_v)_{(\mathrm{ab})}$$

is injective.

(2) If $K_v = \mathbb{R}$, then the (abelian) section map factors through an injective map

$$\pi_0(Y(\mathbb{R})) \hookrightarrow \mathrm{Sec}(Y/\mathbb{R})_{(\mathrm{ab})}\,.$$

(3) If $K_v = \mathbb{C}$, then the (abelian) section map $Y(\mathbb{C}) \to \mathrm{Sec}(Y/\mathbb{C})_{(\mathrm{ab})}$ is constant.

Proof. Part (3) is trivial; for part (1) it suffices to prove that the abelian section map is injective, which is [Sti12, Proposition 73]. For part (2), the section map factors through $\pi_0(Y(\mathbb{R}))$ e.g. by the discussion in [Sti12, §16.1], so it suffices to prove injectivity of the abelian section map

$$\pi_0(Y(\mathbb{R})) \to \mathrm{Sec}(Y/\mathbb{R})_{\mathrm{ab}}\,.$$

This follows from a theorem of Wickelgren [Wic14, Theorem 1.1]. □

Now let us write

$$Y(\mathbb{A}_K)_\bullet = {\prod_{v\nmid\infty}}' Y(K_v) \times \prod_{v \text{ real}} \pi_0(Y(K_v))$$

for the set of *modified adelic points* of Y [Sto07; Sto17]. It follows from Lemma 2.2 that the adelic point $(y_v)_v \in Y(\mathbb{A}_K)$ associated to a Selmer section or abelian Selmer section is unique when viewed as an element of $Y(\mathbb{A}_K)_\bullet$. Thus the assignment $s \mapsto (y_v)_v$ defines functions

$$\mathrm{loc}\colon \mathrm{Sec}(Y/K)^{\mathrm{Sel}} \to Y(\mathbb{A}_K)_\bullet \quad \text{and} \quad \mathrm{loc}\colon \mathrm{Sec}(Y/K)^{\mathrm{Sel}}_{\mathrm{ab}} \to Y(\mathbb{A}_K)_\bullet\,,$$

[5] Our convention is that curves are required to be smooth and geometrically connected, but not necessarily projective.

which we call the *localisation map*.

2.1. **Finite descent.** The S-Selmer section set is closely related to the finite descent obstruction. If $(P, \mathcal{G})$ is a pair consisting of a finite étale K-group scheme $\mathcal{G}$ and a (right) $\mathcal{G}$-torsor P over Y in the étale topology, then we say that a modified adelic point $y \in Y(\mathbb{A}_K)_\bullet$ *survives* $(P, \mathcal{G})$ just when there is a continuous Galois cocycle $\xi\colon G_K \to \mathcal{G}(\overline{K})$ such that y lies in the image of $P_\xi(\mathbb{A}_K)_\bullet \to Y(\mathbb{A}_K)_\bullet$ where P_ξ denotes the ξ-twisted torsor [Sto07, Definition 5.2].

Definition 2.3 ([Sto07, Definition 5.4][6])**.** We say that $y \in Y(\mathbb{A}_K)_\bullet$ survives the *finite descent obstruction* if it survives every pair $(P, \mathcal{G})$. We say it survives the *finite abelian descent obstruction* if it survives every pair $(P, \mathcal{G})$ where $\mathcal{G}$ is abelian.

The sets of modified adelic points surviving the various descent obstructions are denoted by

$$Y(\mathbb{A}_K)_\bullet^{\text{f-cov}} \subseteq Y(\mathbb{A}_K)_\bullet^{\text{f-ab}} \subseteq Y(\mathbb{A}_K)_\bullet .$$

The set of rational points $Y(K)$ is contained in each of these sets.

The relationship between Selmer sections and finite descent was studied by Harari and Stix.

Theorem 2.4. *Let Y be a hyperbolic curve over a number field K. Then:*

1. *the image of the localisation map* $\mathrm{loc}\colon \mathrm{Sec}(Y/K)^{\mathrm{Sel}} \to Y(\mathbb{A}_K)_\bullet$ *on Selmer sections is the finite descent locus* $Y(\mathbb{A}_K)_\bullet^{\text{f-cov}}$*; and*
2. *the image of the localisation map* $\mathrm{loc}\colon \mathrm{Sec}(Y/K)^{\mathrm{Sel}}_{\mathrm{ab}} \to Y(\mathbb{A}_K)_\bullet$ *on abelian Selmer sections is the finite abelian descent locus* $Y(\mathbb{A}_K)_\bullet^{\text{f-ab}}$.

The first part is a particular case of [HS12, Theorem 11] (cf. the cohomological formulation of the finite descent locus in [Sto07, §5]). For the second part, [HS12, Theorem 11] shows that the image of the localisation map on abelian Selmer sections is the set of all modified adelic points which survive all pairs $(P, \mathcal{G})$ with *abelian geometric monodromy*, meaning that the image of the induced map $\pi_1^{\text{ét}}(Y_{\overline{K}}) \to \mathcal{G}(\overline{K})$ is abelian. So to prove the second part, we have to verify the following.

Lemma 2.5. *Let Y be a geometrically connected quasi-projective variety over a number field K and $y \in Y(\mathbb{A}_K)$ an adelic point. Then the following are equivalent*

(i) *y survives all pairs $(P, \mathcal{G})$ with $\mathcal{G}$ abelian;*
(ii) *y survives all pairs $(P, \mathcal{G})$ with abelian geometric monodromy.*

Proof. The implication (ii)⇒(i) is trivial since any pair $(P, \mathcal{G})$ with $\mathcal{G}$ abelian automatically has abelian geometric monodromy. For the converse implication, suppose that $(P, \mathcal{G})$ is a pair with abelian geometric monodromy, and let L/K be a finite extension such that P_L is a disjoint union of geometrically connected L-varieties. If P_0 is a connected component of P_L and $\mathcal{G}_0 \leq \mathcal{G}_L$ is its stabiliser, then $\mathcal{G}_0$ is the image of the monodromy homomorphism $\pi_1^{\text{ét}}(Y_{\overline{K}}) \to \mathcal{G}(\overline{K})$, and hence is abelian.

Now we use a construction from [Sto07, p. 365]. Applying the Weil restriction functor provides a finite *abelian* K-group scheme $\mathcal{G}_1 := \mathrm{Res}^L_K \mathcal{G}_0$ and a $\mathcal{G}_1$-torsor $\mathrm{Res}^L_K P_0$ over $\mathrm{Res}^L_K Y_L$. Restricting along the canonical map $Y \to \mathrm{Res}^L_K Y_L$ provides us with a $\mathcal{G}_1$-torsor P_1 over Y which admits a morphism $(P_{1,L}, \mathcal{G}_{1,L}) \to (P_0, \mathcal{G}_0)$. In particular, we have a morphism $f\colon (P_1, \mathcal{G}_1) \to (P, \mathcal{G})$ defined over L. [Sto07,

[6] Stoll only defines this set in the case that Y is projective, but the same definition carries over in general.

Lemma 5.6] shows that, after twisting the pair $(P, \mathcal{G})$, the morphism f is even defined over K.

By [Sto07, Lemma 5.3(1) & (5)], if an adelic point y survives $(P_1, \mathcal{G}_1)$ then it also survives $(P, \mathcal{G})$. This gives the implication (i)⇒(ii). □

2.1.1. *Theorem of the diagonal.* We will need to use one key property of the finite abelian descent set in what follows, which in the projective case is essentially due to Stoll.

Theorem 2.6 (cf. [Sto07, Theorem 8.2])**.** *Let Y be a curve of non-positive Euler characteristic over a number field K, and let $Z \subset Y_{\mathrm{cl}}$ be a finite closed subscheme. Then the image of $Z(\mathbb{A}_K)$ in $Y(\mathbb{A}_K)_\bullet$ meets the finite abelian descent locus $Y(\mathbb{A}_K)_\bullet^{\text{f-ab}}$ in $Z(K)$. More generally, if $\mathfrak{P}$ is a set of places of K of Dirichlet density 1 and $y \in Y(\mathbb{A}_K)_\bullet^{\text{f-ab}}$ is such that $y_v \in Z(K_v)$ for all $v \in \mathfrak{P}$, then $y \in Z(K)$.*

Remark 2.7. Since the finite descent set $Y(\mathbb{A}_K)_\bullet^{\text{f-cov}}$ is contained in $Y(\mathbb{A}_K)_\bullet^{\text{f-ab}}$, the analogous statement to Theorem 2.6 holds *a posteriori* for $Y(\mathbb{A}_K)_\bullet^{\text{f-cov}}$.

We call Theorem 2.6 the *theorem of the diagonal*, since it says that a point in $Z(\mathbb{A}_K) = \prod_v Z(K_v)$ which is unobstructed by the finite descent obstruction in Y lies diagonally in the product (with the usual caveats at infinite places).

In the proof, we will frequently use the fact that we can safely ignore the components of elements of $Y(\mathbb{A}_K)_\bullet^{\text{f-ab}}$ at a sparse set of places v.

Lemma 2.8. *Let Y/K be a curve of non-positive Euler characteristic, and let $\mathfrak{P}$ be a set of finite places of K of Dirichlet density 1. Then the projection map*

$$Y(\mathbb{A}_K)_\bullet^{\text{f-ab}} \to \prod_{v\in\mathfrak{P}} Y(K_v)$$

is injective.

Proof. Let $y, y' \in Y(\mathbb{A}_K)_\bullet^{\text{f-ab}}$ be such that $y_v = y'_v$ for all $v \in \mathfrak{P}$. By Theorem 2.6, we know that y and y' are the images of abelian Selmer sections $s, s' \in \mathrm{Sec}(Y/K)_{\mathrm{ab}}^{\mathrm{Sel}}$ under the localisation map.

Now a theorem of Saïdi [Saï22, Proposition 3.2] implies that the restriction map

$$\mathrm{H}^1(G_K, \pi_1^{\mathrm{ab}}(Y_{\overline{K}})) \to \prod_{v\in\mathfrak{P}} \mathrm{H}^1(G_v, \pi_1^{\mathrm{ab}}(Y_{\overline{K}_v}))$$

is injective. Since the set $\mathrm{Sec}(Y/K)_{\mathrm{ab}}$ of abelian sections is either empty or a torsor under $\mathrm{H}^1(G_K, \pi_1^{\mathrm{ab}}(Y_{\overline{K}}))$, this implies that the restriction map

$$\mathrm{Sec}(Y/K)_{\mathrm{ab}} \to \prod_{v\in\mathfrak{P}} \mathrm{Sec}(Y_{K_v}/K_v)_{\mathrm{ab}}$$

is also injective. But by assumption we have

$$s|_{G_v} = \kappa_{\mathrm{ab}}(y_v) = \kappa_{\mathrm{ab}}(y'_v) = s'|_{G_v}$$

for all $v \in \mathfrak{P}$, and hence we deduce that $s = s'$.

In particular, we have

$$\kappa_{\mathrm{ab}}(y_v) = s|_{G_v} = s'|_{G_v} = \kappa_{\mathrm{ab}}(y'_v)$$

for *all* places v. So by Lemma 2.2 we have $y_v = y'_v$ for all finite places v, and y_v and y'_v lie in the same component of $Y(K_v)$ for all real places v, i.e. the points y and y' are equal as elements of $Y(\mathbb{A}_K)_\bullet$. Thus we have proven injectivity. □

Corollary 2.9. *Let $Y \hookrightarrow X$ be a locally closed immersion of a curve Y/K of non-positive Euler characteristic in a K-variety X. Then the induced map*

$$Y(\mathbb{A}_K)_\bullet^{\text{f-ab}} \hookrightarrow X(\mathbb{A}_K)_\bullet^{\text{f-ab}}$$

is injective (even though the map $Y(\mathbb{A}_K)_\bullet \to X(\mathbb{A}_K)_\bullet$ may not be).

Proof. Follows from the commuting square

$$\begin{array}{ccc} Y(\mathbb{A}_K)_\bullet^{\text{f-ab}} & \longrightarrow & X(\mathbb{A}_K)_\bullet^{\text{f-ab}} \\ \big\downarrow & & \big\downarrow \\ \prod_{v\nmid\infty} Y(K_v) & \hookrightarrow & \prod_{v\nmid\infty} X(K_v)\,. \end{array}$$

□

We now begin the proof of Theorem 2.6 with some preliminary reductions. Firstly, we may enlarge the curve Y by filling in punctures:

Lemma 2.10. *Let X be a curve over K containing Y as an open subset. If Theorem 2.6 holds for X, then it also holds for Y.*

Proof. Let $y \in Y(\mathbb{A}_K)_\bullet^{\text{f-ab}}$ be such that $y_v \in Z(K_v)$ for all $v \in \mathfrak{P}$. By Corollary 2.9 we have

$$Y(\mathbb{A}_K)_\bullet^{\text{f-ab}} \subseteq X(\mathbb{A}_K)_\bullet^{\text{f-ab}}\,,$$

so we can regard y as an element of $X(\mathbb{A}_K)_\bullet^{\text{f-ab}}$ without ambiguity. If Theorem 2.6 holds for X then we deduce that y lies in $Z(K)$ and are done. □

Secondly, we are free to enlarge the base field:

Lemma 2.11. *Let L/K be a finite extension. If Theorem 2.6 holds for Y_L, it also holds for Y.*

Proof. Let $y \in Y(\mathbb{A}_K)_\bullet^{\text{f-ab}}$ be such that $y_v \in Z(K_v)$ for all $v \in \mathfrak{P}$. By [Sto07, Proposition 5.15] and Corollary 2.9 (applied to the closed embedding $Y \hookrightarrow \mathrm{Res}_K^L Y_L$) we have

$$Y(\mathbb{A}_K)_\bullet^{\text{f-ab}} \subseteq Y(\mathbb{A}_L)_\bullet^{\text{f-ab}}\,.$$

The set of places of L lying over places in $\mathfrak{P}$ is again of Dirichlet density 1. If Theorem 2.6 holds for Y_L, we deduce that y lies in

$$Z(L) \cap Y(\mathbb{A}_K)_\bullet^{\text{f-ab}} = Z(K)$$

and are done. □

Proof of Theorem 2.6. Using the two preceding reduction steps, we see that it suffices to deal with the cases that $Y = \mathbb{G}_m$ or Y is smooth projective of genus at least 1. The latter case was dealt with in [Sto07, Theorem 8.2]; we deal with the former by following the strategy of [Sto07, Proposition 3.6]. For this, we need a preliminary cohomological calculation.

Lemma 2.12. *Let m denote the number of roots of unity in K. Then for any positive integer N, the cohomology group $\mathrm{H}^1(G_{K(\mu_N)|K}, \mu_N)$ is $2m$-torsion.*

Proof. By the Chinese Remainder Theorem, it suffices to show that the cohomology group $\mathrm{H}^1(G_{K(\mu_N)|K}, \mu_{p^r})$ is $2m$-torsion for all prime powers p^r dividing N. We have the inflation-restriction exact sequence

$$0 \to \mathrm{H}^1(G_{K(\mu_{p^r})|K}, \mu_{p^r}) \to \mathrm{H}^1(G_{K(\mu_N)|K}, \mu_{p^r}) \to \mathrm{H}^1(G_{K(\mu_N)|K(\mu_{p^r})}, \mu_{p^r})^{G_{K(\mu_N)|K}},$$

in which the left-hand group is 2-torsion by [NSW13, Proposition 9.1.6]. On the other hand, the right-hand group is $\mathrm{Hom}(G_{K(\mu_N)|K(\mu_{p^r})}, \mu_{p^r}(K))$, which is m-torsion since $\mu_{p^r}(K)$ is. Thus the middle group is $2m$-torsion. □

Now we complete the proof of Theorem 2.6 in the case $Y = \mathbb{G}_m$. We may suppose, enlarging K if necessary using Lemma 2.11, that Z^{red} consists of a finite set of K-rational points. Let $y \in \mathbb{G}_m(\mathbb{A}_K)_\bullet^{\text{f-ab}}$ be such that $y_v \in Z(K_v) = Z(K)$ for all $v \in \mathfrak{P}$. Since y lies in the finite abelian descent set, we have that y is the image of an element $\hat{y} = (y_N)_N \in \mathrm{H}^1(G_K, \hat{\mathbb{Z}}(1)) = \varprojlim_N (K^\times/K^{\times N})$ under the localisation map

$$\varprojlim_N (K^\times/K^{\times N}) \to \prod_v \varprojlim_N (K_v^\times/K_v^{\times N}) .$$

Now fix a positive integer N, and consider the fields $K_z := K(\mu_N, \sqrt[N]{y_N/z})$ for $z \in Z(K)$, where $y_N \in K^\times/K^{\times N}$ is the image of $\hat{y}$. If $v \in \mathfrak{P}$ is a place which splits completely in $K(\mu_N)$, then it splits completely in some K_z, since y_N is congruent to $y_v \in Z(K_v) = Z(K)$ modulo $K_v^{\times N}$. By the Chebotarev density theorem, the density of places which split completely in a fixed K_z is $\frac{1}{[K_z:K]}$, while the density of primes which split completely in $K(\mu_N)$ is $\frac{1}{[K(\mu_N):K]}$, from which we deduce the inequality

$$\sum_{z\in Z(K)} \frac{1}{[K_z : K]} \geq \frac{1}{[K(\mu_N) : K]} .$$

Multiplying the inequality by $[K(\mu_N) : K]$, we obtain

$$\sum_{z\in Z(K)} \frac{1}{[K_z : K(\mu_N)]} \geq 1.$$

So there is some $z \in Z(K)$ such that $[K_z : K(\mu_N)] \leq \#Z(K)$. In other words, there is some $z \in Z(K)$ such that y_N/z has order at most $\#Z(K)$ in $K(\mu_N)^\times/K(\mu_N)^{\times N}$. So $(y_N/z)^{\#Z(K)!}$ is an Nth power in $K(\mu_N)$. Since the the kernel of

$$K^\times/K^{\times N} \to K(\mu_N)^\times/K(\mu_N)^{\times N}$$

is $2m$-torsion by Lemma 2.12 and inflation-restriction, we deduce that $(y_N/z)^M$ is an Nth power in K, where $M := 2m \cdot \#Z(K)!$.

So we have proved that for every positive integer N there exists some $z \in Z(K)$ such that $(y_N/z)^M$ is an Nth power in K, where M does not depend on N. Since the set $Z(K)$ is finite, there is a single z that works for all N, so we deduce that $(\hat{y}/z)^M = 1$. But the torsion subgroup of $\varprojlim_N (K^\times/K^{\times N})$ is the group $\mu_\infty(K)$ of roots of unity in K, so we deduce that $\hat{y} \in \mu_\infty(K) \cdot Z(K)$ and in particular $\hat{y} \in K^\times$. Since we have $\hat{y} \equiv y_v \bmod K_v^{\times N}$ for all N and all v, the points y_v must all be equal to $\hat{y}$ and so in fact $\hat{y} \in Z(K)$ and we are done. □

2.1.2. *The S-Selmer Section Conjecture and strong sufficiency of finite descent.* Suppose now that $\mathcal{Y}/\mathcal{O}_{K,S}$ is a regular S-integral model of a hyperbolic curve Y/K, and write

$$\mathcal{Y}(\mathbb{A}_{K,S})_\bullet^{\text{f-cov}} \subseteq \mathcal{Y}(\mathbb{A}_{K,S})_\bullet$$

for the intersection of the finite descent locus $Y(\mathbb{A}_K)_\bullet^{\text{f-cov}}$ with

$$\mathcal{Y}(\mathbb{A}_{K,S})_\bullet := \prod_{v\notin S}\mathcal{Y}(\mathcal{O}_v) \times \prod_{v\in S\,,\,v\nmid\infty} Y(K_v) \times \prod_{v\text{ real}} \pi_0(Y(K_v))\,.$$

This is the set appearing in the statement of Conjecture 1.13, strong sufficiency of finite descent. We will now show that this conjecture is equivalent to the S-Selmer Section Conjecture 1.6.

Proposition 2.13. *Let Y be a hyperbolic curve over K, and let $\mathcal{Y}$ be a regular S-integral model of Y. Then Conjecture 1.6 holds for $(\mathcal{Y}, S)$ if and only Conjecture 1.13 holds for $(\mathcal{Y}, S)$.*

One of the two implications in Proposition 2.13 is an easy consequence of Theorem 2.4. If $y \in \mathcal{Y}(\mathbb{A}_{K,S})_\bullet^{\text{f-cov}}$ is in the finite descent locus, then we know that $y = \mathrm{loc}(s)$ for some Selmer section s, necessarily S-Selmer. So if the S-Selmer Section Conjecture held for $(\mathcal{Y}, S)$, then we would deduce that s comes from an $\mathcal{O}_{K,S}$-integral point $y' \in \mathcal{Y}(\mathcal{O}_{K,S})$, and hence $y = y'$ is $\mathcal{O}_{K,S}$-integral. This gives the implication

$$\text{Conjecture 1.6 for } (\mathcal{Y}, S) \Rightarrow \text{Conjecture 1.13 for } (\mathcal{Y}, S)$$

The converse implication is rather more tricky, given that we do not know the localisation map

$$\mathrm{loc}\colon \mathrm{Sec}(Y/K)^{\mathrm{Sel}} \to Y(\mathbb{A}_K)_\bullet$$

to be injective. In place of injectivity, we instead prove the following lemma, from which the converse implication in Proposition 2.13 is immediate.

Lemma 2.14. *Let Y be a hyperbolic curve over K and let $s \in \mathrm{Sec}(Y/K)^{\mathrm{Sel}}$ be a Selmer section. Then $\mathrm{loc}(s) \in Y(K)$ if and only if s comes from a K-rational point (namely the K-rational point $\mathrm{loc}(s)$).*

For the proof of Lemma 2.14, we will need a certain lifting property for Selmer sections along finite étale coverings.

Lemma 2.15. *Let $f\colon Y' \to Y$ be a finite étale covering of hyperbolic curves over a number field K, and let $s' \in \mathrm{Sec}(Y'/K)$ be a section. Then s' is Selmer if and only if $f(s') \in \mathrm{Sec}(Y/K)$ is Selmer.*

Proof. One direction is obvious: if s' is Selmer then $f(s')$ is Selmer. Conversely, suppose that $s = f(s')$ is a Selmer section with associated adelic point $(y_v)_v \in Y(\mathbb{A}_K)$. Then [Sti12, Corollary 34] shows that the restricted section $s'|_{G_v}$ comes from a K_v-rational point $y'_v \in Y'(K_v)$ for all places v, which may be chosen to lie over y_v. If we choose a regular integral model $\mathcal{Y}/\mathcal{O}_K$ of Y and let $\mathcal{Y}'$ denote the normalisation of $\mathcal{Y}$ in the function field of Y', then $\mathcal{Y}' \to \mathcal{Y}$ is a finite morphism and the valuative criterion for properness ensures that $y'_v \in \mathcal{Y}'(\mathcal{O}_v)$ if and only if $y_v \in \mathcal{Y}(\mathcal{O}_v)$. In particular, we have $y'_v \in \mathcal{Y}'(\mathcal{O}_v)$ for all but finitely many v and so $(y'_v)_v$ is an adelic point of Y'. Hence s' is Selmer. □

Proof of Lemma 2.14. The "if" implication is trivial. For the "only if" implication, recall that there is a natural pro-discrete topology on $\mathrm{Sec}(Y/K)$, whose basic opens are the sets

$$U_{Y'} := \mathrm{im}(\mathrm{Sec}(Y'/K) \to \mathrm{Sec}(Y/K))$$

for $Y' \to Y$ a finite étale covering with Y' geometrically connected [Sti12, §4.2]. The subset $Y(K) \subseteq \mathrm{Sec}(Y/K)$ of sections arising from K-rational points is closed in this topology: if we choose a finite étale covering $Y'_{\overline{K}} \to Y_{\overline{K}}$ with $Y'_{\overline{K}}$ connected of genus ≥ 2, then the sets $U_{Y'}$ for $Y' \to Y$ a K-form of $Y'_{\overline{K}} \to Y_{\overline{K}}$ constitute an open covering of $\mathrm{Sec}(Y/K)$ with

$$Y(K) \cap U_{Y'} = \mathrm{im}(Y'(K) \to Y(K))$$

finite, hence closed, for every Y' by [Sti12, Corollary 34] and Faltings' Theorem.

