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Optimal temperature distribution for a nonisothermal Cahn–Hilliard system with source term

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Abstract

In this note, we study the optimal control of a nonisothermal phase field system of Cahn–Hilliard type that constitutes an extension of the classical Caginalp model for nonisothermal phase transitions with a conserved order parameter. The system couples a Cahn–Hilliard type equation with source term for the order parameter with the universal balance law of internal energy. In place of the standard Fourier form, the constitutive law of the heat flux is assumed in the form given by the theory developed by Green and Naghdi, which accounts for a possible thermal memory of the evolution. This has the consequence that the balance law of internal energy becomes a second-order in time equation for the *thermal displacement* or *freezing index*, that is, a primitive with respect to time of the temperature. Another particular feature of our system is the presence of the source term in the equation for the order parameter, which entails additional mathematical difficulties because the mass conservation of the order parameter, typical of the classic Cahn–Hilliard equation, is no longer satisfied. In this paper, we analyze the case that the double-well potential driving the evolution of the phase transition is differentiable, either (in the regular case) on the whole set of reals or (in the singular logarithmic case) on a finite open interval; nondifferentiable cases like the double obstacle potential are excluded from the analysis. We prove the Fréchet differentiability of the control-to-state operator between suitable Banach spaces for both the regular and the logarithmic cases and establish the solvability of the corresponding adjoint systems in order to derive the associated first-order necessary optimality conditions for the optimal control problem. Crucial for the whole analysis to work is the so-called “strict separation property”, which states that the order parameter attains its values in a compact subset of the interior of the effective domain of the nonlinearity. While this separation property turns out to be generally valid for regular potentials in three dimensions of space, it can be shown for the logarithmic case only in two dimensions.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be some open, bounded, and connected set having a smooth boundary $\Gamma := \partial\Omega$ and the outward unit normal field $\boldsymbol{n}$. Denoting by $\partial_{\boldsymbol{n}}$ the directional derivative in the direction of $\boldsymbol{n}$, and putting, with a fixed final time $T > 0$,

$$Q_t := \Omega \times (0,t) \text{ and } \Sigma_t := \Gamma \times (0,t) \text{ for } t \in (0,T], \text{ as well as } Q := Q_T \text{ and } \Sigma := \Sigma_T,$$

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we study in this paper as *state system* the following initial-boundary value problem:

$$\partial_t\varphi - \Delta\mu + \gamma\varphi = f \qquad \text{in } Q, \tag{1.1}$$
$$\mu = -\Delta\varphi + F'(\varphi) + a - b\partial_t w \qquad \text{in } Q, \tag{1.2}$$
$$\partial_t^2 w - \Delta(\kappa_1\partial_t w + \kappa_2 w) + \lambda\partial_t\varphi = u \qquad \text{in } Q, \tag{1.3}$$
$$\partial_{\boldsymbol{n}}\varphi = \partial_{\boldsymbol{n}}\mu = \partial_{\boldsymbol{n}}(\kappa_1\partial_t w + \kappa_2 w) = 0 \qquad \text{on } \Sigma, \tag{1.4}$$
$$\varphi(0) = \varphi_0, \quad w(0) = w_0, \quad \partial_t w(0) = w_1 \quad \text{in } \Omega. \tag{1.5}$$

The *cost functional* under consideration is given by

$$\begin{aligned}\mathcal{J}((\varphi,w),u) := {}& \frac{\alpha_1}{2}\int_0^T\!\!\int_\Omega |\varphi-\varphi_Q|^2 + \frac{\alpha_2}{2}\int_\Omega |\varphi(T)-\varphi_\Omega|^2 \\ & + \frac{\alpha_3}{2}\int_0^T\!\!\int_\Omega |w-w_Q|^2 + \frac{\alpha_4}{2}\int_\Omega |w(T)-w_\Omega|^2 \\ & + \frac{\alpha_5}{2}\int_0^T\!\!\int_\Omega |\partial_t w-w'_Q|^2 + \frac{\alpha_6}{2}\int_\Omega |\partial_t w(T)-w'_\Omega|^2 + \frac{\nu}{2}\int_0^T\!\!\int_\Omega |u|^2, \end{aligned} \tag{1.6}$$

with nonnegative constants α_i, $1 \le i \le 6$, which are not all zero, and where $\varphi_\Omega, w_\Omega, w'_\Omega \in L^2(\Omega)$ and $\varphi_Q, w_Q, w'_Q \in L^2(Q)$ denote given target functions.

For the control variable u, we choose as control space

$$\mathcal{U} := L^\infty(Q), \tag{1.7}$$

and the related set of admissible controls is given by

$$\mathcal{U}_{\rm ad} := \bigl\{u \in \mathcal{U} : u_{\min} \le u \le u_{\max} \quad \text{a.e. in } Q\bigr\}, \tag{1.8}$$

where $u_{\min}, u_{\max} \in L^\infty(Q)$ satisfy $u_{\min} \le u_{\max}$ almost everywhere in Q.

In summary, the control problem under investigation can be reformulated as follows:

$$(\mathbf{P}) \quad \min_{u\in\mathcal{U}_{\rm ad}} \mathcal{J}((\varphi,w),u) \quad \text{subject to the constraint that } (\varphi,\mu,w) \text{ solves (1.1)–(1.5).}$$

The state system (1.1)–(1.5) is a formal extension of the nonisothermal Cahn–Hilliard system introduced by Caginalp in [4] to model the phenomenon of nonisothermal phase segregation in binary mixtures (see also [3,5] and the derivation in [2, Ex. 4.4.2, (4.44), (4.46)]); it corresponds to the Allen–Cahn counterpart analyzed recently in [13]. The unknowns in the state system have the following physical meaning: φ is a normalized difference between the volume fractions of pure phases in the binary mixture (the dimensionless *order parameter* of the phase transformation, which should attain its values in the interval $[-1,1]$), μ is the associated *chemical potential*, and w is the so-called *thermal displacement* (or *freezing index*), which is directly connected to the temperature ϑ (which in the case of the Caginalp model is actually a temperature difference) through the relation

$$w(\cdot,t) = w_0 + \int_0^t \vartheta(\cdot,s)\,ds, \quad t \in [0,T]. \tag{1.9}$$

Moreover, κ_1 and κ_2 in (1.3) stand for prescribed positive coefficients related to the heat flux, which is here assumed in the Green–Naghdi form (see [19–21,26])

$$\mathbf{q} = -\kappa_1\nabla(\partial_t w) - \kappa_2\nabla w \quad \text{where } \kappa_i > 0,\ i = 1,2, \tag{1.10}$$

which accounts for a possible previous thermal history of the phenomenon. Moreover, γ is a positive physical constant related to the intensity of the mass absorption/production of the source, where the source term in (1.1) is $S := f - \gamma\varphi$. This term reflects the fact that the system may not be isolated and the loss or production of mass is possible, which happens, e.g., in numerous liquid-liquid phase segregation problems that arise in cell biology [15] and in tumor growth models [17]. Notice that the presence of the source term entails that the property of mass conservation of the order parameter is no longer valid; in fact, from (1.1) it directly follows that the mass balance has the form

$$\frac{d}{dt}\Big(\frac{1}{|\Omega|}\int_\Omega \varphi(t)\Big) = \frac{1}{|\Omega|}\int_\Omega S(t), \quad \text{for a.e. } t \in (0,T), \tag{1.11}$$

where $|\Omega|$ denotes the volume of Ω. To this concern, we would like to quote the paper [8], where a comparable Cahn–Hilliard system without mass conservation was examined from the optimal control viewpoint. Moreover, we refer to [6, 7, 12, 23, 25, 27, 31], where similar systems have been analyzed. For optimal control problems involving sparsity effects, let us mention [14, 16, 28, 30]. Also, let us incidentally point out that the differential structure of equation (1.3), with respect to w, is sometimes also referred to as the *strongly damped wave equation*, see, e.g., [24] and the references therein.

In addition to the quantities already introduced, λ stands for the latent heat of the phase transformation, a, b are physical constants, and the control variable u is a distributed heat source/sink. Besides, φ_0, w_0, and w_1 indicate some given initial values. Finally, the function F is assumed to have a double-well shape. Prototypical choices for the double-well shaped nonlinearity F are the regular and singular *logarithmic potential* and its common (nonsingular) polynomial approximation, the *regular potential*. In the order, they are defined as

$$F_{log}(r) := \begin{cases} (1+r)\ln(1+r) + (1-r)\ln(1-r) - c_1 r^2 & \text{if } |r| \le 1, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.12}$$

$$F_{reg}(r) := \frac{1}{4}\left(r^2 - 1\right)^2, \quad r \in \mathbb{R}, \tag{1.13}$$

with the convention that $0\ln(0) := \lim_{r\searrow 0} r\ln(r) = 0$ and $c_1 > 1$ so that F_{log} is nonconvex. Another important example is the nonregular and singular *double obstacle potential*, given by

$$F_{2obs}(r) := -c_2 r^2 \quad \text{if } |r| \le 1 \quad \text{and} \quad F_{2obs}(r) := +\infty \quad \text{if } |r| > 1, \tag{1.14}$$

with $c_2 > 0$. However, the double obstacle case is not included in the subsequent analysis, although we expect that, with similar techniques as those employed in [10], it is possible to extend the analysis also to this kind of nonregular potentials.

The state system (1.1)–(1.5) was recently in [11] analyzed concerning well-posedness and regularity (see the results cited below in Section 2), where also the double obstacle case was included. Here, we concentrate on the optimal control problem. While the existence of optimal controls is not too difficult to show, the derivation of first-order necessary optimality conditions is a much more challenging task, since it makes the derivation of differentiability properties of the associated control-to-state operator necessary. This, however, requires that the order parameter φ satisfies the so-called *strict separation property*, which means that φ attains its values in a compact subset of the interior of the effective domain of the derivative F' of F. While for regular potentials this condition turns out to be generally satisfied, it cannot be guaranteed for singular potentials. In fact, following the ideas of the recent paper [9] on the isothermal case, one is just able to ensure the validity of the strict separation property for the logarithmic potential $F_{\log}$ in the two-dimensional case $d = 2$. Correspondingly, the analysis leading to first-order necessary optimality conditions will be restricted to either the regular case for $d \leq 3$ or the logarithmic case in two dimensions of space. In this sense, our results apply to similar cases as those studied in [9] in the isothermal situation. Observe, however, that the control problem considered in [9] differs considerably from that studied in this paper: indeed, in [9] the control u occurs in the order parameter equation resembling (1.1), while in our case it appears in the energy balance (1.3); for this reason, the set of admissible controls $\mathcal{U}_{\rm ad}$ had to be assumed in [9] as a subset of the space $H^1(0,T;L^2(\Omega)) \cap L^\infty(Q)$, which is cumbersome from the viewpoint of optimal control, instead of the much better space $L^\infty(Q)$ used here.

