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The asymptotics for the perfect conductivity problem with stiff $C^{1,\alpha}$-inclusions

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ABSTRACT

This paper is devoted to an investigation of blow-up phenomena occurring in high-contrast fiber-reinforced composites. When the distance between perfect conductors or between the conductors and the matrix boundary tends to zero, the electric field may appear blow-up. The major objective of this paper is to give a precise description for the singular behavior of such a high concentration in the presence of $C^{1,\alpha}$-inclusions with extreme conductivities. Our results contain the boundary and interior asymptotics of the concentrated field in all dimensions. In particular, the blow-up factor for each dimension is accurately captured.

1. Background

This work is concerned with studying the asymptotic behavior of the electric field concentration in the thin gaps between inclusions or the narrow regions between the inclusions and the external boundary. We focus on high-contrast fiber-reinforced composite materials when the concentrated field of inclusions is close to maximal, which means that the distance between neighboring fibers or between the fibers and the matrix boundary is much smaller than their sizes.

Initially our interest is motivated by the issue of material failure initiation. It is well known that elliptic equations with discontinuous coefficients can be used to describe heterogeneous media with fibers close to touching. Stimulated by the great work [6,20,35] on damage analysis in composite materials, there has been a long list of literature, beginning with [12,32,33], on gradient estimates for solutions of elliptic equations and systems with piecewise coefficients. The estimates in [32,33] depend on the ellipticity of the coefficients. When elliptic constants degenerate to infinity, we consider a mathematical model of a composite of a

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homogeneous medium in which perfectly conducting inclusions are close to touching or the inclusions are nearly touching the matrix boundary. The key quantity of interest in describing the singularities of the concentration is the blow-up rate of the gradient of a solution to the perfect conductivity problem. Denote by ε the distance between two inclusions or between the inclusions and the external boundary. In the close touching regime, Ammari, Kang and Lim [3] were the first to show the blow-up rate of the field is $\varepsilon^{-1/2}$ for two circular fibers in two dimensions by constructing a lower bound on the gradient and then the authors, collaborated with H. Lee and J. Lee, proved its optimality in [4]. Subsequently, many mathematicians made use of different methods to demonstrate that the blow-up rate of the gradient of a solution to the perfect conductivity problem is $\varepsilon^{-1/2}$ in two dimensions, $|\varepsilon\ln\varepsilon|^{-1}$ in three dimensions, and ε^{-1} in dimensions greater than or equal to four. See Bao, Li and Yin [8,9], as well as Lim and Yun [34], and Yun [37,38].

Further, for the purpose of driving the development of numerical approaches to multiscale problem, it is critical to give a precise characterization for the singularities of the concentrated field. Kang, Lim and Yun [21,22] established an asymptotic formula of the gradient for two circular inclusions in two dimensions and spherical inclusions in three dimensions, respectively. Ammari et al. [2] used the technique of disks osculating to convex domains to generalize the result in [21] to the case when inclusions are strictly convex domains in two dimensions. Recently, Li, Li and Yang [28] gave a sharp description of the electric field in dimensions two and three for two arbitrarily 2-convex inclusions and explicitly revealed the effect of relative principal curvatures of inclusions. When the relative curvature of inclusions degenerates to zero, that is, we consider m-convex inclusions with $m > 2$, Li [31] extended the asymptotics in [28] to the case of m-convex inclusions in dimensions two and three and captured a blow-up factor different from that in [28]. Recently, Zhao [39] established a boundary asymptotic formula of the concentrated field for m-convex inclusions and the boundary data of k-order growth in all dimensions and showed that this type of boundary data can strengthen the singularity of the electric field. Additionally, it is worthwhile to mention that for core-shell geometry with circular boundaries, Kim and Lim [25] derived an asymptotic formula of the potential function by using the single and double layer potentials with image line charges. For high-contrast composites with the matrix described by nonlinear p-Laplace equation, Gorb and Novikov [18] utilized the method of barriers to obtain a qualitative characterization of the concentration. For more related results and for an extensive bibliography we also refer to papers [1,5,7,10,11,13,14,16,19,23,26,29,30,36] and references therein.

In the present work, we consider the following two situations: when one inclusion is very close to touching the external boundary and when two inclusions are very close but not touching. This paper is based on the work [15] completed by Chen, Li and Xu, where they used De Giorgi-Nash estimates and Campanato's approach to create an adapted version of the iteration technique with respect to the energy in the presence of $C^{1,\alpha}$-inclusions and then established the optimal gradient estimates of the concentrated field. It is worth emphasizing that they overcome the difficulty that the constructed auxiliary function is not smooth enough to apply the $W^{2,p}$-estimates as in [30] for the case of $C^{2,\alpha}$-inclusions. Moreover, this paper, as a continuation of [28,39], extends the precise characterization of the electric field for $C^{2,\alpha}$-inclusions there to $C^{1,\alpha}$-inclusions here.

The outline of this paper is as follows. We establish the boundary asymptotics of the concentrated field in Sections 2–5 and the interior asymptotics in Sections 6–7. Specifically, in Section 2, we first list our main results of the boundary asymptotics and then use the linear decomposition (2.14) below to reduce the original problem to establishing the following three types of asymptotic expansions, that is, (i) asymptotics of ∇v_i, $i = 0, 1$ in Section 3, where v_1 and v_0 solve the following equations (2.5) and (2.12), respectively; (ii) asymptotic of the blow-up factor $Q[\varphi]$ defined by (2.13) below in Section 4; (iii) asymptotic of the energy $\int_\Omega |\nabla v_1|^2$ in Section 5. In Section 6 we state the interior asymptotic results in Theorem 6.1 and then complete its proof in Section 7.

2. The boundary asymptotics for the perfect conductivity problem

2.1. Governing equation

Consider a bounded domain $D \subset \mathbb{R}^n$ $(n \geq 2)$ with its boundary being of $C^{1,\alpha}$ $(0 < \alpha < 1)$. Assume that there is a $C^{1,\alpha}$-subdomain D_1^* inside D such that D_1^* touches the external boundary ∂D only at one point. By a translation and rotation of the coordinates, if necessary, we let

$$\partial D_1^* \cap \partial D = \{0'\} \subset \mathbb{R}^n, \quad D \subset \{(x', x_n) \in \mathbb{R}^n |\, x_n > 0\}.$$

Here and below, we use superscript prime to denote $(n-1)$-dimensional domains and variables, such as B' and x'. After translating D_1^* by ε along x_n-axis, we have

$$D_1^\varepsilon := D_1^* + (0', \varepsilon),$$

where $\varepsilon > 0$ is an arbitrarily small constant. For simplicity, denote

$$D_1 := D_1^\varepsilon, \quad \text{and} \quad \Omega := D \setminus \overline{D}_1.$$

In this paper we first consider the following boundary value problem:

$$\begin{cases} \Delta u = 0, & \text{in } D \setminus D_1, \\ u = C_1, & \text{in } \overline{D}_1, \\ \int_{\partial D_1} \frac{\partial u}{\partial \nu}\big|_+ = 0, & \\ u = \varphi, & \text{on } \partial D, \end{cases} \tag{2.1}$$

where the free constant C_1 is determined later by the third line of (2.1) and

$$\frac{\partial u}{\partial \nu}\Big|_+ := \lim_{\tau \to 0} \frac{u(x + \nu\tau) - u(x)}{\tau}.$$

Here and below ν represents the outward unit normal to the domain and the subscript $\pm$ indicates the limit from outside and inside the domain, respectively.

We further describe our domain. Suppose that there exists a small constant $R > 0$ independent of ε, such that the partial boundaries of ∂D_1 and ∂D near the origin can be represented as follows:

$$x_n = \varepsilon + h_1(x') \quad \text{and} \quad x_n = h(x'), \qquad x' \in B'_{2R},$$

where h_1 and h satisfy that for $\beta > 0$,

(***A1***) $h_1(x') - h(x') = \lambda |x'|^{1+\alpha} + O(|x'|^{1+\alpha+\beta})$,
(***A2***) $|\nabla_{x'} h_1(x')|, |\nabla_{x'} h(x')| \leq \kappa_1 |x'|^\alpha$,
(***A3***) $\|h_1\|_{C^{1,\alpha}(B'_{2R})} + \|h\|_{C^{1,\alpha}(B'_{2R})} \leq \kappa_2$.

For $z' \in B'_R$, $0 < t \leq 2R$, define

$$\Omega_t(z') := \left\{ x \in \mathbb{R}^n \;\middle|\; h(x') < x_n < \varepsilon + h_1(x'),\ |x' - z'| < t \right\}.$$

For simplicity, we let Ω_t be the abbreviated notation for the domain $\Omega_t(0')$ in the following. Construct two scalar auxiliary functions $\bar{u} \in C^{1,\alpha}(\mathbb{R}^n)$ and $\bar{u}_0 \in C^{1,\alpha}(\mathbb{R}^n)$ satisfying that $\bar{u} = 1$ on ∂D_1, $\bar{u} = 0$ on ∂D and

$$\bar{u}(x) = \frac{x_n - h(x')}{\varepsilon + h_1(x') - h(x')}, \text{ in } \Omega_{2R}, \quad \|\bar{u}\|_{C^{1,\alpha}(\Omega\setminus\Omega_R)} \leq C, \tag{2.2}$$

and $\bar{u}_0 = 0$ on ∂D_1, $\bar{u}_0 = \varphi(x)$ on ∂D, and

$$\bar{u}_0 = \varphi(x', h(x'))(1 - \bar{u}), \text{ in } \Omega_{2R}, \quad \|\bar{u}_0\|_{C^{1,\alpha}(\Omega\setminus\Omega_R)} \leq C\|\varphi\|_{C^{1,\alpha}(\partial D)}. \tag{2.3}$$

To simplify notations, we denote

$$\Gamma_\alpha = \Gamma\left(1 - \frac{1}{1+\alpha}\right)\Gamma\left(\frac{1}{1+\alpha}\right),$$

where $\Gamma(s) = \int_0^{+\infty} t^{s-1}e^{-t}\,dt$, $s > 0$ is the Gamma function. Denote by ω_{n-1} the area of the surface of unit sphere in $(n-1)$-dimension. For $(x', x_n) \in \Omega_{2R}$, denote

$$\delta(x') := \varepsilon + h_1(x') - h(x'). \tag{2.4}$$

Note that in the following the universal constant C or order $O(1)$, whose values may vary from line to line, depends only on $\lambda, \kappa_1, \kappa_2, R$ and an upper bound of the $C^{1,\alpha}$ norms of ∂D_1 and ∂D, but not on ε. For the sake of simplicity, we let $\varphi(0) = 0$. Otherwise, we replace u by $u - \varphi(0)$ throughout the paper.