Thus, according to [Sti12, Lemma 53], we have the following criterion for a section $s \in \mathrm{Sec}(Y/K)$ to come from a K-rational point: $s \in Y(K)$ if and only if $Y'(K) \neq \emptyset$ for every finite étale covering $Y' \to Y$ with Y' geometrically connected such that $s \in U_{Y'}$. So suppose that s is a Selmer section such that $\mathrm{loc}(s) = y \in Y(K)$ and that $f\colon Y' \to Y$ is such that $s \in U_{Y'}$. This means that s is the image of a section $s' \in \mathrm{Sec}(Y'/K)$, which is also Selmer by Lemma 2.15. Since $f(\mathrm{loc}(s')) = \mathrm{loc}(s) = y \in Y(K)$, we have that $\mathrm{loc}(s')$ lies in the image of $Y'_y(\mathbb{A}_K) \to Y'(\mathbb{A}_K)_\bullet$, and also in the finite descent locus by Theorem 2.4. So the theorem of the diagonal applied to Y'_y implies that $\mathrm{loc}(s') \in Y'_y(K)$, and in particular $Y'(K) \neq \emptyset$. Via the criterion of [Sti12, Lemma 53], this implies that the original section s came from a K-rational point, completing the proof. □

This completes the proof of Proposition 2.13. □

2.2. **Refined Chabauty–Kim.** For the remainder of this section, we restrict to the case $K = \mathbb{Q}$, and assume moreover that the smooth compactification of our hyperbolic curve Y has a rational point. We recall in outline the refined Chabauty–Kim method of [BD20].

Let $\pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}, b)$ be the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of Y based at either a rational basepoint or rational tangential basepoint b, and let $\pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}, b) \twoheadrightarrow \Pi$ be a $G_{\mathbb{Q}}$-equivariant quotient. For every rational point $y \in Y(\mathbb{Q})$, we write ${}_y\Pi_b$ for the pushout of the $\mathbb{Q}_p$-pro-unipotent étale path torsor $\pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}; b, y)$ along the map $\pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}, b) \to \Pi$. The map

$$j\colon Y(\mathbb{Q}) \to \mathrm{H}^1(G_{\mathbb{Q}}, \Pi(\mathbb{Q}_p)) \quad , \quad y \mapsto [{}_y\Pi_b]$$

is called the *pro-unipotent Kummer map*. Exactly the same construction also provides a local pro-unipotent Kummer map

$$j_\ell\colon Y(\mathbb{Q}_\ell) \to \mathrm{H}^1(G_\ell, \Pi(\mathbb{Q}_p)) \quad , \quad y \mapsto [{}_y\Pi_b]$$

for each prime ℓ.

The local non-abelian continuous cohomology sets $\mathrm{H}^1(G_\ell, \Pi(\mathbb{Q}_p))$ are the $\mathbb{Q}_p$-points of certain local non-abelian continuous cohomology schemes $\mathrm{H}^1(G_\ell, \Pi)$, which are affine $\mathbb{Q}_p$-schemes that are of finite type if Π is finite-dimensional [Kim05, Proposition 2]. The corresponding global object is called a *Selmer scheme*. Several different versions of this object appear in the literature; we will primarily be interested in the *refined Selmer scheme* of [BD20].

Definition 2.16. Suppose that S is a finite set of primes not containing p, and that $\mathcal{Y}/\mathbb{Z}_S$ is a regular S-integral model of Y. We define the *refined Selmer scheme*

$$\mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y})$$

to be the $\mathbb{Q}_p$-scheme parametrising continuous $G_\mathbb{Q}$-cohomology classes $\xi \in \mathrm{H}^1(G_\mathbb{Q}, \Pi)$ such that

$$\xi|_{G_\ell} \in \begin{cases} j_\ell(\mathcal{Y}(\mathbb{Z}_\ell))^{\mathrm{Zar}} & \text{if } \ell \notin S \\ j_\ell(Y(\mathbb{Q}_\ell))^{\mathrm{Zar}} & \text{if } \ell \in S \end{cases}$$

for all primes ℓ (including p), where $(-)^{\mathrm{Zar}}$ denotes Zariski-closure inside $\mathrm{H}^1(G_\ell, \Pi)$.

The refined Selmer scheme sits in a commuting square

$$\begin{array}{ccc} \mathcal{Y}(\mathbb{Z}_S) & \hookrightarrow & \mathcal{Y}(\mathbb{Z}_p) \\ \downarrow{\scriptstyle j_S} & & \downarrow{\scriptstyle j_p} \\ \mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y})(\mathbb{Q}_p) & \xrightarrow{\mathrm{loc}_p} & \mathrm{H}^1(G_p, \Pi(\mathbb{Q}_p)) \end{array} \tag{2.1}$$

where the localisation map loc_p is just restriction to the subgroup $G_p \subseteq G_\mathbb{Q}$. The *refined Chabauty–Kim locus*

$$\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\Pi} \subseteq \mathcal{Y}(\mathbb{Z}_p)$$

is defined to be the set of all points $y \in \mathcal{Y}(\mathbb{Z}_p)$ such that $j_p(y)$ lies in the scheme-theoretic image of $\mathrm{loc}_p\colon \mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y}) \to \mathrm{H}^1(G_p, \Pi)$. In the special case that Π is the whole fundamental group $\pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}, b)$, we write $\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\infty}$ for the refined Chabauty–Kim locus.

Remark 2.17. We recall for the benefit of the reader a few facts that are known about the images of the local Kummer maps above.

- If $\ell \notin S \cup \{p\}$, then $j_\ell(\mathcal{Y}(\mathbb{Z}_\ell))$ is finite [KT08, Corollary 0.2]. If moreover ℓ is a prime of good reduction for $(\mathcal{Y}, b)$ – meaning that $\mathcal{Y}_{\mathbb{Z}_\ell}$ is the complement of an étale divisor in a smooth proper $\mathbb{Z}_\ell$-scheme and b is $\mathbb{Z}_\ell$-integral – then
$$j_\ell(\mathcal{Y}(\mathbb{Z}_\ell)) \subseteq \{*\}$$
where $*$ is the basepoint in $\mathrm{H}^1(G_\ell, \Pi)$. (It can happen that $\mathcal{Y}(\mathbb{Z}_\ell)$ is empty when b is tangential, in which case the containment is strict.)
- If $\ell \in S$, then the Zariski-closure of $j_\ell(Y(\mathbb{Q}_\ell))$ has dimension ≤ 1 [BD20, Proposition 1.2.1(2)].
- If $\ell = p$ and p is a prime of good reduction for $(\mathcal{Y}, b)$, then
$$j_p(\mathcal{Y}(\mathbb{Z}_p))^{\mathrm{Zar}} \subseteq \mathrm{H}^1_f(G_p, \Pi)$$
is contained in the subscheme parametrising crystalline Π-torsors. We have equality provided $\mathcal{Y}(\mathbb{Z}_p) \neq \emptyset$ [Kim09, Theorem 1][Kim12, Proposition 1.4].

Remark 2.18. We will later see one other kind of Selmer scheme. Namely, suppose that the model $\mathcal{Y}/\mathbb{Z}_S$ of the hyperbolic curve Y is the complement of an étale divisor in a smooth proper $\mathbb{Z}_S$-scheme and that the basepoint b is $\mathbb{Z}_S$-integral. Then we also have the *Bloch–Kato Selmer scheme*

$$\mathrm{H}^1_{f,S}(G_\mathbb{Q}, \Pi)$$

parametrising Π-torsors with a compatible continuous action of $G_\mathbb{Q}$ which is unramified outside $S \cup \{p\}$ and crystalline at p (assumed $\notin S$). One can use the Bloch–Kato

Selmer scheme to cut out a (non-refined) *Chabauty–Kim locus* $\mathcal{Y}(\mathbb{Z}_p)_{S,\Pi} \subseteq \mathcal{Y}(\mathbb{Z}_p)$ via the commuting square

$$\begin{array}{ccc} \mathcal{Y}(\mathbb{Z}_S) & \hookrightarrow & \mathcal{Y}(\mathbb{Z}_p) \\ \downarrow{\scriptstyle j_S} & & \downarrow{\scriptstyle j_p} \\ \mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, \Pi(\mathbb{Q}_p)) & \xrightarrow{\mathrm{loc}_p} & \mathrm{H}^1_f(G_p, \Pi(\mathbb{Q}_p)) \, , \end{array} \tag{2.2}$$

namely the set of points $y \in \mathcal{Y}(\mathbb{Z}_p)$ such that $j_p(y)$ lies in the scheme-theoretic image of $\mathrm{loc}_p \colon \mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, \Pi) \to \mathrm{H}^1_f(G_p, \Pi)$. This is the Chabauty–Kim locus as defined in [Kim05; Kim09].

Under the above assumptions, we have the containment

$$\mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y}) \subseteq \mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, \Pi) \, ,$$

and hence the Chabauty–Kim locus contains the refined Chabauty–Kim locus: $\mathcal{Y}(\mathbb{Z}_p)_{S,\Pi} \supseteq \mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\Pi}$.

2.2.1. *Finite descent and refined Chabauty–Kim.* The $\mathbb{Q}_p$-pro-unipotent étale fundamental group $\pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}, b)$ is the Malčev completion of the profinite étale fundamental group $\pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}, b)$, meaning that there is a continuous group homomorphism

$$\pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}, b) \to \pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}, b)(\mathbb{Q}_p)$$

which is initial among continuous group homomorphisms from $\pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}, b)$ into the $\mathbb{Q}_p$-points of pro-unipotent groups over $\mathbb{Q}_p$. Moreover, for any rational point $y \in Y(\mathbb{Q})$ the $\mathbb{Q}_p$-pro-unipotent étale path torsor $\pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}; b, y)$ is the pushout of the profinite étale path torsor $\pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}; b, y)$ from $\pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}, b)$ to $\pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}, b)$, meaning that we have

$$\pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}; b, y)(R) = \pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}; b, y) \times^{\pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}, b)} \pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}, b)(R)$$

for all $\mathbb{Q}_p$-algebras R. The discussion in [Sti12, §2.4] shows that the class of the $\pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}, b)$-torsor $\pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}; b, y)$, endowed with its natural $G_{\mathbb{Q}}$-action induced from the action on $Y_{\overline{\mathbb{Q}}}$, is equal to the class of the section $\kappa(y)$ under the canonical identifications

$$\mathrm{Sec}(Y/\mathbb{Q}) = \mathrm{H}^1(G_{\mathbb{Q}}, \pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}, b)) = \{G_{\mathbb{Q}}\text{-equivariant } \pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}, b)\text{-torsors}\}/\sim \, .$$

Hence the composite

$$Y(\mathbb{Q}) \xrightarrow{\kappa} \mathrm{H}^1(G_{\mathbb{Q}}, \pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}, b)) \to \mathrm{H}^1(G_{\mathbb{Q}}, \Pi(\mathbb{Q}_p))$$

is equal to the pro-unipotent Kummer map j. Similarly, the local pro-unipotent Kummer map j_ℓ is the composite

$$Y(\mathbb{Q}_\ell) \xrightarrow{\kappa_\ell} \mathrm{H}^1(G_\ell, \pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}, b)) \to \mathrm{H}^1(G_\ell, \Pi(\mathbb{Q}_p)) \, .$$

As an easy consequence, we obtain the following compatibility between the finite descent locus and the refined Chabauty–Kim locus.

Lemma 2.19. *Let $Y/\mathbb{Q}$ be a hyperbolic curve whose smooth compactification has a rational point, let $\mathcal{Y}/\mathbb{Z}_S$ be a regular S-integral model of Y and let $p \notin S$ be prime. Then the image of the projection*

$$\mathcal{Y}(\mathbb{A}_{\mathbb{Q},S})^{\text{f-cov}}_{\bullet} \to \mathcal{Y}(\mathbb{Z}_p)$$

is contained in the refined Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\Pi}$ for every quotient Π.

Proof. Let $y = (y_v)_v \in \mathcal{Y}(\mathbb{A}_{\mathbb{Q},S})_\bullet^{\text{f-cov}}$ be an S-integral adelic point which survives the finite descent obstruction, and let $s \in \mathrm{Sec}(\mathcal{Y}/\mathbb{Z}_S)^{\mathrm{Sel}}$ be a Selmer section such that $\mathrm{loc}(s) = y$ by Theorem 2.4. It follows from the above discussion that we have a commuting diagram

$$\begin{array}{ccccc} & & \mathrm{H}^1(G_\mathbb{Q}, \pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}, b)) & \longrightarrow & \mathrm{H}^1(G_\mathbb{Q}, \Pi(\mathbb{Q}_p)) \\ & & \downarrow & & \downarrow \\ Y(\mathbb{Q}_\ell) & \xrightarrow{\kappa_\ell} & \mathrm{H}^1(G_\ell, \pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}, b)) & \longrightarrow & \mathrm{H}^1(G_\ell, \Pi(\mathbb{Q}_p)) \end{array}$$

in which the composite along the bottom row is the local pro-unipotent Kummer map j_ℓ. So if $\xi \in \mathrm{H}^1(G_\mathbb{Q}, \Pi(\mathbb{Q}_p))$ is the image of $s \in \mathrm{Sec}(Y/\mathbb{Q}) = \mathrm{H}^1(G_\mathbb{Q}, \pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}, b))$, then $\xi|_{G_\ell} = j_\ell(y_\ell)$ for all primes ℓ. It follows that $\xi \in \mathrm{Sel}_{S,\Pi}(\mathcal{Y})^{\min}(\mathbb{Q}_p)$, and hence that $y_p \in \mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\Pi}$ as desired. □

Corollary 2.20. *Let $Y/\mathbb{Q}$ be a hyperbolic curve and let $\mathcal{Y}/\mathbb{Z}_S$ be a regular S-integral model of Y which has an S-integral point or S-integral tangential point. If Conjecture 1.7 holds for $(\mathcal{Y}, S, p)$ for all primes p in a set of Dirichlet density 1, then Conjecture 1.13 holds for $(\mathcal{Y}, S)$.*

Proof. Suppose that Conjecture 1.7 holds for all primes p in a set $\mathfrak{P}$ of Dirichlet density 1. Let $Z \subset Y$ denote the reduced closed subscheme of Y consisting of the finite set $\mathcal{Y}(\mathbb{Z}_S)$ of S-integral points. If $y = (y_v)_v \in \mathcal{Y}(\mathbb{A}_{\mathbb{Q},S})_\bullet^{\text{f-cov}}$ lies in the finite descent locus, then by Lemma 2.19 we must have

$$y_p \in \mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\infty} = Z(\mathbb{Q})$$

for all $p \in \mathfrak{P}$. So we deduce by the theorem of the diagonal that

$$y \in Y(\mathbb{Q}) \cap \mathcal{Y}(\mathbb{A}_{\mathbb{Q},S})_\bullet = \mathcal{Y}(\mathbb{Z}_S)$$

and we are done. □

Combining Proposition 2.13 and Corollary 2.20 completes the proof of Theorem A. □

3. Comparison of étale and motivic Selmer schemes

The Chabauty–Kim method for $\mathcal{Y} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is understood quite explicitly thanks to work of Corwin, Dan-Cohen and Wewers [DCW16; DC20; DCW15; CDC20a; CDC20b]. Their work uses a different foundation for the theory, using the motivic fundamental groupoid of the thrice-punctured line in place of the étale fundamental groupoid. In this section, we will recall their method and explain how it relates to the Chabauty–Kim method as originally formulated by Kim. We advise the reader to skip this section on a first reading: we will not use the motivic approach in the following sections, and the point of this section is simply to justify why we can import results from the papers of Corwin, Dan-Cohen and Wewers despite the different foundations of the method.

Let S be a finite set of primes and denote by $\mathbb{Z}_S$ the ring of S-integers. Let

$$\mathrm{MT}(\mathbb{Z}_S, \mathbb{Q})$$

be the category of *mixed Tate motives* over $\mathbb{Z}_S$ with $\mathbb{Q}$-coefficients [DG05, §1]. This is a Tannakian category over $\mathbb{Q}$. Its simple objects are the *Tate motives* $\mathbb{Q}(n)$ for $n \in \mathbb{Z}$, which satisfy

$$\mathrm{Ext}^1_{\mathrm{MT}(\mathbb{Z}_S,\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n)) = K_{2n-1}(\mathbb{Z}_S)_{\mathbb{Q}} = \begin{cases} 0 & \text{if } n \leq 0, \\ \mathbb{Z}_S^\times \otimes \mathbb{Q} & \text{if } n = 1, \\ K_{2n-1}(\mathbb{Q}) \otimes \mathbb{Q} & \text{if } n \geq 2, \end{cases} \tag{3.1}$$

and

$$\mathrm{Ext}^2_{\mathrm{MT}(\mathbb{Z}_S,\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n)) = 0 \quad \text{for all } n\,.$$

A neutral fibre functor for $\mathrm{MT}(\mathbb{Z}_S,\mathbb{Q})$ is given by the *de Rham realisation functor*

$$\rho_{\mathrm{dR}}\colon \mathrm{MT}(\mathbb{Z}_S,\mathbb{Q}) \to \mathbb{Q}\text{-}\mathrm{Vect}_{\mathrm{f}}\,.$$

We write $G^{\mathrm{MT}}_{\mathbb{Q},S}$ for the Tannakian fundamental group of $\mathrm{MT}(\mathbb{Z}_S,\mathbb{Q})$ based at ρ_{dR}, which is a semidirect product of $\mathbb{G}_m$ by a pro-unipotent group $U^{\mathrm{MT}}_{\mathbb{Q},S}$.

Let $\mathcal{Y}/\mathbb{Z}_S$ be the thrice-punctured line with generic fibre $Y/\mathbb{Q}$, and let b be an S-integral base point. By this we mean either an S-integral point $b \in \mathcal{Y}(\mathbb{Z}_S)$ or an S-integral tangent vector, i.e. a nowhere vanishing section of the tangent bundle $y^*\mathcal{T}_{\mathbb{P}^1/\mathbb{Z}_S}$ over $\mathbb{Z}_S$ at a cusp $y \in \{0,1,\infty\}$. Deligne and Goncharov [DG05, Théorème 4.4] construct the "motivic fundamental group" $\pi_1(Y,b)$ of $\mathbb{P}^1 \smallsetminus \{0,1,\infty\}$, which is a pro-algebraic group in $\mathrm{MT}(\mathbb{Z}_S,\mathbb{Q})$ (in the sense of [Del89, §5]), whose de Rham realisation is the de Rham fundamental group $\pi_1^{\mathrm{dR}}(Y,b)$. We let Π be a quotient of $\pi_1(Y,b)$ in the category $\mathrm{MT}(\mathbb{Z}_S,\mathbb{Q})$. The *motivic Selmer scheme* is defined to be the affine $\mathbb{Q}$-scheme parametrising Π-torsors in $\mathrm{MT}(\mathbb{Z}_S,\mathbb{Q})$ as follows.

Definition 3.1. For a $\mathbb{Q}$-algebra R, we write

$$\mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S},\Pi^{\mathrm{dR}})(R)$$

for the set of algebraic cocycles $\xi\colon G^{\mathrm{MT}}_{\mathbb{Q},S,R} \to \Pi^{\mathrm{dR}}_R$ defined over R, modulo the natural twisting action of $\Pi^{\mathrm{dR}}(R)$. The functor

$$R \mapsto \mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S},\Pi^{\mathrm{dR}})(R)$$

is representable by an affine $\mathbb{Q}$-scheme, which is called the *motivic Selmer scheme* $\mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S},\Pi^{\mathrm{dR}})$. Equivalently (and without reference to the de Rham fibre functor), the motivic Selmer scheme can be described as the affine $\mathbb{Q}$-scheme representing the functor

$$R \mapsto \{\Pi\text{-torsors over } R \text{ in } \mathrm{MT}(\mathbb{Z}_S,\mathbb{Q})\}/\mathrm{iso}\,,$$

where R is viewed as a ring in ind-$\mathrm{MT}(\mathbb{Z}_S,\mathbb{Q})$ in the obvious way (as a direct sum of copies of the unit object).