The plan of the paper is as follows. The next section is devoted to collect previous results concerning the well-posedness of the state system (1.1)–(1.5). Then, under suitable conditions, we provide some stronger analytic results in terms of regularity and stability properties of the state system with respect to the control variable u appearing in (1.3). The proof of these new results are addressed in Section 3. Then, using these results, we analyze in Section 4 the optimal control problem **(P)**.

2 Notation, assumptions and analytic results

First, let us set some notation and general assumptions. For any Banach space X, we employ the notation $\|\cdot\|_X$, X^*, and $\langle\cdot,\cdot\rangle_X$, to indicate the corresponding norm, its dual space, and the related duality pairing between X^* and X. For two Banach spaces X and Y continuously embedded in some topological vector space Z, we introduce the linear space $X \cap Y$, which becomes a Banach space when equipped with its natural norm $\|v\|_{X\cap Y} := \|v\|_X + \|v\|_Y$, for $v \in X \cap Y$.

A special notation is used for the standard Lebesgue and Sobolev spaces defined on Ω. For every $1 \leq p \leq \infty$ and $k \geq 0$, they are denoted by $L^p(\Omega)$ and $W^{k,p}(\Omega)$, with the associated norms $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p$ and $\|\cdot\|_{W^{k,p}(\Omega)}$, respectively. If $p = 2$, they become Hilbert spaces, and we employ the standard convention $H^k(\Omega) := W^{k,2}(\Omega)$. For convenience, we also set

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{v \in H^2(\Omega) : \ \partial_{\boldsymbol{n}} v = 0 \text{ on } \Gamma\}.$$

For simplicity, we use the symbol $\|\cdot\|$ for the norm in H and in any power of it. Observe that the embeddings $W \hookrightarrow V \hookrightarrow H \hookrightarrow V^* \hookrightarrow W^*$ are dense and compact. As usual, H is identified with a subspace of V^* to have the Hilbert triplet (V, H, V^*) along with identity

$$\langle u, v\rangle = (u, v) \quad \text{for every } u \in H \text{ and } v \in V,$$

where we employ the special notation $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_V$.

Next, for a generic element $v \in V^*$, we define its generalized mean value $\overline{v}$ by

$$\overline{v} := \frac{1}{|\Omega|} \langle v, \mathbf{1}\rangle, \tag{2.1}$$

where $\mathbf{1}$ stands for the constant function that takes the value 1 in Ω. It is clear that $\overline{v}$ reduces to the usual mean value if $v \in H$. The same notation $\overline{v}$ is employed also if v is a time-dependent function.

To conclude, for normed spaces X and $v \in L^1(0,T;X)$, we define the convolution products

$$(\mathbf{1} * v)(t) := \int_0^t v(s)\,{\rm d}s, \quad (\mathbf{1} \circledast v)(t) := \int_t^T v(s)\,{\rm d}s, \qquad t \in [0,T]. \tag{2.2}$$

For the remainder of this paper, we make the following general assumptions.

(A1) The structural constants γ, a, b, κ_1, κ_2, and λ are positive.

(A2) The double-well potential F can be written as $F = \widehat{\beta} + \widehat{\pi}$, where

$$\widehat{\beta} : \mathbb{R} \to [0, +\infty] \text{ is convex and lower semicontinuous with } \widehat{\beta}(0) = 0.$$

This entails that $\beta := \partial\widehat{\beta}$ is a maximal monotone graph with $\beta(0) \ni 0$. Moreover, we assume that

$$\widehat{\pi} \in C^3(\mathbb{R}), \text{ where } \pi := \widehat{\pi}' : \mathbb{R} \to \mathbb{R} \text{ is a Lipschitz continuous function.}$$

Besides, denoting the effective domain of β by $D(\beta)$, we assume that $D(\beta) = (r_-, r_+)$ with $-\infty \le r_- < 0 < r_+ \le +\infty$ and that the restriction of $\widehat{\beta}$ to (r_-, r_+) belongs to $C^3(r_-, r_+)$. There, β reduces to the derivative of $\widehat{\beta}$, and we require that

$$\lim_{r \searrow r_-} \beta(r) = -\infty \text{ and } \lim_{r \nearrow r_+} \beta(r) = +\infty.$$

Please note that F' in (1.2) has to be understood as $\beta + \pi$.

(A3) Let $f \in L^\infty(Q)$. We set $\rho := \frac{\|f\|_\infty}{\gamma}$ and assume the compatibility condition that all of the quantities

$$\inf_{x\in\Omega} \varphi_0(x),\ \sup_{x\in\Omega} \varphi_0(x),\ -\rho - (\overline{\varphi_0})^-\ ,\ \rho + (\overline{\varphi_0})^+ \quad \text{belong to the interior of } D(\beta),$$

where $(\cdot)^+$ and $(\cdot)^-$ denote the positive and negative part functions, respectively.

The analysis of the above system (1.1)–(1.5) has been the subject of investigation in [11]. There, weak and strong well-posedness has been addressed for general potentials and source terms. Since here we aim at solving the optimal control problem **(P)**, we are forced to work under the framework of strong solutions. This, in particular, forces us to restrict the investigation to differentiable potentials, more precisely, to either regular ones like (1.13) or, under the further restriction that $d = 2$, to the logarithmic potential from (1.12). Since we are going to assume **(A1)**–**(A3)** in any case, we state the following results under these assumptions, even if some of the conditions may be relaxed (cf. [11]).

As a consequence of [11, Thms. 2.2, 2.3, and 2.5], we have the following well-posedness result for the initial-boundary value problem (1.1)–(1.5).

Theorem 2.1 (Well-posedness of the state system)**.** *Suppose that* **(A1)**–**(A3)** *hold true, and let the data of the system fulfill*

$$f \in H^1(0,T;V^*), \quad u \in L^2(0,T;H), \tag{2.3}$$
$$\varphi_0 \in H^3(\Omega) \cap W, \quad w_0 \in V, \quad w_1 \in V. \tag{2.4}$$

Then, there exists a unique solution (φ, μ, w) *to the system* (1.1)–(1.5) *satisfying*

$$\varphi \in H^1(0,T;V) \cap L^\infty(0,T;W^{2,6}(\Omega)) \quad \textit{with} \quad \beta(\varphi) \in L^\infty(0,T;L^6(\Omega)), \tag{2.5}$$
$$\mu \in L^\infty(0,T;V), \tag{2.6}$$
$$w \in H^2(0,T;H) \cap C^1([0,T];V), \tag{2.7}$$

as well as the estimate

$$\|\varphi\|_{H^1(0,T;V)\cap L^\infty(0,T;W^{2,6}(\Omega))} + \|\mu\|_{L^\infty(0,T;V)} + \|\beta(\varphi)\|_{L^\infty(0,T;L^6(\Omega))} \\ + \|w\|_{H^2(0,T;H)\cap C^1([0,T];V)} \le K_1\,, \tag{2.8}$$

with some constant $K_1 > 0$ that depends only on the structure of the system, Ω, T, and upper bounds for the norms of the data and the quantities related to assumptions (2.3)–(2.4)*. Besides, let $u_i \in L^2(0,T;H)$, $i = 1,2$, and let (φ_i, μ_i, w_i) be the corresponding solutions. Then it holds that*

$$\|\varphi_1 - \varphi_2\|_{L^\infty(0,T;V^*)\cap L^2(0,T;V)} + \|w_1 - w_2\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \\ \le K_2\|\mathbf{1}*(u_1 - u_2)\|_{L^2(0,T;H)}\,, \tag{2.9}$$

with some $K_2 > 0$ that depends only on the structure of the system, Ω, T, and an upper bound for the norms of $\beta(\varphi_1)$ and $\beta(\varphi_2)$ in $L^1(Q)$.

Let us remark that, due to (2.5), the compact embedding $W^{2,6}(\Omega) \hookrightarrow C^0(\overline{\Omega})$, and classical compactness results (see, e.g., [29, Sect. 8, Cor. 4]), it follows that $\varphi \in C^0(\overline{Q})$.

Remark 2.2. The above well-posedness result refers to the natural variational form of the homogeneous Neumann problem for equation (1.1), due to the low regularity of μ specified in (2.6). However, it is clear that, thanks to (2.5), **(A3)** and the elliptic regularity theory, we also have that $\mu \in L^2(0,T;W)$, so that we actually can write (1.1) in its strong form. A similar consideration can be repeated for the linear combination $\kappa_1\partial_t w + \kappa_2 w$ in (1.3) as you can find in the remark below.

Remark 2.3. We point out that the regularity $C^1([0,T];V)$ for the variable w stated in (2.7) does not directly follow from [11, Thms. 2.2, 2.3, 2.5], where just the regularity $W^{1,\infty}(0,T;V)$ was noticed. This, however, can be deduced with the help of (1.3), rewritten as the parabolic equation

$$\frac{1}{\kappa_1}\partial_t y - \Delta y = f_w, \quad \text{with } y := \kappa_1\partial_t w + \kappa_2 w \text{ and } f_w := u - \lambda\partial_t\varphi + \frac{\kappa_2}{\kappa_1}\partial_t w, \tag{2.10}$$

where, due to the previous results, it readily follows that $f_w \in L^2(0,T;H)$. Note that y satisfies (2.10) along with the Neumann homogeneous boundary condition in (1.4), and the initial condition (cf. (1.5))

$$y(0) = (\kappa_1\partial_t w + \kappa_2 w)(0) = \kappa_1 w_1 + \kappa_2 w_0 \in V.$$

Then, by a straightforward application of the parabolic regularity theory (see, e.g., [1,22]), it turns out that

$$y = \kappa_1\partial_t w + \kappa_2 w \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W).$$

At this point, it is not difficult to check that $w \in C^1([0,T];V)$, whereas we cannot infer the regularity $w \in H^1(0,T;W)$ unless when $w(0) = w_0 \in W$.

As will be clear in the forthcoming Section 4, the analytic framework encapsulated in Theorem 2.1 does not suffice to rigorously prove the Fréchet differentiability of the solution operator associated with the system (1.1)–(1.5) (cf. Theorem 4.4 further on) which is a key point to formulate the first-order necessary conditions for optimality addressed in Section 4.3. For this reason, before entering the study of the optimal control problem **(P)**, we present some refined analytic results which are now possible by virtue of the more restricting condition we are assuming on the potentials. In particular, a key regularity property to include singular and regular potentials in the analysis of the optimal control problem is the so-called *strict separation property* for the order parameter φ. This means that the values of φ are always confined in a compact subset of the interior of $D(\beta)$. Notice that, if $D(\beta) = \mathbb{R}$, then the boundedness of φ that follows from the previous theorem already guarantees this property. For singular potentials, when $D(\beta)$ is an open interval, that means that the singularities of the potential at the end-points of $D(\beta)$ are not reached by φ at any time, meaning that the potential and its derivative actually are globally Lipschitz continuous functions. The proof of the following result, sketched in Section 3, is derived with minor modifications arguing as done in [9, Prop. 2.6]. It ensures both more regularity for the solution and the desired separation property in the important case of the logarithmic potential (1.12) in two dimensions.