2.2. Main results

To derive a precise characterization for the gradient of a solution to the perfect conductivity problem (2.1), the key issue lies in calculating the energy of the harmonic function v_1 defined by the following

$$\begin{cases} \Delta v_1 = 0, & \text{in } D \setminus \overline{D}_1, \\ v_1 = 1, & \text{on } \partial D_1, \\ v_1 = 0, & \text{on } \partial D. \end{cases} \tag{2.5}$$

That is,

Theorem 2.1 *(The energy). Assume that $D_1 \subset D \subseteq \mathbb{R}^n$ $(n \geq 2)$ are defined as above, conditions (***A1***)–(***A3***) hold. Let $v_1 \in H^1(D \setminus \overline{D}_1)$ be the solution of (2.5). Then, for a sufficiently small $\varepsilon > 0$,*

(i) for $n = 2$,

$$\int_\Omega |\nabla v_1|^2 = \frac{2\Gamma_\alpha}{(1+\alpha)\lambda^{\frac{1}{1+\alpha}}}\varepsilon^{-\frac{\alpha}{1+\alpha}} \begin{cases} 1 + O(1)\varepsilon^{\frac{\beta}{1+\alpha}}, & \alpha > \beta, \\ 1 + O(1)\varepsilon^{\frac{\alpha}{1+\alpha}}|\ln\varepsilon|, & 0 < \alpha \leq \beta; \end{cases}$$

(ii) for $n \geq 3$,

$$\int_\Omega |\nabla v_1|^2 = \int_{\Omega^*} |\nabla v_1^*|^2 + O(1) \begin{cases} \varepsilon^{\frac{\alpha^2(1-\alpha)}{2(2+\alpha)(1+\alpha)^2}}, & n = 3, \\ \varepsilon^{\frac{\alpha^2}{2(2+\alpha)(1+\alpha)^2}\min\{1+\alpha, 2-\alpha\}}, & n = 4, \\ \varepsilon^{\frac{\alpha^2}{2(2+\alpha)(1+\alpha)}}, & n \geq 5, \end{cases}$$

where v_1^* satisfies

$$\begin{cases} \Delta v_1^* = 0, & in\ D \setminus \overline{D_1^*}, \\ v_1^* = 1, & on\ \partial D_1^* \setminus \{0\}, \\ v_1^* = 0, & on\ \partial D. \end{cases} \tag{2.6}$$

Denote $\Omega^* := D \setminus \overline{D_1^*}$. We define a linear functional with respect to φ,

$$Q^*[\varphi] := \int_{\partial D_1^*} \frac{\partial v_0^*}{\partial \nu}, \tag{2.7}$$

where v_0^* is a solution of the following problem:

$$\begin{cases} \Delta v_0^* = 0, & \text{in } \Omega^*, \\ v_0^* = 0, & \text{on } \partial D_1^*, \\ v_0^* = \varphi(x), & \text{on } \partial D. \end{cases} \tag{2.8}$$

For the order of the rest term, we denote

$$r_\varepsilon = \begin{cases} \varepsilon^{\min\{\frac{\beta}{1+\alpha}, \frac{(1-\alpha)\alpha}{2(2+\alpha)}\}}, & \alpha > \beta,\ n = 2, \\ \varepsilon^{\frac{(1-\alpha)\alpha}{2(2+\alpha)}}, & 0 < \alpha \le \beta,\ n = 2, \\ \varepsilon^{\frac{\alpha^2(1-\alpha)}{2(2+\alpha)(1+\alpha)^2}}, & n = 3, \\ \varepsilon^{\frac{\alpha^2}{2(2+\alpha)(1+\alpha)^2}\min\{1+\alpha, 2-\alpha\}}, & n = 4, \\ \varepsilon^{\frac{\alpha^2}{2(2+\alpha)(1+\alpha)}}, & n \ge 5. \end{cases} \tag{2.9}$$

Theorem 2.2. *Assume that $D_1 \subset D \subseteq \mathbb{R}^n$ $(n \ge 2)$ are defined as above, conditions (**A1**)–(**A3**) hold. For $\varphi \in C^{1,\alpha}(\partial D)$, let $u \in H^1(D;\mathbb{R}^n) \cap C^1(\overline{\Omega};\mathbb{R}^n)$ be the solution of (2.1). Then for a sufficiently small $\varepsilon > 0$ and $x \in \Omega_R$, if $Q^*[\varphi] \ne 0$,*

(i) for $n = 2$,

$$\nabla u = \frac{(1+\alpha)\lambda^{\frac{1}{1+\alpha}} Q^*[\varphi]}{2\Gamma_\alpha}(1 + O(r_\varepsilon))\varepsilon^{\frac{\alpha}{1+\alpha}} \nabla \bar{u} + \nabla \bar{u}_0 + O(1)\delta^{-\frac{1-\alpha}{1+\alpha}} \|\varphi\|_{C^1(\partial D)};$$

(ii) for $n \ge 3$,

$$\nabla u = \frac{Q^*[\varphi]}{\int_{\Omega^*} |\nabla v_1^*|^2}(1 + O(r_\varepsilon))\nabla \bar{u} + \nabla \bar{u}_0 + O(1)\delta^{-\frac{1}{1+\alpha}} \|\varphi\|_{C^1(\partial D)}, \tag{2.10}$$

where $\bar{u}$ and $\bar{u}_0$ are defined by (2.2) and (2.3), respectively, δ is defined by (2.4), $Q^*[\varphi]$ is defined by (2.7), v_1^* solves (2.6) and r_ε is defined in (2.9).

Remark 2.3. Let $\varphi \in C^{1,\alpha}(\partial D)$ satisfy that $\varphi(0) = 0$, $\varphi \not\equiv 0$ and $\varphi \ge 0$ on ∂D. For example, take $\varphi = |x'|^{1+\alpha}$ on ∂D. We now claim that this type of boundary data can make $Q^*[\varphi] \ne 0$. Recalling the definition of $Q^*[\varphi]$ and using integration by parts, we obtain that $Q^*[\varphi] = -\int_{\partial D \setminus \{0\}} \frac{\partial v_1^*}{\partial \nu}\varphi$. Then applying the Hopf Lemma for v_1^*, we have $\frac{\partial v_1^*}{\partial \nu}\big|_{\partial D \setminus \{0\}} < 0$. Since $\varphi \not\equiv 0$ and $\varphi \ge 0$ on ∂D, we then obtain that $Q^*[\varphi] \ne 0$.

Remark 2.4. It is worth mentioning that in contrast to the boundary asymptotics of [39] in the presence of m-convex inclusions for $m \ge 2$, the blow-up factor $Q[\varphi]$ here can't strengthen the singularities of the concentrated field for any boundary data φ. In addition, we improve the upper and lower bounds of the gradient in [15] to the exact asymptotic expansions here and below.

Remark 2.5. We would like to remark that the major singularity of $|\nabla\bar{u}_0|$ lies in $|\partial_n\bar{u}_0| = \frac{|\varphi(x',h(x'))|}{\varepsilon+h_1-h} \leq \frac{C|x'|\|\varphi\|_{C^1(\partial D)}}{\varepsilon+|x'|^{1+\alpha}}$ and is thus no more than $\varepsilon^{-\frac{\alpha}{1+\alpha}}$ on the cylinder surface $\{|x'| = \varepsilon^{\frac{1}{1+\alpha}}\} \cap \Omega$. Similarly, the singularity of $|\nabla\bar{u}|$ is determined by $\partial_{x_n}\bar{u} = \frac{1}{\varepsilon+\lambda|x'|^{1+\alpha}+O(|x'|^{1+\alpha+\beta})}$ with its greatest blow-up rate ε^{-1} arriving at the $(n-1)$-dimension sphere $\{|x'| \leq \varepsilon^{\frac{1}{1+\alpha}}\} \cap \Omega$. Then in view of the linear decomposition (2.14) below, we see from the asymptotics of ∇u in Theorem 2.2 that the greatest blow-up rate arises from the first part $\frac{Q[\varphi]}{\int_\Omega |\nabla v_1|^2}\nabla v_1$ of (2.14) with its blow-up rate being $\varepsilon^{-\frac{1}{1+\alpha}}$ in two dimensions and ε^{-1} in dimensions greater than or equal to three.

Remark 2.6. If $n \geq 3$, since the energy degenerates to no singularity, we can obtain the same asymptotic result as (2.10) for more general $C^{1,\alpha}$-inclusions as follows:

$$\lambda_1|x'|^{1+\alpha} \leq h_1 - h \leq \lambda_2|x'|^{1+\alpha}, \quad \text{in } \Omega_{2R},$$

where λ_1 and λ_2 are two positive constants independent of ε.

As shown in [30] and [39], we have a linear decomposition of the solution u to problem (2.1) as follows:

$$u(x) = C_1 v_1(x) + v_0(x), \qquad \text{in } \Omega, \tag{2.11}$$

where v_1 is defined by (2.5) and v_0 solves

$$\begin{cases} \Delta v_0 = 0, & \text{in } D \setminus \overline{D}_1, \\ v_0 = 0, & \text{on } \partial D_1, \\ v_0 = \varphi(x), & \text{on } \partial D. \end{cases} \tag{2.12}$$

Analogous to (2.7) and (2.8), we introduce a linear functional with respect to φ as follows:

$$Q[\varphi] = \int\limits_{\partial D_1} \frac{\partial v_0}{\partial \nu}. \tag{2.13}$$

From the third line of (2.1) and (2.11), we get

$$C_1 \int\limits_{\partial D_1} \frac{\partial v_1}{\partial \nu} + \int\limits_{\partial D_1} \frac{\partial v_0}{\partial \nu} = 0.$$

Then it follows from integration by parts that

$$\nabla u = \frac{Q[\varphi]}{\int_\Omega |\nabla v_1|^2}\nabla v_1 + \nabla v_0. \tag{2.14}$$

3. Identification of the leading terms

Before expanding ∇v_0 with respect to the distance ε, we first state a result with its detailed proofs seen in [15].

Theorem 3.1. *Let Q be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a $C^{1,\alpha}$ $(0 < \alpha < 1)$ boundary portion $\Gamma \subset \partial Q$. Let $w \in H^1(Q) \cap C^1(Q \cup \Gamma)$ be the solution of*

$$\begin{cases} -\Delta w = \mathrm{div}\mathbf{f}, & \text{in } Q, \\ w = 0, & \text{on } \Gamma, \end{cases}$$

where $\mathbf{f} \in C^\alpha(Q; \mathbb{R}^n)$. Then for any domain $Q' \subset\subset Q \cup \Gamma$,

$$\|w\|_{C^{1,\alpha}(Q')} \leq C(\|w\|_{L^\infty(Q)} + [\mathbf{f}]_{\alpha,Q}),$$

where $C = C(n, \alpha, Q', Q)$.

Here, the Hölder semi-norm of $\mathbf{f} = (f_1, f_2, \cdots, f_n)$ is defined by

$$[\mathbf{f}]_{\alpha,Q} := \max_{1\leq i\leq n} \sup_{x,y\in Q} \frac{|f_i(x) - f_i(y)|}{|x - y|^\alpha}.$$

Remark 3.2. Theorem 3.1 actually improves the classical $C^{1,\alpha}$ estimates [17] in the setting with partially zero boundary condition, which plays a key role in the iteration scheme with respect to the energy.

We now demonstrate that $\nabla \bar{u}_0$ is the main term of ∇v_0.

Theorem 3.3. *Assume as above. Let v_0 be the weak solution of (2.12). Then, for a sufficiently small $\varepsilon > 0$,*

$$\nabla v_0 = \nabla \bar{u}_0 + O(1)\delta^{-\frac{1}{1+\alpha}}(|\varphi(x', h(x'))| + \delta^{\frac{1}{1+\alpha}}\|\varphi\|_{C^1(\partial D)}), \quad \text{in } \Omega_R, \tag{3.1}$$

and

$$\|\nabla v_0\|_{L^\infty(\Omega\setminus\Omega_R)} \leq C\|\varphi\|_{C^{1,\alpha}(\partial D)},$$

where $\bar{u}_0$ is defined by (2.3).