This cohomology scheme can be understood quite explicitly [Cor21, Theorem A.4]. Every cohomology class $[\xi] \in \mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S},\Pi^{\mathrm{dR}})(R)$ can be represented by a unique cocycle $\xi\colon G^{\mathrm{MT}}_{\mathbb{Q},S,R} \to \Pi^{\mathrm{dR}}_R$ whose restriction to $\mathbb{G}_{m,R}$ is trivial. The restriction of ξ to the pro-unipotent part $U^{\mathrm{MT}}_{S,R}$ is then $\mathbb{G}_{m,R}$-equivariant, and the assignment $[\xi] \mapsto \xi|_{U^{\mathrm{MT}}_{\mathbb{Q},S,R}}$ gives an identification

$$\mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S},\Pi^{\mathrm{dR}}) = \mathrm{Z}^1(U^{\mathrm{MT}}_{\mathbb{Q},S},\Pi^{\mathrm{dR}})^{\mathbb{G}_m}$$

between the motivic Selmer scheme and the affine $\mathbb{Q}$-scheme parametrising $\mathbb{G}_m$-equivariant cocycles for the action of $U^{\mathrm{MT}}_{\mathbb{Q},S}$ on Π^{dR}. The affine ring $\mathcal{O}(U^{\mathrm{MT}}_{\mathbb{Q},S})$ of $U^{\mathrm{MT}}_{\mathbb{Q},S}$ is known as the *ring of unipotent motivic periods* [Bro17, §3.4.1], and the p-adic period map $\mathcal{O}(U^{\mathrm{MT}}_{\mathbb{Q},S}) \to \mathbb{Q}_p$ defines a point in $U^{\mathrm{MT}}_{\mathbb{Q},S}(\mathbb{Q}_p)$, which is the inverse of the point η_p^{ur} from [CÜ13, Lemma 2.2.5]. Evaluation at $(\eta_p^{\mathrm{ur}})^{-1}$ defines a morphism

$$\mathrm{ev}_p\colon \mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S}, \Pi^{\mathrm{dR}})_{\mathbb{Q}_p} = \mathrm{Z}^1(U^{\mathrm{MT}}_{\mathbb{Q},S}, \Pi^{\mathrm{dR}})^{\mathbb{G}_m}_{\mathbb{Q}_p} \to \Pi^{\mathrm{dR}}_{\mathbb{Q}_p} \tag{3.2}$$

of affine $\mathbb{Q}_p$-schemes, also called the motivic localisation map loc_Π in [CDC20a, §2.4.2].[7]

For any S-integral point $y \in \mathcal{Y}(\mathbb{Z}_S)$, Deligne and Goncharov also define a *motivic path torsor* $\pi_1(Y; b, y)$, which is a $\pi_1(Y, b)$-torsor over $\mathbb{Q}$ in $\mathrm{MT}(\mathbb{Z}_S, \mathbb{Q})$ [DG05]. Its pushout along the quotient map $\pi_1(Y, b) \twoheadrightarrow \Pi$ yields a Π-torsor

$$_y\Pi_b \coloneqq \pi_1(Y; b, y) \overset{\pi_1(Y,b)}{\times} \Pi$$

over $\mathbb{Q}$ in $\mathrm{MT}(\mathbb{Z}_S, \mathbb{Q})$, i.e. a $\mathbb{Q}$-point of $\mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S}, \Pi^{\mathrm{dR}})$. We thus have a *motivic Kummer map*

$$j_S^{\mathrm{dR}}\colon \mathcal{Y}(\mathbb{Z}_S) \to \mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S}, \Pi^{\mathrm{dR}})(\mathbb{Q}) \quad , \quad y \mapsto [_y\Pi_b]\,.$$

Additionally, when $\Pi = \pi_1(Y, b)$ is the full fundamental group, the affine ring of Π^{dR} can be identified with the shuffle algebra on the space of differential forms on Y with at worst logarithmic poles along the boundary [CDC20a, §2.2.2], so one can define a *de Rham Kummer map*

$$j_p^{\mathrm{dR}}\colon \mathcal{Y}(\mathbb{Z}_p) \to \pi_1^{\mathrm{dR}}(Y, b)(\mathbb{Q}_p)$$

sending a point $y \in \mathcal{Y}(\mathbb{Z}_p)$ to the point corresponding to the map

$$\mathcal{O}(\pi_1^{\mathrm{dR}}(Y, b)) \to \mathbb{Q}_p, \quad \omega \mapsto \int_b^y \omega\,,$$

where $\int_b^y(-)$ denotes the iterated Coleman integral. For a general quotient Π, composing with the projection $\pi_1(Y, b) \twoheadrightarrow \Pi$ yields a de Rham Kummer map

$$j_p^{\mathrm{dR}}\colon \mathcal{Y}(\mathbb{Z}_p) \to \Pi^{\mathrm{dR}}(\mathbb{Q}_p)\,.$$

These two Kummer maps fit into a commuting square [CDC20a, §2.4.2]

$$\begin{array}{ccc} \mathcal{Y}(\mathbb{Z}_S) & \hookrightarrow & \mathcal{Y}(\mathbb{Z}_p) \\ \downarrow{\scriptstyle j_S^{\mathrm{dR}}} & & \downarrow{\scriptstyle j_p^{\mathrm{dR}}} \\ \mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S}, \Pi^{\mathrm{dR}})(\mathbb{Q}_p) & \xrightarrow{\mathrm{ev}_p} & \Pi^{\mathrm{dR}}(\mathbb{Q}_p)\,, \end{array} \tag{3.3}$$

strongly reminiscent of the commuting square (2.2) in the classical Chabauty–Kim method. What we wish to explain carefully in this section is how to identify these two squares with one another, in order that the results of [CDC20a] can be applied to the classical Chabauty–Kim square (2.2) and its refined variant (2.1).

[7][CDC20a] actually defines loc_Π to be evaluation at η_p^{ur} rather than its inverse. However, it is evaluation at $(\eta_p^{\mathrm{ur}})^{-1}$ that is needed for the theory [CDC20a], as it is this choice which makes the square (3.3) below commute.

3.1. **Étale realisation.** Now let $p \notin S$ be a prime. We fix an algebraic closure $\overline{\mathbb{Q}}/\mathbb{Q}$, let $G_{\mathbb{Q}} \coloneqq \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and denote by $\mathrm{Rep}^{f,S}_{\mathbb{Q}_p}(G_{\mathbb{Q}})$ the category of continuous $\mathbb{Q}_p$-linear $G_{\mathbb{Q}}$-representations which are unramified outside $S \cup \{p\}$ and crystalline at p. This is a Tannakian category over $\mathbb{Q}_p$. There is a p-adic étale realisation functor

$$\rho_{\text{ét}}\colon \mathrm{MT}(\mathbb{Z}_S, \mathbb{Q}) \to \mathrm{Rep}^{f,S}_{\mathbb{Q}_p}(G_{\mathbb{Q}}), \quad M \mapsto M^{\text{ét}},$$

which is an exact $\mathbb{Q}$-linear tensor-functor. The étale realisation functor takes the motivic fundamental group $\pi_1(Y, b)$ to the $\mathbb{Q}_p$-pro-unipotent étale fundamental group $\pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}, b)$, and takes a motivic path torsor $\pi_1(Y; b, y)$ to the corresponding $\mathbb{Q}_p$-pro-unipotent étale path torsor $\pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}; b, y)$ [DG05, Théorème 4.4]. The étale realisation $\Pi^{\text{ét}}$, therefore, is a $G_{\mathbb{Q}}$-equivariant quotient of $\pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}, b)$ and the étale realisation of ${}_y\Pi_b$ is isomorphic to ${}_y\Pi_b^{\text{ét}}$ for all $y \in \mathcal{Y}(\mathbb{Z}_S)$, i.e. the following diagram commutes

$$\begin{array}{ccc}
\mathcal{Y}(\mathbb{Z}_S) & \longrightarrow & \mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S}, \Pi^{\mathrm{dR}})(\mathbb{Q}) \\
\| & & \big\downarrow \rho_{\text{ét}} \\
\mathcal{Y}(\mathbb{Z}_S) & \longrightarrow & \mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, \Pi^{\text{ét}})(\mathbb{Q}_p)\,.
\end{array}$$

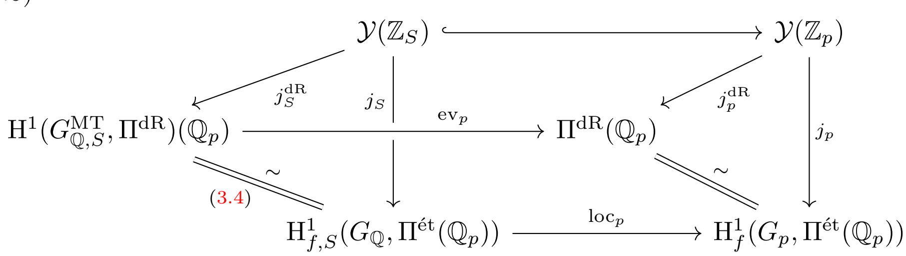
Now if R is a $\mathbb{Q}_p$-algebra and P is a Π-torsor over R in $\mathrm{MT}(\mathbb{Z}_S, \mathbb{Q})$, then the étale realisation $P^{\text{ét}}$ is a $\Pi^{\text{ét}}$-torsor over $R \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Base-changing along the multiplication map $R \otimes_{\mathbb{Q}} \mathbb{Q}_p \to R$ yields a $\Pi^{\text{ét}}$-torsor over R. This construction provides a morphism

$$\mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S}, \Pi^{\mathrm{dR}})_{\mathbb{Q}_p} \to \mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, \Pi^{\text{ét}}) \tag{3.4}$$

of affine $\mathbb{Q}_p$-schemes. The compatibility between the classical Chabauty–Kim method and the motivic Chabauty–Kim method is made precise in the following theorem.

Theorem 3.2. *Suppose that p is odd. Let $\pi_1(Y, b) \twoheadrightarrow \Pi$ be a quotient in $\mathrm{MT}(\mathbb{Z}_S, \mathbb{Q})$ and $\Pi^{\text{ét}}$ its p-adic étale realisation. Then the map (3.4) is an isomorphism, fitting into a commuting diagram*

(3.5)



where the identification $\Pi^{\mathrm{dR}}_{\mathbb{Q}_p} \cong \mathrm{H}^1_f(G_p, \Pi^{\text{ét}})$ is the one coming from the Bloch–Kato exponential [Kim12, Proposition 1.4].

The vertical face of the commuting prism above is the usual (unrefined) Chabauty–Kim square (2.2), and the top face is the motivic Chabauty–Kim square (3.3).

Remark 3.3. The commutativity of the above diagram implies that the unrefined Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)_{S,\Pi^{\text{ét}}}$ is exactly the set of points $y \in \mathcal{Y}(\mathbb{Z}_p)$ such that $j_p^{\mathrm{dR}}(y)$ lies in the scheme-theoretic image of the evaluation map ev_p. We remark that this could *a priori* be a different set to the motivic Chabauty–Kim locus of [CDC20a, Definition 2.27], which we here denote $\mathcal{Y}(\mathbb{Z}_p)^{\mathrm{mot}}_{S,\Pi}$. Indeed, one can consider the universal cocycle evaluation map [CDC20a, Definition 2.20]

$$\mathfrak{ev}\colon \mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S}, \Pi^{\mathrm{dR}}) \times U^{\mathrm{MT}}_{\mathbb{Q},S} \to \Pi^{\mathrm{dR}} \times U^{\mathrm{MT}}_{\mathbb{Q},S}\,,$$

and the definition of the motivic Chabauty–Kim locus is equivalent to saying that $\mathcal{Y}(\mathbb{Z}_p)^{\mathrm{mot}}_{S,\Pi}$ is the set of points $y \in \mathcal{Y}(\mathbb{Z}_p)$ such that $j_p^{\mathrm{dR}}(y)$ lies in the fibre of the scheme-theoretic image of $\mathfrak{ev}$ over the point $(\eta_p^{\mathrm{ur}})^{-1} \in U^{\mathrm{MT}}_{\mathbb{Q},S}(\mathbb{Q}_p)$. The fibre of the Zariski-closure of the image of $\mathfrak{ev}$ could *a priori* be larger than the Zariski-closure of the image of ev_p (which is the fibre of $\mathfrak{ev}$ at $(\eta_p^{\mathrm{ur}})^{-1}$), so we only *a priori* have the containment

$$\mathcal{Y}(\mathbb{Z}_p)_{S,\Pi^{\text{ét}}} \subseteq \mathcal{Y}(\mathbb{Z}_p)^{\mathrm{mot}}_{S,\Pi}\,,$$

see [CDC20a, Remark 2.28]. Note that the justification of [CDC20a, Remark 2.28] implicitly uses the compatibility between étale and motivic Selmer schemes made precise in Theorem 3.2, but this is not proved in [CDC20a].

3.2. **Mixed Tate categories.** The remainder of this section is devoted to a proof of Theorem 3.2. We start by recalling some general definitions and properties of mixed Tate categories. For a reference, see for instance [GZ18, Appendix A].

Definition 3.4. Let K be a field. Let $\mathcal{T}$ be a Tannakian K-category, and let $K(1)$ be a fixed rank 1 object of $\mathcal{T}$. A pair $(\mathcal{T}, K(1))$ is called a *mixed Tate category* if the objects $K(n) \coloneqq K(1)^{\otimes n}$ for $n \in \mathbb{Z}$, are mutually non-isomorphic, any irreducible object in $\mathcal{T}$ is isomorphic to some $K(n)$, and one has

$$\mathrm{Ext}^1_{\mathcal{T}}(K(0), K(n)) = 0$$

for all $n \leq 0$.

If $(\mathcal{T}, K(1))$ is a mixed Tate category, then every object $M \in \mathcal{T}$ admits a unique increasing filtration $W_\bullet M$ indexed by the even integers, called the *weight filtration*, such that $\mathrm{gr}^W_{2n} M$ is a direct sum of copies of $K(-n)$. There exists a canonical fibre functor to the category of finite-dimensional K-vector spaces

$$\omega = \omega_{\mathcal{T}}\colon \mathcal{T} \to K\text{-}\mathrm{Vect}_{\mathrm{f}}, \quad M \mapsto \bigoplus_n \mathrm{Hom}(K(-n), \mathrm{gr}^W_{2n} M)\,. \tag{3.6}$$ [8]

By the Tannakian formalism, $\mathcal{T}$ is equivalent to the category of representations of the Tannakian fundamental group

$$G_{\mathcal{T}} \coloneqq \pi_1(\mathcal{T}, \omega) \coloneqq \underline{\mathrm{Aut}}^{\otimes}(\omega)\,.$$

The subcategory of semisimple objects (direct sums of the Tate objects $K(n)$) is isomorphic to the category of graded finite-dimensional K-vector spaces (with $K(n)$ sitting in degree $-n$), which defines a surjective homomorphism $G_{\mathcal{T}} \twoheadrightarrow \mathbb{G}_m$. In other words, the homomorphism $G_{\mathcal{T}} \to \mathbb{G}_m$ is given by the action of $G_{\mathcal{T}}$ on the fibre of $K(-1)$.[9] Its kernel is a pro-unipotent group $U_{\mathcal{T}}$. The grading on $\omega(M)$ for all $M \in \mathcal{T}$ defines a splitting, so that $G_{\mathcal{T}}$ is canonically a semi-direct product:

$$G_{\mathcal{T}} = U_{\mathcal{T}} \rtimes \mathbb{G}_m\,.$$

[8] In the case of the category $\mathrm{MT}(\mathbb{Z}_S, \mathbb{Q})$ of mixed Tate motives, the canonical fibre functor and the de Rham realisation functor are canonically isomorphic [DG05, Proposition 2.10].

[9] This is the opposite of the convention in [DG05], where the homomorphism is given by the action on $K(1)$.

The Lie algebra of $U_\mathcal{T}$ has a grading via the $\mathbb{G}_m$-action. It is (non-canonically) isomorphic to the graded pro-nilpotent Lie algebra generated by the graded vector space

$$\mathrm{Ext}^1_\mathcal{T}(K(0),K(n))^\vee \text{ in degree } -n, \text{ where } n=1,2,\dots$$

with relations coming from $\mathrm{Ext}^2_\mathcal{T}(K(0),K(n))$ for $n\geq 1$. In particular, if all of these Ext^2 spaces vanish, then $\mathrm{Lie}(U_\mathcal{T})$ is a free pro-nilpotent Lie algebra. The abelianisation of $\mathrm{Lie}(U_\mathcal{T})$ is *canonically* isomorphic to the product

$$\prod_{n>0}\mathrm{Ext}^1_\mathcal{T}(K(0),K(n))^\vee$$

3.2.1. *Tate functors.*

Definition 3.5. Let K_2/K_1 be an extension of fields and let $(\mathcal{T}_1,K_1(1))$ and $(\mathcal{T}_2,K_2(1))$ be mixed Tate categories over K_1 and K_2, respectively. A *Tate functor*

$$\rho\colon\mathcal{T}_1\to\mathcal{T}_2$$

is a K_1-linear exact tensor functor equipped with an isomorphism $\rho(K_1(1))=K_2(1)$.

Any Tate functor $\rho\colon\mathcal{T}_1\to\mathcal{T}_2$ is automatically compatible with the canonical fibre functors, hence induces a homomorphism of Tannakian fundamental groups

$$\rho^*\colon G_{\mathcal{T}_2}\to G_{\mathcal{T}_1,K_2}\,.$$

This homomorphism is compatible with the semidirect product structure $G_{\mathcal{T}_i}=U_{\mathcal{T}_i}\rtimes\mathbb{G}_m$, so it induces a graded Lie algebra homomorphism

$$\rho^*\colon\mathrm{Lie}(U_{\mathcal{T}_2})\to\mathrm{Lie}(U_{\mathcal{T}_1})_{K_2}\,.$$

Lemma 3.6. *Let K_2/K_1 be an extension of fields and $\rho\colon\mathcal{T}_1\to\mathcal{T}_2$ a Tate functor. Suppose that the induced map*

$$\rho_*\colon\mathrm{Ext}^1_{\mathcal{T}_1}(K_1(0),K_1(n))\otimes_{K_1}K_2\to\mathrm{Ext}^1_{\mathcal{T}_2}(K_2(0),K_2(n))$$

is an isomorphism for all $n>0$, and that $\mathrm{Ext}^2_{\mathcal{T}_1}(K_1,K_1(n))=0$ for all n. Then the induced map

$$\rho^*\colon G_{\mathcal{T}_2}\to G_{\mathcal{T}_1,K_2}$$

is an isomorphism.

Proof. It suffices to show that the induced map

$$\rho^*\colon\mathrm{Lie}(U_{\mathcal{T}_2})\to\mathrm{Lie}(U_{\mathcal{T}_1})_{K_2}$$

is an isomorphism. The first assumption ensures that $(\rho^*)^{\mathrm{ab}}$ is an isomorphism, and this implies inductively that ρ^* is surjective modulo each step in the descending central series. So ρ^* is surjective. Since $\mathrm{Lie}(U_{\mathcal{T}_1})$ is free, ρ^* must have a right inverse $s\colon\mathrm{Lie}(U_{\mathcal{T}_1})_{K_2}\to\mathrm{Lie}(U_{\mathcal{T}_2})$. Since s^{ab} is also an isomorphism, exactly the same argument establishes that s is surjective. So ρ^*, having a surjective right inverse, is an isomorphism. □

3.3. **Comparison of Selmer schemes.** We say that a representation V of $G_{\mathbb{Q}}$ is *mixed Tate* just when it admits a finite ascending filtration $W_\bullet V$ supported in even degrees, called the *weight filtration*, such that $\mathrm{gr}^W_{2n}V$ is isomorphic to a direct sum of copies of $\mathbb{Q}_p(-n)$ for all n. The weight filtration on any mixed Tate representation is unique, and any $G_{\mathbb{Q}}$-equivariant map between mixed Tate representations is automatically strict for the weight filtrations. We write

$$\mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_{\mathbb{Q}}) \subseteq \mathrm{Rep}^{f,S}_{\mathbb{Q}_p}(G_{\mathbb{Q}})$$

for the category of *mixed Tate representations* which are unramified outside $S\cup\{p\}$ and crystalline at p. The category $\mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_{\mathbb{Q}})$ is a mixed Tate category over $\mathbb{Q}_p$: the Ext-groups

$$\mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_{\mathbb{Q}})}(\mathbb{Q}_p(0),\mathbb{Q}_p(n))$$

vanish for $n<0$ since any extension of $\mathbb{Q}_p(0)$ by $\mathbb{Q}_p(n)$ is split by the weight filtration, and vanish for $n=0$ since any continuous homomorphism $G_{\mathbb{Q}}\to\mathbb{Q}_p$ unramified outside $S\cup\{p\}$ must factor through the finite group $G^{\mathrm{ab}}_{S\cup\{p\}}$. We write $G^{\mathrm{MTR}}_{\mathbb{Q},S}$ for the Tannakian fundamental group of $\mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_{\mathbb{Q}})$ based at its canonical fibre functor. As in the category of mixed Tate motives, the canonical fibre functor is canonically $\otimes$-isomorphic to the de Rham fibre functor $\mathsf{D}_{\mathrm{dR}}(-|_{G_p})$, and we permit ourselves to identify the two.

The étale realisation functor

$$\rho_{\text{ét}}\colon \mathrm{MT}(\mathbb{Z}_S,\mathbb{Q})\to \mathrm{Rep}^{f,S}_{\mathbb{Q}_p}(G_{\mathbb{Q}})$$

has image contained inside the mixed Tate representations. So there is an induced homomorphism

$$G^{\mathrm{MTR}}_{\mathbb{Q},S}\to G^{\mathrm{MT}}_{\mathbb{Q},S,\mathbb{Q}_p} \tag{3.7}$$

of affine group schemes over $\mathbb{Q}_p$. In the appendix, we will show that the map

$$\mathrm{Ext}^1_{\mathrm{MT}(\mathbb{Z}_S,\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n))\otimes_{\mathbb{Q}}\mathbb{Q}_p\to \mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_{\mathbb{Q}})}(\mathbb{Q}_p(0),\mathbb{Q}_p(n))$$

induced by étale realisation is an isomorphism for p odd.[10] Since Ext^2's in $\mathrm{MT}(\mathbb{Z}_S,\mathbb{Q})$ vanish [DG05, Proposition 1.9], this implies by Lemma 3.6 that (3.7) is an isomorphism for p odd (as per Theorem 1.12 in the introduction).

We use this to finish the proof of Theorem 3.2. The map

$$\mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S},\Pi^{\mathrm{dR}})_{\mathbb{Q}_p}\to \mathrm{H}^1_{f,S}(G_{\mathbb{Q}},\Pi^{\text{ét}})$$

described earlier can be factored as

$$\mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S,\mathbb{Q}_p},\Pi^{\mathrm{dR}}_{\mathbb{Q}_p})\xrightarrow{\sim}\mathrm{H}^1(G^{\mathrm{MTR}}_{\mathbb{Q},S},\Pi^{\mathrm{dR}}_{\mathbb{Q}_p})\to \mathrm{H}^1_{f,S}(G_{\mathbb{Q}},\Pi^{\text{ét}})$$

where the first map is pullback along the isomorphism (3.7) and the second corresponds to the inclusion

$$\{\Pi^{\text{ét}}\text{-torsors over } R \text{ in } \mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_{\mathbb{Q}})\}/\mathrm{iso}\subseteq\{\Pi^{\text{ét}}\text{-torsors over } R \text{ in } \mathrm{Rep}^{f,S}_{\mathbb{Q}_p}(G_{\mathbb{Q}})\}/\mathrm{iso}$$

for a $\mathbb{Q}_p$-algebra R.

[10] It is a theorem of Soulé [Sou81, Theorem 1] that the domain and codomain of this map are isomorphic as $\mathbb{Q}_p$-vector spaces. However, it is not obvious – at least to us – that the isomorphism constructed by Soulé agrees with the map induced by étale realisation. This, ultimately, is the reason we need the appendix.