Theorem 2.4 (Regularity and separation principle)**.** *Suppose that* **(A1)–(A3)** *hold, let* $d = 2$*, and* F *be the logarithmic potential defined in* (1.12)*. Moreover, in addition to* (2.3)–(2.4)*, let* f *and the auxiliary datum* μ_0 *fulfill*

$$f \in H^1(0,T;H), \quad \mu_0 := -\Delta\varphi_0 + F'(\varphi_0) + a - bw_1 \in W. \tag{2.11}$$

Then, the unique solution (φ, μ, w) *obtained from Theorem 2.1 additionally enjoys the regularity properties*

$$\partial_t\varphi \in L^\infty(0,T;H) \cap L^2(0,T;W), \quad \mu \in L^\infty(Q), \quad \beta(\varphi) \in L^\infty(Q), \tag{2.12}$$

as well as

$$\|\partial_t\varphi\|_{L^\infty(0,T;H)\cap L^2(0,T;W)} + \|\mu\|_{L^\infty(Q)} + \|\beta(\varphi)\|_{L^\infty(Q)} \le K_4,$$

for some $K_4 > 0$ *that depends only on the structure of the system, the initial data,* Ω*, and* T*. Furthermore, assume that*

$$r_- < \min_{x\in\overline{\Omega}} \varphi_0(x) \le \max_{x\in\overline{\Omega}} \varphi_0(x) < r_+.$$

Then, the order parameter φ *enjoys the strict separation property, that is, there exist real numbers* r_* *and* r^* *depending only on the structure of the system such that*

$$r_- < r_* \le \varphi(x,t) \le r^* < r_+ \quad \text{for } a.e.\ (x,t) \in Q.$$

Remark 2.5. We point out that the regularity for μ in (2.12) is a consequence of the regularity $\mu \in L^\infty(0,T;W)$ and of the Sobolev embedding $W \hookrightarrow L^\infty(\Omega)$, which holds up to the three-dimensional case. Notice also that a class of potentials slightly more general than the logarithmic one in (1.12) may be possibly considered: for this aim we refer to [18, Thm. 5.1], where a strict separation property has been derived in a suitable framework.

As a straightforward consequence of the above results, we have the following.

Corollary 2.6. *Suppose that either* $D(\beta) = \mathbb{R}$ *or that the assumptions of Theorem 2.4 are fulfilled. Then, there exists a positive constant* K_5 *just depending on the structure and an upper bound for the norms of the data of the system such that*

$$\|\varphi\|_{L^\infty(Q)} + \max_{i=0,1,2,3} \|F^{(i)}(\varphi)\|_{L^\infty(Q)} \le K_5. \tag{2.13}$$

With the above regularity improvement, we are now in a position to obtain a stronger continuous dependence estimate concerning the controls.

Theorem 2.7 (Refined continuous dependence result)**.** *Suppose that* **(A1)–(A3)** *hold. Moreover, assume that the first and second derivatives of the potential F are Lipschitz continuous. Consider $u_i \in L^2(0,T;H)$, $i = 1,2$, and let (φ_i, μ_i, w_i), $i = 1,2$, be the corresponding solutions. Then, it holds that*

$$\|\varphi_1 - \varphi_2\|_{H^1(0,T;V^*)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} + \|\mu_1 - \mu_2\|_{L^2(0,T;V)} \\ + \|w_1 - w_2\|_{H^2(0,T;V^*)\cap W^{1,\infty}(0,T;V)\cap H^1(0,T;W)} \leq K_6 \|u_1 - u_2\|_{L^2(0,T;H)}, \tag{2.14}$$

with some $K_6 > 0$ that depends only on the structure of the system, Ω, and T.

Notice that the above result holds for regular potentials both in dimensions two and three, as for these the Lipschitz continuity of F' follows as a consequence of Theorem 2.1. On the other hand, the logarithmic potential can be considered just in dimension two as a consequence of the separation principle established by Theorem 2.4. It is worth pointing out that the regularity improvement obtained in Theorem 2.4 does not require more regularity of the control variable u. In particular, the strong well-posedness for the system is guaranteed for any control $u \in L^2(0,T;H)$ (in which the control space $\mathcal{U}$ is embedded, see (1.7)).

Let us conclude this section by collecting some useful tools that will be employed later on. We often owe to the Young, Poincaré and compactness inequalities:

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for every } a, b \in \mathbb{R} \text{ and } \delta > 0, \tag{2.15}$$

$$\|v\|_V \leq C_\Omega \left(\|\nabla v\| + |\overline{v}|\right) \quad \text{for every } v \in V, \tag{2.16}$$

$$\|v\| \leq \delta \|\nabla v\| + C_{\Omega,\delta} \|v\|_* \quad \text{for every } v \in V \text{ and } \delta > 0, \tag{2.17}$$

where C_Ω depends only on Ω, $C_{\Omega,\delta}$ depends on δ, in addition, and $\|\cdot\|_*$ is the norm in V^* to be introduced below (see (2.20)).

Next, we recall an important tool which is commonly used when working with problems connected to the Cahn–Hilliard equation. Consider the weak formulation of the Poisson equation $-\Delta z = \psi$ with homogeneous Neumann boundary conditions. Namely, for a given $\psi \in V^*$ (and not necessarily in H), we consider the problem:

$$\text{find} \quad z \in V \quad \text{such that} \quad \int_\Omega \nabla z \cdot \nabla v = \langle \psi, v \rangle \quad \text{for every } v \in V. \tag{2.18}$$

Since Ω is connected and regular, it is well known that the above problem admits a unique solution z if and only if ψ has zero mean value. Hence, we can introduce the associated solution operator $\mathcal{N}$, which turns out to be an isomorphism between the following spaces, as

$$\mathcal{N} : \mathrm{dom}(\mathcal{N}) := \{\psi \in V^* : \overline{\psi} = 0\} \to \{z \in V : \overline{z} = 0\}, \quad \mathcal{N} : \psi \mapsto z, \tag{2.19}$$

where z is the unique solution to (2.18) satisfying $\overline{z} = 0$. Moreover, it follows that the formula

$$\|\psi\|_*^2 := \|\nabla \mathcal{N}(\psi - \overline{\psi})\|^2 + |\overline{\psi}|^2 \quad \text{for every } \psi \in V^* \tag{2.20}$$

defines a Hilbert norm in V^* that is equivalent to the standard dual norm of V^*. From the above properties, one can obtain the following identities:

$$\int_\Omega \nabla \mathcal{N}\psi \cdot \nabla v = \langle \psi, v \rangle \quad \text{for every } \psi \in \mathrm{dom}(\mathcal{N}),\, v \in V, \tag{2.21}$$

$$\langle \psi, \mathcal{N}\zeta \rangle = \langle \zeta, \mathcal{N}\psi \rangle \quad \text{for every } \psi, \zeta \in \mathrm{dom}(\mathcal{N}), \tag{2.22}$$

$$\langle \psi, \mathcal{N}\psi \rangle = \int_\Omega |\nabla \mathcal{N}\psi|^2 = \|\psi\|_*^2 \quad \text{for every } \psi \in \mathrm{dom}(\mathcal{N}), \tag{2.23}$$

as well as

$$\int_0^t \langle \partial_t v(s), \mathcal{N}v(s) \rangle \, ds = \int_0^t \langle v(s), \mathcal{N}(\partial_t v(s)) \rangle \, ds = \frac{1}{2} \|v(t)\|_*^2 - \frac{1}{2} \|v(0)\|_*^2 \,, \tag{2.24}$$

which holds for every $t \in [0, T]$ and every $v \in H^1(0, T; \mathrm{dom}(\mathcal{N}))$.

Finally, without further reference later on, we are going to employ the following convention: the capital-case symbol C is used to denote every constant that depends only on the structural data of the problem such as Ω, T, a, b, κ_1, κ_2, γ, λ, the shape of the nonlinearities, and the norms of the involved functions. Therefore, its meaning may vary from line to line and even within the same line. In addition, when a positive constant δ enters the computation, the related symbol C_δ, in place of a general C, denotes constants that depend on δ, in addition.

3 Regularity and continuous dependence results

This section is devoted to the proofs of Theorem 2.4 and Theorem 2.7. The first result is propedeutic to the second one which will play a key role in proving that the solution operator associated with the system enjoys some differentiability properties.

Proof of Theorem 2.4. We can follow exactly the same argument as that used in [9, Sect. 5.2] to prove the analogous result [9, Prop. 2.6]. However, although we should perform the estimates in a rigorous way on a suitable discrete scheme designed on a proper approximating problem as done in the quoted paper, we proceed formally, for simplicity, by directly acting on problem (1.1)–(1.5), and point out the few differences arising from the presence of the additional variable w. We differentiate both (1.1) and (1.2) with respect to time and test the resulting inequalities by $\partial_t\varphi$ and $\Delta\partial_t\varphi$, respectively. If we sum up and integrate by parts and over $(0, t)$, then a cancellation occurs, and we obtain that

$$\frac{1}{2} \int_\Omega |\partial_t\varphi(t)|^2 + \gamma \int_{Q_t} |\partial_t\varphi|^2 + \int_{Q_t} |\Delta\partial_t\varphi|^2$$
$$= \frac{1}{2} \int_\Omega |\partial_t\varphi(0)|^2 + \int_{Q_t} \partial_t f \, \partial_t\varphi + \int_{Q_t} (\beta' + \pi')(\varphi) \, \partial_t\varphi \, \Delta\partial_t\varphi - b \int_{Q_t} \partial_t^2 w \, \Delta\partial_t\varphi \,.$$

This is the analogue of [9, formula (5.16)] and essentially differs from it just for the presence of the last term. However, this term can be easily dealt with by using Young's inequality and the regularity of w ensured by (2.7). Indeed, we have that

$$-b \int_{Q_t} \partial_t^2 w \, \Delta\partial_t\varphi \le \frac{1}{4} \int_{Q_t} |\Delta\partial_t\varphi|^2 + C \int_{Q_t} |\partial_t^2 w|^2 \le \frac{1}{4} \int_{Q_t} |\Delta\partial_t\varphi|^2 + C \,.$$