Remark 3.4. Since $\varphi \in C^1(\partial D)$ and $\varphi(0) = 0$, we can further refine the result (3.1) as follows:

$$\nabla v_0 = \nabla \bar{u}_0 + O(1)\|\varphi\|_{C^1(\partial D)}.$$

Proof of Theorem 3.3. From hypotheses (***A1***)–(***A2***), we deduce that for $0 < s \leq \frac{1}{8\kappa_1 \max\{1, \lambda^{-\frac{1}{1+\alpha}}\}}\delta^{\frac{1}{1+\alpha}}$,

$$[\nabla \bar{u}_0]_{\alpha,\Omega_s(z')} \leq C\left(|\varphi(z', h(z'))|\delta^{-\frac{2+\alpha}{1+\alpha}} + \|\varphi\|_{C^1(\partial D)}\delta^{-1}\right)s^{1-\alpha}. \tag{3.2}$$

To simplify the notation, we use $\|\varphi\|_{C^1}$ to denote $\|\varphi\|_{C^1(\partial D)}$ in the following. Define

$$w_0 := v_0 - \bar{u}_0. \tag{3.3}$$

Step 1. Proof of

$$\|\nabla w_0\|_{L^2(\Omega)} \leq C\|\varphi\|_{C^1}. \tag{3.4}$$

From (3.3), we see that w_0 solves

$$\begin{cases} \Delta w_0 = -\mathrm{div}(\nabla \bar{u}_0), & \text{in } \Omega, \\ w_0 = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Picking the test function w_0 in equation (3.5) and integrating by parts, it follows from (2.3) and $\partial_{nn}\bar{u}_0 = 0$ in Ω_R that

$$\begin{aligned} \int_{\Omega} |\nabla w_0|^2 = & -\int_{\Omega\setminus\Omega_R} \nabla\bar{u}_0 \nabla w_0 - \int_{\Omega_R} \nabla_{x'}\bar{u}_0 \nabla_{x'} w_0 - \int_{\substack{|x'|=R, \\ h(x')<x_n<\varepsilon+h_1(x')}} w_0 \partial_n \bar{u}_0 \nu_n \\ \leq & \|\nabla\bar{u}_0\|_{L^2(\Omega\setminus\Omega_R)} \|\nabla w_0\|_{L^2(\Omega\setminus\Omega_R)} + \|\nabla_{x'}\bar{u}_0\|_{L^2(\Omega_R)} \|\nabla_{x'} w_0\|_{L^2(\Omega_R)} \\ & + \int_{\substack{|x'|=R, \\ h(x')<x_n<\varepsilon+h_1(x')}} C\|\varphi\|_{C^0(\partial D)} |w_0| \\ \leq & C\|\varphi\|_{C^1} \|w_0\|_{L^2(\Omega)}, \end{aligned}$$

where in last line we use the Sobolev trace embedding theorem as follows:

$$\int_{\substack{|x'|=R, \\ h(x')<x_n<\varepsilon+h_1(x')}} |w_0| \leq C \left(\int_{\Omega\setminus\Omega_R} |\nabla w_0|^2 dx \right)^{\frac{1}{2}}.$$

Thus,

$$\|\nabla w_0\|_{L^2(\Omega)} \leq C\|\varphi\|_{C^1}.$$

That is, (3.4) holds.
Step 2. Proof of

$$\int_{\Omega_\delta(z')} |\nabla w_0|^2 dx \leq C\delta^{n-\frac{2}{1+\alpha}} \left(|\varphi(z', h(z'))|^2 + \delta^{\frac{2}{1+\alpha}} \|\varphi\|_{C^1}^2 \right), \tag{3.6}$$

where δ is defined by (2.4). For $0 < t < s < R$, let η be a smooth cutoff function such that $\eta = 1$ if $|x' - z'| < t$, $\eta = 0$ if $|x' - z'| > s$, $0 \leq \eta \leq 1$ if $t \leq |x' - z'| \leq s$, and $|\nabla_{x'}\eta| \leq \frac{2}{s-t}$. Observe that w_0 also satisfies

$$\begin{cases} -\Delta w_0 = \mathrm{div}(\nabla\bar{u}_0 - (\nabla\bar{u}_0)_{\Omega_s(z')}), & \text{in } \Omega_{2R}, \\ w_0 = 0, & \text{on } \Gamma^{\pm}_{2R}, \end{cases} \tag{3.7}$$

where

$$(\nabla\bar{u}_0)_{\Omega_s(z')} = \frac{1}{|\Omega_s(z')|} \int_{\Omega_s(z')} \nabla\bar{u}_0(x) dx.$$

Taking the test function $w_0\eta^2$ in (3.7) and integrating by parts, we obtain the following iteration formula:

$$\int\limits_{\Omega_t(z')} |\nabla w_0|^2 dx \le \frac{C}{(s-t)^2} \int\limits_{\Omega_s(z')} |w_0|^2 dx + C \int\limits_{\Omega_s(z')} |\nabla \bar{u}_0 - (\nabla \bar{u}_0)_{\Omega_s(z')}|^2 dx.$$

For $|z'| < R$, $\delta(z') < s < \frac{2}{3}\max\{\varepsilon^{\frac{1}{1+\alpha}}, |z'|\}$, we have $\frac{\delta(z')}{C} \le \delta(x') \le C\delta(z')$ in $\Omega_s(z')$. Since $w_0 = 0$ on Γ_R^-, we deduce that

$$\int\limits_{\Omega_s(z')} |w_0|^2 \le C\delta^2 \int\limits_{\Omega_s(z')} |\nabla w_0|^2, \tag{3.8}$$

and due to (3.2), we have

$$\begin{aligned} &\int\limits_{\Omega_s(z')} |\nabla \bar{u}_0 - (\nabla \bar{u}_0)_{\Omega_s(z')}|^2 dx \\ &\le C s^{n+1} \delta^{-\frac{3+\alpha}{1+\alpha}} (|\varphi(z', h(z'))|^2 + \delta^{\frac{2}{1+\alpha}} \|\varphi\|_{C^1}^2). \end{aligned} \tag{3.9}$$

Denote

$$F(t) := \int\limits_{\Omega_t(z')} |\nabla w_0|^2.$$

From (3.8)–(3.9), we derive

$$F(t) \le \left(\frac{c\delta}{s-t}\right)^2 F(s) + C s^{n+1} \delta^{-\frac{3+\alpha}{1+\alpha}} (|\varphi(z', h(z'))|^2 + \delta^{\frac{2}{1+\alpha}} \|\varphi\|_{C^1}^2), \tag{3.10}$$

where c and C are universal constants.

Pick $k = \left[\frac{1}{4c\delta^{\frac{\alpha}{2(1+\alpha)}}}\right] + 1$ and $t_i = \delta + 2ci\delta$, $i = 0, 1, 2, \cdots, k$. Then, (3.10), together with $s = t_{i+1}$ and $t = t_i$, leads to

$$F(t_i) \le \frac{1}{4} F(t_{i+1}) + C(i+1)^{n+1} \delta^{n - \frac{2}{1+\alpha}} \left(|\varphi(z', h(z'))|^2 + \delta^{\frac{2}{1+\alpha}} \|\varphi\|_{C^1}^2\right).$$

It follows from k iterations and (3.4) that for a sufficiently small $\varepsilon > 0$,

$$F(t_0) \le C\delta^{n - \frac{2}{1+\alpha}} \left(|\varphi(z', h(z'))|^2 + \delta^{\frac{2}{1+\alpha}} \|\varphi\|_{C^1}^2\right).$$

Thus, (3.6) is proved.

Step 3. Proof of

$$|\nabla w_0| \le C\delta^{-\frac{1}{1+\alpha}} (|\varphi(x', h(x'))| + \delta^{\frac{1}{1+\alpha}} \|\varphi\|_{C^1(\partial D)}), \quad \text{in } \Omega_R. \tag{3.11}$$

We first make a change of variables in $\Omega_\delta(z')$ as follows:

$$\begin{cases} x' - z' = \delta y', \\ x_n = \delta y_n, \end{cases}$$

which turns it into Q_1 of nearly unit size, where, for $0 < r \le 1$,

$$Q_r = \left\{ y \in \mathbb{R}^n \,\Big|\, \frac{1}{\delta} h(\delta y' + z') < y_n < \frac{\varepsilon}{\delta} + \frac{1}{\delta} h_1(\delta y' + z'),\ |y'| < r \right\},$$

with its top and bottom boundaries denoted, respectively, by

$$\Sigma_r^+ = \left\{ y \in \mathbb{R}^n \,\Big|\, y_n = \frac{\varepsilon}{\delta} + \frac{1}{\delta} h_1(\delta y' + z'),\ |y'| < r \right\},$$

and

$$\Sigma_r^- = \left\{ y \in \mathbb{R}^n \,\Big|\, y_n = \frac{1}{\delta} h(\delta y' + z'),\ |y'| < r \right\}.$$

For $y \in Q_1$, write

$$W_0(y', y_n) = w(\delta y' + z', \delta y_n), \quad V_0(y', y_n) := \bar{u}_0(\delta y' + z', \delta y_n).$$

From (3.5), we see that W_0 satisfies

$$\begin{cases} \Delta W_0 = -\mathrm{div}(\nabla V_0), & \text{in } Q_1, \\ W_0 = 0, & \text{on } \Sigma_1^{\pm}. \end{cases}$$

Step 3.1. We first utilize De Giorgi-Nash approach to establish the L^∞ estimate of W_0 as follows:

$$\|W_0\|_{L^\infty(Q_{1/2})} \leq C(\|W_0\|_{L^2(Q_1)} + [\nabla V_0]_{\alpha, Q_1}). \tag{3.12}$$

For $\theta \geq 1$, $N \geq k > 0$, we define a function $H \in C^1([k, \infty))$ such that $H(t) = t^\beta - k^\beta$ for $t \in [k, N]$ and H is linear for $t \in [N, \infty)$. Let $\psi = W_0^+ + k$ and $v = G(\psi) = \int_k^\psi |H'(s)| ds$. Pick the test function $\eta^2 v$, where the smooth cut-off function η satisfies that for $\frac{1}{2} \leq r_1 < r_2 \leq 1$, $\eta = 1$ for $|y'| \leq r_1$, $\eta = 0$ for $|y'| \geq r_2$, and $|\nabla \eta| \leq \frac{2}{r_2 - r_1}$.

In light of the fact that $G(s) \leq G'(s)s$ and $\nabla W_0 = \nabla \psi$ when $v = G(\psi) > 0$, it follows from integration by parts and Young's inequality that

$$\int_{Q_1} \eta^2 G'(\psi) |\nabla \psi| \leq C \int_{Q_1} |\nabla \eta|^2 G'(\psi) \psi^2 + C \int_{Q_1} \eta^2 \frac{|\nabla V_0|^2}{k^2} G'(\psi) \psi^2.$$

Set $k = \|\nabla V_0\|_{L^q(Q_1)}$ for $q > n$. From the definition of G and the Hölder inequality, we have

$$\|\eta \nabla H(\psi)\|_{L^2(Q_1)}^2 \leq C \|\nabla \eta H'(\psi) \psi\|_{L^2(Q_1)}^2 + C \|\eta H'(\psi) \psi\|_{L^{\frac{2q}{q-2}}(Q_1)}^2. \tag{3.13}$$

By virtue of the interpolation inequality, we obtain that for any $\tau > 0$,

$$\|\eta H'(\psi) \psi\|_{L^{\frac{2q}{q-2}}(Q_1)} \leq \tau \|\eta H'(\psi) \psi\|_{L^{\frac{2\hat{n}}{\hat{n}-2}}(Q_1)} + C \tau^{-\frac{\hat{n}}{q - \hat{n}}} \|\eta H'(\psi) \psi\|_{L^2(Q_1)}, \tag{3.14}$$

where $\hat{n} \in (2, q)$ for $n = 2$ and $\hat{n} = n$ for $n > 2$. Since $\eta H(\psi) \in H_0^1(Q_1)$, it follows from the Sobolev inequality that

$$\|\eta H(\psi)\|_{L^{\frac{2\hat{n}}{\hat{n}-2}}(Q_1)} \leq C(\|\eta \nabla H(\psi)\|_{L^2(Q_1)} + \|H(\psi) \nabla \eta\|_{L^2(Q_1)}). \tag{3.15}$$

Substituting (3.13)–(3.14) into (3.15), we obtain

$$\|\eta H(\psi)\|_{L^{\frac{2\hat{n}}{\hat{n}-2}}(Q_1)} \leq C(\tau\|\eta H'(\psi)\psi\|_{L^{\frac{2\hat{n}}{\hat{n}-2}}(Q_1)} + C\tau^{-\frac{\hat{n}}{q-\hat{n}}}\|\eta H'(\psi)\psi\|_{L^2(Q_1)})$$
$$+ C\left(\int_{Q_1} |\nabla\eta|^2(|H'(\psi)\psi|^2 + H^2(\psi))\right)^{\frac{1}{2}}.$$