To show that the second map is also an isomorphism we need to show that any $\Pi^{\text{ét}}$-torsor $P^{\text{ét}}$ over a $\mathbb{Q}_p$-algebra R in $\mathrm{Rep}^{f,S}_{\mathbb{Q}_p}(G_{\mathbb{Q}})$ is automatically mixed Tate, i.e. is in $\mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_{\mathbb{Q}})$. For this, let

$$\mathbb{Q}_p = W_0\mathcal{O}(\Pi^{\text{ét}}) \subseteq W_2\mathcal{O}(\Pi^{\text{ét}}) \subseteq W_4\mathcal{O}(\Pi^{\text{ét}}) \subseteq \dots$$

denote the weight filtration on $\mathcal{O}(\Pi^{\text{ét}}) \in \text{ind-}\mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_{\mathbb{Q}})$. Picking an element $\gamma \in P^{\text{ét}}(R)$ (which exists since $\Pi^{\text{ét}}$ is pro-unipotent) induces an isomorphism $\Pi^{\text{ét}}_R \simeq P^{\text{ét}}$ and hence an isomorphism $\mathcal{O}(P^{\text{ét}}) \simeq \mathcal{O}(\Pi^{\text{ét}}) \otimes_{\mathbb{Q}_p} R$. We define

$$R = W_0\mathcal{O}(P^{\text{ét}}) \subseteq W_2\mathcal{O}(P^{\text{ét}}) \subseteq W_4\mathcal{O}(P^{\text{ét}}) \subseteq \dots$$

to be the R-linear filtration on $\mathcal{O}(P^{\text{ét}})$ corresponding to the weight filtration on $\mathcal{O}(\Pi^{\text{ét}})$ under this identification. Arguing as in [Bet22, Proposition 2.2.6], one sees that this filtration is independent of the choice of γ, as is the induced identification

$$\mathrm{gr}^W_\bullet \mathcal{O}(P^{\text{ét}}) \cong \mathrm{gr}^W_\bullet \mathcal{O}(\Pi^{\text{ét}}) \otimes_{\mathbb{Q}_p} R$$

on graded pieces. In particular, the filtration on $\mathcal{O}(P^{\text{ét}})$ is $G_{\mathbb{Q}}$-invariant, and each $\mathrm{gr}^W_{2n} \mathcal{O}(P^{\text{ét}})$ is $G_{\mathbb{Q}}$-equivariantly isomorphic to a direct sum of copies of $\mathbb{Q}_p(-n) \otimes_{\mathbb{Q}_p} R$. So $\mathcal{O}(P^{\text{ét}}) \in \text{ind-}\mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_{\mathbb{Q}})$ and we are done. This completes the proof that (3.4) is an isomorphism.

It remains to prove that (3.5) commutes. The commutativity of the triangle

$$\begin{array}{ccc}
 & & \mathcal{Y}(\mathbb{Z}_S) \\
 & \swarrow^{j_S^{\mathrm{dR}}} & \downarrow{\scriptstyle j_S} \\
\mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S}, \Pi^{\mathrm{dR}})(\mathbb{Q}_p) & & \\
 & \overset{\sim}{\underset{(3.4)}{=\!=}} & \mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, \Pi^{\text{ét}}(\mathbb{Q}_p))
\end{array}$$

follows from the fact that the $\mathbb{Q}_p$-pro-unipotent étale path torsor $\pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}; b, y)$ is the étale realisation of the motivic path torsor $\pi_1(Y; b, y)$. The commutativity of the other triangle in (3.5) is part of the usual Chabaty–Kim diagram. So it remains to justify why the square

$$\begin{array}{ccc}
\mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S}, \Pi^{\mathrm{dR}})(\mathbb{Q}_p) & \xrightarrow{\mathrm{ev}_p} & \Pi^{\mathrm{dR}}(\mathbb{Q}_p) \\
{\scriptstyle (3.4)}\Vert\wr & & \Vert\wr \\
\mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, \Pi^{\text{ét}}(\mathbb{Q}_p)) & \xrightarrow{\mathrm{loc}_p} & \mathrm{H}^1_f(G_p, \Pi^{\text{ét}}(\mathbb{Q}_p))
\end{array} \tag{3.8}$$

commutes. For this, we need to recall the definition of both the Bloch–Kato logarithm

$$\mathrm{H}^1_f(G_p, \Pi^{\text{ét}}(\mathbb{Q}_p)) \cong \Pi^{\mathrm{dR}}(\mathbb{Q}_p)$$

and the p-adic period point

$$\eta_p^{\mathrm{ur}} \in U_S^{\mathrm{MT}}(\mathbb{Q}_p)\,.$$

For the former, $\mathrm{H}^1_f(G_p, \Pi^{\text{ét}}(\mathbb{Q}_p))$ is the set of isomorphism classes of crystalline G_p-equivariant $\Pi^{\text{ét}}$-torsors over $\mathbb{Q}_p$. Given such a torsor P, we know that $\mathsf{D}_{\mathrm{dR}}(P) :=$

$\mathrm{Spec}(\mathsf{D}_{\mathrm{dR}}(\mathcal{O}(P)))$ is a $\mathsf{D}_{\mathrm{dR}}(\Pi^{\text{ét}}) \cong \Pi^{\mathrm{dR}}$-torsor in the category of weakly admissible filtered φ-modules (where the crystalline Frobenius comes from the comparison $\mathsf{D}_{\mathrm{dR}} \cong \mathsf{D}_{\mathrm{cris}}$); in particular, $\mathsf{D}_{\mathrm{dR}}(P)$ carries a Frobenius automorphism and a Hodge filtration on its affine ring. There is a unique Frobenius-invariant path $\gamma_{\mathrm{cris}} \in \mathsf{D}_{\mathrm{dR}}(P)(\mathbb{Q}_p)$ and a unique Hodge-filtered path $\gamma_{\mathrm{dR}} \in \mathsf{D}_{\mathrm{dR}}(P)(\mathbb{Q}_p)$, the latter meaning that the map $\mathcal{O}(\mathsf{D}_{\mathrm{dR}}(P)) \to \mathbb{Q}_p$ given by evaluation at γ_{dR} is compatible with Hodge filtrations. The Bloch–Kato logarithm is then given by

$$\log_{\mathrm{BK}}([P]) = \gamma_{\mathrm{dR}}^{-1}\gamma_{\mathrm{cris}}\,.$$

For the latter, given a mixed Tate filtered φ-module in the sense of [CÜ13, Definition 1.1.3], there are two $\otimes$-functorial splittings of the weight filtration on M. The first splitting s_{dR} comes from the fact that the Hodge and weight filtrations on M are opposed, so

$$M = \bigoplus_i W_{2i}M \cap F^iM\,.$$

The second splitting s_{cris} is the one coming from the Dieudonné–Manin classification of isocrystals, i.e. (in our case) the decomposition of M into φ-eigenspaces. So the automorphism of M given by

$$M \xrightarrow[\sim]{s_{\mathrm{cris}}^{-1}} \bigoplus_i \mathrm{gr}^W_{2i} M \xrightarrow[\sim]{s_{\mathrm{dR}}} M$$

is $\otimes$-natural in M, and so defines an element η_p^{ur} in the Tannaka group of the category of mixed Tate filtered φ-modules. Since the $\mathbb{Q}_p$-linear de Rham realisation $M^{\mathrm{dR}}_{\mathbb{Q}_p} \coloneqq \mathbb{Q}_p \otimes_{\mathbb{Q}} M^{\mathrm{dR}}$ of any mixed Tate motive $M \in \mathrm{MT}(\mathbb{Z}_S, \mathbb{Q})$ is a mixed Tate filtered φ-module, η_p^{ur} also defines an element of $G^{\mathrm{MT}}_{\mathbb{Q},S}(\mathbb{Q}_p)$. This element lies in $U^{\mathrm{MT}}_{\mathbb{Q},S}(\mathbb{Q}_p)$ since it acts trivially on $M = \mathbb{Q}(1)$.

The definition of η_p^{ur} makes the following clear.

Lemma 3.7. *Let M be a mixed filtered φ-module. Then $\eta_p^{\mathrm{ur}}\varphi(\eta_p^{\mathrm{ur}})^{-1}$ is the $\mathbb{Q}_p$-linear automorphism of M which acts on $W_{2i}M \cap F^iM$ by multiplication by p^{-i}.*

Remark 3.8. In fact, η_p^{ur} is uniquely characterised by this property and the fact that it acts trivially on $\mathbb{Q}_p(1)$.

Now we show that (3.8) commutes. Let P be a Π-torsor over $\mathbb{Q}_p$ in the category $\mathrm{MT}(\mathbb{Z}_S, \mathbb{Q})$, representing an element of $\mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S}, \Pi^{\mathrm{dR}})(\mathbb{Q}_p)$. We let P^{dR} and $P^{\text{ét}}$ denote the torsors obtained by applying the $\mathbb{Q}_p$-linear de Rham and étale realisation functors, respectively, and then base-changing along the multiplication map $\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathbb{Q}_p$. So $[P^{\text{ét}}] \in \mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, \Pi^{\text{ét}}(\mathbb{Q}_p))$ is the element corresponding to $[P] \in \mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S}, \Pi^{\mathrm{dR}})(\mathbb{Q}_p)$ under (3.4), and P^{dR} can equivalently be described as the $\mathbb{Q}$-linear de Rham realisation of P (which is a $G^{\mathrm{MT}}_{\mathbb{Q},S}$-equivariant Π^{dR}-torsor over $\mathbb{Q}_p$). The usual comparison isomorphisms between realisations imply that we have a canonical isomorphism

$$\mathsf{D}_{\mathrm{dR}}(P^{\text{ét}}) \cong P^{\mathrm{dR}}$$

of $\mathsf{D}_{\mathrm{dR}}(\Pi^{\text{ét}}) \cong \Pi^{\mathrm{dR}}$-torsors in the category of mixed Tate φ-modules. By mild abuse of notation, we also write $\gamma_{\mathrm{cris}} \in P^{\mathrm{dR}}(\mathbb{Q}_p)$ and $\gamma_{\mathrm{dR}} \in P^{\mathrm{dR}}(\mathbb{Q}_p)$ for the unique Frobenius-invariant and Hodge-filtered paths, respectively.

Lemma 3.9. *$\gamma_{\mathrm{dR}} \in P^{\mathrm{dR}}(\mathbb{Q}_p)$ is the unique element such that the associated cocycle*

$$\xi_{\gamma_{\mathrm{dR}}}\colon G^{\mathrm{MT}}_{\mathbb{Q},S,\mathbb{Q}_p} \to \Pi^{\mathrm{dR}}_{\mathbb{Q}_p} \quad \text{given by} \quad \xi_{\gamma_{\mathrm{dR}}}(\sigma) \coloneqq \gamma_{\mathrm{dR}}^{-1}\sigma(\gamma_{\mathrm{dR}})$$

restricts trivially to $\mathbb{G}_{m,\mathbb{Q}_p}$. Additionally, we have

$$(\eta_p^{\mathrm{ur}})^{-1}(\gamma_{\mathrm{dR}}) = \gamma_{\mathrm{cris}}\,.$$

Proof. The map $\mathrm{ev}_{\gamma_{\mathrm{dR}}}\colon \mathcal{O}(P^{\mathrm{dR}}) \to \mathbb{Q}_p$ given by evaluation at γ_{dR} is compatible with Hodge filtrations, and also (trivially) compatible with weight filtrations. Hence $\mathrm{ev}_{\gamma_{\mathrm{dR}}}$ is projection onto the 0th factor in the grading

$$\mathcal{O}(P^{\mathrm{dR}}) = \bigoplus_i W_{2i}\mathcal{O}(P^{\mathrm{dR}}) \cap F^i\mathcal{O}(P^{\mathrm{dR}})$$

associated to the torus $\mathbb{G}_{m,\mathbb{Q}_p}$. So γ_{dR} is $\mathbb{G}_{m,\mathbb{Q}_p}$-invariant, and hence its associated cocycle vanishes on $\mathbb{G}_{m,\mathbb{Q}_p}$.

Lemma 3.7 implies that $\eta_p^{\mathrm{ur}}\varphi(\eta_p^{\mathrm{ur}})^{-1}$ is a $\mathbb{Q}_p$-point of this torus, so fixes γ_{dR}. This implies that

$$(\eta_p^{\mathrm{ur}})^{-1}(\gamma_{\mathrm{dR}}) = \varphi(\eta_p^{\mathrm{ur}})^{-1}(\gamma_{\mathrm{dR}})$$

is Frobenius-invariant, and hence

$$(\eta_p^{\mathrm{ur}})^{-1}(\gamma_{\mathrm{dR}}) = \gamma_{\mathrm{cris}}\,. \qquad \square$$

Now using the above lemma, we see that the image of $[P^{\text{ét}}|_{G_p}] \in \mathrm{H}^1_f(G_p, \Pi^{\text{ét}}(\mathbb{Q}_p))$ under the Bloch–Kato logarithm is

$$\gamma_{\mathrm{dR}}^{-1}\gamma_{\mathrm{cris}} = \gamma_{\mathrm{dR}}^{-1}(\eta_p^{\mathrm{ur}})^{-1}(\gamma_{\mathrm{dR}}) = \xi_{\gamma_{\mathrm{dR}}}((\eta_p^{\mathrm{ur}})^{-1})\,,$$

which is equal to the image of $[P] \in \mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S}, \Pi^{\mathrm{dR}}(\mathbb{Q}_p)) = \mathrm{Z}^1(U^{\mathrm{MT}}_{\mathbb{Q},S}, \Pi^{\mathrm{dR}}(\mathbb{Q}_p))^{\mathbb{G}_m}$ under the map ev_p given by evaluation at $(\eta_p^{\mathrm{ur}})^{-1}$. This concludes the proof of Theorem 3.2. $\square$

4. Refined Selmer scheme of the thrice-punctured line

Let S be a finite set of primes and let $\mathcal{Y}/\mathbb{Z}_S$ be the thrice-punctured line with generic fibre $Y/\mathbb{Q}$. Let $p \notin S$ be prime and let $U = \pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}, b)$ be the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of Y at an S-integral point or tangent vector b. In this section we want to describe the refined Selmer scheme $\mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y})$ of the thrice-punctured line, for any $G_{\mathbb{Q}}$-equivariant quotient Π of U dominating the abelianisation:

$$U \twoheadrightarrow \Pi \twoheadrightarrow U^{\mathrm{ab}}.$$

Assuming $2 \in S$ (otherwise, $\mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y})$ is empty), we find that the refined Selmer scheme can be written as a union of closed subschemes $\mathrm{Sel}^{\Sigma}_{S,\Pi}(\mathcal{Y})$ over all *refinement conditions* $\Sigma \in \{0, 1, \infty\}^S$:

$$\mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y}) = \bigcup_{\Sigma} \mathrm{Sel}^{\Sigma}_{S,\Pi}(\mathcal{Y}) \tag{4.1}$$

(Corollary 4.8 below). This induces a similar description of the refined Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\Pi}$:

$$\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\Pi} = \bigcup_{\Sigma} \mathcal{Y}(\mathbb{Z}_p)^{\Sigma}_{S,\Pi}. \tag{4.2}$$

Each refinement condition $\Sigma = (\Sigma_\ell)_{\ell \in S} \in \{0,1,\infty\}^S$ can be seen as a collection of mod-ℓ congruence conditions on S-integral points for $\ell \in S$. Namely, if we denote by $\mathcal{Y}(\mathbb{Z}_S)_\Sigma$ the set of those S-integral points whose image under the mod-ℓ reduction map

$$\mathrm{red}_\ell \colon \mathcal{Y}(\mathbb{Q}_\ell) \subseteq \mathbb{P}^1(\mathbb{Q}_\ell) = \mathbb{P}^1(\mathbb{Z}_\ell) \twoheadrightarrow \mathbb{P}^1(\mathbb{F}_\ell)$$

lies in $\mathcal{Y}(\mathbb{F}_\ell) \cup \{\Sigma_\ell\}$ for all $\ell \in S$, then we can write

$$\mathcal{Y}(\mathbb{Z}_S) = \bigcup_\Sigma \mathcal{Y}(\mathbb{Z}_S)_\Sigma, \tag{4.3}$$

and the inclusion $\mathcal{Y}(\mathbb{Z}_S) \subseteq \mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\Pi}$ of S-integral points (4.3) in the refined Chabauty–Kim locus (4.2) can be refined to inclusions

$$\mathcal{Y}(\mathbb{Z}_S)_\Sigma \subseteq \mathcal{Y}(\mathbb{Z}_p)^{\Sigma}_{S,\Pi}$$

for each refinement condition $\Sigma \in \{0,1,\infty\}^S$.

In order to prove these facts, we show a general statement to the effect that the equations defining the refined subscheme $\mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y})$ inside the full (unrefined) Selmer scheme $\mathrm{H}^1_{f,S}(G_\mathbb{Q},\Pi)$ all come from the abelian (or depth 1) Selmer scheme, i.e. they are pulled back along the canonical map

$$\mathrm{H}^1_{f,S}(G_\mathbb{Q},\Pi) \to \mathrm{H}^1_{f,S}(G_\mathbb{Q},U^{\mathrm{ab}})$$

induced by the abelianisation map $\Pi \twoheadrightarrow U^{\mathrm{ab}}$ (Corollary 4.3 below). We are then reduced to the case $\Pi = U^{\mathrm{ab}}$ where we already know how to describe the refined Selmer scheme thanks to [BBK+23]. We end this section by noting in §4.5 and §4.6 how refined Chabauty–Kim loci interact with a change of the fundamental group quotient and with the S_3-action on the thrice-punctured line, which will be useful for determining these loci in concrete examples in Section 5.

Recall (Definition 2.16) that the *refined Selmer scheme* $\mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y})$ of Y with respect to Π is the scheme parametrising those continuous cohomology classes in $\mathrm{H}^1(G_\mathbb{Q},\Pi)$ whose image under the localisation map

$$\mathrm{loc}_\ell \colon \mathrm{H}^1(G_\mathbb{Q},\Pi) \to \mathrm{H}^1(G_\ell,\Pi)$$

is contained in the Zariski closure of $j_\ell(Y(\mathbb{Q}_\ell))$ for $\ell \in S$, respectively of $j_\ell(\mathcal{Y}(\mathbb{Z}_\ell))$ for $\ell \notin S$, where j_ℓ is the local Kummer map

$$j_\ell \colon Y(\mathbb{Q}_\ell) \to \mathrm{H}^1(G_\ell,\Pi(\mathbb{Q}_p)).$$

We will now determine these local Selmer images.

4.1. **Local conditions at primes outside S.** Let us first consider the primes $\ell \notin S$. The prime $\ell = 2$ has a special role: Since $\mathcal{Y}(\mathbb{F}_2) = \mathbb{P}^1(\mathbb{F}_2) \smallsetminus \{0,1,\infty\} = \emptyset$, we have $\mathcal{Y}(\mathbb{Z}_2) = \emptyset$. So in particular

$$j_2(\mathcal{Y}(\mathbb{Z}_2)) = \emptyset,$$

which leads to the local condition at 2 being unsatisfiable if $2 \notin S$:

Proposition 4.1. *If $2 \notin S$, then the refined Selmer scheme is empty:*

$$\mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y}) = \emptyset.$$

If $\ell \neq 2$, then we always have $-1 \in \mathcal{Y}(\mathbb{Z}_\ell)$, so that in particular $j_\ell(\mathcal{Y}(\mathbb{Z}_\ell))$ is non-empty. For any $y \in \mathcal{Y}(\mathbb{Z}_\ell)$, the étale path torsor $\pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}; b, y)$ is unramified ($\ell \neq p$) respectively crystalline ($\ell = p$). For $\ell \neq p$, the only unramified class in $\mathrm{H}^1(G_\ell, \Pi(\mathbb{Q}_p))$ is the trivial one [KT08, Proof of Cor. 0.3 in §2], i.e. we have

$$j_\ell(\mathcal{Y}(\mathbb{Z}_\ell)) = \{*\}.$$

At $\ell = p$, the Zariski closure of $j(\mathcal{Y}(\mathbb{Z}_p))$ in $\mathrm{H}^1(G_p, \Pi)$ is precisely the subspace of crystalline classes:

$$j_p(\mathcal{Y}(\mathbb{Z}_p))^{\mathrm{Zar}} = \mathrm{H}^1_f(G_p, \Pi).$$

This follows from the Bloch–Kato isomorphism $\log_{\mathrm{BK}}\colon \mathrm{H}^1_f(G_p, \Pi) \cong F^0\backslash\Pi^{\mathrm{dR}}$ and the fact that the de Rham Kummer map

$$j_p^{\mathrm{dR}}\colon \mathcal{Y}(\mathbb{Z}_p) \to F^0\backslash\Pi^{\mathrm{dR}}(\mathbb{Q}_p)$$

has Zariski-dense image [Kim09, Theorem 1].

In conclusion, if $2 \in S$, then the local conditions at $\ell \notin S$ defining the refined Selmer scheme $\mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y})$ are the same as those defining the unrefined Selmer scheme $\mathrm{H}^1_{f,S}(G_\mathbb{Q}, \Pi)$ (cf. Remark 2.18). We have therefore an inclusion

$$\mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y}) \subseteq \mathrm{H}^1_{f,S}(G_\mathbb{Q}, \Pi),$$

with $\mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y})$ defined as a closed subscheme by the remaining local conditions at $\ell \in S$. We now determine those local conditions at primes contained in S.