As the other terms can be treated as in the quoted paper, we arrive at the analogue of [9, formula (5.17)], i.e.,

$$\|\partial_t\varphi\|^2_{L^\infty(0,T;H)} + \|\Delta\partial_t\varphi\|^2_{L^2(0,T;H)} \\ \le C\left(\|\partial_t\varphi(0)\|^2 + \|\beta'(\varphi)\|^2_{L^2(0,T;L^3(\Omega))} + 1\right) e^{C\,\|\beta'(\varphi)\|^4_{L^4(0,T;L^3(\Omega))}}\,.$$

At this point, the new variable w just enters the computation of $\partial_t\varphi(0)$. By still proceeding formally, we recover the initial value for $\mu(0) = \mu_0$ from (1.2) at the time $t = 0$, then, using the regularity of μ_0 (and f) stated in (2.11), we find out that

$$\partial_t\varphi(0) = f(0) + \Delta\mu_0 - \gamma\varphi_0 \in H$$

from (1.1), also written for $t = 0$. Then, we obtain that

$$\|\partial_t\varphi(0)\|^2 \le \|f(0) + \Delta\mu_0 - \gamma\varphi_0\|^2 \le C.$$

At this point, w does not enter the argument any longer, and we can proceed and then conclude exactly as in [9]. □

Proof of Theorem 2.7. To begin with, let us set the following notation for the differences involved in the statement:

$$\varphi := \varphi_1 - \varphi_2, \quad \mu := \mu_1 - \mu_2, \quad u := u_1 - u_2, \quad w := w_1 - w_2.$$

Next, we write the system solved by the differences that, in its strong form, reads as

$$\partial_t\varphi - \Delta\mu + \gamma\varphi = 0 \quad \text{in } Q, \tag{3.1}$$
$$\mu = -\Delta\varphi + (F'(\varphi_1) - F'(\varphi_2)) - b\partial_t w \quad \text{in } Q, \tag{3.2}$$
$$\partial_t^2 w - \Delta(\kappa_1\partial_t w + \kappa_2 w) + \lambda\partial_t\varphi = u \quad \text{in } Q, \tag{3.3}$$
$$\partial_{\boldsymbol n}\varphi = \partial_{\boldsymbol n}\mu = \partial_{\boldsymbol n}(\kappa_1\partial_t w + \kappa_2 w) = 0 \quad \text{on } \Sigma, \tag{3.4}$$
$$\varphi(0) = w(0) = \partial_t w(0) = 0 \quad \text{in } \Omega. \tag{3.5}$$

First estimate. First, we recall that F' is now assumed to be Lipschitz continuous. Then, testing (3.1) by φ, (3.2) by μ, and adding the resulting equations lead us to

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 + \gamma\|\varphi\|^2 + \|\mu\|^2 = \int_\Omega (F'(\varphi_1) - F'(\varphi_2))\mu - b\int_\Omega \partial_t w\mu \\ \le \frac{1}{2}\|\mu\|^2 + C\big(\|\varphi\|^2 + \|\partial_t w\|^2\big).$$

Now, recalling the continuous dependence estimate already proved in Theorem 2.1, we infer, after integrating over time, that

$$\|\varphi_1 - \varphi_2\|_{L^\infty(0,T;H)} + \|\mu_1 - \mu_2\|_{L^2(0,T;H)} \le C\|1 * (u_1 - u_2)\|_{L^2(0,T;H)}. \tag{3.6}$$

Second estimate. First, let us establish an auxiliary estimate. Since F' and F'' are supposed to be Lipschitz continuous and (2.8) ensures a uniform bound for $\|\nabla\varphi_2\|_\infty$, we have, almost everywhere in $(0,T)$, that

$$\begin{aligned}\|F'(\varphi_1)-F'(\varphi_2)\|_V &\le \|F'(\varphi_1)-F'(\varphi_2)\| + \|F''(\varphi_1)\nabla\varphi_1 - F''(\varphi_2)\nabla\varphi_2\| \\ &\le C\,\|\varphi\| + \|F''(\varphi_1)\nabla\varphi\| + \|(F''(\varphi_1)-F''(\varphi_2))\nabla\varphi_2\| \\ &\le C\,\|\varphi\| + C\,\|\nabla\varphi\| \le C\,\|\varphi\|_V\,.\end{aligned}$$

Next, we multiply (3.1) by $1/|\Omega|$ to obtain that

$$\frac{d}{dt}\,\overline{\varphi}(t) + \gamma\,\overline{\varphi}(t) = 0 \quad \text{for a.a. } t\in(0,T), \tag{3.7}$$

which entails that $\overline{\varphi}(t)=0$ for every $t\in[0,T]$ since $\overline{\varphi}(0)=0$. In particular, besides φ, even $\partial_t\varphi$ has zero mean value. Thus, we are allowed to test (3.1) by $\mathcal{N}(\partial_t\varphi)$, (3.2) by $-\partial_t\varphi$, and (3.3) by $\frac{b}{\lambda}\partial_t w$, and add the resulting identities. By also accounting for the Lipschitz continuity of F' and the Young inequality, we deduce that, a.e. in $(0,T)$,

$$\begin{aligned}&\|\partial_t\varphi\|_*^2 + \frac{\gamma}{2}\frac{d}{dt}\|\varphi\|_*^2 + \frac12\frac{d}{dt}\|\nabla\varphi\|^2 + \frac{b}{2\lambda}\frac{d}{dt}\|\partial_t w\|^2 + \frac{\kappa_1 b}{2\lambda}\|\nabla(\partial_t w)\|^2 + \frac{\kappa_2 b}{2\lambda}\frac{d}{dt}\|\nabla w\|^2 \\ &= \int_\Omega (F'(\varphi_1)-F'(\varphi_2))\partial_t\varphi + \frac{b}{\lambda}\int_\Omega u\,\partial_t w \\ &\le C\big(\|\varphi\|_V\,\|\partial_t\varphi\|_* + \|u\|\,\|\partial_t w\|\big) \le \frac12\|\partial_t\varphi\|_*^2 + C\big(\|\varphi\|_V^2 + \|u\|^2 + \|\partial_t w\|^2\big).\end{aligned}$$

Hence, integrating over time and using (2.9), we may conclude that

$$\begin{aligned}&\|\varphi_1-\varphi_2\|_{H^1(0,T;V^*)\cap L^\infty(0,T;V)} + \|w_1-w_2\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \\ &\quad\le C\|u_1-u_2\|_{L^2(0,T;H)}.\end{aligned} \tag{3.8}$$

Third estimate. By testing (3.1) by μ, we have that

$$\int_\Omega \partial_t\varphi\,\mu + \int_\Omega |\nabla\mu|^2 + \gamma\int_\Omega \varphi\mu = 0\,.$$

Now, we recall that φ and $\partial_t\varphi$ have zero mean value. Hence, by also accounting for the Poincaré inequality (2.16), we deduce that

$$\int_\Omega |\nabla\mu|^2 = -\int_\Omega \partial_t\varphi\,(\mu-\overline{\mu}) - \gamma\int_\Omega \varphi(\mu-\overline{\mu}) \le \frac12\int_\Omega|\nabla\mu|^2 + C\big(\|\partial_t\varphi\|_*^2 + \|\varphi\|^2\big).$$

Therefore, thanks to (3.6) and (3.8), it readily follows that

$$\|\mu_1-\mu_2\|_{L^2(0,T;V)} \le C\|u_1-u_2\|_{L^2(0,T;H)}. \tag{3.9}$$

Fourth estimate. A simple comparison argument in (3.2), along with the above estimates and elliptic regularity theory, entails that

$$\|\varphi_1-\varphi_2\|_{L^2(0,T;W)} \le C\|u_1-u_2\|_{L^2(0,T;H)}. \tag{3.10}$$

Fifth estimate. We take an arbitrary $v \in L^2(0,T;V)$, multiply (3.3) by v, and integrate over Q and by parts. By rearranging and estimating, we easily obtain that

$$\int_Q \partial_t^2 w\, v \le C\big(\|u\|_{L^2(0,T;H)} + \|\partial_t w\|_{L^2(0,T;V)} + \|w\|_{L^2(0,T;V)} + \|\partial_t \varphi\|_{L^2(0,T;V^*)}\big)\|v\|_{L^2(0,T;V)} \,.$$

On account of the previous estimates, we conclude that

$$\|\partial_t^2 w_1 - \partial_t^2 w_2\|_{L^2(0,T;V^*)} \le C\|u_1 - u_2\|_{L^2(0,T;H)}. \tag{3.11}$$

Sixth estimate. Arguing as in Remark 2.3, we now rewrite (3.3) as a parabolic equation in the auxiliary variable $y := \kappa_1 \partial_t w + \kappa_2 w + \kappa_1 \lambda \varphi$ obtaining that

$$\frac{1}{\kappa_1}\partial_t y - \Delta y = u + \frac{\kappa_2}{\kappa_1}\partial_t w - \kappa_1 \lambda \Delta \varphi.$$

Besides, let us underline that y satisfies homogeneous Neumann boundary conditions and null initial conditions, as it can be realized from (3.4) and (3.5). Then, using a well-known parabolic regularity result and the already found estimates (3.8) and (3.10), it is straightforward to deduce that

$$\|y\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W)} \le C\Big\|u + \frac{\kappa_2}{\kappa_1}\partial_t w - \kappa_1\lambda\Delta\varphi\Big\|_{L^2(0,T;H)} \le C\|u\|_{L^2(0,T;H)} \,.$$

Thus, by solving the Cauchy problem for the ordinary differential equation $\kappa_1 \partial_t w + \kappa_2 w = y - \kappa_1 \lambda \varphi$ in terms of w, and recalling again (3.8) and (3.10), we find out that

$$\|w_1 - w_2\|_{H^2(0,T;V^*)\cap W^{1,\infty}(0,T;V)\cap H^1(0,T;W)} \le C\|u_1 - u_2\|_{L^2(0,T;H)}. \tag{3.12}$$

This completes the proof, as collecting the above estimates yields (2.14). □

4 The optimal control problem

In this section, we study the optimal control problem introduced at the beginning, which we recall here for the reader's convenience:

$$(\mathbf{P}) \qquad \min_{u\in\mathcal{U}_{\rm ad}} \mathcal{J}((\varphi,w),u) \quad \text{subject to the constraint that } (\varphi,\mu,w) \text{ solves (1.1)–(1.5),}$$

where the cost functional $\mathcal{J}$ is given by (1.6).