For a small constant $\tau > 0$, we have

$$\|\eta H(\psi)\|_{L^{\frac{2\hat{n}}{\hat{n}-2}}(Q_1)} \leq C\left(\int_{Q_1} |\eta H'(\psi)\psi|^2 + \int_{Q_1} |\nabla\eta|^2(|H'(\psi)\psi|^2 + H^2(\psi))\right)^{\frac{1}{2}},$$

where $C = C(n, q)$. Further, by letting N tend to infinity, we obtain

$$\|\eta\psi^\beta\|_{L^{\frac{2\hat{n}}{\hat{n}-2}}(Q_1)} \leq C\beta\left(\int_{Q_1} (\eta^2 + |\nabla\eta|^2)\psi^{2\beta}\right)^{\frac{1}{2}}.$$

Denote $\chi = \frac{\hat{n}}{\hat{n}-2}$. Recalling the definition of η, we have

$$\|\psi\|_{L^{2\beta\chi}(Q_{r_1})} \leq \left(\frac{C\beta}{r_2 - r_1}\right)^{\beta^{-1}} \|\psi\|_{L^{2\beta}(Q_{r_2})}.$$

Thus, we iterate by $\beta = \chi^i$ and $r_i = \frac{1}{2} + \frac{1}{2^{i+1}}$, $i = 0, 1, \cdots$,

$$\|\psi\|_{L^{2\chi^{i+1}}(Q_{r_{i+1}})} \leq \left(\frac{C\chi^i}{r_i - r_{i+1}}\right)^{\chi^{-i}} \|\psi\|_{L^{2\chi^i}(Q_{r_i})} \leq (C\chi)^{\sum_{j=0}^{i} \frac{j}{\chi^j}} \|\psi\|_{L^2(Q_1)}.$$

By letting $i \to \infty$, we obtain

$$\|\psi\|_{L^\infty(Q_{1/2})} \leq C\|\psi\|_{L^2(Q_1)},$$

where $C = C(n, q, Q_1)$. In view of the definition of ψ, we have

$$\|W_0^+\|_{L^\infty(Q_{1/2})} \leq C(\|W_0\|_{L^2(Q_1)} + \|\nabla V_0\|_{L^q(Q_1)}). \tag{3.16}$$

By replacing W_0 by $-W_0$, (3.16) also holds. Consequently,

$$\|W_0\|_{L^\infty(Q_{1/2})} \leq C(\|W_0\|_{L^2(Q_1)} + \|\nabla V_0\|_{L^q(Q_1)}).$$

Observe that W_0 also solves

$$-\Delta W_0 = \operatorname{div}(\nabla V_0 - (\nabla V_0)_{Q_1}).$$

Arguing as before, we obtain

$$\|W_0\|_{L^\infty(Q_{1/2})} \leq C(\|W_0\|_{L^2(Q_1)} + \|\nabla V_0 - (\nabla V_0)_{Q_1}\|_{L^q(Q_1)})$$
$$\leq C(\|W_0\|_{L^2(Q_1)} + [\nabla V_0]_{\alpha,Q_1}).$$

Step 3.2. By making use of Theorem 3.1, the Poincaré inequality and (3.12), we have

$$\|\nabla W_0\|_{L^\infty(Q_{1/2})} \le C(\|\nabla W_0\|_{L^2(Q_1)} + [\nabla V_0]_{\alpha,Q_1}).$$

Then rescaling back to the original region $\Omega_\delta(z')$, we derive

$$\|\nabla w_0\|_{L^\infty(\Omega_{\delta/4}(z'))} \le \frac{C}{\delta}\left(\delta^{1-\frac{n}{2}}\|\nabla w_0\|_{L^2(\Omega_\delta(z'))} + \delta^{1+\alpha}[\nabla \bar{u}_0]_{\alpha,\Omega_\delta(z')}\right). \tag{3.17}$$

Substituting (3.2) and (3.6) into (3.17), we obtain that (3.11) holds. On the other hand, it follows from the standard interior estimates and boundary estimates for the Laplace equation that

$$\|\nabla v_0\|_{L^\infty(\Omega\setminus\Omega_R)} \le C\|\varphi\|_{C^{1,\alpha}(\partial D)}.$$

Thus, Theorem 3.3 is proved. □

Applying Theorem 3.3, we have

Lemma 3.5. *Assume as above. Let $v_1 \in H^1(\Omega)$ be a weak solution of (2.5). Then, for a sufficiently small $\varepsilon > 0$,*

$$\nabla v_1 = \nabla \bar{u} + O(1)\delta^{-\frac{1}{1+\alpha}}, \qquad in\ \Omega_R, \tag{3.18}$$

and

$$\|\nabla v_1\|_{L^\infty(\Omega\setminus\Omega_R)} \le C.$$

4. Expansion of the blow-up factor $Q[\varphi]$

In this section our major goal is to give an expansion of $Q[\varphi]$ with respect to ε as follows.

Lemma 4.1. *Assume as above. Then, for a sufficiently small $\varepsilon > 0$,*

$$Q[\varphi] = Q^*[\varphi] + O(1)\|\varphi\|_{C^1(\partial D)}\varepsilon^{\frac{(n-1-\alpha)\alpha}{n(2+\alpha)}},$$

where $Q^*[\varphi]$ and $Q[\varphi]$ are defined by (2.7) and (2.13), respectively.

Proof. Recalling the definitions of $Q[\varphi]$ and $Q^*[\varphi]$ and integrating by parts, we have

$$Q[\varphi] = \int_{\partial D} \frac{\partial v_1}{\partial \nu}\varphi(x), \qquad Q^*[\varphi] = \int_{\partial D} \frac{\partial v_1^*}{\partial \nu}\varphi(x),$$

where v_1 and v_1^* satisfy (2.5) and (2.6), respectively. Thus,

$$Q[\varphi] - Q^*[\varphi] = \int_{\partial D} \frac{\partial (v_1 - v_1^*)}{\partial \nu}\cdot\varphi(x).$$

Observe that the unit outward normal ν to ∂D is given by

$$\nu = \frac{(\nabla_{x'}h(x'), -1)}{\sqrt{1+|\nabla_{x'}h(x')|^2}}, \quad \text{for } x \in \Omega_R.$$

It follows from (***A2***) that for $i = 1, \cdots, n-1$,

$$|\nu_i| \le C|x'|^\alpha, \quad |\nu_n| \le 1, \quad \text{in } \Omega_R. \tag{4.1}$$

For $0 < r < R$, denote

$$\mathcal{C}_r := \left\{ x \in \mathbb{R}^n \middle| \, |x'| < r, \, \frac{1}{2} \min_{|x'| \le r} h(x') \le x_n \le \varepsilon + 2 \max_{|x'| \le r} h_1(x') \right\}.$$

We next divide into two steps to estimate the difference between $Q[\varphi]$ and $Q^*[\varphi]$.

Step 2.1. Note that $v_1 - v_1^*$ solves

$$\begin{cases} \Delta(v_1 - v_1^*) = 0, & \text{in } D \setminus (\overline{D_1 \cup D_1^*}), \\ v_1 - v_1^* = 1 - v_1^*, & \text{on } \partial D_1 \setminus D_1^*, \\ v_1 - v_1^* = v_1 - 1, & \text{on } \partial D_1^* \setminus (D_1 \cup \{0\}), \\ v_1 - v_1^* = 0, & \text{on } \partial D. \end{cases}$$

We first estimate $|v_1 - v_1^*|$ on $\partial(D_1 \cup D_1^*) \setminus \mathcal{C}_{\varepsilon^\gamma}$, where $0 < \gamma < 1/2$ to be determined later. For $x \in \partial D_1 \setminus D_1^*$, it follows from the definition of v_1^* that

$$|v_1 - v_1^*| = |v_1^*(x', x_n - \varepsilon) - v_1^*(x', x_n)| \le C\varepsilon. \tag{4.2}$$

From (3.18), we see that for $x \in \partial D_1^* \setminus (D_1 \cup \mathcal{C}_{\varepsilon^\gamma})$,

$$|v_1 - v_1^*| \le C\varepsilon^{1-(1+\alpha)\gamma}. \tag{4.3}$$

Introduce an auxiliary function $\bar{u}^*$ such that $\bar{u}^* = 1$ on $\partial D_1^* \setminus \{0\}$, $\bar{u}^* = 0$ on ∂D, and

$$\bar{u}^* = \frac{x_n - h(x')}{h_1(x') - h(x')}, \quad \text{in } \Omega_{2R}^*, \qquad \|\bar{u}^*\|_{C^2(\Omega^* \setminus \Omega_R^*)} \le C,$$

where $\Omega_r^* := \Omega^* \cap \{|x'| < r\}$, $0 < r \le 2R$. From (***A1***)–(***A2***), we derive that for $x \in \Omega_R^*$,

$$|\nabla_{x'}(\bar{u} - \bar{u}^*)| \le \frac{C}{|x'|}, \quad |\partial_n(\bar{u} - \bar{u}^*)| \le \frac{C\varepsilon}{|x'|^{1+\alpha}(\varepsilon + |x'|^{1+\alpha})}. \tag{4.4}$$

Applying Theorem 3.3 to (2.6), it follows that for $x \in \Omega_R^*$,

$$|\nabla(v_1^* - \bar{u}^*)| \le \frac{C}{|x'|}. \tag{4.5}$$

Then using Theorem 3.3 and (4.4)–(4.5), we deduce that for $x \in \Omega_R^* \cap \{|x'| = \varepsilon^\gamma\}$,

$$\begin{aligned} |\partial_n(v_1 - v_1^*)| \le & |\partial_n(v_1 - \bar{u})| + |\partial_n(\bar{u} - \bar{u}^*)| + |\partial_n(v_1^* - \bar{u}^*)| \\ \le & C\left(\frac{1}{\varepsilon^{2(1+\alpha)\gamma-1}} + \frac{1}{\varepsilon^\gamma}\right). \end{aligned}$$

This, in combination with $v_1 - v_1^* = 0$ on ∂D, reads that

$$\begin{aligned} |(v_1 - v_1^*)(x', x_n)| = & |(v_1 - v_1^*)(x', x_n) - (v_1 - v_1^*)(x', h(x'))| \\ \le & C\big(\varepsilon^{1-(1+\alpha)\gamma} + \varepsilon^{\alpha\gamma}\big). \end{aligned} \tag{4.6}$$

Making use of (4.2)–(4.3) and (4.6) and choosing $\gamma = \frac{1}{2+\alpha}$, we derive

$$|v_1 - v_1^*| \le C\varepsilon^{\frac{\alpha}{2+\alpha}}, \qquad \text{on } \partial\big(D \setminus (\overline{D_1 \cup D_1^* \cup \mathcal{C}_{\varepsilon^{\frac{1}{2+\alpha}}}})\big),$$

which in combination with the maximum principle yields that

$$|v_1 - v_1^*| \le C\varepsilon^{\frac{\alpha}{2+\alpha}}, \qquad \text{in } D \setminus \big(\overline{D_1 \cup D_1^* \cup \mathcal{C}_{\varepsilon^{\frac{1}{2+\alpha}}}}\big). \tag{4.7}$$

In view of (4.7), it follows from the standard interior and boundary estimates that $\frac{1}{(1+\alpha)(2+\alpha)} < \tilde{\gamma} < \frac{1}{2+\alpha}$,

$$|\nabla(v_1 - v_1^*)| \le C\varepsilon^{(1+\alpha)\tilde{\gamma} - \frac{1}{2+\alpha}}, \qquad \text{on } \partial D \setminus \mathcal{C}_{\varepsilon^{\frac{1}{2+\alpha}-\tilde{\gamma}}}.$$