4.2. **Reduction to abelianised fundamental group.**

Lemma 4.2. *For $\ell \neq p$, the morphism $\mathrm{H}^1(G_\ell, \Pi) \to \mathrm{H}^1(G_\ell, U^{\mathrm{ab}})$ is an isomorphism.*

Proof. We have to show that the map $\mathrm{H}^1(G_\ell, \Pi(R)) \to \mathrm{H}^1(G_\ell, U^{\mathrm{ab}}(R))$ is an isomorphism for any $\mathbb{Q}_p$-algebra R. Let Π_n denote the nth quotient of Π along the lower central series. Let V_n be the kernel of the natural projection $\Pi_n \to \Pi_{n-1}$, so that we have a short exact sequence of $\mathbb{Q}_p$-group schemes

$$1 \to V_n \to \Pi_n \to \Pi_{n-1} \to 1. \tag{4.4}$$

The abelianisation $\Pi^{\mathrm{ab}} = U^{\mathrm{ab}} = V_1$ is a vector group whose $\mathbb{Q}_p$-points as a Galois representation are given by $\mathbb{Q}_p(1) \oplus \mathbb{Q}_p(1)$. In particular, V_1 is pure of weight -2. For larger n, we have a surjection

$$V_1(\mathbb{Q}_p)^{\otimes n} \twoheadrightarrow V_n(\mathbb{Q}_p), \quad x_1 \otimes \ldots \otimes x_n \mapsto [x_1, [x_2, [\ldots, x_n]]],$$

so $V_n(\mathbb{Q}_p)$ is pure of weight $-2n$ and therefore a direct sum of copies of $\mathbb{Q}_p(n)$:

$$V_n(\mathbb{Q}_p) = \bigoplus \mathbb{Q}_p(n).$$

Since $\ell \neq p$, we have

$$\mathrm{H}^1(G_\ell, \mathbb{Q}_p(n)) = 0 \quad \text{and} \quad \mathrm{H}^2(G_\ell, \mathbb{Q}_p(n)) = 0$$

for all $n \geq 2$ by [NSW13, Prop. (7.3.10) and remark], and hence $\mathrm{H}^1(G_\ell, V_n(\mathbb{Q}_p)) = \mathrm{H}^2(G_\ell, V_n(\mathbb{Q}_p)) = 0$. Since $\mathrm{H}^i(G_\ell, V_n(R)) = \mathrm{H}^i(G_\ell, V_n(\mathbb{Q}_p)) \otimes_{\mathbb{Q}_p} R$, the same holds on the level of R-valued points:

$$\mathrm{H}^1(G_\ell, V_n(R)) = 0 \quad \text{and} \quad \mathrm{H}^2(G_\ell, V_n(R)) = 0.$$

The long exact cohomology sequence in nonabelian Galois cohomology for the central extension (4.4) now implies that the map

$$(p_n)_*\colon \mathrm{H}^1(G_\ell, \Pi_n(R)) \xrightarrow{\cong} \mathrm{H}^1(G_\ell, \Pi_{n-1}(R)) \tag{4.5}$$

is an isomorphism for $n \geq 2$. Composing these isomorphisms we find that the map $\mathrm{H}^1(G_\ell, \Pi_n(R)) \to \mathrm{H}^1(G_\ell, U^{\mathrm{ab}}(R))$ induced by the abelianisation map $\Pi_n \twoheadrightarrow \Pi_1 = U^{\mathrm{ab}}$ is an isomorphism for all $n \geq 1$. Finally, to go from all Π_n to their limit Π we use the isomorphism

$$\mathrm{H}^1(G_\ell, \Pi(R)) \cong \varprojlim_n \mathrm{H}^1(G_\ell, \Pi_n(R)),$$

which is shown in [Bet22, Lemma 4.0.5] for $R = \mathbb{Q}_p$ but the same proof works for any $\mathbb{Q}_p$-algebra. □

Corollary 4.3. *Let Π be a $G_{\mathbb{Q}}$-equivariant intermediate quotient of $U \twoheadrightarrow U^{\mathrm{ab}}$. Then the refined Selmer scheme $\mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y}) \subseteq \mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, \Pi)$ is the inverse image of the refined Selmer scheme $\mathrm{Sel}^{\min}_{S,U^{\mathrm{ab}}}(\mathcal{Y})$ for the abelianisation U^{ab}:*

$$\begin{array}{ccc} \mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y}) & \subseteq & \mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, \Pi) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Sel}^{\min}_{S,U^{\mathrm{ab}}}(\mathcal{Y}) & \subseteq & \mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, U^{\mathrm{ab}}). \end{array}$$

Proof. By definition, the local conditions at primes $\ell \in S$ defining the refined Selmer scheme are determined by the Zariski closure of the image of the ℓ-adic Kummer map $j_\ell\colon Y(\mathbb{Q}_\ell) \to \mathrm{H}^1(G_\ell, \Pi(\mathbb{Q}_p))$. By Lemma 4.2, these local conditions can be checked on the level of the abelianised fundamental group. □

4.3. **Refined Selmer scheme for the abelianised fundamental group.** The refined Selmer scheme $\mathrm{Sel}^{\min}_{S,U^{\mathrm{ab}}}(\mathcal{Y})$ for the abelianised fundamental group was previously studied in [BBK+23], so we only give a brief summary of the facts we need here. We have $U^{\mathrm{ab}}(\mathbb{Q}_p) = \mathbb{Q}_p(1) \oplus \mathbb{Q}_p(1)$ as a Galois representation, with the two factors corresponding to loops around 0 and 1, respectively. By Kummer theory, we have

$$\mathrm{H}^1(G_\ell, \mathbb{Q}_p(1)) = \big(\varprojlim_n \mathbb{Q}_\ell^\times/(\mathbb{Q}_\ell^\times)^{p^n}\big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathbb{Q}_p,$$

for $\ell \neq p$, with the last isomorphism induced by the ℓ-adic valuation map $v_\ell\colon \mathbb{Q}_\ell^\times \to \mathbb{Z}$. As a consequence, we get

$$\mathrm{H}^1(G_\ell, U^{\mathrm{ab}}) \cong \mathbb{A}^2_{\mathbb{Q}_p}$$

for $\ell \neq p$. We choose our coordinates (x_ℓ, y_ℓ) on $\mathrm{H}^1(G_\ell, U^{\mathrm{ab}})$ to be those from the canonical isomorphism above, except for adding a minus sign in the y_ℓ-coordinate. Then we get following explicit description of the local Kummer map $j_\ell\colon Y(\mathbb{Q}_\ell) \to \mathrm{H}^1(G_\ell, U^{\mathrm{ab}}(\mathbb{Q}_p))$ for the quotient U^{ab}.

Lemma 4.4. *For $\ell \neq p$, the local Kummer map j_ℓ is given as $z \mapsto (v_\ell(z), -v_\ell(1-z))$ in the coordinates (x_ℓ, y_ℓ).* □

Remark 4.5. Adding a minus sign in the second coordinate means that our choice of coordinates differs from that in [BBK+23]. However it agrees with the choice made in [CDC20a], which we will adopt for the discussion in Section 5, where we make explicit calculations for small S working in the polylogarithmic quotient.

Taking the choice of coordinates into account, we showed in [BBK+23, Lemma 2.9] that the Zariski closure of the image $j_\ell(Y(\mathbb{Q}_\ell))$ equals the union $R_0 \cup R_1 \cup R_\infty$ of the three following linear subspaces:

$$\begin{aligned} R_0 &= \{(x,y) \in \mathbb{A}^2 \mid y = 0\}, \\ R_1 &= \{(x,y) \in \mathbb{A}^2 \mid x = 0\}, \\ R_\infty &= \{(x,y) \in \mathbb{A}^2 \mid x + y = 0\}, \end{aligned}$$

where the naming convention comes from the fact that these linear conditions correspond to the three cusps.

Again by Kummer theory, the unrefined Selmer scheme $\mathrm{H}^1_{f,S}(G_\mathbb{Q}, U^{\mathrm{ab}})$ is an affine space of the form $\mathbb{A}^S_{\mathbb{Q}_p} \times \mathbb{A}^S_{\mathbb{Q}_p}$, as discussed in [BBK+23, §2.4].[11] We choose coordinates $((x_\ell)_{\ell\in S}, (y_\ell)_{\ell\in S})$ such that the global Kummer map

$$j_S \colon \mathcal{Y}(\mathbb{Z}_S) \to \mathrm{H}^1_{f,S}(G_\mathbb{Q}, U^{\mathrm{ab}}) = \mathbb{A}^S_{\mathbb{Q}_p} \times \mathbb{A}^S_{\mathbb{Q}_p}$$

is given by

$$z \mapsto ((v_\ell(z))_{\ell\in S}, (-v_\ell(1-z))_{\ell\in S}). \tag{4.6}$$

The coordinates y_ℓ thus differ again by a sign from the ones used in [BBK+23].

The localisation map

$$\mathrm{loc}_\ell \colon \mathrm{H}^1_{f,S}(G_\mathbb{Q}, U^{\mathrm{ab}}) \to \mathrm{H}^1(G_\ell, U^{\mathrm{ab}})$$

for $\ell \neq p$ is now simply the projection $\mathbb{A}^S \times \mathbb{A}^S \to \mathbb{A}^2$ onto the (x_ℓ, y_ℓ)-components. A point $((x_\ell)_{\ell\in S}, (y_\ell)_{\ell\in S})$ of the Selmer scheme is therefore contained in the *refined* Selmer scheme if and only if for all $\ell \in S$ we have $(x_\ell, y_\ell) \in R_0 \cup R_1 \cup R_\infty$. This yields the following explicit description of the refined Selmer scheme.

Proposition 4.6. *Assume $2 \in S$. Then the refined Selmer scheme $\mathrm{Sel}^{\min}_{S,U^{\mathrm{ab}}}(\mathcal{Y})$ for the abelianised fundamental group is the subscheme of $\mathbb{A}^S_{\mathbb{Q}_p} \times \mathbb{A}^S_{\mathbb{Q}_p}$ defined by the equations*

$$x_\ell y_\ell (x_\ell + y_\ell) = 0 \quad \textit{for all } \ell \in S.$$

It can be expressed as the union of the following $3^{\#S}$ linear subspaces

$$\mathrm{Sel}^{\min}_{S,U^{\mathrm{ab}}}(\mathcal{Y}) = \bigcup_{\Sigma} \mathrm{Sel}^{\Sigma}_{S,U^{\mathrm{ab}}}(\mathcal{Y}), \tag{4.7}$$

where

$$\mathrm{Sel}^{\Sigma}_{S,U^{\mathrm{ab}}}(\mathcal{Y}) := \{((x_\ell)_{\ell\in S}, (y_\ell)_{\ell\in S}) \mid (x_\ell, y_\ell) \in R_{\Sigma_\ell} \textit{ for } \ell \in S\} \subseteq \mathbb{A}^S_{\mathbb{Q}_p} \times \mathbb{A}^S_{\mathbb{Q}_p}$$

and $\Sigma = (\Sigma_\ell)_{\ell\in S} \in \{0, 1, \infty\}^S$. □

4.4. **Refined Selmer scheme for general fundamental group quotients.** Via Corollary 4.3, we obtain a similar description of the refined Selmer scheme for larger quotients. Assume $2 \in S$ and let $U \twoheadrightarrow \Pi \twoheadrightarrow U^{\mathrm{ab}}$ be any $G_\mathbb{Q}$-equivariant intermediate quotient.

[11] The Selmer scheme $\mathrm{H}^1_{f,S}(G_\mathbb{Q}, U^{\mathrm{ab}})$ is denoted $\mathrm{Sel}_{S,1}$ in [BBK+23].

Definition 4.7. For each $\Sigma = (\Sigma_\ell)_{\ell \in S} \in \{0, 1, \infty\}^S$ define $\mathrm{Sel}^{\Sigma}_{S,\Pi}(\mathcal{Y})$ as the pull-back of $\mathrm{Sel}^{\Sigma}_{S,U^{\mathrm{ab}}}(\mathcal{Y})$ from the abelian Selmer scheme:

$$\begin{array}{ccc} \mathrm{Sel}^{\Sigma}_{S,\Pi}(\mathcal{Y}) & \subseteq & \mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, \Pi) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Sel}^{\Sigma}_{S,U^{\mathrm{ab}}}(\mathcal{Y}) & \subseteq & \mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, U^{\mathrm{ab}}). \end{array}$$

We call $\mathrm{Sel}^{\Sigma}_{S,\Pi}(\mathcal{Y})$ the refined Selmer scheme for the *refinement condition* Σ.

Moreover, define the refined Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)^{\Sigma}_{S,\Pi} \subseteq \mathcal{Y}(\mathbb{Z}_p)$ for the refinement condition Σ as the set of all $z \in \mathcal{Y}(\mathbb{Z}_p)$ whose image under the p-adic Kummer map j_p is contained in the scheme-theoretic image of $\mathrm{Sel}^{\Sigma}_{S,\Pi}(\mathcal{Y})$ under the localisation map loc_p.

Then Corollary 4.3 yields the following:

Corollary 4.8. *The refined Selmer scheme* $\mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y})$ *can be written as a union*

$$\mathrm{Sel}^{\min}_{S,\Pi}(\mathcal{Y}) = \bigcup_{\Sigma} \mathrm{Sel}^{\Sigma}_{S,\Pi}(\mathcal{Y})$$

over all refinement conditions $\Sigma \in \{0, 1, \infty\}^S$*. Similarly, the refined Chabauty–Kim locus* $\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\Pi}$ *can be written as a union*

$$\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\Pi} = \bigcup_{\Sigma} \mathcal{Y}(\mathbb{Z}_p)^{\Sigma}_{S,\Pi}.$$

Remark 4.9. As discussed at the beginning of this section, the presentation of the refined Selmer scheme and the refined Chabauty–Kim locus as a union over refinement conditions $\Sigma = (\Sigma_\ell)_{\ell \in S} \in \{0, 1, \infty\}^S$ corresponds to writing the S-integral points as a union

$$\mathcal{Y}(\mathbb{Z}_S) = \bigcup_{\Sigma} \mathcal{Y}(\mathbb{Z}_S)_{\Sigma},$$

with $\mathcal{Y}(\mathbb{Z}_S)_{\Sigma}$ denoting the set of those S-integral points whose mod-ℓ reduction lies in $\mathcal{Y}(\mathbb{F}_\ell) \cup \{\Sigma_\ell\}$ for all $\ell \in S$. We have the inclusion

$$\mathcal{Y}(\mathbb{Z}_S)_{\Sigma} \subseteq \mathcal{Y}(\mathbb{Z}_p)^{\Sigma}_{S,\Pi}$$

for all $\Sigma \in \{0, 1, \infty\}^S$.

Remark 4.10. We have worked in this section with the p-adic étale Selmer scheme rather than the motivic Selmer scheme because the refinement conditions are phrased in terms of the localisation maps

$$\mathrm{loc}_\ell \colon \mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, \Pi) \to \mathrm{H}^1(G_\ell, \Pi),$$

and these are only defined over $\mathbb{Q}_p$. However, the equations which we computed for the refined Selmer scheme have coefficients in $\mathbb{Q}$ and are independent of p, so they could be used a posteriori to define a refined Selmer subscheme of the motivic Selmer scheme $\mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S}, \Pi^{\mathrm{dR}})$ over $\mathbb{Q}$. It would be interesting to find a motivic definition of the refined Selmer scheme that is independent of p by construction.

For later use we note two functoriality properties of refined Chabauty–Kim loci: its behaviour with respect to a change of the fundamental group quotient, and its interaction with the natural S_3-action on the thrice-punctured line.

4.5. **Change of fundamental group quotient.** Smaller quotients of the fundamental group result in larger corresponding Chabauty–Kim loci:

Lemma 4.11. *Let $U \twoheadrightarrow \Pi_1 \twoheadrightarrow \Pi_2 \twoheadrightarrow U^{\mathrm{ab}}$ be two $G_{\mathbb{Q}}$-equivariant quotients of U dominating the abelianisation. Then we have the inclusion $\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\Pi_1} \subseteq \mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\Pi_2}$. More precisely, for each refinement condition $\Sigma \in \{0,1,\infty\}^S$, we have the inclusion $\mathcal{Y}(\mathbb{Z}_p)^{\Sigma}_{S,\Pi_1} \subseteq \mathcal{Y}(\mathbb{Z}_p)^{\Sigma}_{S,\Pi_2}$.*

Proof. The quotient map $\Pi_1 \twoheadrightarrow \Pi_2$ induces the vertical maps in the commutative diagram

$$\begin{array}{ccc} \mathrm{Sel}^{\Sigma}_{S,\Pi_1}(\mathcal{Y}) & \xrightarrow{\mathrm{loc}_p} & \mathrm{H}^1_f(G_p,\Pi_1) \\ \downarrow & & \downarrow \\ \mathrm{Sel}^{\Sigma}_{S,\Pi_2}(\mathcal{Y}) & \xrightarrow{\mathrm{loc}_p} & \mathrm{H}^1_f(G_p,\Pi_2). \end{array}$$

The inclusion $\mathcal{Y}(\mathbb{Z}_p)^{\Sigma}_{S,\Pi_1} \subseteq \mathcal{Y}(\mathbb{Z}_p)^{\Sigma}_{S,\Pi_2}$ now follows from the definitions. □

4.6. **The S_3-action.** The thrice-punctured line $\mathcal{Y} = \mathbb{P}^1 \smallsetminus \{0,1,\infty\}$ carries a natural S_3-action given by the Möbius transformations

$$z, \quad \frac{1}{z}, \quad 1-z, \quad \frac{1}{1-z}, \quad \frac{z-1}{z}, \quad \frac{z}{z-1},$$

which permute the three cusps $\{0,1,\infty\}$. (We identify S_3 with the symmetric group $S_{\{0,1,\infty\}}$.) This action can be exploited for calculating refined Chabauty–Kim loci.

For $1 \leq n \leq \infty$, denote by U_n the depth-n quotient of the $\mathbb{Q}_p$-pro-unipotent étale fundamental group U of the thrice-punctured line $\mathcal{Y}$, and let

$$\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,n} = \bigcup_{\Sigma} \mathcal{Y}(\mathbb{Z}_p)^{\Sigma}_{S,n}$$

be the associated refined Chabauty–Kim locus, written as a union over refinement conditions $\Sigma \in \{0,1,\infty\}^S$ as in Corollary 4.8.

Lemma 4.12. *The refined Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,n}$ is stable under the S_3-action. More precisely, the S_3-action permutes the subsets $\mathcal{Y}(\mathbb{Z}_p)^{\Sigma}_{S,n}$ according to the action of $S_3 = S_{\{0,1,\infty\}}$ on $\{0,1,\infty\}^S$, in the sense that*

$$\sigma(\mathcal{Y}(\mathbb{Z}_p)^{\Sigma}_{S,n}) = \mathcal{Y}(\mathbb{Z}_p)^{\sigma(\Sigma)}_{S,n}$$

for all $\Sigma \in \{0,1,\infty\}^S$ and $\sigma \in S_3$.

Proof. The first statement is proved in [BBK+23, Corollary 2.15(i)] as a consequence of more general functoriality properties of Chabauty–Kim loci. The second statement is proved in [BBK+23, Corollary 2.15(ii)]. It is stated there only for $n \leq 2$ because the sets $\mathcal{Y}(\mathbb{Z}_p)^{\Sigma}_{S,n}$, had only been defined in depth at most 2 in loc. cit., but the same proof works for arbitrary $n \leq \infty$. □

Remark 4.13. Lemma 4.12 does not hold for arbitrary fundamental group quotients, i.e., if $U \twoheadrightarrow \Pi$ is any $G_{\mathbb{Q}}$-equivariant quotient and σ an automorphism of $\mathcal{Y}/\mathbb{Z}_S$, then the refined Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\Pi}$ is in general not stable under σ. The fact that the loci for the descending central series quotients U_n are stable under σ relies on those quotients being σ-equivariant, in a suitable sense that takes different base points into account. However, if for example $\Pi = U_{\mathrm{PL}}$ is the polylogarithmic quotient studied in [CDC20a], the corresponding locus $\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,\mathrm{PL}}$ will be stable only under the involution $z \mapsto 1/z$ but not under all automorphisms of the thrice-punctured line. This is the motivation for defining an S_3-symmetrised (unrefined) Chabauty–Kim locus in [CDC20a, §5.2].

5. Determination of Chabauty–Kim loci

The goal of this section is to prove that the thrice-punctured line satisfies the refined Kim's conjecture for $S = \{2\}$ and all odd primes p (Theorem B). The main steps are the following:

(1) We obtain from [CDC20a] a formula for the localisation map

$$\mathrm{loc}_p \colon \mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, U_{\mathrm{PL}}) \to \mathrm{H}^1_f(G_p, U_{\mathrm{PL}}),$$

where U_{PL} is the *polylogarithmic quotient* of the fundamental group (Definition 5.1). Corwin–Dan-Cohen work in the motivic setting, so we use our motivic-étale comparison theorem from Section 3.3 to transfer their results to the usual étale setting.

(2) From Section 4 we know the equations cutting out the *refined* Selmer scheme inside the full Selmer scheme of U_{PL}.

(3) Specialising to $S = \{2\}$, we find functions that vanish on the scheme-theoretic image of the refined Selmer scheme $\mathrm{Sel}^{(1)}_{\{2\},U_{\mathrm{PL}}}(\mathcal{Y})$ under the p-adic localisation map, for the particular refinement condition $\Sigma = (1)$. As a result, we find infinitely many functions vanishing on the associated refined Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)^{(1)}_{\{2\},U_{\mathrm{PL}}}$:

$$\log(z) = 0, \quad \mathrm{Li}_k(z) = 0 \quad \text{for } k \geq 2 \text{ even}.$$

(4) We then show that the only common solution in $\mathcal{Y}(\mathbb{Z}_p)$ of these equations is the $\{2\}$-integral point $z = -1$.

(5) By exploiting the S_3-action, we show that those calculations for the particular refinement condition $\Sigma = (1)$ are enough to determine the complete refined Chabauty–Kim locus, thus confirming that it consists exactly of the $\{2\}$-integral points $\{2, -1, 1/2\}$.

In §5.6 we show how the same methods can be used to prove the classical (non-refined) Kim's conjecture in the case $S = \emptyset$ (Theorem D). This had previously been proved for half the primes in [BDCKW18, §6] in depth 2; we can prove it for all primes by going into higher depth. Finally, we end by showing in §5.7 that certain higher genus curves also satisfy the a Kim-like conjecture, strong enough to deduce the Selmer Section Conjecture for those curves (Theorem E).

5.1. The polylogarithmic quotient.

Let S be any finite set of primes, fix a prime $p \notin S$ and let $U = \pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}, b)$ the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of Y with respect to the tangential base point $b = \vec{1}_0$. Here, $\vec{1}_0$ denotes the tangent vector at 0 corresponding to 1 under the canonical identification $T_0\mathbb{P}^1 \cong \mathbb{A}^1$. The inclusion $Y \hookrightarrow \mathbb{G}_m$ induces a $G_{\mathbb{Q}}$-equivariant homomorphism of fundamental groups $U \to \mathbb{Q}_p(1)$.

Definition 5.1. The *polylogarithmic quotient* of U is defined as

$$U_{\mathrm{PL}} = U/[N, N],$$

where N is the kernel of the homomorphism $U \to \mathbb{Q}_p(1)$ induced by the inclusion $Y \hookrightarrow \mathbb{G}_m$.

The de Rham variant of the polylogarithmic quotient was studied extensively in [CDC20a; CDC20b]. In order to define it, one can start with the full $\mathbb{Q}$-pro-unipotent de Rham fundamental group U^{dR}, defined as the Tannaka group of the category of unipotent $\mathbb{Q}$-vector bundles with connection on Y, and define its polylogarithmic quotient $U^{\mathrm{dR}} \twoheadrightarrow U^{\mathrm{dR}}_{\mathrm{PL}}$ in the same way as U_{PL} is defined as a quotient of the étale fundamental group U above.

Remark 5.2. The polylogarithmic quotient is of motivic origin: let $\pi_1(Y, b)$ the $\mathbb{Q}$-prounipotent motivic fundamental group in $\mathrm{MT}(\mathbb{Z}_S, \mathbb{Q})$. Then a definition similar to Definition 5.1 defines the polylogarithmic quotient $\pi_1(Y, b) \twoheadrightarrow \pi_1(Y, b)_{\mathrm{PL}}$ in the category $\mathrm{MT}(\mathbb{Z}_S, \mathbb{Q})$. It specialises to U_{PL} under the p-adic étale realisation functor, and to $U^{\mathrm{dR}}_{\mathrm{PL}}$ under the de Rham realisation functor.

Let $U_{\mathrm{PL},n}$ (resp. $U^{\mathrm{dR}}_{\mathrm{PL},n}$) denote the depth-$n$ quotient of the polylogarithmic étale (resp. de Rham) fundamental group, for $n \leq \infty$. The group $U^{\mathrm{dR}}_{\mathrm{PL},n}$ has coordinate ring given by

$$\mathcal{O}(U^{\mathrm{dR}}_{\mathrm{PL},n}) = \mathbb{Q}[\log, \mathrm{Li}_1, \mathrm{Li}_2, \ldots, \mathrm{Li}_n]. \tag{5.1}$$

In terms of these coordinates, the p-adic de Rham Kummer map

$$j_p^{\mathrm{dR}} \colon \mathcal{Y}(\mathbb{Z}_p) \to U^{\mathrm{dR}}_{\mathrm{PL}}(\mathbb{Q}_p)$$

is given by

$$\begin{aligned} \log(j_p^{\mathrm{dR}}(z)) &= \log^p(z), \\ \mathrm{Li}_k(j_p^{\mathrm{dR}}(z)) &= \mathrm{Li}_k^p(z) \quad \text{for } k \geq 1. \end{aligned}$$

In other words, the algebraic functions log and Li_k on the de Rham fundamental group pull back to the p-adic logarithm and polylogarithms on $\mathcal{Y}(\mathbb{Z}_p)$. We sometimes drop the superscript $(-)^p$ and write simply $\log(z)$ and $\mathrm{Li}_k(z)$ when no confusion with the algebraic functions on $U^{\mathrm{dR}}_{\mathrm{PL},n}$ by the same name is likely to arise.