To begin with, let us fix some notation concerning the solution operator $\mathcal{S}$ associated with the system (1.1)–(1.5). As a consequence of the Theorems 2.1, 2.4, and 2.7, the *control-to-state operator*

$$\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) : L^2(Q)(\supset \mathcal{U}) \to \mathcal{Y}, \quad \mathcal{S} : u \mapsto (\varphi,\mu,w),$$

is well defined, where $(\varphi,\mu,w) \in \mathcal{Y}$ is the unique solution to the state system, and the Banach space $\mathcal{Y}$, referred to as the *state space*, is defined by the regularity specified in (2.5)–(2.7) and partially in (2.12), that is,

$$\mathcal{Y} := \big(W^{1,\infty}(0,T;H) \cap H^1(0,T;W) \cap L^\infty(0,T;W^{2,6}(\Omega))\big) \times L^\infty(0,T;V) \\ \times \big(H^2(0,T;H) \cap C^1([0,T];V)\big).$$

Moreover, the continuous dependence estimate provided by Theorem 2.7 allows us to infer that the solution operator is Lipschitz continuous in the sense that, for any pair (u_1, u_2) of controls, it holds that

$$\|\mathcal{S}(u_1) - \mathcal{S}(u_2)\|_{\mathcal{X}} \le K_6 \|u_1 - u_2\|_{L^2(0,T;H)},$$

where $\mathcal{X}$ is the space defined by

$$\begin{aligned}\mathcal{X} := & \big(H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W)\big) \times L^2(0,T;V) \\ & \times \big(H^2(0,T;V^*) \cap W^{1,\infty}(0,T;V) \cap H^1(0,T;W)\big).\end{aligned} \tag{4.1}$$

Furthermore, we introduce the *reduced cost functional*, given by

$$\mathcal{J}_{\rm red} : L^2(Q) \subset \mathcal{U} \to \mathbb{R}, \quad \mathcal{J}_{\rm red} : u \mapsto \mathcal{J}(\mathcal{S}_1(u), \mathcal{S}_3(u), u), \tag{4.2}$$

which allows us to reduce the optimization problem **(P)** to the form

$$\min_{u \in \mathcal{U}_{\rm ad}} \mathcal{J}_{\rm red}(u).$$

In what follows, we are working in the framework of Theorem 2.1 (and possibly in the sense of Theorem 2.4). For this reason, the following conditions will be in order:

- **(C1)** The source f fulfills (2.3), and the initial data $\varphi_0, w_0,$ and w_1 satisfy (2.4). Moreover, if we consider the logarithmic potential and $d = 2$, they additionally fulfill (2.11).
- **(C2)** The functions $u_{\min}, u_{\max}$ belong to $\mathcal{U}$ with $u_{\min} \le u_{\max}$ a.e. in Q.
- **(C3)** $\alpha_1, \dots, \alpha_6,$ and ν are nonnegative constants, not all zero.
- **(C4)** The target functions fulfill $\varphi_Q, w_Q, w'_Q \in L^2(Q)$, $\varphi_\Omega \in V, w_\Omega \in H$, and $w'_\Omega \in V$.

4.1 Existence of optimal controls

The first result we are going to address concerns the existence of optimal controls.

Theorem 4.1 (Existence of optimal controls)**.** *We suppose that the assumptions* **(A1)**–**(A3)** *and* **(C1)**–**(C4)** *are fulfilled. Then, the optimal control problem* **(P)** *admits a solution.*

Proof of Theorem 4.1. As the proof is an immediate consequence of the direct method of the calculus of variations, we just briefly outline the crucial steps. Consider a minimizing sequence $\{u_n\}_n \subset \mathcal{U}_{\rm ad}$ for the reduced cost functional $\mathcal{J}_{\rm red}$ defined by (4.2). Let us introduce also the sequence of the associated states $\{(\varphi_n, \mu_n, w_n)\}_n$, where $(\varphi_n, \mu_n, w_n) = \mathcal{S}(u_n)$ for every $n \in \mathbb{N}$. Namely, we have that

$$\lim_{n\to\infty} \mathcal{J}_{\rm red}(u_n) = \lim_{n\to\infty} \mathcal{J}\big((\mathcal{S}_1(u_n), \mathcal{S}_3(u_n)), u_n\big) = \inf_{u \in \mathcal{U}_{\rm ad}} \mathcal{J}_{\rm red}(u) \ge 0.$$

Thus, as $\mathcal{U}_{\rm ad}$ is bounded in $\mathcal{U}$, by standard compactness arguments, using also that $\mathcal{U}_{\rm ad}$ is closed and convex, we obtain a limit function $u^* \in \mathcal{U}_{\rm ad}$ and a nonrelabelled subsequence such that, as $n \to \infty$,

$$u_n \to u^* \quad \text{weakly-star in } L^\infty(Q).$$

On the other hand, by the boundedness property (2.8) stated in Theorem 2.1, along with standard compactness results (see, e.g., [29, Sect. 8, Cor. 4]), we also have that

$$\begin{aligned}
\varphi_n \to \varphi^* \quad & \text{weakly-star in } H^1(0,T;V)\cap L^\infty(0,T;W^{2,6}(\Omega)),\\
& \quad \text{and strongly in } C^0(\overline{Q}),\\
\mu_n \to \mu^* \quad & \text{weakly-star in } L^\infty(0,T;V),\\
w_n \to w^* \quad & \text{weakly-star in } H^2(0,T;H)\cap W^{1,\infty}(0,T;V),\\
& \quad \text{and strongly in } C^1([0,T];H),\\
F'(\varphi_n) \to \xi^* \quad & \text{weakly-star in } L^\infty(0,T;L^6(\Omega)),
\end{aligned}$$

for some limits φ^*, μ^*, w^*, and ξ^*. The first strong convergence follows from the compact embedding $W^{2,6}(\Omega) \hookrightarrow C^0(\overline{\Omega})$. Besides, as

$$\pi(\varphi_n) \to \pi(\varphi^*) \quad \text{strongly in } C^0(\overline{Q}) \quad \text{and} \quad \beta(\varphi_n) \to \xi^* - \pi(\varphi^*) \quad \text{weakly in } L^1(Q),$$

by maximal monotonicity arguments it is not difficult to conclude that $\xi^* = F'(\varphi^*)$. Then, using the above weak, weak-star and strong convergence properties, it is a standard matter to pass to the limit as n tends to infinity in the variational formulation associated with system (1.1)–(1.5), written for φ_n, μ_n, w_n, and u_n. This will also prove that (φ^*, μ^*, w^*) is nothing but $\mathcal{S}(u^*)$. Finally, the lower semicontinuity of norms entails that

$$\mathcal{J}_{\rm red}(u^*) \le \liminf_{n\to\infty} \mathcal{J}_{\rm red}(u_n) = \lim_{n\to\infty} \mathcal{J}_{\rm red}(u_n) = \inf_{u\in\mathcal{U}_{\rm ad}} \mathcal{J}_{\rm red}(u)\,,$$

meaning that u^* is a global minimizer for $\mathcal{J}_{\rm red}$. □

4.2 Differentiability of the solution operator

In the following, we are going to prove some differentiability properties for the solution operator $\mathcal{S}$. Since these have to be analyzed in open sets, let us take an open ball in the L^∞-topology that contains the set of admissible controls $\mathcal{U}_{\rm ad}$, namely, let $R > 0$ be chosen such that

$$\mathcal{U}_R := \{u \in \mathcal{U} : \|u\|_{\mathcal{U}} < R\} \supset \mathcal{U}_{\rm ad}.$$

Now, we fix $u \in \mathcal{U}_R$ and denote by $(\varphi, \mu, w) = \mathcal{S}(u)$ the unique corresponding state. Then, the linearized system to (1.1)–(1.5) at the fixed control u is given, for any $h \in L^2(Q)$, as follows:

$$\partial_t\xi - \Delta\eta + \gamma\xi = 0 \quad \text{in } Q, \tag{4.3}$$

$$\eta = -\Delta\xi + F''(\varphi)\xi - b\partial_t\zeta \quad \text{in } Q, \tag{4.4}$$

$$\partial_t^2\zeta - \Delta(\kappa_1\partial_t\zeta + \kappa_2\zeta) + \lambda\partial_t\xi = h \quad \text{in } Q, \tag{4.5}$$

$$\partial_{\boldsymbol n}\xi = \partial_{\boldsymbol n}\eta = \partial_{\boldsymbol n}(\kappa_1\partial_t\zeta + \kappa_2\zeta) = 0 \quad \text{on } \Sigma, \tag{4.6}$$

$$\xi(0) = \zeta(0) = \partial_t\zeta(0) = 0 \quad \text{in } \Omega. \tag{4.7}$$

The proof of the well-posedness of the above system is very similar (and, in fact, easier, as the system is linear) to the proof of Theorem 2.1. We have the following result.

Theorem 4.2 (Well-posedness of the linearized system)**.** *Assume that* **(A1)–(A3)** *and* **(C1)** *hold, and let $u \in \mathcal{U}_R$ with associated state $(\varphi, \mu, w) = \mathcal{S}(u)$ be given. Then, for every $h \in L^2(Q)$, the* *linearized system* (4.3)–(4.7) *admits a unique solution* $(\xi, \eta, \zeta) \in \mathcal{X}$*, where* $\mathcal{X}$ *is the Banach space introduced by* (4.1)*. Furthermore, there exists some* $K_7 > 0$*, which depends only on the structure of the system and an upper bound for the norm of* f *and those of the initial data, such that*

$$\begin{aligned} &\|\xi\|_{H^1(0,T;V^*)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} + \|\eta\|_{L^2(0,T;V)} \\ &\quad + \|\zeta\|_{H^2(0,T;V^*)\cap W^{1,\infty}(0,T;V)\cap H^1(0,T;W)} \le K_7 \|h\|_{L^2(0,T;H)}. \end{aligned} \tag{4.8}$$

Remark 4.3. Due to the low regularity level given by the definition (4.1) of $\mathcal{X}$, the above result must refer to a proper variational formulation of the linearized problem. For instance, (4.3) with the homogeneous Neumann boundary condition for η has to be read as

$$\langle \partial_t \xi, v\rangle + \int_\Omega \nabla\eta\cdot\nabla v + \gamma\int_\Omega \xi v = 0 \quad \text{a.e. in } (0,T), \text{ for every } v\in V.$$

Proof of Theorem 4.2. As the system is linear, the uniqueness of solutions readily follows once (4.8) has been shown for a special solution. Indeed, suppose that there are two solutions (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2). It is then enough to repeat the procedure used below with $\xi = \xi_1 - \xi_2$, $\eta = \eta_1 - \eta_2$ and $\zeta = \zeta_1 - \zeta_2$ to realize that the same as (4.8) holds with the right-hand side equal to 0 so that $(\xi_1, \eta_1, \zeta_1) \equiv (\xi_2, \eta_2, \zeta_2)$, i.e., the uniqueness.