Therefore,

$$|\mathcal{A}^{out}| := \left| \int\limits_{\partial D \setminus \mathcal{C}_{\varepsilon^{\frac{1}{2+\alpha}-\tilde{\gamma}}}} \frac{\partial(v_1 - v_1^*)}{\partial \nu} \cdot \varphi(x) \right| \le C\|\varphi\|_{L^\infty(\partial D)} \varepsilon^{(1+\alpha)\tilde{\gamma} - \frac{1}{2+\alpha}}. \tag{4.8}$$

Step 2.2. We now estimate the remainder as follows:

$$\begin{aligned}\mathcal{A}^{in} &:= \int\limits_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{2+\alpha}-\tilde{\gamma}}}} \frac{\partial(v_1 - v_1^*)}{\partial \nu} \cdot \varphi(x) \\ &= \int\limits_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{2+\alpha}-\tilde{\gamma}}}} \frac{\partial(w_1 - w_1^*)}{\partial \nu} \cdot \varphi(x) + \int\limits_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{2+\alpha}-\tilde{\gamma}}}} \frac{\partial(\bar{u} - \bar{u}^*)}{\partial \nu} \cdot \varphi(x) \\ &=: \mathcal{A}_w + \mathcal{A}_u,\end{aligned}$$

where $w_1 = v_1 - \bar{u}$ and $w_1^* = v_1^* - \bar{u}^*$. A direct application of Theorem 3.3 yields that

$$|\mathcal{A}_w| \le C \int\limits_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{2+\alpha}-\tilde{\gamma}}}} \|\varphi\|_{C^1(\partial D)} \le C\|\varphi\|_{C^1(\partial D)} \varepsilon^{(\frac{1}{2+\alpha}-\tilde{\gamma})(n-1)}. \tag{4.9}$$

As for $\mathcal{A}_u$, combining (4.1) and (4.4), we have

$$\begin{aligned}|\mathcal{A}_u| \le& \left| \int\limits_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{2+\alpha}-\tilde{\gamma}}}} \sum_{i=1}^{n-1} \partial_i(\bar{u} - \bar{u}^*)\nu_i \varphi(x) \right| + \left| \int\limits_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{2+\alpha}-\tilde{\gamma}}}} \partial_n(\bar{u} - \bar{u}^*)\nu_n \varphi(x) \right| \\ \le& C\|\varphi\|_{C^1(\partial D)} \varepsilon^{(\frac{1}{2+\alpha}-\tilde{\gamma})(n-1-\alpha)}.\end{aligned} \tag{4.10}$$

Consequently, by taking $\tilde{\gamma} = \frac{n-\alpha}{n(2+\alpha)}$ and using (4.8)–(4.10), we obtain

$$|Q[\varphi] - Q^*[\varphi]| \le C\|\varphi\|_{C^1(\partial D)} \varepsilon^{\frac{(n-1-\alpha)\alpha}{n(2+\alpha)}}.$$

That is, Lemma 4.1 holds. □

5. Proofs of Theorem 2.1 and Theorem 2.2

5.1. Proof of Theorem 2.1

Fix $\bar{\gamma} = \frac{\alpha^2}{2(2+\alpha)(1+\alpha)^2}$. Decompose the energy $\int_\Omega |\nabla v_1|^2$ as follows:

$$\int\limits_{\Omega} |\nabla v_1|^2 = \int\limits_{\Omega_{\varepsilon^{\bar{\gamma}}}} |\nabla v_1|^2 + \int\limits_{\Omega_R \setminus \Omega_{\varepsilon^{\bar{\gamma}}}} |\nabla v_1|^2 + \int\limits_{\Omega \setminus \Omega_R} |\nabla v_1|^2 =: \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

Step 1. For the first term I, in light of the definition of $\bar{u}$, it follows from Theorem 3.3 that

$$\begin{aligned} \mathrm{I} &= \int\limits_{\Omega_{\varepsilon^{\bar{\gamma}}}} |\partial_n \bar{u}|^2 + \int\limits_{\Omega_{\varepsilon^{\bar{\gamma}}}} |\partial_{x'} \bar{u}|^2 + 2\int\limits_{\Omega_{\varepsilon^{\bar{\gamma}}}} \nabla \bar{u} \cdot \nabla (v_1 - \bar{u}) + \int\limits_{\Omega_{\varepsilon^{\bar{\gamma}}}} |\nabla (v_1 - \bar{u})|^2 \\ &= \int\limits_{|x'|<\varepsilon^{\bar{\gamma}}} \frac{dx'}{\varepsilon + h_1(x') - h(x')} + O(1) \begin{cases} |\ln \varepsilon|, & n = 2, \\ \varepsilon^{(n-2)\bar{\gamma}}, & n \geq 3. \end{cases} \end{aligned} \tag{5.1}$$

As for the second term II, it can be split as follows

$$\begin{aligned} \mathrm{II}_1 &= \int\limits_{(\Omega_R \setminus \Omega_{\varepsilon^{\bar{\gamma}}}) \setminus (\Omega^*_R \setminus \Omega^*_{\varepsilon^{\bar{\gamma}}})} |\nabla v_1|^2, \\ \mathrm{II}_2 &= \int\limits_{\Omega^*_R \setminus \Omega^*_{\varepsilon^{\bar{\gamma}}}} |\nabla v_1^*|^2, \\ \mathrm{II}_3 &= \int\limits_{\Omega^*_R \setminus \Omega^*_{\varepsilon^{\bar{\gamma}}}} |\nabla (v_1 - v_1^*)|^2 + 2 \int\limits_{\Omega^*_R \setminus \Omega^*_{\varepsilon^{\bar{\gamma}}}} \nabla v_1^* \cdot \nabla (v_1 - v_1^*). \end{aligned}$$

In view of (3.18) and the fact that the thickness of $(\Omega_R \setminus \Omega_{\varepsilon^{\bar{\gamma}}}) \setminus (\Omega^*_R \setminus \Omega^*_{\varepsilon^{\bar{\gamma}}})$ is ε, we have

$$\mathrm{II}_1 \leq C\varepsilon \int\limits_{\varepsilon^{\bar{\gamma}}<|x'|<R} \frac{dx'}{|x'|^{2(1+\alpha)}} \leq C \begin{cases} \varepsilon, & n > 1 + 2(1+\alpha), \\ \varepsilon |\ln \varepsilon|, & n = 1 + 2(1+\alpha), \\ \varepsilon^{\frac{(n+5)\alpha^2 + 10\alpha + 4}{2(2+\alpha)(1+\alpha)^2}}, & n < 1 + 2(1+\alpha). \end{cases} \tag{5.2}$$

From (4.5), we obtain that for $n = 2$,

$$\begin{aligned} \mathrm{II}_2 &= \int\limits_{\Omega^*_R \setminus \Omega^*_{\varepsilon^{\bar{\gamma}}}} |\nabla \bar{u}^*|^2 + 2 \int\limits_{\Omega^*_R \setminus \Omega^*_{\varepsilon^{\bar{\gamma}}}} \nabla \bar{u}^* \cdot \nabla (v_1^* - \bar{u}^*) + \int\limits_{\Omega^*_R \setminus \Omega^*_{\varepsilon^{\bar{\gamma}}}} |\nabla (v_1^* - \bar{u}^*)|^2 \\ &= \int\limits_{\varepsilon^{\bar{\gamma}}<|x_1|<R} \frac{dx_1}{h_1(x_1) - h(x_1)} + O(1)|\ln \varepsilon|; \end{aligned} \tag{5.3}$$

for $n \geq 3$,

$$\mathrm{II}_2 = \int\limits_{\Omega^*_R \setminus \Omega^*_{\varepsilon^{\bar{\gamma}}}} |\nabla \bar{u}^*|^2 + 2 \int\limits_{\Omega^*_R \setminus \Omega^*_{\varepsilon^{\bar{\gamma}}}} \nabla \bar{u}^* \cdot \nabla (v_1^* - \bar{u}^*) + \int\limits_{\Omega^*_R \setminus \Omega^*_{\varepsilon^{\bar{\gamma}}}} |\nabla (v_1^* - \bar{u}^*)|^2$$

$$= \int\limits_{\varepsilon^{\bar{\gamma}}<|x'|<R} \frac{dx'}{h_1(x')-h(x')} + \int\limits_{\Omega_R^*} \left(|\nabla(v_1^*-\bar{u}^*)|^2 + |\nabla_{x'}\bar{u}^*|^2\right)$$
$$+ 2\int\limits_{\Omega_R^*} \nabla\bar{u}^* \cdot \nabla(v_1^*-\bar{u}^*) + O(1)\varepsilon^{(n-2)\bar{\gamma}}. \tag{5.4}$$

For $\varepsilon^{\bar{\gamma}} \leq |z'| \leq R$, we utilize the change of variable

$$\begin{cases} x'-z' = |z'|^{1+\alpha}y', \\ x_n = |z'|^{1+\alpha}y_n, \end{cases}$$

to rescale $\Omega_{|z'|+|z'|^{1+\alpha}} \setminus \Omega_{|z'|}$ and $\Omega^*_{|z'|+|z'|^{1+\alpha}} \setminus \Omega^*_{|z'|}$ into two nearly unit-size squares (or cylinders) Q_1 and Q_1^*, respectively. Let

$$V_1(y) = v_1(z'+|z'|^{1+\alpha}y', |z'|^{1+\alpha}y_n), \quad \text{in } Q_1,$$

and

$$V_1^*(y) = v_1^*(z'+|z'|^{1+\alpha}y', |z'|^{1+\alpha}y_n), \quad \text{in } Q_1^*.$$

Since $0 < V_1, V_1^* < 1$, we see from the standard elliptic estimate that

$$\|V_1\|_{C^{1,\alpha}(Q_1)} \leq C, \quad \text{and } \|V_1^*\|_{C^{1,\alpha}(Q_1^*)} \leq C.$$

Applying an interpolation with (4.7), we derive

$$|\nabla(V_1-V_1^*)| \leq C\varepsilon^{\frac{\alpha}{2+\alpha}(1-\frac{1}{1+\alpha})} \leq C\varepsilon^{\frac{\alpha^2}{(2+\alpha)(1+\alpha)}}.$$

Hence, rescaling it back to $v_1 - v_1^*$ and in light of $\varepsilon^{\frac{\alpha^2}{2(2+\alpha)(1+\alpha)^2}} \leq |z'| \leq R$, we have

$$|\nabla(v_1-v_1^*)(x)| \leq C\varepsilon^{\frac{\alpha^2}{(2+\alpha)(1+\alpha)}}|z'|^{-1-\alpha} \leq C\varepsilon^{\frac{\alpha^2}{2(2+\alpha)(1+\alpha)}}, \quad x \in \Omega^*_{|z'|+|z'|^{1+\alpha}} \setminus \Omega_{|z'|}.$$

Consequently,

$$|\nabla(v_1-v_1^*)| \leq C\varepsilon^{\frac{\alpha^2}{2(2+\alpha)(1+\alpha)}}, \quad \text{in } D \setminus \left(\overline{D_1 \cup D_1^* \cup \mathcal{C}_{\varepsilon^{\frac{\alpha^2}{2(2+\alpha)(1+\alpha)^2}}}}\right). \tag{5.5}$$

Combining (4.5) and (5.5), we obtain that

$$|\mathrm{II}_3| \leq C\varepsilon^{\frac{\alpha^2}{2(2+\alpha)(1+\alpha)}}. \tag{5.6}$$