5.2. **Coordinates on the polylogarithmic Selmer scheme.** In order to carry out the Chabauty–Kim method for the polylogarithmic quotient, we need to understand the Bloch–Kato Selmer scheme $\mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, U_{\mathrm{PL},n})$. Its motivic analogue was studied in [CDC20a], so we apply our motivic-étale comparison theorem from Section 3.3 to transfer their results to the étale setting.

Proposition 5.3. *The Selmer scheme $\mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, U_{\mathrm{PL},n})$ is an affine space over $\mathbb{Q}_p$ of dimension*

$$\dim \mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, U_{\mathrm{PL},n}) = 2\#S + \lfloor (n-1)/2 \rfloor,$$

with coordinate ring isomorphic to

$$\mathcal{O}(\mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, U_{\mathrm{PL},n})) \cong \mathbb{Q}_p[(x_\ell)_{\ell \in S}, (y_\ell)_{\ell \in S}, z_3, z_5, \ldots] \tag{5.2}$$

where the coordinates z_{2i+1} are indexed by the odd integers in the interval $[3, n]$.

Proof. By Theorem 3.2, the étale Selmer scheme is isomorphic to the motivic one, basechanged from $\mathbb{Q}$ to $\mathbb{Q}_p$:

$$\mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, U_{\mathrm{PL},n}) \cong \mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S}, U^{\mathrm{dR}}_{\mathrm{PL},n})_{\mathbb{Q}_p}.$$

By [CDC20a, Corollary 3.11 and §3.3.1], the motivic Selmer scheme is an affine space with specified coordinates. The translation between their and our naming convention is as follows:

$$x_\ell = \Phi^{\tau_\ell}_{e_0}, \quad y_\ell = \Phi^{\tau_\ell}_{e_1}, \quad z_{2i+1} = \Phi^{\sigma_{2i+1}}_{e_1 e_0 \cdots e_0}.$$ ∎

Remark 5.4. The coordinates x_ℓ and y_ℓ on $\mathrm{H}^1_{f,S}(G_\mathbb{Q}, U_{\mathrm{PL},n})$ from Proposition 5.3 agree with those defined in Section 4.3 on the abelian Selmer scheme $\mathrm{H}^1_{f,S}(G_\mathbb{Q}, U^{\mathrm{ab}})$ when they are pulled back along the natural morphism

$$\mathrm{H}^1_{f,S}(G_\mathbb{Q}, U_{\mathrm{PL},n}) \to \mathrm{H}^1_{f,S}(G_\mathbb{Q}, U^{\mathrm{ab}})$$

induced by the abelianisation map $U_{\mathrm{PL},n} \twoheadrightarrow U_{\mathrm{PL},1} = U^{\mathrm{ab}}$.

Remark 5.5. Unlike the x_ℓ and y_ℓ, the coordinates z_{i+1} on $\mathrm{H}^1_{f,S}(G_\mathbb{Q}, U_{\mathrm{PL},n})$ from Proposition 5.3 are not canonical. As discussed in [CDC20a, §4.1], they depend on a choice of free generators $\{\tau_\ell : \ell \in S\} \cup \{\sigma_3, \sigma_5, \ldots\}$ of the Lie algebra of $U^{\mathrm{MT}}_{\mathbb{Q},S}$, the unipotent radical of the mixed Tate Galois group of $\mathbb{Z}_S$. There is a natural choice for the τ_ℓ, which determines the coordinates x_ℓ and y_ℓ, but there is no (obvious) natural choice for the σ_{2i+1}.

5.3. **The Chabauty–Kim diagram for the polylog quotient.** Consider the Chabauty–Kim diagram (2.2) for the depth-n polylogarithmic quotient $U_{\mathrm{PL},n}$:

$$\begin{array}{ccc} \mathcal{Y}(\mathbb{Z}_S) & \hookrightarrow & \mathcal{Y}(\mathbb{Z}_p) \\ \downarrow{\scriptstyle j_S} & & \downarrow{\scriptstyle j_p} \\ \mathrm{H}^1_{f,S}(G_\mathbb{Q}, U_{\mathrm{PL},n})(\mathbb{Q}_p) & \xrightarrow{\mathrm{loc}_p} & \mathrm{H}^1_f(G_p, U_{\mathrm{PL},n})(\mathbb{Q}_p)\,. \end{array} \tag{5.3}$$

Thanks to our motivic-étale comparison theorem and the work of [CDC20a], we understand the localisation map quite explicitly. We have the coordinates (5.2) on $\mathrm{H}^1_{f,S}(G_\mathbb{Q}, U_{\mathrm{PL},n})$ and the cordinates (5.1) on $\mathrm{H}^1_f(G_p, U_{\mathrm{PL},n})$ via the Bloch–Kato logarithm $\mathrm{H}^1_f(G_p, U_{\mathrm{PL},n}) \cong (U^{\mathrm{dR}}_{\mathrm{PL},n})_{\mathbb{Q}_p}$.

Theorem 5.6. *The localisation map* $\mathrm{loc}_p\colon \mathrm{H}^1_{f,S}(G_\mathbb{Q}, U_{\mathrm{PL},n}) \to \mathrm{H}^1_f(G_p, U_{\mathrm{PL},n})$ *in diagram* (5.3) *corresponds to the homomorphism on coordinate rings*

$$\mathrm{loc}_p^\sharp\colon \mathbb{Q}_p[\log, \mathrm{Li}_1, \ldots, \mathrm{Li}_n] \to \mathbb{Q}_p[(x_\ell)_{\ell\in S}, (y_\ell)_{\ell\in S}, (z_{2i+1})_{1\le i\le \lfloor (n-1)/2\rfloor}],$$

which is explicitly given as follows:

$$\mathrm{loc}_p^\sharp(\log) = \sum_{\ell\in S} a_{\tau_\ell} x_\ell, \tag{5.4}$$

$$\begin{aligned} \mathrm{loc}_p^\sharp(\mathrm{Li}_k) = &\sum_{\ell_1,\ldots,\ell_{k-1},q\in S} a_{\tau_{\ell_1}\cdots\tau_{\ell_{k-1}}\tau_q} x_{\ell_1}\cdots x_{\ell_{k-1}} y_q \\ &+ \sum_{i=1}^{\lfloor (k-1)/2\rfloor} \sum_{\ell_1,\ldots,\ell_{k-2i-1}\in S} a_{\sigma_{2i+1}\tau_{\ell_1}\cdots\tau_{\ell_{k-2i-1}}} x_{\ell_1}\cdots x_{\ell_{k-2i-1}} z_{2i+1}. \end{aligned} \tag{5.5}$$

Here, the coefficients a_w whose subscripts are words in the symbols $\{\tau_\ell : \ell \in S\} \cup \{\sigma_3, \sigma_5, \ldots\}$ are constants in $\mathbb{Q}_p$.

Proof. By Theorem 3.2, the étale localisation map is isomorphic to the motivic localisation map (3.2)

$$\mathrm{ev}_p\colon \mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S}, U^{\mathrm{dR}}_{\mathrm{PL},n})_{\mathbb{Q}_p} \to (U^{\mathrm{dR}}_{\mathrm{PL},n})_{\mathbb{Q}_p}.$$

As discussed in Section 3, this map is given by identifying $\mathrm{H}^1(G^{\mathrm{MT}}_{\mathbb{Q},S}, U^{\mathrm{dR}}_{\mathrm{PL},n})_{\mathbb{Q}_p}$ with the space $\mathrm{Z}^1(U^{\mathrm{MT}}_{\mathbb{Q},S}, U^{\mathrm{dR}}_{\mathrm{PL},n})^{\mathbb{G}_m}_{\mathbb{Q}_p}$ of $\mathbb{G}_m$-equivariant cocycles and evaluating at a certain $\mathbb{Q}_p$-point $(\eta_p^{\mathrm{ur}})^{-1} \in U^{\mathrm{MT}}_{\mathbb{Q},S}(\mathbb{Q}_p)$. This is equivalent to specialising the "universal cocycle evaluation map" [CDC20a, Definition 2.20] to the point $(\eta_p^{\mathrm{ur}})^{-1}$. A formula for the universal cocycle evaluation map is proved in [CDC20a, Corollary 3.11]. That formula involves various motivic periods $f_w \in \mathcal{O}(U^{\mathrm{MT}}_{\mathbb{Q},S})$; specialising to $(\eta_p^{\mathrm{ur}})^{-1}$ amounts to evaluating them at this point, or equivalently applying the p-adic period map $\mathrm{per}_p\colon \mathcal{O}(U^{\mathrm{MT}}_{\mathbb{Q},S}) \to \mathbb{Q}_p$:

$$a_w \coloneqq f_w((\eta_p^{\mathrm{ur}})^{-1}) = \mathrm{per}_p(f_w) \in \mathbb{Q}_p.$$

In this way the formula by Corwin–Dan-Cohen specialises to one stated here, up to using different names for the coordinates. □

Remark 5.7. As discussed in Remark 5.5 above, the choice of coordinates on the polylogarithmic Selmer scheme is not canonical but depends on a choice of free generators $\{\tau_\ell : \ell \in S\} \cup \{\sigma_3, \sigma_5, \dots\}$ of the Lie algebra of $U^{\mathrm{MT}}_{\mathbb{Q},S}$. As a consequence, the p-adic constants a_w appearing as coefficients in the localisation map (5.4)–(5.5) also depend on this choice. The subscripts of the a_w are in fact words in those chosen generators.

Partly due to the choices involved in their definition, the p-adic periods a_w are difficult to determine in practice. As explained in [CDC20a, §4.1], there are however choices for which at least those a_w subscripted by a single letter τ_ℓ or σ_{2i+1} have particular known values. These values are p-adic logarithms of the primes in S in the case of a_{τ_ℓ} and p-adic zeta values in the case of $a_{\sigma_{2i+1}}$:

$$a_{\tau_\ell} = \log^p(\ell) \quad \text{for } \ell \in S, \tag{5.6}$$

$$a_{\sigma_{2i+1}} = \zeta^p(2i+1) \quad \text{for } i \geq 1. \tag{5.7}$$

We shall adopt this choice in the following.

For concreteness, let us spell out the localisation map in depth 4:

$$\mathrm{loc}^\sharp(\log) = \sum_{\ell \in S} \log^p(\ell) x_\ell, \tag{5.8}$$

$$\mathrm{loc}^\sharp(\mathrm{Li}_1) = \sum_{\ell \in S} \log^p(\ell) y_\ell, \tag{5.9}$$

$$\mathrm{loc}^\sharp(\mathrm{Li}_2) = \sum_{\ell, q \in S} a_{\tau_\ell \tau_q} x_\ell y_q, \tag{5.10}$$

$$\mathrm{loc}^\sharp(\mathrm{Li}_3) = \sum_{\ell_1, \ell_2, q \in S} a_{\tau_{\ell_1} \tau_{\ell_2} \tau_q} x_{\ell_1} x_{\ell_2} y_q + \zeta^p(3) z_3, \tag{5.11}$$

$$\mathrm{loc}^\sharp(\mathrm{Li}_4) = \sum_{\ell_1, \ell_2, \ell_3, q \in S} a_{\tau_{\ell_1} \tau_{\ell_2} \tau_{\ell_3} \tau_q} x_{\ell_1} x_{\ell_2} x_{\ell_3} y_q + \sum_{\ell \in S} a_{\sigma_3 \tau_\ell} x_\ell z_3. \tag{5.12}$$

Remark 5.8. The first three formulas (5.8)–(5.10) describe the localisation map in depth 2 that was studied in [DCW15] and [BBK+23, §2.3]. The coefficients $a_{\tau_\ell \tau_q}$ of the bilinear polynomial (5.10) agree with those denoted $a_{\ell,q}$ in [BBK+23].[12] They do not depend on choices. While there is no closed formula for them, there exists an algorithm based on Tate's calculation of the Milnor K-group $K_2(\mathbb{Q})$ [Mil71, Theorem 11.6] to express the $a_{\tau_\ell \tau_q}$ as $\mathbb{Q}$-linear combinations of dilogarithms of rational numbers (see [KLS22] for our implementation of this algorithm in SageMath).

[12] Compared to [BBK+23], our y_ℓ-coordinates on the Selmer scheme as well as the third coordinate on the de Rham fundamental group differ by a sign. These two signs cancel out, so that the resulting coefficients for the bilinear map are the same.

The following proposition summarises what we know about the polylogarithmic Chabauty–Kim diagram:

Proposition 5.9. *The Chabauty–Kim diagram for the polylogarithmic quotient* (5.3) *is isomorphic to the following diagram*

(5.13)

$$\begin{array}{ccc} \mathcal{Y}(\mathbb{Z}_S) & \hookrightarrow & \mathcal{Y}(\mathbb{Z}_p) \\ \downarrow{\scriptstyle j_S} & & \downarrow{\scriptstyle j_p} \\ \mathrm{Spec}(\mathbb{Q}_p[(x_\ell)_{\ell\in S},(y_\ell)_{\ell\in S},z_3,z_5,\dots])(\mathbb{Q}_p) & \xrightarrow{\mathrm{loc}_p} & \mathrm{Spec}(\mathbb{Q}_p[\log,\mathrm{Li}_1,\dots,\mathrm{Li}_n])(\mathbb{Q}_p)\,. \end{array}$$

The localisation map loc_p *is given by Eqs.* (5.4)–(5.5)*, the local Kummer map* j_p *is given by*

$$\begin{aligned} \log(j_p(z)) &= \log^p(z), \\ \mathrm{Li}_k(j_p(z)) &= \mathrm{Li}_k^p(z) \quad \text{for } k \geq 1, \end{aligned}$$

and the x_ℓ*- and* y_ℓ*-components of the global Kummer map* j_S *are given by*

$$\begin{aligned} x_\ell(j_S(z)) &= v_\ell(z), \\ y_\ell(j_S(z)) &= -v_\ell(1-z). \end{aligned}$$

Remark 5.10. As mentioned in Remark 5.5, the z_{2i+1}-coordinates on the Selmer scheme depend on choices. As a consequence, the z_{2i+1}-components of the global Kummer map j_S are difficult to understand. It follows however from its motivic origin that the numbers $z_{2i+1}(j_S(z))$ (along with $x_\ell(j_S(z))$ and $y_\ell(j_S(z))$) are contained in $\mathbb{Q} \subset \mathbb{Q}_p$ and independent of p.

From Section 4 we also know the equations defining the *refined* Selmer scheme inside the full Selmer scheme:

Proposition 5.11. *Assume* $2 \in S$*. Then, in terms of the coordinates from Proposition* 5.3*, the refined Selmer scheme* $\mathrm{Sel}^{\min}_{S,\mathrm{PL},n}(\mathcal{Y})$ *for the polylogarithmic quotient is defined inside* $\mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, U_{\mathrm{PL},n})$ *by the equations*

$$x_\ell y_\ell (x_\ell + y_\ell) = 0 \quad \text{for } \ell \in S.$$

It can be written as a union

$$\mathrm{Sel}^{\min}_{S,\mathrm{PL},n}(\mathcal{Y}) = \bigcup_{\Sigma} \mathrm{Sel}^{\Sigma}_{S,\mathrm{PL},n}(\mathcal{Y})$$

over the $3^{\#S}$ *refinement conditions* $\Sigma \in \{0,1,\infty\}^S$*, where* $\mathrm{Sel}^{\Sigma}_{S,\mathrm{PL},n}(\mathcal{Y})$ *is defined by the equations*

$$\begin{aligned} y_\ell &= 0 \quad \text{if } \Sigma_\ell = 0, \\ x_\ell &= 0 \quad \text{if } \Sigma_\ell = 1, \\ x_\ell + y_\ell &= 0 \quad \text{if } \Sigma_\ell = \infty, \end{aligned}$$

for $\ell \in S$*.*

Proof. This is a special case of Corollary 4.8, combined with the fact that the coordinates x_ℓ and y_ℓ on the polylogarithmic Selmer scheme are those coming from the abelian Selmer scheme (see Remark 5.4). □

5.4. **Equations for a refined polylogarithmic Chabauty–Kim locus.** Now let $S = \{2\}$ and let p be any odd prime. We carry out the refined Chabauty–Kim method in this case, starting by determining the functions cutting out the refined Chabauty–Kim locus for the particular refinement condition $\Sigma = (1)$. Abbreviate $x \coloneqq x_2$, $y \coloneqq y_2$, and $\tau \coloneqq \tau_2$. Then by Theorem 5.6, the localisation map for the polylogarithmic quotient $U_{\mathrm{PL},n}$ corresponds on coordinate rings to the homomorphism

$$\mathrm{loc}_p^\sharp \colon \mathbb{Q}_p[\log, \mathrm{Li}_1, \ldots, \mathrm{Li}_n] \to \mathbb{Q}_p[x, y, (z_{2i+1})_{1\le i\le \lfloor (n-1)/2\rfloor}]$$

given by

$$\mathrm{loc}_p^\sharp(\log) = a_\tau x, \tag{5.14}$$

$$\mathrm{loc}_p^\sharp(\mathrm{Li}_k) = a_{\tau^k} x^{k-1} y + \sum_{i=1}^{\lfloor (k-1)/2\rfloor} a_{\sigma_{2i+1}\tau^{k-2i-1}} x^{k-2i-1} z_{2i+1} \quad (1 \le k \le n). \tag{5.15}$$

By Proposition 5.11, the *refined* Selmer scheme is the union of the three hyperplanes

$$\mathrm{Sel}^{\min}_{\{2\},\mathrm{PL},n}(\mathcal{Y}) = \mathrm{Sel}^{(0)}_{\{2\},\mathrm{PL},n}(\mathcal{Y}) \cup \mathrm{Sel}^{(1)}_{\{2\},\mathrm{PL},n}(\mathcal{Y}) \cup \mathrm{Sel}^{(\infty)}_{\{2\},\mathrm{PL},n}(\mathcal{Y})$$

defined by $y = 0$, $x = 0$, and $x + y = 0$, respectively. We calculate the restriction of the localisation map to the refined subscheme for the refinement condition $\Sigma = (1)$. Denote by $i^{(1)}$ the inclusion of the refined subscheme $\mathrm{Sel}^{(1)}_{\{2\},\mathrm{PL},n}(\mathcal{Y})$ in the full Selmer scheme.

Proposition 5.12. *On the subspace* $\mathrm{Sel}^{(1)}_{\{2\},\mathrm{PL},n}(\mathcal{Y})$ *defined by* $x = 0$ *the localisation map* (5.14)–(5.15) *restricts as follows:*

$$\begin{aligned}
(i^{(1)} \circ \mathrm{loc}_p)^\sharp(\log) &= 0,\\
(i^{(1)} \circ \mathrm{loc}_p)^\sharp(\mathrm{Li}_1) &= a_\tau y,\\
(i^{(1)} \circ \mathrm{loc}_p)^\sharp(\mathrm{Li}_k) &= 0 \quad \textit{for } k \ge 2 \textit{ even},\\
(i^{(1)} \circ \mathrm{loc}_p)^\sharp(\mathrm{Li}_k) &= a_{\sigma_k} z_k \quad \textit{for } k \ge 3 \textit{ odd}.
\end{aligned}$$

Proof. For $k \ge 2$, all terms in (5.15) contain factors of x, except the very last summand if k is odd. □

Proposition 5.12 tells us immediately that the functions log and Li_k for $k \ge 2$ even vanish on the scheme-theoretic image of $\mathrm{Sel}^{(1)}_{\{2\},\mathrm{PL},n}(\mathcal{Y})$ under the localisation map. Pulling these back along the local Kummer map gives us many p-adic analytic functions that vanish on the corresponding refined Chabauty–Kim locus:

Proposition 5.13. *On the polylogarithmic refined Chabauty–Kim locus* $\mathcal{Y}(\mathbb{Z}_p)^{(1)}_{\{2\},\mathrm{PL},n}$ *of depth* n*, the following equations hold:*

$$\log(z) = 0, \tag{5.16}$$

$$\mathrm{Li}_k(z) = 0 \quad \textit{for } 2 \le k \le n \textit{ even}. \tag{5.17}$$

Remark 5.14. Proposition 5.13 generalises [BBK+23, Theorem A] from depth $n = 2$ to arbitrary depth.

5.5. **Proof of refined Kim's conjecture for $S = \{2\}$.** Recall that the refined Kim's conjecture for $S = \{2\}$ and any odd prime p states that the inclusion

$$\{2, -1, 1/2\} = \mathcal{Y}(\mathbb{Z}[1/2]) \subseteq \mathcal{Y}(\mathbb{Z}_p)^{\min}_{\{2\},\infty}$$

is an equality. The refined Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)^{(1)}_{\{2\},\infty}$ for the refinement condition $\Sigma = (1)$ should therefore consist only of -1, the unique $\{2\}$-integral point reducing to the cusp 1 modulo 2 (see Remark 4.9). We first show that this is indeed the case already for the polylogarithmic quotient (Corollary 5.16 below), and then we exploit the S_3-action to deduce the refined Kim's conjecture for the full refined locus.

Note that the element $z = -1$ satisfies the equations from Proposition 5.13:

$$\log(-1) = 0 \quad \text{and} \quad \mathrm{Li}_k(-1) = 0 \text{ for } k \geq 2 \text{ even}$$

since -1 is contained in $\mathcal{Y}(\mathbb{Z}_p)^{(1)}_{\{2\},\mathrm{PL},\infty}$. We show that $z = -1$ is the only solution. The key to this is the following result:

Proposition 5.15. *Let $p \geq 5$ be a prime, and let $\mathcal{Y} = \mathbb{P}^1_{\mathbb{Z}_p} \smallsetminus \{0, 1, \infty\}$. Then the only element $z \in \mathcal{Y}(\mathbb{Z}_p)$ satisfying $\log(z) = 0$ and $\mathrm{Li}_{p-3}(z) = 0$ is $z = -1$.*

Proof. The modified p-adic polylogarithm

$$\mathrm{Li}^{(p)}_n(z) \coloneqq \mathrm{Li}_n(z) - \frac{1}{p^n}\,\mathrm{Li}_n(z^p)\,,$$

is $\mathbb{Z}_p$-valued and is congruent modulo p to $\frac{1}{1-z^p}\,\mathrm{li}_n(z)$ where

$$\mathrm{li}_n(z) \coloneqq \sum_{k=1}^{p-1} \frac{z^k}{k^n} \in \mathbb{F}_p[z]$$

is the finite polylogarithm [Bes02, Proposition 2.1]. In the particular case that $n = p-3$, we have

$$\mathrm{li}_{p-3}(z) = \sum_{k=1}^{p-1} k^2 z^k = \frac{z(z+1)(z^p-1)}{(z-1)^3} = z(z+1)(z-1)^{p-3}$$

and so the only solution to $\mathrm{li}_{p-3}(z) = 0$ in $\mathbb{F}_p \smallsetminus \{0,1\}$ is $z = -1 \bmod p$.