Since the proof of existence is standard, we avoid introducing any approximation scheme and just provide formal estimates. The rigorous argument can be straightforwardly reproduced, e.g., on a suitable Faedo–Galerkin scheme.

First estimate. We aim at proving that

$$\|\xi\|_{L^\infty(0,T;V)} + \|\eta\|_{L^2(0,T;V)} + \|\zeta\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \le C\|h\|_{L^2(0,T;H)}\,. \tag{4.9}$$

We preliminarily observe that

$$\|\partial_t\xi\|_{L^2(0,t;V^*)} \le C\big(\|\xi\|_{L^2(0,t;H)} + \|\eta\|_{L^2(0,t;V)}\big) \quad \text{for every } t\in(0,T]\,, \tag{4.10}$$

as one immediately sees by multiplying (4.3) by any $v \in L^2(0,t;V)$ and integrating over Q_t and by parts. Moreover, we recall (2.8) and (2.13) and observe that the former yields a uniform L^∞ bound for $\nabla\varphi$ since $W^{1,6}(\Omega) \hookrightarrow L^\infty(\Omega)$. It then follows that

$$\|F''(\varphi)\xi\|_V \le C\|\xi\|_V \quad \text{a.e. in } (0,T)\,. \tag{4.11}$$

At this point, we are ready to perform the desired estimate. We test (4.3) by $\eta+\xi$, (4.4) by $-\partial_t\xi+\eta$, (4.5) by $\frac{b}{\lambda}\partial_t\zeta$, and add the resulting equalities to infer that a.e. in $(0,T)$ it holds

$$\begin{aligned} &\|\eta\|_V^2 + \frac12\frac{d}{dt}\|\xi\|_V^2 + \gamma\|\xi\|^2 + \frac{b}{2\lambda}\frac{d}{dt}\|\partial_t\zeta\|^2 + \frac{\kappa_1 b}{\lambda}\|\nabla\partial_t\zeta\|^2 + \frac{\kappa_2 b}{2\lambda}\frac{d}{dt}\|\nabla\zeta\|^2 \\ &\quad = -\gamma\int_\Omega \xi\eta + \int_\Omega F''(\varphi)\xi\,(\eta - \partial_t\xi) - b\int_\Omega \partial_t\zeta\eta + \frac{b}{\lambda}\int_\Omega h\,\partial_t\zeta\,, \end{aligned}$$

thanks to a number of cancellations. Now, the whole right-hand side can easily be bounded from above by

$$\frac14\|\eta\|_V^2 + C\big(\|\xi\|_V^2 + \|\partial_t\zeta\|^2 + \|h\|^2\big) - \int_\Omega F''(\varphi)\xi\,\partial_t\xi\,,$$

and it is clear that (4.9) follows upon integrating in time and invoking Gronwall's lemma provided we can properly estimate the time integral of the last term. Using also (4.10) and (4.11), we have that

$$
\begin{aligned}
&-\int_{Q_t} F''(\varphi)\xi\,\partial_t\xi \le C\|F''(\varphi)\xi\|_{L^2(0,t;V)}\|\partial_t\xi\|_{L^2(0,t;V^*)} \\
&\le C\|\xi\|_{L^2(0,t;V)}\bigl(\|\xi\|_{L^2(0,t;H)} + \|\eta\|_{L^2(0,t;V)}\bigr) \le \frac{1}{4}\,\|\eta\|^2_{L^2(0,t;V)} + C\|\xi\|^2_{L^2(0,t;V)}\,,
\end{aligned}
$$

and this is sufficient to conclude.

Second Estimate. We now readily deduce from (4.10) that

$$
\|\partial_t\xi\|_{L^2(0,T;V^*)} \le C\|h\|_{L^2(0,T;H)}\,. \tag{4.12}
$$

On the other hand, by comparing the terms in (4.4) and taking advantage of (4.9) and (4.11), well-known elliptic regularity results allow us to infer that

$$
\|\xi\|_{L^2(0,T;W)} \le C\|h\|_{L^2(0,T;H)}\,. \tag{4.13}
$$

Third Estimate. Now, let us rewrite equation (4.5) in terms of the auxiliary variable $z := \kappa_1\partial_t\zeta + \kappa_2\zeta + \kappa_1\lambda\xi$. We obtain

$$
\frac{1}{\kappa_1}\partial_t z - \Delta z = h + \frac{\kappa_2}{\kappa_1}\partial_t\zeta - \kappa_1\lambda\Delta\xi,
$$

and observe that, in view of (4.6)–(4.7), z satisfies Neumann homogeneous boundary conditions and null initial conditions. Then, by known parabolic regularity results, (4.9), and (4.13), we easily deduce that

$$
\|z\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W)} \le C\Bigl\|h + \frac{\kappa_2}{\kappa_1}\partial_t\zeta - \kappa_1\lambda\Delta\xi\Bigr\|_{L^2(0,T;H)} \le C\|h\|_{L^2(0,T;H)}\,.
$$

Hence, by recalling the definition of z and the already proved bounds (4.9), (4.12), and (4.13), we arrive at

$$
\|\zeta\|_{H^2(0,T;V^*)\cap W^{1,\infty}(0,T;V)\cap H^1(0,T;W)} \le C\|h\|_{L^2(0,T;H)}\,. \tag{4.14}
$$

Due to the embeddings $V^* \hookrightarrow W^*$ and $W \hookrightarrow H \equiv H^* \hookrightarrow W^*$, by interpolation we have that

$$
H^2(0,T;V^*) \cap H^1(0,T;W) \hookrightarrow C^1([0,T];H),
$$

whence (4.14) entails, in particular, that

$$
\|\zeta\|_{C^1([0,T];H)} \le C\|h\|_{L^2(0,T;H)}. \tag{4.15}
$$

This concludes the sketch of the proof. □

We now expect that – provided we select the correct Banach spaces – the linearized system encapsulates the behavior of the Fréchet derivative of the solution operator $\mathcal{S}$. This is stated rigorously in the next theorem, but prior to this, let us introduce the following Banach space:

$$
\begin{aligned}
\mathcal{Z} := &\bigl(H^1(0,T;W^*) \cap C^0([0,T];H) \cap L^2(0,T;W)\bigr) \times L^2(0,T;H) \\
&\times \bigl(H^2(0,T;W^*) \cap C^1([0,T];H) \cap H^1(0,T;W)\bigr).
\end{aligned} \tag{4.16}
$$

Theorem 4.4 (Fréchet differentiability of the solution operator). *Let the set of assumptions* **(A1)–(A3)** *and* **(C1)** *be fulfilled. Then, the control-to-state operator $\mathcal{S}$ is Fréchet differentiable at any $u \in \mathcal{U}_R$ as a mapping from $L^2(Q)$ into $\mathcal{Z}$. Moreover, for $u \in \mathcal{U}_R$, the mapping $D\mathcal{S}(u) \in \mathcal{L}(L^2(Q), \mathcal{Z})$ acts as follows: for every $h \in L^2(Q)$, $D\mathcal{S}(u)h$ is the unique solution (ξ, η, ζ) to the linearized system* (4.3)–(4.7) *associated with h.*

Proof of Theorem 4.4. We fix $u \in \mathcal{U}_R$ and first notice that the map $h \mapsto (\xi, \eta, \zeta)$ of the statement actually belongs to $\mathcal{L}(L^2(Q), \mathcal{Z})$ as a consequence of (4.8). Then, we proceed with a direct check of the claim by showing that

$$\frac{\|\mathcal{S}(u+h) - \mathcal{S}(u) - (\xi, \eta, \zeta)\|_{\mathcal{Z}}}{\|h\|_{L^2(Q)}} \to 0 \quad \text{as } \|h\|_{L^2(Q)} \to 0. \tag{4.17}$$

This will imply both the Fréchet differentiability of $\mathcal{S}$ in the sense specified in the statement and the validity of the identity $D\mathcal{S}(u)h = (\xi, \eta, \zeta)$.

At this place, we remark that the following argumentation will be formal, because of the low regularity of the linearized variables (recall Remark 4.3). Nevertheless, we adopt it for brevity, in order to avoid any approximation, like a Faedo–Galerkin scheme based on the eigenfunctions of the Laplace operator with homogeneous Neumann boundary conditions (in which case, e.g., the Laplacian of the components of the discrete solution could actually be used as test functions).

Without loss of generality, we may assume that $\|h\|_{L^2(Q)}$ is small enough. In particular, we owe to the estimates proved for the solutions to the nonlinear problem corresponding to both u and $u + h$. For convenience, let us set

$$\psi := \varphi^h - \varphi - \xi, \quad \sigma := \mu^h - \mu - \eta, \quad \omega := w^h - w - \zeta,$$

with $(\varphi^h, \mu^h, w^h) := \mathcal{S}(u+h)$, $(\varphi, \mu, w) := \mathcal{S}(u)$, and where (ξ, η, ζ) is the unique solution to (4.3)–(4.7) associated with h. Due to the previous results, we already know that $(\psi, \sigma, \omega) \in \mathcal{X} \hookrightarrow \mathcal{Z}$ and that, by difference, it yields a weak solution to the system

$$\partial_t \psi - \Delta\sigma + \gamma\psi = 0 \qquad \text{in } Q, \tag{4.18}$$

$$\sigma = -\Delta\psi + [F'(\varphi^h) - F'(\varphi) - F''(\varphi)\xi] - b\partial_t \omega \qquad \text{in } Q, \tag{4.19}$$

$$\partial_t^2 \omega - \Delta(\kappa_1 \partial_t \omega + \kappa_2 \omega) + \lambda \partial_t \psi = 0 \qquad \text{in } Q, \tag{4.20}$$

$$\partial_{\boldsymbol{n}} \psi = \partial_{\boldsymbol{n}} \sigma = \partial_{\boldsymbol{n}}(\kappa_1 \partial_t \omega + \kappa_2 \omega) = 0 \qquad \text{on } \Sigma, \tag{4.21}$$

$$\psi(0) = \omega(0) = \partial_t \omega(0) = 0 \qquad \text{in } \Omega. \tag{4.22}$$

Besides, with the above notation, (4.17) amounts show that

$$\|(\psi, \sigma, \omega)\|_{\mathcal{Z}} = o(\|h\|_{L^2(Q)}) \quad \text{as } \|h\|_{L^2(Q)} \to 0. \tag{4.23}$$

Moreover, Theorems 2.1 and 2.7 entail that

$$\|\varphi^h\|_{H^1(0,T;V) \cap L^\infty(0,T;W^{2,6}(\Omega))} + \|\mu^h\|_{L^\infty(0,T;V)} + \|w^h\|_{H^2(0,T;H) \cap W^{1,\infty}(0,T;V)} \le K_1, \tag{4.24}$$

as well as

$$\|\varphi^h - \varphi\|_{H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} + \|\mu^h - \mu\|_{L^2(0,T;V)} \\ + \|w^h - w\|_{H^2(0,T;V^*) \cap W^{1,\infty}(0,T;H) \cap H^1(0,T;V)} \le K_6 \|h\|_{L^2(0,T;H)}. \tag{4.25}$$

Actually, for the logarithmic potential in the two-dimensional setting, we also have a stronger version of (4.24) arising as a consequence of Theorem 2.4.