Thus, it follows from (5.2)–(5.4) and (5.6) that for $n = 2$,

$$\mathrm{II} = \int\limits_{\varepsilon^{\bar{\gamma}}<|x_1|<R} \frac{dx_1}{h_1(x_1)-h(x_1)} + O(1)|\ln\varepsilon|; \tag{5.7}$$

for $n \geq 3$,

$$\mathrm{II} = \int\limits_{\varepsilon^{\bar{\gamma}}<|x'|<R} \frac{dx'}{h_1(x')-h(x')} + \int\limits_{\Omega_R^*} \left(|\nabla(v_1^*-\bar{u}^*)|^2 + |\partial_{x'}\bar{u}^*|^2\right)$$
$$+ 2\int\limits_{\Omega_R^*} \nabla\bar{u}^*\cdot\nabla(v_1^*-\bar{u}^*) + O(1)\begin{cases}\varepsilon^{\bar{\gamma}}, & n=3,\\ \varepsilon^{(1+\alpha)\bar{\gamma}}, & n\geq 4.\end{cases} \tag{5.8}$$

For the term III, (5.5), together with the fact that $|\nabla v_1|$ is bounded in $D_1^*\setminus(D_1\cup\Omega_R)$ and $D_1\setminus D_1^*$ and the volume of $D_1^*\setminus(D_1\cup\Omega_R)$ and $D_1\setminus D_1^*$ is of order $O(\varepsilon)$, leads to that

$$\begin{aligned}\mathrm{III} &= \int\limits_{D\setminus(D_1\cup D_1^*\cup\Omega_R)} |\nabla v_1|^2 + O(1)\varepsilon\\ &= \int\limits_{D\setminus(D_1\cup D_1^*\cup\Omega_R)} |\nabla v_1^*|^2 + 2\int\limits_{D\setminus(D_1\cup D_1^*\cup\Omega_R)} \nabla v_1^*\cdot\nabla(v_1-v_1^*)\\ &\quad + \int\limits_{D\setminus(D_1\cup D_1^*\cup\Omega_R)} |\nabla(v_1-v_1^*)|^2 + O(1)\varepsilon\\ &= \int\limits_{\Omega^*\setminus\Omega_R^*} |\nabla v_1^*|^2 + O(1)\varepsilon^{\frac{\alpha^2}{2(2+\alpha)(1+\alpha)}}.\end{aligned} \tag{5.9}$$

From (5.1) and (5.7)–(5.9), we see

$$\int\limits_{\Omega}|\nabla v_1|^2 = \int\limits_{\varepsilon^{\bar{\gamma}}<|x'|<R}\frac{dx'}{h_1(x')-h(x')} + \int\limits_{|x'|<\varepsilon^{\bar{\gamma}}}\frac{dx'}{\varepsilon+h_1(x')-h(x')}$$
$$+\begin{cases}O(1)|\ln\varepsilon|, & n=2,\\ \mathcal{M}_R^* + O(1)\begin{cases}\varepsilon^{\bar{\gamma}}, & n=3,\\ \varepsilon^{(1+\alpha)\bar{\gamma}}, & n\geq 4,\end{cases}\end{cases}$$

where

$$\mathcal{M}_R^* = \int\limits_{\Omega^*\setminus\Omega_R^*}|\nabla v_1^*|^2 + 2\int\limits_{\Omega_R^*}\nabla\bar{u}^*\cdot\nabla(v_1^*-\bar{u}^*)$$
$$+\int\limits_{\Omega_R^*}\left(|\nabla(v_1^*-\bar{u}^*)|^2 + |\partial_{x'}\bar{u}^*|^2\right).$$

Step 2. For $n=2$,

$$\begin{aligned}&\int\limits_{|x_1|<R}\frac{dx_1}{\varepsilon+h_1-h} + \int\limits_{\varepsilon^{\bar{\gamma}}<|x_1|<R}\frac{\varepsilon\, dx_1}{(h_1-h)(\varepsilon+h_1-h)}\\ =&\int\limits_{|x_1|<R}\frac{1}{\varepsilon+\lambda|x_1|^{1+\alpha}} + \int\limits_{|x_1|<R}\left(\frac{1}{\varepsilon+h_1-h} - \frac{1}{\varepsilon+\lambda|x_1|^{1+\alpha}}\right) + O(1)\varepsilon^{\frac{2+3\alpha}{(2+\alpha)(1+\alpha)}}\\ =&2\int_0^R\frac{1}{\varepsilon+\lambda s^{1+\alpha}} + O(1)\int_0^R\frac{s^{\beta}}{\varepsilon+\lambda s^{1+\alpha}}\end{aligned}$$

$$
=\frac{2\Gamma_\alpha}{(1+\alpha)\lambda^{\frac{1}{1+\alpha}}}\varepsilon^{-\frac{\alpha}{1+\alpha}}\begin{cases}1+O(1)\varepsilon^{\frac{\beta}{1+\alpha}}, & \alpha>\beta,\\ 1+O(1)\varepsilon^{\frac{\alpha}{1+\alpha}}|\ln\varepsilon|, & \alpha=\beta,\\ 1+O(1)\varepsilon^{\frac{\alpha}{1+\alpha}}, & 0<\alpha<\beta;\end{cases}
$$

for $n\geq 3$,

$$
\int\limits_{|x'|<R}\frac{dx'}{h_1-h}-\int\limits_{|x'|<\varepsilon^{\bar{\gamma}}}\frac{\varepsilon\,dx'}{(h_1-h)(\varepsilon+h_1-h)}
$$
$$
=\int\limits_{\Omega_R^*}|\partial_n\bar{u}^*|^2+O(1)\varepsilon^{(n-2-\alpha)\bar{\gamma}}.
$$

Consequently, Theorem 2.1 is proved by combining **Step 1** and **Step 2**.

5.2. Proof of Theorem 2.2

Recalling decomposition (2.14), it follows from Theorem 2.1, Theorem 3.3, Lemma 3.5 and Lemma 4.1 that Theorem 2.2 holds.

6. The interior asymptotics for the perfect conductivity problem

In the following, we mainly deal with the asymptotic expansions for the electric field in the thin gap between two inclusions. The corresponding assumptions for the case of two adjacent inclusions can be made similarly. We would like to emphasize the following differences in comparison with previous assumptions. Let D_1^* and D_2^* be two $C^{1,\alpha}$-subdomains of a bounded open set $D\subset\mathbb{R}^n$ $(n\geq 2)$ with $C^{1,\alpha}$ boundary, where $0<\alpha<1$. Assume that D_1^* and D_2^* touch only at one point and they are far away from ∂D. By a translation and rotation of the coordinates, if necessary, we let their intersection point be located at the origin, and

$$
D_1^*\subset\{(x',x_n)\in\mathbb{R}^n|\,x_n>0\},\quad D_2^*\subset\{(x',x_n)\in\mathbb{R}^n|\,x_n<0\}.
$$

Translating D_i^*, $i=1,2$, by $\pm\frac{\varepsilon}{2}$ along x_n-axis, respectively, we denote

$$
D_1:=D_1^*+(0',\frac{\varepsilon}{2}),\quad D_2:=D_2^*+(0',\frac{\varepsilon}{2}),
$$

where $\varepsilon>0$ is a sufficiently small constant.

The perfect conductivity problem with two adjacent inclusions is modeled as follows:

$$
\begin{cases}\Delta u=0, & \text{in } \Omega,\\ u=C_i, & \text{on } \partial D_i,\ i=1,2,\\ \int_{\partial D_i}\frac{\partial u}{\partial\nu}\big|_+=0, & i=1,2,\\ u=\varphi, & \text{on } \partial D,\end{cases}\tag{6.1}
$$

where the free constants C_1 and C_2 are determined later by the third line of (6.1). Similarly as above, we assume that there exists a small constant $R>0$ independent of ε, such that the portions of ∂D_1 and ∂D_2 near the origin can be written as

$$
x_n=\frac{\varepsilon}{2}+h_1(x')\quad\text{and}\quad x_n=-\frac{\varepsilon}{2}+h_2(x'),\qquad x'\in B'_{2R},
$$

where h_i, $i = 1, 2$ satisfy

(*H1*) $h_1(x') - h_2(x') = \lambda|x'|^{1+\alpha}$, if $x' \in B'_{2R}$,
(*H2*) $|\nabla_{x'} h_i(x')| \leq \kappa_1 |x'|^{\alpha}$, if $x' \in B'_{2R}$, $i = 1, 2$,
(*H3*) $\|h_1\|_{C^{1,\alpha}(B'_{2R})} + \|h_2\|_{C^{1,\alpha}(B'_{2R})} \leq \kappa_2$,

where λ and $\kappa_i, i = 1, 2$, are three positive constants independent of ε. Note that we get rid of the remainder $O(|x'|^{1+\alpha+\beta})$ in assumption (***H1***) compared with the preceding condition (***A1***).

With slight abuse of notations, we also use the preceding notations to define

$$\Omega := D \setminus \overline{(D_1 \cup D_2)}, \quad \Omega^* := D \setminus \overline{(D_1^* \cup D_2^*)},$$

and

$$\Omega_t := \left\{ x \in \mathbb{R}^n \;\Big|\; -\frac{\varepsilon}{2} + h_2(x') < x_n < \frac{\varepsilon}{2} + h_1(x'),\; |x'| < t \right\}, \quad 0 < t \leq 2R,$$

and

$$\delta(x') := \varepsilon + h_1(x') - h_2(x'), \quad (x', x_n) \in \Omega_{2R}. \tag{6.2}$$

Similarly as before, we introduce a scalar auxiliary functions $\bar{u} \in C^{1,\alpha}(\mathbb{R}^n)$ satisfying that $\bar{u} = 1$ on ∂D_1, $\bar{u} = 0$ on $\partial D \cup \partial D_2$ and

$$\bar{u}(x) = \frac{x_n - h_2(x') + \frac{\varepsilon}{2}}{\varepsilon + h_1(x') - h_2(x')}, \text{ in } \Omega_{2R}, \quad \|\bar{u}\|_{C^{1,\alpha}(\Omega \setminus \Omega_R)} \leq C. \tag{6.3}$$

Before stating our main result, we first define

$$\mathcal{Q}^*[\varphi] = \int\limits_{\partial D_1^*} \frac{\partial v_0^*}{\partial \nu} \int\limits_{\partial D} \frac{\partial v_2^*}{\partial \nu} - \int\limits_{\partial D_2^*} \frac{\partial v_0^*}{\partial \nu} \int\limits_{\partial D} \frac{\partial v_1^*}{\partial \nu}, \tag{6.4}$$

$$\Theta^* = -\mathcal{K} \int\limits_{\partial D} \frac{\partial (v_1^* + v_2^*)}{\partial \nu}, \tag{6.5}$$

$$\mathfrak{S}^* = -\int\limits_{\partial D_1^*} \frac{\partial v_1^*}{\partial \nu} \int\limits_{\partial D} \frac{\partial v_2^*}{\partial \nu} + \int\limits_{\partial D_1^*} \frac{\partial v_2^*}{\partial \nu} \int\limits_{\partial D} \frac{\partial v_1^*}{\partial \nu}, \tag{6.6}$$

where $\mathcal{K} = \frac{2\Gamma_\alpha}{(1+\alpha)\lambda^{\frac{1}{1+\alpha}}}$ and v_i^*, $i = 0, 1, 2$, verify

$$\begin{cases} \Delta v_0^* = 0, & \text{in } \Omega^*, \\ v_0^* = 0, & \text{on } \partial D_1^* \cup \partial D_2^*, \\ v_0^* = \varphi(x), & \text{on } \partial D, \end{cases} \qquad \begin{cases} \Delta v_i^* = 0, & \text{in } \Omega^*, \\ v_i^* = \delta_{ij}, & \text{on } \partial D_j^* \setminus \{0\}, \\ v_i^* = 0, & \text{on } \partial D, \end{cases} \quad i = 1, 2, \tag{6.7}$$

respectively. We would like to point out that the definition of $\mathfrak{S}^*$ is only valid under the case of $n \geq 3$. Denote

$$\tilde{r}_\varepsilon = \begin{cases} \varepsilon^{\frac{\alpha}{2+\alpha}}, & n=2, \\ \varepsilon^{\frac{\alpha^2(1-\alpha)}{2(2+\alpha)(1+\alpha)^2}}, & n=3, \\ \varepsilon^{\frac{\alpha^2}{2(2+\alpha)(1+\alpha)^2}\min\{1+\alpha,2-\alpha\}}, & n=4, \\ \varepsilon^{\frac{\alpha^2}{2(2+\alpha)(1+\alpha)}}, & n\geq 5. \end{cases} \tag{6.8}$$