If $z \in \mathcal{Y}(\mathbb{Z}_p)$ satisfies $\log(z) = 0$ and $\mathrm{Li}_{p-3}(z) = 0$, then z is a root of unity in $\mathbb{Z}_p$, so satisfies $z^p = z$. Hence z also satisfies $\mathrm{Li}^{(p)}_{p-3}(z) = 0$, and so $\mathrm{li}_{p-3}(\bar{z}) = 0$, where $\bar{z} \in \mathbb{F}_p$ is the reduction of z. But we've seen that this implies that $\bar{z} = -1$, which implies that $z = -1$ as desired. □

Corollary 5.16. *For any odd prime p we have*

$$\mathcal{Y}(\mathbb{Z}_p)^{(1)}_{\{2\},\mathrm{PL},\infty} = \{-1\}.$$

Proof. Let $z \in \mathcal{Y}(\mathbb{Z}_p)^{(1)}_{\{2\},\mathrm{PL},\infty}$, so z satisfies the equations from Proposition 5.13: $\log(z) = 0$ and $\mathrm{Li}_k(z) = 0$ for $k \geq 2$ even. For $p \geq 5$, we get $z = -1$ from Proposition 5.15. For $p = 3$, already the first equation $\log(z) = 0$ implies $z = -1$ since that is the only root of unity in $\mathcal{Y}(\mathbb{Z}_3)$. □

We can now determine the full refined Chabauty–Kim locus by exploiting the S_3-action on the thrice-punctured line. At this point we have to use the full fundamental group rather than its polylogarithmic quotient in order to have the S_3-symmetries present on the refined Chabauty–Kim locus (see Remark 4.13).

Corollary 5.17. *The thrice-punctured line satisfies the refined Chabauty–Kim conjecture for $S = \{2\}$ and all odd primes p:*

$$\mathcal{Y}(\mathbb{Z}_p)^{\min}_{\{2\},\infty} = \{2, -1, 1/2\}.$$

Proof. Let z be an element of the refined Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)^{\min}_{\{2\},\infty}$. By Corollary 4.8, the refined Selmer scheme is a union over three refinement conditions corresponding to the three cusps:

$$\mathcal{Y}(\mathbb{Z}_p)^{\min}_{\{2\},\infty} = \mathcal{Y}(\mathbb{Z}_p)^{(0)}_{\{2\},\infty} \cup \mathcal{Y}(\mathbb{Z}_p)^{(1)}_{\{2\},\infty} \cup \mathcal{Y}(\mathbb{Z}_p)^{(\infty)}_{\{2\},\infty}.$$

By Lemma 4.12, there is an S_3-conjugate $\sigma(z)$ of z contained in $\mathcal{Y}(\mathbb{Z}_p)^{(1)}_{\{2\},\infty}$. By Lemma 4.11 applied to the quotient map $U \twoheadrightarrow U_{\mathrm{PL}}$, that refined locus is contained in the corresponding one for the polylogarithmic quotient:

$$\mathcal{Y}(\mathbb{Z}_p)^{(1)}_{\{2\},\infty} \subseteq \mathcal{Y}(\mathbb{Z}_p)^{(1)}_{\{2\},\mathrm{PL},\infty}.$$

That latter locus consists only of -1 by Corollary 5.16. In particular we have $\sigma(z) = -1$. The original point z is thus an element of the S_3-orbit of -1, which is precisely $\{2, -1, 1/2\}$, the set of $\{2\}$-integral points of $\mathcal{Y}$. □

5.6. **Proof of Kim's conjecture for $S = \emptyset$.** We have focused in this paper on the refined Chabauty–Kim method due to its role in providing instances of the S-Selmer Section Conjecture. It is interesting to observe, however, that the same method for proving the refined conjecture for $S = \{2\}$ can be used to prove the original (non-refined) Kim's conjecture in the case $S = \emptyset$. This is Theorem D from the introduction, which we shall prove now.

As explained in Remark 2.18, if $\pi_1^{\text{ét},\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}, b) \twoheadrightarrow \Pi$ is any $G_{\mathbb{Q}}$-equivariant quotient of the $\mathbb{Q}_p$-pro-unipotent étale fundamental group, we can use the Bloch–Kato Selmer scheme $\mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, \Pi)$ to define the (non-refined) Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)_{S,\Pi}$ as the set of points $y \in \mathcal{Y}(\mathbb{Z}_p)$ such that $j_p(z)$ lies in the scheme-theoretic image of $\mathrm{H}^1_{f,S}(G_{\mathbb{Q}}, \Pi)$ under the localisation map j_p [Kim05; Kim09]. The classical formulation of Kim's conjecture states that the Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)_{S,n}$ for the depth-n quotient of the fundamental group should consist exactly of the S-integral points for sufficiently large n:

Conjecture 5.18 (Kim's conjecture [BDCKW18, Conjecture 3.1])**.** $\mathcal{Y}(\mathbb{Z}_S) = \mathcal{Y}(\mathbb{Z}_p)_{S,n}$ *for $n \gg 0$.*

Now let $S = \emptyset$. In this case there are no S-integral points: $\mathcal{Y}(\mathbb{Z}) = \emptyset$. It is shown in [BDCKW18, §6] that Kim's conjecture holds in depth 1 for $p = 3$ and all primes $p \equiv 2 \bmod 3$; moreover, it is conjectured (and verified numerically for $p < 10^5$) that Kim's conjecture holds in depth 2 for all primes $p \equiv 1 \bmod 3$. Using our methods we can show the conjecture for all primes by going into higher depth:

Theorem 5.19. *Let $p \geq 5$ be any prime. Then Kim's conjecture for $S = \emptyset$ holds in depth $p - 3$:*

$$\mathcal{Y}(\mathbb{Z}_p)_{\emptyset,p-3} = \emptyset.$$

Proof. It is enough to show the conjecture for the polylogarithmic quotient of the fundamental group: $\mathcal{Y}(\mathbb{Z}_p)_{\emptyset,\mathrm{PL},p-3} = \emptyset$. Specialising Theorem 5.6 to $S = \emptyset$, the localisation map $\mathrm{loc}_p \colon \mathrm{H}^1_{f,\emptyset}(G_{\mathbb{Q}}, U_{\mathrm{PL},n}) \to \mathrm{H}^1_f(G_p, U_{\mathrm{PL},n})$ corresponds to the homomorphism on coordinate rings

$$\mathrm{loc}_p^\sharp \colon \mathbb{Q}_p[\log, \mathrm{Li}_1, \ldots, \mathrm{Li}_n] \to \mathbb{Q}_p[(z_{2i+1})_{1 \leq i \leq \lfloor (n-1)/2 \rfloor}]$$

given by

$$\begin{aligned}
\mathrm{loc}_p^\sharp(\log) &= 0,\\
\mathrm{loc}_p^\sharp(\mathrm{Li}_1) &= 0,\\
\mathrm{loc}_p^\sharp(\mathrm{Li}_k) &= 0 \quad \text{for } k \geq 2 \text{ even},\\
\mathrm{loc}_p^\sharp(\mathrm{Li}_k) &= a_{\sigma_k} z_k \quad \text{for } k \geq 3 \text{ odd}.
\end{aligned}$$

This implies that all elements of the Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)_{\emptyset,\mathrm{PL},n}$ satisfy the equations

$$\log(z) = 0, \quad \mathrm{Li}_1(z) = 0, \quad \mathrm{Li}_k(z) = 0 \text{ for } 2 \leq k \leq n \text{ even}. \tag{5.18}$$

By Proposition 5.15, the only element of $\mathcal{Y}(\mathbb{Z}_p)$ satisfying $\log(z) = 0$ and $\mathrm{Li}_{p-3}(z) = 0$ is $z = -1$. However, that element is not a root of Li_1:

$$\mathrm{Li}_1(-1) = -\log(1-(-1)) = -\log(2) \neq 0.$$

Thus, there are no solutions to the equations (5.18) if $n \geq p-3$. □

This proves Theorem D from the introduction.

Remark 5.20. The equations (5.18) have previously been derived in [DCW16, Theorem 1.13].[13] The statement of that theorem contains the assumption that $\zeta^p(n) \neq 0$ for $n \geq 3$ odd (as is conjectured); this assumption is only necessary for saying that the Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)_{\emptyset,\mathrm{PL},\infty}$ is *precisely* the one cut out by the given equations, whereas we only need that the Chabauty–Kim locus is contained in their vanishing set.

Remark 5.21. It was shown in [BDCKW18, §6] that the depth 1 locus $\mathcal{Y}(\mathbb{Z}_p)_{\emptyset,1}$, defined by the equations $\log(z) = 0$ and $\mathrm{Li}_1(z) = 0$, consists precisely of the two primitive sixth roots of unity in $\mathbb{Z}_p$ if $p \equiv 1 \bmod 3$, and is empty otherwise. Adding in the higher depth equation $\mathrm{Li}_{p-3}(z) = 0$ cuts the set down to the empty set for all p.

5.7. **Curves of higher genus.** We conclude this paper with the proof of Theorem E, i.e., the observation that there are also some special higher genus curves where one can prove a Kim-like conjecture, strong enough to deduce the Selmer Section Conjecture. Let $X/\mathbb{Q}$ be a smooth projective curve of genus ≥ 2 with a rational point, and suppose that A is a quotient of the Jacobian of X of dimension ≥ 2, Mordell–Weil rank 0 and finite Tate–Shafarevich group. Since A is a quotient of the Jacobian, its $\mathbb{Q}_p$-linear Tate module V_pA is an abelian quotient of the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of X. The assumptions on the Mordell–Weil rank and Tate–Shafarevich group imply that

$$\mathrm{H}^1_f(G_{\mathbb{Q}}, V_pA) = 0\,,$$

and hence that the Chabauty–Kim locus $X(\mathbb{Q}_p)_{V_pA}$ associated to this quotient is the kernel of the map

$$j_p\colon X(\mathbb{Q}_p) \to \mathrm{H}^1_f(G_p, V_pA)$$

(when p is a prime of good reduction).

[13] The statement of [DCW16, Theorem 1.13] contains a typo: it is the $\mathrm{Li}_k(z)$ for k *even* not *odd* which vanish on the Chabauty–Kim locus.

This can be reinterpreted geometrically. Let

$$f\colon X\to A$$

be the composite of the Abel–Jacobi map with the projection from the Jacobian to A. The induced map on $\mathbb{Q}_p$-pro-unipotent fundamental groups is exactly the quotient map from the fundamental group of X to V_pA, so the map above factors as the composite

$$X(\mathbb{Q}_p)\xrightarrow{f}A(\mathbb{Q}_p)\xrightarrow{\kappa_p}\mathrm{H}^1_f(G_p,V_pA)$$

where κ_p is the $\mathbb{Q}_p$-linear Kummer map for the abelian variety A. Now the kernel of κ_p is exactly the torsion subgroup $A(\mathbb{Q}_p)_{\mathrm{tors}}$, so the Chabauty–Kim locus $X(\mathbb{Q}_p)_{V_pA}$ is the set of points $x\in X(\mathbb{Q}_p)$ such that $f(x)$ is a torsion point on A.

Now the Manin–Mumford Conjecture implies that the set of points on $X_{\overline{\mathbb{Q}}}$ mapping to a torsion point on $A_{\overline{\mathbb{Q}}}$ is finite, so there is a finite closed subscheme $Z\subset X_{\mathrm{cl}}$ such that

$$X(\mathbb{Q}_p)_{V_pA}=Z(\mathbb{Q}_p)$$

for all primes p of good reduction. This in particular implies Theorem E.

Finally, let us explain why the conclusion of Theorem E, although weaker than Kim's Conjecture, is nonetheless strong enough to deduce the Selmer Section Conjecture for curves X as above. The containment $X(\mathbb{Q}_p)_{V_pA}\subseteq Z(\mathbb{Q}_p)$ for good primes p implies by Lemma 2.19 that any point $x=(x_v)_v\in X(\mathbb{A}_{\mathbb{Q},S})_\bullet^{\text{f-cov}}$ in the finite descent locus satisfies $x_p\in Z(\mathbb{Q}_p)$ for all good primes p. So we deduce from the theorem of the diagonal that $x\in X(\mathbb{Q})$, i.e. strong sufficiency of finite descent (Conjecture 1.13) holds for X. So the Selmer Section Conjecture holds for X. □

We remark that strong sufficiency of finite descent for such curves X (in slightly greater generality) was proven by Stoll [Sto07, Theorem 8.6]. The above argument essentially shows that Stoll's theorem lifts to a statement about Chabauty–Kim loci.

Appendix A. Mixed Tate motives vs. Galois representations

This appendix is devoted to the proof of the following

Proposition A.1. *Let K be a number field, S a finite set of places of K, and p a rational prime not divisible by any element of S. Let $\mathrm{MT}(\mathcal{O}_{K,S},\mathbb{Q})$ denote the $\mathbb{Q}$-linear category of mixed Tate motives over $\mathcal{O}_{K,S}$ with coefficients in $\mathbb{Q}$, and let $\mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_K)$ denote the $\mathbb{Q}_p$-linear category of mixed Tate representations of G_K, unramified outside $S\cup\{\mathfrak{p}\mid p\}$ and crystalline at all $\mathfrak{p}\mid p$. Suppose that p is odd. Then the map*

$$\mathrm{Ext}^1_{\mathrm{MT}(\mathcal{O}_{K,S},\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n))\otimes_{\mathbb{Q}}\mathbb{Q}_p\to\mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_K)}(\mathbb{Q}_p(0),\mathbb{Q}_p(n))\tag{A.1}$$

induced by étale realisation is an isomorphism for all $n>0$.

Since Ext^2's in $\mathrm{MT}(\mathcal{O}_{K,S},\mathbb{Q})$ vanish [DG05, Proposition 1.9], this implies by Lemma 3.6 that the induced map

$$G^{\mathrm{MTR}}_{K,S}\to G^{\mathrm{MT}}_{K,S,\mathbb{Q}_p}$$

on Tannaka groups is an isomorphism, and thus completes the proof of Theorem 1.12 from the introduction. As remarked in Section 3, it is a result of Soulé [Sou81, Theorem 1] that the domain and codomain of the map (A.1) are isomorphic (for p odd; the case $K = \mathbb{Q}$ and $p = 2$ was dealt with in unpublished work of Sharifi [Sha00, Theorem 1]). What is unclear to us – and what necessitates this appendix – is why étale realisation in particular is an isomorphism.

We now begin the proof. The Ext-group on the left-hand side of (A.1) is given by

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{MT}(\mathcal{O}_{K,S},\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n)) &\cong K_{2n-1}(\mathcal{O}_K)\otimes_{\mathbb{Z}}\mathbb{Q} \\ &= \begin{cases}\mathcal{O}_{K,S}^{\times}\otimes_{\mathbb{Z}}\mathbb{Q} & n=1\,,\\ K_{2n-1}(K)\otimes_{\mathbb{Z}}\mathbb{Q} & n>1\,,\end{cases}\end{aligned}$$

by [DG05, (2.1.2) & (1.6.3)]. The Ext-group on the right-hand side is given by

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_K)}(\mathbb{Q}_p(0),\mathbb{Q}_p(n)) &\cong \mathrm{H}^1_{f,S}(G_K,\mathbb{Q}_p(n)) \\ &\cong \begin{cases}\mathcal{O}_{K,S}^{\times}\otimes_{\mathbb{Z}}\mathbb{Q}_p & n=1\,,\\ \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_K[1/p]),\mathbb{Q}_p(n)) & n>1\,,\end{cases}\end{aligned}$$

by Kummer theory for $n = 1$ and for $n > 1$ using the fact that any extension of $\mathbb{Q}_p(0)$ by $\mathbb{Q}_p(n)$ is automatically unramified away from p and crystalline at p [BK90, Example 3.9]. Soulé's result then tells us that that

$$\dim_{\mathbb{Q}}\mathrm{Ext}^1_{\mathrm{MT}(\mathcal{O}_{K,S},\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n)) = \dim_{\mathbb{Q}_p}\mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_K)}(\mathbb{Q}_p(0),\mathbb{Q}_p(n)) \tag{A.2}$$

for $n > 0$, so the domain and codomain of (A.1) have the same dimension. In fact, Soulé constructs an isomorphism

$$K_{2n-1}(\mathcal{O}_K)\otimes_{\mathbb{Z}}\mathbb{Q}_p \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_K[1/p]),\mathbb{Q}_p(n))$$

for $n > 1$: the *Chern class map*. It is presumably the case that the Chern class map agrees with étale realisation (3.7) under the above identifications, but this is not the strategy we take. Instead, following an approach suggested to us by Marc Levine, we will show directly that (A.1) is surjective for $n = 1$ and injective for $n > 1$, deducing isomorphy from (A.2).

In the case $n = 1$, the proof of [DG05, Proposition 1.8] shows that the Kummer class

$$\kappa(z) \in \mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_K)}(\mathbb{Q}_p(0),\mathbb{Q}_p(1))$$

of an element $z \in \mathcal{O}_{K,S}^{\times}$ is the realisation of an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$ in $\mathrm{MT}(\mathcal{O}_{K,S},\mathbb{Q})$. Since this Ext-group is the $\mathbb{Q}_p$-linear span of the Kummer classes of the elements $z \in \mathcal{O}_{K,S}^{\times}$, we see that (A.1) is surjective for $n = 1$; it is then bijective by dimension considerations.

In the case $n > 1$, we know that any extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ in the category $\mathrm{MT}(K,\mathbb{Q})$ of mixed Tate motives over K with coefficients in $\mathbb{Q}$ automatically lies in $\mathrm{MT}(\mathcal{O}_{K,S},\mathbb{Q})$ [DG05, Proposition 1.9], and that any extension of $\mathbb{Q}_p(0)$ by $\mathbb{Q}_p(n)$ in the category of G_K-representations automatically lies in $\mathrm{Rep}^{\mathrm{MT},S}_{\mathbb{Q}_p}(G_K)$ (i.e. is unramified outside $S \cup \{\mathfrak{p} \mid p\}$ and crystalline at all $\mathfrak{p} \mid p$) by [BK90, Example 3.9]. So we want to show that the map

$$\mathrm{Ext}^1_{\mathrm{MT}(K,\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n))\otimes_{\mathbb{Q}}\mathbb{Q}_p \to \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Q}_p}(G_K)}(\mathbb{Q}_p(0),\mathbb{Q}_p(n)) \tag{A.3}$$

induced by étale realisation is an isomorphism.

The Ext-groups on both sides of (A.3) are computed by Hom-groups in certain triangulated categories. On the one hand, the category $\mathrm{MT}(K,\mathbb{Q})$ is, by definition, a full and extension-closed subcategory of $\mathrm{DM}^-_{\mathrm{Nis}}(K,\mathbb{Z})_{\mathbb{Q}}$, where $\mathrm{DM}^-_{\mathrm{Nis}}(K,\mathbb{Z})$ denotes Voevodsky's bounded above triangulated category of motives and the subscript $\mathbb{Q}$ means that we tensor the Hom-groups with $\mathbb{Q}$ [DG05, §1.1]. So the first Ext-group in (A.3) is given by

$$\mathrm{Ext}^1_{\mathrm{MT}(K,\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n)) = \mathrm{Hom}_{\mathrm{DM}^-_{\mathrm{Nis}}(K,\mathbb{Z})}(\mathbb{Z}(0),\mathbb{Z}(n)[1])\otimes_{\mathbb{Z}}\mathbb{Q}\,.$$

Since $n\geq 2$, one could equally take the right-hand Hom-group inside the category $\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K,\mathbb{Z})$ of bounded above effective Voevodsky motives.

Giving a corresponding description of the second Ext-group in (A.3) is a little subtle, and was worked out by Ekedahl [Eke07], see also [Lev98, §I.V.2]. Let $\acute{\mathrm{E}}\mathrm{t}_K$ denote the small étale site of K, and let

$$\mathrm{Sh}(\acute{\mathrm{E}}\mathrm{t}_K,\mathbb{Z}/p^*)$$

denote the category of inverse systems

$$\cdots\to\mathcal{F}_3\to\mathcal{F}_2\to\mathcal{F}_1$$

of sheaves of $\mathbb{Z}$-modules on $\acute{\mathrm{E}}\mathrm{t}_K$, where $\mathcal{F}_r$ is p^r-torsion for all r. An object of $\mathrm{D}^-(\mathrm{Sh}(\acute{\mathrm{E}}\mathrm{t}_K,\mathbb{Z}/p^*))$ is called *normalised* just when the maps

$$\mathbb{Z}/p^r\overset{\mathrm{L}}{\otimes}_{\mathbb{Z}/p^{r+1}}\mathcal{F}_{r+1}\to\mathcal{F}_r$$

is a quasi-isomorphism for all r [Eke07, Definition 2.1(ii)], cf. [Eke07, Proposition 2.2]. The category

$$\mathrm{D}^-(\lim\mathrm{Sh}(\acute{\mathrm{E}}\mathrm{t}_K,\mathbb{Z}_p))$$

is defined to be the full subcategory of $\mathrm{D}^-(\mathrm{Sh}(\acute{\mathrm{E}}\mathrm{t}_K,\mathbb{Z}/p^*))$ consisting of the normalised objects. There is a canonical functor

$$\mathrm{Rep}_{\mathbb{Z}_p}(G_K)\to\mathrm{D}^-\lim\mathrm{Sh}(\acute{\mathrm{E}}\mathrm{t}_K,\mathbb{Z}_p)$$

sending a continuous representation of G_K on a finitely generated free $\mathbb{Z}_p$-module Λ to the inverse system

$$\cdots\to\Lambda/p^3\to\Lambda/p^2\to\Lambda/p\,,$$

where we permit ourselves the usual conflation between finite sets with a continuous action of G_K and sheaves of finite sets on $\acute{\mathrm{E}}\mathrm{t}_K$. This embeds $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ fully faithfully as an extension-closed subcategory of $\mathrm{D}^-\lim\mathrm{Sh}(\acute{\mathrm{E}}\mathrm{t}_K,\mathbb{Z}_p)$, and so $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ also embeds as a full extension-closed subcategory of the isogeny category $\mathrm{D}^-\lim\mathrm{Sh}(\acute{\mathrm{E}}\mathrm{t}_K,\mathbb{Z}_p)_{\mathbb{Q}_p}$. In particular, the second Ext-group in (A.3) is given by

$$\mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Q}_p}(G_K)}(\mathbb{Q}_p(0),\mathbb{Q}_p(n)) = \mathrm{Hom}_{\mathrm{D}^-\lim\mathrm{Sh}(\acute{\mathrm{E}}\mathrm{t}_K,\mathbb{Z}_p)}(\mathbb{Z}_p(0),\mathbb{Z}_p(n))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\,.$$

So Proposition A.1 is reduced to the following.