Before entering the details, we recall that Taylor's formula yields that

$$F'(\varphi^h) - F'(\varphi) - F''(\varphi)\xi = F''(\varphi)\psi + R^h\,(\varphi^h-\varphi)^2, \tag{4.26}$$

where the remainder R^h is given by

$$R^h = \int_0^1 F^{(3)}\big(\varphi + s(\varphi^h-\varphi)\big)(1-s)\,ds\,.$$

Due to (2.13), we have that

$$\|R^h\|_{L^\infty(Q)} \le C. \tag{4.27}$$

First estimate. We notice that ψ has zero mean value as can be easily checked by testing (4.18) by $1/|\Omega|$ and using (4.22). Hence, we can test (4.18) by $\mathcal{N}\psi$ and (4.19) by $-\psi$. Moreover, we integrate (4.20) in time and test the resulting equation by $\frac{b}{\lambda}\partial_t\omega$. Finally, we sum up and add the same term $\frac{\kappa_1 b}{2\lambda}\frac{d}{dt}\|\omega\|^2 = \frac{\kappa_1 b}{2\lambda}\int_\Omega \omega\,\partial_t\omega$ to both sides. We obtain that

$$\begin{aligned}
&\frac12\,\frac{d}{dt}\,\|\psi\|_*^2 + \gamma\|\psi\|_*^2 + \|\nabla\psi\|^2 + \frac{b}{\lambda}\,\|\partial_t\omega\|^2 + \frac{\kappa_1 b}{2\lambda}\,\frac{d}{dt}\,\|\omega\|_V^2 \\
&\quad = \int_\Omega [F'(\varphi^h) - F'(\varphi) - F''(\varphi)\xi]\psi - \frac{b\kappa_2}{\lambda}\int_\Omega \nabla(\mathbf{1}*\omega)\cdot\nabla\partial_t\omega + \frac{\kappa_1 b}{2\lambda}\int_\Omega \omega\,\partial_t\omega.
\end{aligned}$$

Since we aim at applying the Gronwall lemma, we should integrate over $(0,t)$ with respect to time. However, for brevity, we just estimate the first two terms of the right-hand side obtained by integration (the last one can be trivially handled by the Young inequality) and avoid writing the integration variable s in the integrals over $(0,t)$. The first one can be controlled by using the Hölder and Young inequalities, (4.25), the continuous embedding $V \hookrightarrow L^4(\Omega)$, (4.26), (4.27), and the compactness inequality (2.17) as follows:

$$\begin{aligned}
&\int_{Q_t} [F'(\varphi^h) - F'(\varphi) - F''(\varphi)\xi]\psi = \int_{Q_t} [F''(\varphi)\psi + R^h\,(\varphi^h-\varphi)^2]\psi \\
&\quad \le C\int_0^t \|\psi\|^2\,ds + C\int_0^t \|\varphi^h-\varphi\|_4^2\|\psi\|\,ds \le C\int_0^t \|\psi\|^2\,ds + C\int_0^t \|\varphi^h-\varphi\|_V^4\,ds \\
&\quad \le C\int_0^t \|\psi\|^2\,ds + CT\|h\|_{L^2(Q)}^4 \le \frac12\int_0^t \|\nabla\psi\|^2\,ds + C\int_0^t \|\psi\|_*^2\,ds + C\|h\|_{L^2(Q)}^4.
\end{aligned}$$

As for the second term, we integrate by parts both in space and time. By also accounting for the Young inequality, we find that

$$\begin{aligned}
&-\frac{b\kappa_2}{\lambda}\int_{Q_t} \nabla(\mathbf{1}*\omega)\cdot\nabla\partial_t\omega = -\frac{b\kappa_2}{\lambda}\int_\Omega \nabla(\mathbf{1}*\omega)(t)\cdot\nabla\omega(t) + \frac{b\kappa_2}{\lambda}\int_{Q_t}|\nabla\omega|^2 \\
&\quad \le \frac{\kappa_1 b}{4\lambda}\int_\Omega |\nabla\omega(t)|^2 + C\int_\Omega\Big|\int_0^t \nabla\omega\,ds\Big|^2 + C\int_{Q_t}|\nabla\omega|^2 \le \frac{\kappa_1 b}{4\lambda}\int_\Omega |\nabla\omega(t)|^2 + C\int_{Q_t}|\nabla\omega|^2.
\end{aligned}$$

Thus, we can apply the Gronwall lemma and conclude that

$$\|\psi\|_{L^\infty(0,T;V^*)\cap L^2(0,T;V)} + \|\omega\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le C\|h\|_{L^2(Q)}^2. \tag{4.28}$$

Second estimate. We test (4.18) by ψ, (4.19) by $\Delta\psi$, and add the resulting equalities to find that

$$\frac{1}{2}\frac{d}{dt}\|\psi\|^2 + \|\Delta\psi\|^2 + \gamma\|\psi\|^2 = \int_\Omega [F'(\varphi^h) - F'(\varphi) - F''(\varphi)\xi]\Delta\psi - b\int_\Omega \partial_t\omega\Delta\psi.$$

As above, we only estimate the right-hand side of the equality obtained by integrating over $(0,t)$. By also accounting for the previous estimate, we have that

$$\begin{aligned}
&\int_{Q_t}[F'(\varphi^h) - F'(\varphi) - F''(\varphi)\xi]\Delta\psi - b\int_{Q_t}\partial_t\omega\Delta\psi \\
&\leq \int_{Q_t}|F''(\varphi)|\,|\psi|\,|\Delta\psi| + \int_{Q_t}|R^h|\,|\varphi^h-\varphi|^2\,|\Delta\psi| + C\int_0^t \|\partial_t\omega\|\,\|\Delta\psi\|\,ds \\
&\leq \frac{1}{2}\int_0^t \|\Delta\psi\|^2\,ds + C\int_0^t (\|\psi\|^2 + \|\partial_t\omega\|^2)\,ds + C\|h\|^4_{L^2(Q)} \\
&\leq \frac{1}{2}\int_0^t \|\Delta\psi\|^2\,ds + C\|h\|^4_{L^2(Q)}.
\end{aligned}$$

Thus, owing also to the elliptic regularity theory, we conclude that

$$\|\psi\|_{L^\infty(0,T;H)\cap L^2(0,T;W)} \leq C\|h\|^2_{L^2(Q)}. \tag{4.29}$$

Third estimate. Next, we test (4.19) by σ and, arguing as above, we obtain that

$$\|\sigma\|_{L^2(0,T;H)} \leq C\|h\|^2_{L^2(Q)}. \tag{4.30}$$

Fourth estimate. We can now test (4.18) by an arbitrary function $v \in L^2(0,T;W)$ and, in view of (4.29) and (4.30), easily infer that

$$\begin{aligned}
\Big|\int_0^T \langle\partial_t\psi, v\rangle_W\Big| &\leq \|\sigma\|_{L^2(0,T;H)}\|\Delta v\|_{L^2(0,T;H)} + \gamma\|\psi\|_{L^2(0,T;H)}\|v\|_{L^2(0,T;H)} \\
&\leq C\|h\|^2_{L^2(Q)}\|v\|_{L^2(0,T;W)} \quad \text{for all } v \in L^2(0,T;W).
\end{aligned}$$

Hence, $\|\partial_t\psi\|_{L^2(0,T;W^*)}$ is uniformly bounded by a quantity proportional to $\|h\|^2_{L^2(Q)}$, so that from (4.29) and an interpolation argument we recover that

$$\|\psi\|_{H^1(0,T;W^*)\cap C^0([0,T];H)\cap L^2(0,T;W)} \leq C\|h\|^2_{L^2(Q)}. \tag{4.31}$$

Fifth estimate. Next, we rewrite equation (4.20) in terms of the auxiliary variable $\tau := \kappa_1\partial_t\omega + \kappa_2\omega + \kappa_1\lambda\psi$ to obtain

$$\frac{1}{\kappa_1}\partial_t\tau - \Delta\tau = \frac{\kappa_2}{\kappa_1}\partial_t\omega - \kappa_1\lambda\Delta\psi.$$

Thanks to (4.21)–(4.22), it turns out that τ satisfies Neumann homogeneous boundary conditions and null initial conditions. Then, by virtue of parabolic regularity results along with (4.28) and (4.31), we have that

$$\|\tau\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W)} \leq C\Big\|\frac{\kappa_2}{\kappa_1}\partial_t\omega - \kappa_1\lambda\Delta\psi\Big\|_{L^2(0,T;H)} \leq C\|h\|^2_{L^2(Q)}\,.$$

Therefore, observing that $\kappa_1\partial_t\omega + \kappa_2\omega = \tau - \kappa_1\lambda\psi$, it follows that both ω and $\partial_t\omega$ satisfy (at least) the same estimate as (4.31), which yields

$$\|\omega\|_{H^2(0,T;W^*)\cap C^1([0,T];H)\cap H^1(0,T;W)} \leq C\|h\|^2_{L^2(Q)}. \tag{4.32}$$

This concludes the proof since the estimates (4.30)–(4.32) directly lead to (4.23). □

4.3 Adjoint system and first-order optimality conditions

As a final step, we now introduce a suitable adjoint system to (1.1)–(1.5) in order to recover a more practical form of the optimality conditions for **(P)**. Let $u \in \mathcal{U}_{\rm ad}$ be given with its associated state (φ, μ, w). In a strong formulation, the adjoint system is expressed by the *backward-in-time* parabolic system

$$- \partial_t p - \Delta q + \gamma p + F''(\varphi)q - \lambda\partial_t r = \alpha_1(\varphi - \varphi_Q) \quad \text{in } Q, \qquad (4.33)$$

$$q = -\Delta p \quad \text{in } Q, \qquad (4.34)$$

$$\begin{aligned} &- \partial_t r - \Delta(\kappa_1 r - \kappa_2(1 \circledast r)) - bq \\ &\quad = \alpha_3(1 \circledast (w - w_Q)) + \alpha_4(w(T) - w_\Omega) + \alpha_5(\partial_t w - w'_Q) \end{aligned} \quad \text{in } Q, \qquad (4.35)$$

$$\partial_{\boldsymbol n} p = \partial_{\boldsymbol n} q = \partial_{\boldsymbol n}(\kappa_1 r - \kappa_2(1 \circledast r)) = 0 \quad \text{on } \Sigma, \qquad (4.36)$$

$$p(T) = \alpha_2(\varphi(T) - \varphi_\Omega) - \lambda\alpha_6(\partial_t w(T) - w'_\Omega), \quad r(T) = \alpha_6(\partial_t w(T) - w'_\Omega) \quad \text{in } \Omega, \qquad (4.37)$$

where the convolution product $\circledast$ has been introduced in (2.2). Concerning this product, note in particular that $\partial_t(1 \circledast r) = -r$. Let us introduce the following shorthand for the right-hand side of (4.35),

$$f_r := \alpha_3(1 \circledast (w - w_Q)) + \alpha_4(w(T) - w_\Omega) + \alpha_5(\partial_t w - w'_Q)\,.$$

We also notice that the second term is independent of time. Due to the regularity properties in (2.7) and **(C4)**, it holds that

$$\|f_r\|_{L^2(0,T;H)} \le C(\|w\|_{H^2(0,T;H)\cap W^{1,\infty}(0,T;V)} + 1) \le C. \qquad (4.38)$$

Let us remark that the variable r corresponds to the adjoint of the freezing index w. Besides, equation (4.35) is of first-order in time instead of second-order. However, it is worth pointing out that (4.35) may be rewritten in the time-integrated variable $1 \circledast r$ as it holds that $-\partial_t r = \partial_t^2(1 \circledast r)$.