Our main result in this section is presented as follows:

Theorem 6.1. *Assume as above, conditions (**H1**)–(**H3**) hold. For $\varphi \in C^{1,\alpha}(\partial D)$, let $u \in H^1(D) \cap C^1(\overline{\Omega})$ be the solution of (6.1). Then for a sufficiently small $\varepsilon > 0$ and $x \in \Omega_R$, if $\mathcal{Q}^*[\varphi] \neq 0$,*

(*i*) *for $n = 2$,*

$$\nabla u = \frac{\mathcal{Q}^*[\varphi]}{\Theta^*}\left(\frac{1}{1-\widetilde{\mathcal{M}^*}\varepsilon^{\frac{\alpha}{1+\alpha}}} + O(\tilde{r}_\varepsilon)\right)\varepsilon^{\frac{\alpha}{1+\alpha}}\nabla\bar{u} + O(1)\delta^{-\frac{1-\alpha}{1+\alpha}}; \tag{6.9}$$

(*ii*) *for $n \geq 3$,*

$$\nabla u = \frac{\mathcal{Q}^*[\varphi]}{\mathfrak{G}^*}(1 + O(\tilde{r}_\varepsilon))\nabla\bar{u} + O(1)\delta^{-\frac{1}{1+\alpha}}, \tag{6.10}$$

where δ is defined by (6.2), $\bar{u}$ is defined by (6.3), $\mathcal{Q}^*[\varphi]$, Θ^* and $\mathfrak{G}^*$ are defined by (6.4)–(6.6), respectively, and $\widetilde{\mathcal{M}^*}$ is defined by (7.15), and $\tilde{r}_\varepsilon$ is defined by (6.8).

Remark 6.2. We here remark that for $n \geq 3$, the lower bound obtained by Chen, Li and Xu in Section 3.3 of [15] is not rigorous since they didn't capture the explicit blow-up factor $\mathcal{Q}^*[\varphi]$ as in the asymptotic (6.10). So our asymptotic result (6.10) gives a perfect answer to the optimality of the blow-up rate in dimensions greater than or equal to three. For the purpose of demonstrating the validity of assumption condition $\mathcal{Q}^*[\varphi] \neq 0$, we provide some special examples in terms of the domain and the boundary data, which were previously given in [8]. Specifically, let $\Omega^* := D \setminus \overline{(D_1^* \cup D_2^*)}$ be a bounded open set in $\mathbb{R}^n$ with $C^{1,\alpha}$ boundary, which is symmetric with respect to x_n-variable, that is, $(x', x_n) \in \Omega^*$ if and only if $(x', -x_n) \in \Omega^*$. For $(x', x_n) \in (\partial D)^+ := \{(x', x_n)|\, x_n > 0\}$, let $\varphi \in C^{1,\alpha}(\partial D)$ satisfy the following condition:

$$\varphi_{odd}(x', x_n) := \frac{1}{2}(\varphi(x', x_n) - \varphi(x', -x_n)) \leq 0 \text{ (or } \geq 0).$$

For example, take $\alpha = \frac{2}{3}$, $\varphi = x_n^{\frac{5}{3}}$ on ∂D and then $\varphi_{odd} = x_n^{\frac{5}{3}} \geq 0$ on $(\partial D)^+$. Denote

$$(v_0^*)_{odd}(x', x_n) := \frac{1}{2}(v_0^*(x', x_n) - v_0^*(x', -x_n)).$$

In light of symmetry, the strong maximum principle and the Hopf Lemma, we have

$$\int_{\partial D} \frac{\partial v_1^*}{\partial \nu} = \int_{\partial D} \frac{\partial v_2^*}{\partial \nu}.$$

Thus,

$$\mathcal{Q}^*[\varphi] = \int_{\partial D} \frac{\partial v_1^*}{\partial \nu}\left(\int_{\partial D_1^*} \frac{\partial v_0^*}{\partial \nu} - \int_{\partial D_2^*} \frac{\partial v_0^*}{\partial \nu}\right)$$

$$=\int\limits_{\partial D}\frac{\partial v_1^*}{\partial\nu}\left(\int\limits_{\partial D_1^*}\frac{\partial(v_0^*)_{odd}}{\partial\nu}-\int\limits_{\partial D_2^*}\frac{\partial(v_0^*)_{odd}}{\partial\nu}\right)$$
$$=2\int\limits_{\partial D}\frac{\partial v_1^*}{\partial\nu}\int\limits_{\partial D_1^*}\frac{\partial(v_0^*)_{odd}}{\partial\nu}.$$

Observe that $(v_0^*)_{odd}$ is harmonic with $(v_0^*)_{odd}(x',0)=0$ and $(v_0^*)_{odd}=\varphi_{odd}\leq 0$ (or ≥ 0) but not identically zero on $(\partial D)^+$. Then it follows from the strong maximum principle and the Hopf Lemma that $\int_{\partial D_1^*}\frac{\partial(v_0^*)_{odd}}{\partial\nu}\neq 0$. Consequently, $\mathcal{Q}^*[\varphi]\neq 0$.

Remark 6.3. In comparison with the asymptotic results of $C^{2,\alpha}$-inclusions in [28,31,39], the results of $C^{1,\alpha}$-inclusions in this paper together with the blow-up analysis for perfect conductors of a bow-tie structure in [24] show that the singularities of the field will enhance with the deterioration of smoothness of inclusions.

7. The proof of Theorem 6.1

Similarly as in (2.11), we split the solution u of (6.1) as follows:

$$u=\sum_{i=1}^{2}C_iv_i+v_0,\quad \text{in }\Omega, \tag{7.1}$$

where v_i, $i=0,1,2$, solve

$$\begin{cases}\Delta v_0=0, & \text{in }\Omega,\\ v_0=0, & \text{on }\partial D_1\cup\partial D_2,\\ v_0=\varphi(x), & \text{on }\partial D,\end{cases}\qquad \begin{cases}\Delta v_i=0, & \text{in }\Omega,\\ v_i=\delta_{ij}, & \text{on }\partial D_j,\ i,j=1,2,\\ v_i=0, & \text{on }\partial D,\end{cases} \tag{7.2}$$

respectively. Similarly as in [8], we denote

$$a_{ij}:=\int\limits_{\partial D_i}\frac{\partial v_j}{\partial\nu},\quad b_i:=-\int\limits_{\partial D_i}\frac{\partial v_0}{\partial\nu},\quad i,j=1,2.$$

Then from the third line of (6.1) and (7.1), we know

$$\begin{cases}a_{11}C_1+a_{12}C_2=b_1,\\ a_{21}C_1+a_{22}C_2=b_2,\end{cases}$$

which in combination with Cramer's rule yields that

$$C_1-C_2=\frac{b_1(a_{21}+a_{22})-b_2(a_{11}+a_{12})}{a_{11}a_{22}-a_{12}a_{21}}. \tag{7.3}$$

Making use of the Green's formula, we obtain that $a_{12}=a_{21}$,

$$a_{11}+a_{12}=a_{11}+a_{21}=-\int\limits_{\partial D}\frac{\partial v_1}{\partial\nu},\quad a_{21}+a_{22}=a_{12}+a_{22}=-\int\limits_{\partial D}\frac{\partial v_2}{\partial\nu}. \tag{7.4}$$

Since

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{11}+a_{12} \\ a_{21} & a_{21}+a_{22} \end{vmatrix},$$

it follows from (7.3)–(7.4) that

$$C_1 - C_2 = \rho_n(\varepsilon)\frac{\mathcal{Q}[\varphi]}{\Theta},$$

where

$$\rho_n(\varepsilon) = \begin{cases} \varepsilon^{\frac{\alpha}{1+\alpha}}, & n = 2, \\ 1, & n \geq 3, \end{cases}$$

and

$$\mathcal{Q}[\varphi] = \int\limits_{\partial D_1} \frac{\partial v_0}{\partial \nu} \int\limits_{\partial D} \frac{\partial v_2}{\partial \nu} - \int\limits_{\partial D_2} \frac{\partial v_0}{\partial \nu} \int\limits_{\partial D} \frac{\partial v_1}{\partial \nu}, \tag{7.5}$$

$$\Theta = -\left(\rho_n(\varepsilon) \int\limits_{\partial D_1} \frac{\partial v_1}{\partial \nu}\right) \int\limits_{\partial D} \frac{\partial v_2}{\partial \nu} + \left(\rho_n(\varepsilon) \int\limits_{\partial D_1} \frac{\partial v_2}{\partial \nu}\right) \int\limits_{\partial D} \frac{\partial v_1}{\partial \nu}. \tag{7.6}$$

Thus, in view of the decomposition (7.1), we have

$$\nabla u = \frac{\mathcal{Q}[\varphi]}{\Theta}\rho_n(\varepsilon)\nabla v_1 + C_2\nabla(v_1 + v_2) + \nabla v_0. \tag{7.7}$$

Due to the fact that $u = C_i$ on ∂D_i and $\|u\|_{H^1(\Omega)} \leq C$ (independent of ε), from the trace embedding theorem we see

$$|C_1| + |C_2| \leq C.$$

Since $\Delta v_0 = 0$ in Ω with $v_0 = 0$ on $\partial D_1 \cup \partial D_2$, and $\Delta(v_1 + v_2 - 1) = 0$ in Ω with $v_1 + v_2 - 1 = 0$ on $\partial D_1 \cup \partial D_2$, it follows from Theorem 1.1 in [27] and the standard elliptic theory that

$$\|\nabla v_0\|_{L^\infty(\Omega)} \leq C, \quad \|\nabla(v_1 + v_2)\|_{L^\infty(\Omega)} \leq C.$$

On the other hand, applying Theorem 3.3 again, we obtain that Lemma 3.5 also holds for the solution v_1 of (7.2), that is, $\nabla v_1 = \nabla \bar{u} + O(1)\delta^{-\frac{1}{1+\alpha}}$ in Ω_R. Then combining these facts, (7.7) becomes

$$\nabla u = \frac{\mathcal{Q}[\varphi]}{\Theta}\rho_n(\varepsilon)\nabla \bar{u} + O(1)\rho_n(\varepsilon)\delta^{-\frac{1}{1+\alpha}}, \tag{7.8}$$

where $\bar{u}$ is defined by (6.3).

A direct application of Lemma 4.1 yields that

Corollary 7.1. *Assume as in Theorem 2.2. Then, for a sufficiently small $\varepsilon > 0$, $i = 1, 2$,*

$$\int\limits_{\partial D_i} \frac{\partial v_0}{\partial \nu} = \int\limits_{\partial D_i^*} \frac{\partial v_0^*}{\partial \nu} + O(1)\varepsilon^{\frac{\alpha}{2+\alpha}}, \quad \int\limits_{\partial D} \frac{\partial v_i}{\partial \nu} = \int\limits_{\partial D} \frac{\partial v_i^*}{\partial \nu} + O(1)\varepsilon^{\frac{\alpha}{2+\alpha}}. \tag{7.9}$$

Consequently,

$$\mathcal{Q}[\varphi] = \mathcal{Q}^*[\varphi] + O(1)\varepsilon^{\frac{\alpha}{2+\alpha}}, \tag{7.10}$$

where v_i^* and v_i, $i = 0, 1, 2$, are defined by (6.7) and (7.2), respectively, $\mathcal{Q}^*[\varphi]$ and $\mathcal{Q}[\varphi]$ are defined by (6.4) and (7.5), respectively.