Proposition A.2. *For $n\geq 2$, the map*

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K,\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(n)[1])\otimes_{\mathbb{Z}}\mathbb{Q}_p\to\mathrm{Hom}_{\mathrm{D}^-\lim\mathrm{Sh}(\acute{\mathrm{E}}\mathrm{t}_K,\mathbb{Z}_p)}(\mathbb{Z}_p,\mathbb{Z}_p(n)[1])\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$$

induced by étale realisation is an isomorphism of finite-dimensional $\mathbb{Q}_p$-vector spaces.

For the proof of Proposition A.2, we need to recall the definition of the étale realisation functor $\rho_{\acute{\mathrm{e}}\mathrm{t}}$, following the exposition in [Ayo14]. Let k be a perfect field and R a ring, and write

$$\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(\mathrm{Sm}_k,R)\quad\text{resp.}\quad\mathrm{Sh}^{\mathrm{tr}}_{\acute{\mathrm{e}}\mathrm{t}}(\mathrm{Sm}_k,R)$$

for the category of sheaves of R-modules with transfers on Sm_k in the Nisnevich topology, resp. étale topology. For $* \in \{\mathrm{Nis}, \text{ét}\}$, we let

$$\mathrm{DM}^{\mathrm{eff},-}_*(k,R) = \mathrm{D}^-(\mathrm{Sh}^{\mathrm{tr}}_*(\mathrm{Sm}_k,R))[W_{\mathbb{A}}^{-1}]$$

denote the triangulated category of effective Voevodsky motives in the corresponding topology, i.e. the localisation of the bounded above derived category of sheaves with transfers at the $\mathbb{A}^1$-homotopy equivalences [MVW06, Definitions 14.1 & 9.2]. By [MVW06, Theorem 14.11], the natural functor $\mathrm{D}^-(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(\mathrm{Sm}_k,R)) \to \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,R)$ has a left adjoint C_* [MVW06, Remark 14.7], and this identifies $\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,R)$ as the full reflective subcategory of $\mathrm{D}^-(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(\mathrm{Sm}_k,R))$ consisting of the $\mathbb{A}^1$-local objects [MVW06, Theorem 14.11]. The same is true in the étale topology when $R = \mathbb{Z}/l$ for l an integer invertible in k such that k has finite étale cohomological dimension for $\mathbb{Z}/l$-coefficients. When we consider motives with coefficients in $\mathbb{Z}/l$, we will always implicitly make this assumption.

Now the identity functor on Sm_k defines a morphism of sites

$$\pi\colon (\mathrm{Sm}_k)_{\text{ét}} \to (\mathrm{Sm}_k)_{\mathrm{Nis}}\,.$$

If $\mathcal{F}$ is a Nisnevich sheaf of $\mathbb{Z}/l$-modules with transfers, then its étale sheafification $\pi^*\mathcal{F}$ admits transfers in a canonical way [MVW06, Theorem 6.17], so we have a functor

$$\pi^*\colon \mathrm{D}^-(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(\mathrm{Sm}_k,\mathbb{Z}/l)) \to \mathrm{D}^-(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(\mathrm{Sm}_k,\mathbb{Z}/l))\,.$$

This functor preserves $\mathbb{A}^1$-local objects (combine Proposition 14.8, Theorem 7.20, Lemma 9.23 and Proposition 9.30 of [MVW06]), so restricts to a functor

$$\pi^*\colon \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbb{Z}/l) \to \mathrm{DM}^{\mathrm{eff},-}_{\text{ét}}(k,\mathbb{Z}/l)\,.$$

There is also a morphism of sites

$$\iota\colon (\mathrm{Sm}_k)_{\text{ét}} \to \text{Ét}_k$$

given by the inclusion of the small étale site Ét_k of $\mathrm{Spec}(k)$ inside Sm_k. For any sheaf $\mathcal{F}$ of $\mathbb{Z}/l$-modules on Ét_k, the extension $\iota^*\mathcal{F}$ admits transfers in a unique way [MVW06, Corollary 9.25], so we have a functor

$$\iota^*\colon \mathrm{D}^-(\mathrm{Sh}(\text{Ét}_k,\mathbb{Z}/l)) \to \mathrm{D}^-(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(\mathrm{Sm}_k,\mathbb{Z}/l))\,.$$

Theorem A.3 (Suslin–Voevodsky rigidity, [MVW06, Theorem 9.35])**.** *The image of ι^* is contained in the subcategory of $\mathbb{A}^1$-local objects, and defines an equivalence*

$$\iota^*\colon \mathrm{D}^-(\mathrm{Sh}(\text{Ét}_k,\mathbb{Z}/l)) \xrightarrow{\sim} \mathrm{DM}^{\mathrm{eff},-}_{\text{ét}}(k,\mathbb{Z}/l)\,.$$

Remark A.4. An explicit quasi-inverse to ι^* is given by the functor

$$\iota_*\colon \mathrm{DM}^{\mathrm{eff},-}_{\text{ét}}(k,\mathbb{Z}/l) \xrightarrow{\sim} \mathrm{D}^-(\mathrm{Sh}(\text{Ét}_k,\mathbb{Z}/l))$$

which forgets transfers and restricts to the small étale site $\text{Ét}_k \subseteq \mathrm{Sm}_k$.

Definition A.5. The *mod-l étale realisation functor*

$$\bar{\rho}_{\text{ét},l}\colon \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbb{Z}/l) \to \mathrm{D}^-(\mathrm{Sh}(\text{Ét}_k,\mathbb{Z}/l))$$

is the composite of π^* with a quasi-inverse to ι^*.

Now if $\mathbb{Z}/l(n) \in \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k, \mathbb{Z}/l)$ denotes the nth Tate object for $n \geq 0$, then there is a canonical isomorphism

$$\bar{\rho}_{\text{ét},l}(\mathbb{Z}/l(n)) \cong \mu_l^{\otimes n}$$

in $\mathrm{D}^-(\mathrm{Sh}(\text{Ét}_k, \mathbb{Z}/l))$ [MVW06, Theorem 10.3]. So étale realisation induces a map

$$\rho_{\text{ét}}\colon \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbb{Z}/l)}(\mathbb{Z}/l, \mathbb{Z}/l(n)[m]) \to \mathrm{Hom}_{\mathrm{D}^-(\mathrm{Sh}(\text{Ét}_k,\mathbb{Z}/l))}(\mathbb{Z}/l, \mu_l^{\otimes n}[m]) \tag{A.4}$$

for all $n \geq 0$ and all m. The following result is a particular formulation of the Beilinson–Lichtenbaum Conjecture.

Proposition A.6. *The map* (A.4) *is an isomorphism for* $n \geq \max\{m, 0\}$.

Proof. The functor π^* has a right adjoint

$$\mathrm{R}\pi_*\colon \mathrm{D}^-(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(\mathrm{Sm}_k, \mathbb{Z}/l)) \to \mathrm{D}^-(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathbb{Z}/l))$$

given by the right-derived functor of the forgetful functor from étale sheaves with transfers to Nisnevich sheaves with transfers. This functor preserves $\mathbb{A}^1$-local objects (since its left adjoint π^* preserves $\mathbb{A}^1$-homotopy equivalences, cf. [MVW06, Lemma 9.20]), so restricts to a functor

$$\mathrm{R}\pi_*\colon \mathrm{DM}^{\mathrm{eff},-}_{\text{ét}}(k, \mathbb{Z}/l) \to \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k, \mathbb{Z}/l)$$

which is again right adjoint to π^*. According to the Beilinson–Lichtenbaum Conjecture (which is a theorem, see [HW19] for an overview), the unit of the adjunction

$$\mathbb{Z}/l(n) \to \mathrm{R}\pi_*\pi^*(\mathbb{Z}/l(n))$$

induces an isomorphism

$$\mathbb{Z}/l(n) \xrightarrow{\sim} \tau_{\leq n}\mathrm{R}\pi_*\pi^*(\mathbb{Z}/l(n))$$

in $\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k, \mathbb{Z}/l)$ for all $n \geq 0$.[14] This implies that (A.4) is bijective for $n \geq \max\{m, 0\}$: it is equal to the composite of the bijections

$$\begin{aligned}
& \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbb{Z}/l)}(\mathbb{Z}/l, \mathbb{Z}/l(n)[m]) \\
\underset{\text{Beilinson–Lichtenbaum}}{\cong}\; & \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbb{Z}/l)}(\mathbb{Z}/l, \tau_{\leq n-m}(\mathrm{R}\pi_*\pi^*\mathbb{Z}/l(n)[m])) \\
\cong\; & \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbb{Z}/l)}(\mathbb{Z}/l, \mathrm{R}\pi_*\pi^*\mathbb{Z}/l(n)[m]) \\
\underset{\text{adjunction}}{\cong}\; & \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\text{ét}}(k,\mathbb{Z}/l)}(\mathbb{Z}/l, \pi^*\mathbb{Z}/l(n)[m]) \\
\underset{\text{Suslin–Voevodsky rigidity}}{\cong}\; & \mathrm{Hom}_{\mathrm{D}^-(\mathrm{Sh}(\text{Ét}_k,\mathbb{Z}/l))}(\mathbb{Z}/l, \mu_l^{\otimes n}[m])\,,
\end{aligned}$$

where in the second identification we use that $\mathbb{Z}/l$ has vanishing cohomology in degrees $> n - m \geq 0$. □

For any l, there is a change-of-coefficients functor

$$\mathrm{L}\phi_l\colon \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k, \mathbb{Z}) \to \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k, \mathbb{Z}/l)$$

defined as follows. Firstly, for a ring R let $\mathrm{PSh}^{\mathrm{tr}}(\mathrm{Sm}_k, R)$ denote the category of presheaves of R-modules with transfers on Sm_k. Let

$$\mathrm{L}\phi_l^{\mathrm{pre}}\colon \mathrm{D}^-(\mathrm{PSh}^{\mathrm{tr}}(\mathrm{Sm}_k, \mathbb{Z})) \to \mathrm{D}^-(\mathrm{PSh}^{\mathrm{tr}}(\mathrm{Sm}_k, \mathbb{Z}/l))$$

[14] The assertion in [HW19, Lemma 1.29(a)] is that this map is an isomorphism in the derived category of Zariski sheaves with transfers when l is a prime power invertible in k. This immediately implies that the map is an isomorphism in $\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k, \mathbb{Z}/l)$ for any l invertible in k.

denote the left-derived functor of the presheaf tensor product $\mathcal{F} \mapsto (\mathbb{Z}/l) \otimes_{\mathbb{Z}}^{\mathrm{pre}} \mathcal{F}$. This functor has a right adjoint given by the evident forgetful functor

$$u_l \colon \mathrm{D}^-(\mathrm{PSh}^{\mathrm{tr}}(\mathrm{Sm}_k, \mathbb{Z}/l)) \to \mathrm{D}^-(\mathrm{PSh}^{\mathrm{tr}}(\mathrm{Sm}_k, \mathbb{Z}))\,.$$

Lemma A.7. *Let $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ be a morphism of bounded above complexes of presheaves of $\mathbb{Z}$-modules with transfers whose Nisnevich sheafification $\mathcal{F}^\bullet_{\mathrm{Nis}} \to \mathcal{G}^\bullet_{\mathrm{Nis}}$ is a quasi-isomorphism (of complexes of Nisnevich sheaves with transfers). Then the induced map*

$$(\mathrm{L}\phi_l^{\mathrm{pre}}\mathcal{F}^\bullet)_{\mathrm{Nis}} \to (\mathrm{L}\phi_l^{\mathrm{pre}}\mathcal{G}^\bullet)_{\mathrm{Nis}}$$

is a quasi-isomorphism.

Proof. Since $\mathrm{L}\phi_l^{\mathrm{pre}}$ is a triangulated functor, it suffices to prove that if $\mathcal{F}^\bullet$ is a bounded above complex of presheaves of $\mathbb{Z}$-modules with transfers such that $\mathcal{F}^\bullet_{\mathrm{Nis}}$ is acyclic, then $(\mathrm{L}\phi_l^{\mathrm{pre}}\mathcal{F}^\bullet)_{\mathrm{Nis}}$ is acyclic. For this, choose a projective resolution $\mathcal{P}^\bullet \xrightarrow{\sim} \mathcal{F}^\bullet$, so that we have the exact sequence

$$0 \to \mathcal{P}^\bullet \xrightarrow{l} \mathcal{P}^\bullet \to u_l((\mathbb{Z}/l) \otimes_{\mathbb{Z}}^{\mathrm{pre}} \mathcal{P}^\bullet) \to 0$$

(since $\mathcal{P}^\bullet$ is sectionwise free as a $\mathbb{Z}$-module). Sheafifying and using that $\mathcal{P}^\bullet_{\mathrm{Nis}}$ is acyclic shows the result. □

As a corollary, the functor

$$\mathrm{L}\phi_l^{\mathrm{Nis}} \colon \mathrm{D}^-(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathbb{Z})) \to \mathrm{D}^-(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathbb{Z}/l))$$

given by $\mathcal{F}^\bullet \mapsto (\mathrm{L}\phi_l^{\mathrm{pre}}\mathcal{F}^\bullet)_{\mathrm{Nis}}$ is well-defined, and has a right adjoint given by the forgetful functor

$$u_l \colon \mathrm{D}^-(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathbb{Z}/l)) \to \mathrm{D}^-(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathbb{Z}))\,.$$

Since the Nisnevich sheaf with transfers $\mathbb{Z}_{\mathrm{tr}}(X)$ represented by an object $X \in \mathrm{Sm}_k$ is projective as an object of $\mathrm{PSh}^{\mathrm{tr}}(\mathrm{Sm}_k, \mathbb{Z})$ [MVW06, Lemma 8.1], it follows that $\mathrm{L}\phi_l^{\mathrm{Nis}}(\mathbb{Z}_{\mathrm{tr}}(X)) = (\mathbb{Z}/l)_{\mathrm{tr}}(X)$, and hence that $\mathrm{L}\phi_l^{\mathrm{Nis}}$ preserves $\mathbb{A}^1$-homotopy equivalences. So $\mathrm{L}\phi_l^{\mathrm{Nis}}$ induces the desired change-of-coefficients functor

$$\mathrm{L}\phi_l \colon \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k, \mathbb{Z}) \to \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k, \mathbb{Z}/l)\,,$$

which has a right adjoint given by the forgetful functor

$$u_l \colon \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k, \mathbb{Z}/l) \to \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k, \mathbb{Z})\,.$$

More precisely, regarding $\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k, R)$ as the subcategory of $\mathbb{A}^1$-local objects in $\mathrm{D}^-(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(\mathrm{Sm}_k, R))$ [MVW06, Theorem 14.11], the functor $\mathrm{L}\phi_l$ is given by

$$\mathcal{F}^\bullet \mapsto C_*(\mathrm{L}\phi_l^{\mathrm{Nis}}(\mathcal{F}^\bullet))$$

where C_* is as in [MVW06, Definition 2.14], and the forgetful functor u_l automatically restricts to a functor on the subcategories of $\mathbb{A}^1$-local objects. We observe that the proof of Lemma A.7 shows the following.

Lemma A.8. *For any $\mathcal{F}^\bullet \in \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k, \mathbb{Z})$, the unit $\eta_{\mathcal{F}^\bullet} \colon \mathcal{F}^\bullet \to u_l\mathrm{L}\phi_l\mathcal{F}^\bullet$ of adjunction fits into an exact triangle*

$$\mathcal{F}^\bullet \xrightarrow{l} \mathcal{F}^\bullet \xrightarrow{\eta_{\mathcal{F}^\bullet}} u_l\mathrm{L}\phi_l\mathcal{F}^\bullet \xrightarrow{+}$$

functorial in $\mathcal{F}^\bullet$.

Remark A.9. Lemma A.8 shows that there is a canonical natural isomorphism

$$u_l \mathrm{L}\phi_l \mathcal{F}^\bullet \cong (\mathbb{Z}/l) \otimes^{\mathrm{tr}}_{L,\mathrm{Nis}} \mathcal{F}^\bullet \,,$$

where $(-) \otimes^{\mathrm{tr}}_{L,\mathrm{Nis}} (-)$ is the tensor product defined in [MVW06, Definition 14.2]. In particular, this shows that $(\mathbb{Z}/l) \otimes^{\mathrm{tr}}_{L,\mathrm{Nis}} \mathcal{F}^\bullet$ lies in the essential image of the forgetful functor u_l, something which is not obvious from the definition of the tensor product.

Now following [Ayo14], we define the *mod-l* étale realisation functor

$$\rho_{\text{ét},l} \colon \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbb{Z}) \to \mathrm{D}^-(\mathrm{Sh}(\text{Ét}_k,\mathbb{Z}/l))$$

to be the composite

$$\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbb{Z}) \xrightarrow{\mathrm{L}\phi_l} \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbb{Z}/l) \xrightarrow{\bar{\rho}_{\text{ét},l}} \mathrm{D}^-(\mathrm{Sh}(\text{Ét}_k,\mathbb{Z}/l)) \,.$$

Corollary A.10. *If $0 \leq m \leq n$, then the kernel of the mod-l étale realisation*

$$\rho_{\text{ét},l} \colon \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(n)[m]) \to \mathrm{Hom}_{\mathrm{D}^-(\mathrm{Sh}(\text{Ét}_k,\mathbb{Z}/l))}(\mathbb{Z}/l,\mu_l^{\otimes n}[m])$$

is $l\,\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(n)[m])$.

Proof. By Proposition A.6 and the adjunction $\mathrm{L}\phi_l \dashv u_l$, the kernel of the étale realisation is the same as the kernel of the map

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(n)[m]) \to \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbb{Z})}(\mathbb{Z},u_l\mathrm{L}\phi_l\mathbb{Z}(n)[m])$$

given by composition with the unit $\mathbb{Z}(n)[m] \to u_l\mathrm{L}\phi_l\mathbb{Z}(n)[m]$. So we conclude by Lemma A.8. ☐

Now consider $l = p^r$ for p a fixed prime and r varying (so p is invertible in k and k has finite p-cohomological dimension). The mod-p^r étale realisation functors together induce a functor

$$\rho_{\text{ét}} \colon \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbb{Z}) \to \mathrm{D}^- \lim \mathrm{Sh}(\text{Ét}_k,\mathbb{Z}_p)$$

which we call the *p-adic étale realisation* functor. Explicitly, if $\mathcal{F}^\bullet \in \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbb{Z})$, then we choose a projective resolution $\mathcal{P}^\bullet \xrightarrow{\sim} \mathcal{F}^\bullet$ in the category of complexes of presheaves with transfers. Then the mod-p^r étale realisations $\mathrm{L}\phi_{p^r}\mathcal{F}^\bullet$ are represented by the complexes

$$C_*(\mathbb{Z}/p^r \otimes^{\mathrm{pre}}_{\mathbb{Z}} \mathcal{P}^\bullet)_{\text{ét}}|_{\text{Ét}_k} \,,$$

which clearly form a normalised complex in $\mathrm{D}^-(\mathrm{Sh}(\text{Ét}_k,\mathbb{Z}/p^*))$. So this construction defines the desired functor

$$\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbb{Z}) \to \mathrm{D}^-(\lim \mathrm{Sh}(\text{Ét}_k,\mathbb{Z}_p)) \,.$$

Now we specialise the above theory to the case $k = K$ is a number field and p an odd prime. The input we need for the proof of Proposition A.2 is the following.

Lemma A.11. *For $n \geq 2$, the group $\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K,\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(n)[1])$ is finitely generated.*

Proof. The group $\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K,\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(n)[1])$ is isomorphic to the motivic cohomology group $\mathrm{H}^1(\mathrm{Spec}(K),\mathbb{Z}(n))$, which sits in a localisation exact sequence

$$\mathrm{H}^1(\mathrm{Spec}(\mathcal{O}_K),\mathbb{Z}(n)) \to \mathrm{H}^1(\mathrm{Spec}(K),\mathbb{Z}(n)) \to \bigoplus_v \mathrm{H}^0(\mathrm{Spec}(k_v),\mathbb{Z}(n-1)) \,,$$

where v runs over the finite places of K and the k_v denote the residue fields. The left-hand term of this exact sequence is finitely generated by [Kah05, Theorem 40], so it suffices to prove the vanishing of the right-hand term. For this, using the exact sequence

$$\mathrm{H}^{-1}(\mathrm{Spec}(k_v),(\mathbb{Q}/\mathbb{Z})(n-1)) \to \mathrm{H}^0(\mathrm{Spec}(k_v),\mathbb{Z}(n-1)) \to \mathrm{H}^0(\mathrm{Spec}(k_v),\mathbb{Q}(n-1))$$

it suffices to prove the vanishing of

$$\mathrm{H}^0(\mathrm{Spec}(k_v),\mathbb{Q}(n-1)) \quad \text{and} \quad \mathrm{H}^{-1}(\mathrm{Spec}(k_v),\mathbb{F}_q(n-1))$$

for all primes v and q and all $n \geq 2$. For the first of these, $\mathrm{H}^0(\mathrm{Spec}(k_v),\mathbb{Q}(n-1))$ is a direct summand in the rational K-group $K_{2n-2}(k_v)_{\mathbb{Q}}$ which is zero by a computation of Quillen [Qui72]. For the second, when $v \nmid q$, we have

$$\mathrm{H}^{-1}(\mathrm{Spec}(k_v),\mathbb{F}_q(n-1)) = \mathrm{H}^{-1}_{\text{ét}}(\mathrm{Spec}(k_v),\mu_q^{\otimes n-1}) = 0$$

by Beilinson–Lichtenbaum; when $v = q$ the vanishing of $\mathrm{H}^{-1}(\mathrm{Spec}(\mathbb{F}_v),\mathbb{F}_q(n-1))$ is a theorem of Geisser–Levine [GL00, Theorem 1.1]. □

Corollary A.12. *For $n \geq 2$, the map*

$$(*) \quad \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K,\mathbb{Z})}(\mathbb{Z}(0),\mathbb{Z}(n)[1]) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{Hom}_{\mathrm{D}^-\lim \mathrm{Sh}(\text{Ét}_K,\mathbb{Z}_p)}(\mathbb{Z}_p(0),\mathbb{Z}_p(n))$$

induced by étale realisation is injective.

Proof. For all $r \geq 1$, the kernel of $(*)$ is, by definition, contained in the kernel of the map

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K,\mathbb{Z})}(\mathbb{Z}(0),\mathbb{Z}(n)[1]) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{Hom}_{\mathrm{D}^-\mathrm{Sh}(\text{Ét}_K,\mathbb{Z}/p^r)}(\mathbb{Z}/p^r(0),\mathbb{Z}/p^r(n))\,,$$

and this latter kernel is $p^r\,\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K,\mathbb{Z})}(\mathbb{Z}(0),\mathbb{Z}(n)[1]) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ by Corollary A.10. So the kernel of $(*)$ is contained in

$$\bigcap_r p^r\,\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K,\mathbb{Z})}(\mathbb{Z}(0),\mathbb{Z}(n)[1]) \otimes_{\mathbb{Z}} \mathbb{Z}_p = 0\,,$$

using that $\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K,\mathbb{Z})}(\mathbb{Z}(0),\mathbb{Z}(n)[1])$ is finitely generated. □

This completes the proof of Proposition A.2, and hence of Proposition A.1: we know the map (A.3) is injective by the above corollary, and then it is surjective by dimension considerations. □

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