The proof of Corollary 7.1 is almost the same to **Step 2.1** in the proof of Lemma 4.1 with a slight modification and thus omitted here.

Recalling the decomposition (7.8) and in light of the asymptotic results (7.9)–(7.10), we need to establish the asymptotic expansion of the blow-up factor Θ defined by (7.6) for the purpose of proving Theorem 6.1. In order to calculate Θ, it suffices to compute the energy $\int_\Omega |\nabla v_1|^2 = \int_{\partial D_1} \frac{\partial v_1}{\partial \nu}$ by using integration by parts.

Analogously as in Theorem 2.1, we obtain

Lemma 7.2. *Assume as above, conditions (**H1**)–(**H3**) hold. Let v_1 be the solution of (7.2). Then, for a sufficiently small $\varepsilon > 0$,*

(*i*) *for $n = 2$,*

$$\int_\Omega |\nabla v_1|^2 = \mathcal{K}\varepsilon^{-\frac{\alpha}{1+\alpha}} + \mathcal{A}_R + O(1)|\ln\varepsilon|, \tag{7.11}$$

(*ii*) *for $n \geq 3$,*

$$\int_\Omega |\nabla v_1|^2 = \int_{\Omega^*} |\nabla v_1^*|^2 + O(1)\begin{cases} \varepsilon^{\frac{\alpha^2(1-\alpha)}{2(2+\alpha)(1+\alpha)^2}}, & n = 3, \\ \varepsilon^{\frac{\alpha^2}{2(2+\alpha)(1+\alpha)^2}\min\{1+\alpha, 2-\alpha\}}, & n = 4, \\ \varepsilon^{\frac{\alpha^2}{2(2+\alpha)(1+\alpha)}}, & n \geq 5, \end{cases} \tag{7.12}$$

where $\mathcal{K} = \frac{2\Gamma_\alpha}{(1+\alpha)\lambda^{\frac{1}{1+\alpha}}}$ and $\mathcal{A}_R$ is defined by (7.14).

Proof. Denote $\Omega_r^* := \Omega^* \cap \{|x'| < r\}$, $0 < r \leq 2R$. Similar to **Step 1** in the proof of Theorem 2.1, we have

$$\begin{aligned} \int_\Omega |\nabla v_1|^2 = & \int_{\varepsilon^{\bar{\gamma}} < |x'| < R} \frac{dx'}{h_1(x') - h_2(x')} + \int_{|x'| < \varepsilon^{\bar{\gamma}}} \frac{dx'}{\varepsilon + h_1(x') - h_2(x')} \\ & + \begin{cases} O(1)|\ln\varepsilon|, & n = 2, \\ M_R^* + O(1)\begin{cases} \varepsilon^{\bar{\gamma}}, & n = 3, \\ \varepsilon^{(1+\alpha)\bar{\gamma}}, & n \geq 4, \end{cases} \end{cases} \end{aligned} \tag{7.13}$$

where

$$\begin{aligned} M_R^* = & \int_{\Omega^* \setminus \Omega_R^*} |\nabla v_1^*|^2 + 2\int_{\Omega_R^*} \nabla \bar{u}^* \cdot \nabla(v_1^* - \bar{u}^*) \\ & + \int_{\Omega_R^*} \left(|\nabla(v_1^* - \bar{u}^*)|^2 + |\partial_{x'}\bar{u}^*|^2\right). \end{aligned}$$

The case of $n \geq 3$ is the same to **Step 2** in the proof of Theorem 2.1 and thus omitted here. We now consider the case of $n = 2$. Since

$$
\begin{aligned}
&\int\limits_{\varepsilon^{\tilde{\gamma}}<|x_1|<R} \frac{dx'}{h_1(x_1)-h_2(x_1)} + \int\limits_{|x_1|<\varepsilon^{\tilde{\gamma}}} \frac{dx_1}{\varepsilon + h_1(x_1)-h_2(x_1)} \\
=&\int\limits_{\varepsilon^{\tilde{\gamma}}<|x_1|<R} \frac{dx_1}{\lambda|x_1|^{1+\alpha}} + \int\limits_{|x_1|<\varepsilon^{\tilde{\gamma}}} \frac{dx_1}{\varepsilon+\lambda|x_1|^{1+\alpha}} \\
=&2\left(\int\limits_0^{+\infty} \frac{1}{\varepsilon+\lambda x_1^{1+\alpha}} - \int\limits_R^{+\infty} \frac{1}{\lambda x_1^{1+\alpha}} + \int\limits_{\varepsilon^{\tilde{\gamma}}}^{+\infty} \frac{\varepsilon}{\lambda x_1^{1+\alpha}(\varepsilon+\lambda x_1^{1+\alpha})}\right) \\
=&\frac{2\Gamma_\alpha}{(1+\alpha)\lambda^{\frac{1}{1+\alpha}}}\varepsilon^{-\frac{\alpha}{1+\alpha}} - \frac{2}{\alpha\lambda R^\alpha} + O(1)\varepsilon^{\frac{5\alpha+4}{2(1+\alpha)(2+\alpha)}},
\end{aligned}
$$

then from (7.13), we have

$$
\int\limits_\Omega |\nabla v_1|^2 = \mathcal{K}\varepsilon^{-\frac{\alpha}{1+\alpha}} + \mathcal{A}_R + O(1)|\ln\varepsilon|,
$$

where

$$
\mathcal{A}_R = -\frac{2}{\alpha\lambda R^\alpha}. \tag{7.14}
$$

That is, (7.11) holds. □

7.1. The proof of Theorem 6.1

Proof of (6.9). For $n = 2$, recalling the definitions of v_1 and v_2 and making use of the Green's formulas, we have

$$
\int\limits_{\partial D_1} \frac{\partial v_2}{\partial \nu} = \int\limits_{\partial D_2} \frac{\partial v_1}{\partial \nu} = -\int\limits_{\partial D_1} \frac{\partial v_1}{\partial \nu} - \int\limits_{\partial D} \frac{\partial v_1}{\partial \nu},
$$

which implies that

$$
\begin{aligned}
\Theta =& -\left(\rho_2(\varepsilon)\int\limits_{\partial D_1} \frac{\partial v_1}{\partial \nu}\right)\int\limits_{\partial D} \frac{\partial v_2}{\partial \nu} + \left(\rho_2(\varepsilon)\int\limits_{\partial D_1} \frac{\partial v_2}{\partial \nu}\right)\int\limits_{\partial D} \frac{\partial v_1}{\partial \nu} \\
=& -\left(\rho_2(\varepsilon)\int\limits_{\partial D_1} \frac{\partial v_1}{\partial \nu}\right)\int\limits_{\partial D} \frac{\partial (v_1+v_2)}{\partial \nu} - \rho_2(\varepsilon)\left(\int\limits_{\partial D} \frac{\partial v_1}{\partial \nu}\right)^2.
\end{aligned}
$$

Then in view of (7.9), we have

$$
\begin{aligned}
\Theta - \Theta^* =& -\left(\rho_2(\varepsilon)\int\limits_{\partial D_1} \frac{\partial v_1}{\partial \nu} - \mathcal{K}\right)\int\limits_{\partial D} \frac{\partial (v_1^*+v_2^*)}{\partial \nu} \\
&- \rho_2(\varepsilon)\left(\int\limits_{\partial D} \frac{\partial v_1^*}{\partial \nu}\right)^2 + O(\varepsilon^{\frac{\alpha}{2+\alpha}})
\end{aligned}
$$

$$
\begin{aligned}
&= -\rho_2(\varepsilon)(\mathcal{A}_R + O(|\ln\varepsilon|)) \int\limits_{\partial D} \frac{\partial(v_1^* + v_2^*)}{\partial\nu} \\
&\quad - \rho_2(\varepsilon)\left(\int\limits_{\partial D} \frac{\partial v_1^*}{\partial\nu}\right)^2 + O(\varepsilon^{\frac{\alpha}{2+\alpha}}) \\
&= -\rho_2(\varepsilon)\left(\mathcal{A}_R \int\limits_{\partial D} \frac{\partial(v_1^* + v_2^*)}{\partial\nu} + \left(\int\limits_{\partial D} \frac{\partial v_1^*}{\partial\nu}\right)^2\right) + O(\varepsilon^{\frac{\alpha}{2+\alpha}}).
\end{aligned}
$$

Denote

$$
\widetilde{\mathcal{M}^*} = -\frac{\mathcal{A}_R}{\mathcal{K}} + \frac{(\alpha_1^*)^2}{\Theta^*}, \quad \alpha_1^* = \int\limits_{\partial D} \frac{\partial v_1^*}{\partial\nu}. \tag{7.15}
$$

Thus,

$$
\begin{aligned}
\frac{\mathcal{Q}[\varphi]}{\Theta} - \frac{\mathcal{Q}^*[\varphi]}{\Theta^*} &= \frac{\mathcal{Q}^*[\varphi]}{\Theta^*} \frac{\frac{\Theta^* - \Theta}{\Theta^*}}{1 - \frac{\Theta^* - \Theta}{\Theta^*}} + \frac{\mathcal{Q}[\varphi] - \mathcal{Q}^*[\varphi]}{\Theta} \\
&= \frac{\mathcal{Q}^*[\varphi]}{\Theta^*} \frac{\widetilde{\mathcal{M}^*}\rho_2(\varepsilon) + O(\varepsilon^{\frac{\alpha}{2+\alpha}})}{1 - \widetilde{\mathcal{M}^*}\rho_2(\varepsilon) + O(\varepsilon^{\frac{\alpha}{2+\alpha}})} + O(\varepsilon^{\frac{\alpha}{2+\alpha}}) \\
&= \frac{\mathcal{Q}^*[\varphi]}{\Theta^*} \frac{\widetilde{\mathcal{M}^*}\rho_2(\varepsilon)}{1 - \widetilde{\mathcal{M}^*}\rho_2(\varepsilon)} + O(\varepsilon^{\frac{\alpha}{2+\alpha}}),
\end{aligned}
$$

which indicates that

$$
\frac{\mathcal{Q}[\varphi]}{\Theta} = \frac{\mathcal{Q}^*[\varphi]}{\Theta^*} \frac{1}{1 - \widetilde{\mathcal{M}^*}\rho_2(\varepsilon)} + O(\varepsilon^{\frac{\alpha}{2+\alpha}}). \tag{7.16}
$$

Then combining with (7.8) and (7.16), we obtain that (6.9) holds.

Proof of (6.10). For $n \geq 3$, we know that $\rho_n(\varepsilon) = 1$. Then recalling the definitions of Θ and $\mathfrak{S}^*$ above, it follows from (7.9) that

$$
\Theta = \mathfrak{S}^* + O(1)\varepsilon^{\frac{\alpha}{2+\alpha}}. \tag{7.17}
$$

We now claim that $\mathfrak{S}^* \neq 0$. In fact, we see from the Hopf Lemma that

$$
\frac{\partial v_1^*}{\partial\nu}\Big|_{\partial D_1^* \setminus\{0\}} > 0, \quad \frac{\partial v_1^*}{\partial\nu}\Big|_{\partial D} < 0, \quad \frac{\partial v_2^*}{\partial\nu}\Big|_{\partial D_1^* \setminus\{0\}} < 0, \quad \frac{\partial v_2^*}{\partial\nu}\Big|_{\partial D} < 0.
$$

Then, we have

$$
\mathfrak{S}^* = -\int\limits_{\partial D_1^*} \frac{\partial v_1^*}{\partial\nu} \int\limits_{\partial D} \frac{\partial v_2^*}{\partial\nu} + \int\limits_{\partial D_1^*} \frac{\partial v_2^*}{\partial\nu} \int\limits_{\partial D} \frac{\partial v_1^*}{\partial\nu} > 0.
$$

Therefore, it follows from (7.8), (7.10), (7.12) and (7.17) that (6.10) holds.

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