Theorem 4.5 (Well-posedness of the adjoint system)**.** *Let the assumptions* **(A1)–(A3)** *and* **(C1)–(C4)** *hold, and let $u \in \mathcal{U}_{\rm ad}$ with associated state $(\varphi, \mu, w) = \mathcal{S}(u)$ be given. Then, the adjoint system* (4.33)–(4.37) *admits a unique weak solution (p, q, r) such that*

$$\begin{aligned} &p \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \\ &q \in L^2(0,T;V), \\ &r \in H^1(0,T;H) \cap L^\infty(0,T;V). \end{aligned}$$

Remark 4.6. Similarly as in Remark 4.3, we should here speak of a proper variational formulation. For instance, (4.33) with the homogeneous Neumann boundary condition for q has to be read as

$$\begin{aligned} -\langle \partial_t p, v\rangle + \int_\Omega \nabla q \cdot \nabla v + \int_\Omega \big(\gamma p + F''(\varphi)q - \lambda\partial_t r\big)v \\ = \int_\Omega \alpha_1(\varphi - \varphi_Q)v \quad \text{a.e. in } (0,T), \text{ for every } v \in V. \end{aligned}$$

Proof of Theorem 4.5. Again, for existence, we proceed formally but let us underline that the following computations can however be reproduced in a rigorous framework.

First estimate. We test (4.33) by $p+q$, (4.34) by $\partial_t p+(K_5+1)q$, where K_5 is the positive constant arising from (2.13), (4.35) by $-\frac{\lambda}{b}\partial_t r$ and add the resulting identities to each other. Then, we infer that

$$-\frac{1}{2}\frac{d}{dt}\|p\|_V^2+(K_5+1)\|q\|^2+\|\nabla q\|^2+\gamma\|p\|^2+\frac{\lambda}{b}\|\partial_t r\|^2$$
$$-\frac{\kappa_1\lambda}{2b}\frac{d}{dt}\|\nabla r\|^2+\frac{\kappa_2\lambda}{b}\int_\Omega \nabla(1\circledast r)\cdot\nabla\partial_t r$$
$$=-\gamma\int_\Omega pq-\int_\Omega F''(\varphi)q(p+q)+\lambda\int_\Omega \partial_t r\,p+\alpha_1\int_\Omega(\varphi-\varphi_Q)(p+q)$$
$$+K_5\int_\Omega \nabla p\cdot\nabla q-\frac{\lambda}{b}\int_\Omega f_r\,\partial_t r. \tag{4.39}$$

Now, recalling (2.13), the second term on the right-hand side can be bounded from above as

$$-\int_\Omega F''(\varphi)q(p+q)\le\|F''(\varphi)\|_\infty\|p\|\|q\|+\|F''(\varphi)\|_\infty\|q\|^2\le\Big(\frac{1}{2}+K_5\Big)\|q\|^2+C\|p\|^2,$$

and the first term appearing on the right can be absorbed by the corresponding contribution appearing on the left of (4.39). By the Young inequality, we see that the remaining terms on the right-hand side are bounded above by

$$\frac{\lambda}{2b}\|\partial_t r\|^2+\frac{1}{2}\|\nabla q\|^2+\frac{1}{4}\|q\|^2+C(\|p\|_V^2+1)\,,$$

thanks to (2.8) and the estimate (4.38) of f_r. Next, we integrate over (t,T), for any $t\in(0,T)$, and notice that **(C4)** provide uniform bounds for $\|p(T)\|_V^2$ and $\|r(T)\|_V^2$ using their explicit form given by (4.37). Moreover, we treat the integral deriving from the last term on the left-hand side of (4.39) as follows. With the notation $Q^t:=\Omega\times(t,T)$, we have that

$$\frac{\kappa_2\lambda}{b}\int_{Q^t}\nabla(1\circledast r)\cdot\nabla\partial_t r=-\frac{\kappa_2\lambda}{b}\int_\Omega\nabla(1\circledast r)(t)\cdot\nabla r(t)+\frac{\kappa_2\lambda}{b}\int_{Q^t}|\nabla r|^2\,.$$

On the other hand, we also have that

$$\Big|-\frac{\kappa_2\lambda}{b}\int_\Omega\nabla(1\circledast r)(t)\cdot\nabla r(t)\Big|\le\frac{\kappa_1\lambda}{4b}\int_\Omega|\nabla r(t)|^2+C\int_\Omega|\nabla(1\circledast r)(t)|^2$$
$$\le\frac{\kappa_1\lambda}{4b}\int_\Omega|\nabla r(t)|^2+C\int_{Q^t}|\nabla r|^2\,.$$

Thus, from the (backward) Gronwall lemma and the obvious subsequent inequality

$$\|r(t)\|\le C\,\|\partial_t r\|_{L^2(Q)}+C\quad\text{for every } t\in[0,T],$$

we infer that

$$\|p\|_{L^\infty(0,T;V)}+\|q\|_{L^2(0,T;V)}+\|r\|_{H^1(0,T;H)\cap L^\infty(0,T;V)}\le C.$$

Second estimate. Elliptic regularity theory applied to (4.34) then produces

$$\|p\|_{L^2(0,T;W)}\le C.$$

Third estimate. Finally, it is a standard matter to infer from a comparison argument in (4.33), along with the above estimates, that

$$\|\partial_t p\|_{L^2(0,T;V^*)} \le C.$$

This concludes the (formal) proof of the existence of a solution. By performing the same estimates in the case of vanishing right-hand side and final data, we see that the solution must vanish, whence uniqueness in the general case follows by linearity. □

Finally, using the adjoint variables, we present the first-order necessary conditions for an optimal control u^* solving **(P)**. In the following, (p, q, r) and (ξ, η, ζ) denote the solutions of the respective adjoint problem and linearized problem, but written in terms of the associated state $(\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$ that replaces (φ, μ, w) in systems (4.3)–(4.7) and (4.33)–(4.37).

Theorem 4.7 (First-order optimality conditions)**.** *Suppose that* **(A1)**–**(A3)** *and* **(C1)**–**(C4)** *hold. Let* u^* *be an optimal control for* **(P)** *with associated state* $(\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$ *and adjoint* (p, q, r)*. Then, it necessarily fulfills the variational inequality*

$$\int_Q (r + \nu u^*)(u - u^*) \ge 0 \quad \textit{for every } u \in \mathcal{U}_{\rm ad}. \tag{4.40}$$

Proof of Theorem 4.7. From standard results of convex analysis, the first-order necessary optimality condition for every optimal control u^* of **(P)** is expressed in the abstract form as

$$\langle D\mathcal{J}_{\rm red}(u^*), u - u^* \rangle \ge 0 \quad \forall u \in \mathcal{U}_{\rm ad},$$

where $D\mathcal{J}_{\rm red}$ denotes the Fréchet derivative of the reduced cost functional $\mathcal{J}$. As a consequence of the Fréchet differentiability of the control-to-state operator established in Theorem 4.4, and the form of the cost functional $\mathcal{J}$ in (1.6), this entails that any optimal control u^* necessarily fulfills

$$\begin{aligned}
&\alpha_1 \int_Q (\varphi^* - \varphi_Q)\xi + \alpha_2 \int_\Omega (\varphi^*(T) - \varphi_\Omega)\xi(T) + \alpha_3 \int_Q (w^* - w_Q)\zeta \\
&\quad + \alpha_4 \int_\Omega (w^*(T) - w_\Omega)\zeta(T) + \alpha_5 \int_Q (\partial_t w^* - w'_Q)\partial_t \zeta \\
&\quad + \alpha_6 \int_\Omega (\partial_t w^*(T) - w'_\Omega)\partial_t \zeta(T) + \nu \int_Q u^*(u - u^*) \ge 0 \quad \forall u \in \mathcal{U}_{\rm ad},
\end{aligned} \tag{4.41}$$

where (ξ, η, ζ) is the unique solution to the linearized system as obtained from Theorem 4.2 associated with $(\varphi, \mu, w) = (\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$ and $h = u - u^*$. Unfortunately, the above formulation is not very useful in numerical applications as it depends on the linearized variables. However, with the help of the adjoint variables, playing the role of Lagrangian multipliers, the above variational inequality can be simplified. In this direction, we test (4.3) by p, (4.4) by q, (4.5) by r, and add the resulting equalities and integrate over time and by parts. More precisely, we should consider the variational formulations of the linearized and adjoint systems mentioned in Remarks 4.3 and 4.6 in order to avoid writing some Laplacian that does not exist in the usual sense, and we should also owe to (well-known) generalized versions of the integration by parts in time. However, for shortness, we proceed as said above and obtain

$$\begin{aligned}
0 &= \int_Q [\partial_t \xi - \Delta\eta + \gamma\xi]p + \int_Q [-\eta - \Delta\xi + F''(\varphi)\xi - b\partial_t\zeta]q \\
&\quad + \int_Q [\partial_t^2 \zeta - \Delta(\kappa_1 \partial_t \zeta + \kappa_2 \zeta) + \lambda \partial_t \xi - h]r
\end{aligned}$$

$$
\begin{aligned}
&= \int_Q \xi[-\partial_t p - \Delta q + \gamma p + F''(\varphi) q - \lambda \partial_t r] \\
&\quad + \int_Q \eta[-\Delta p - q] + \int_Q \partial_t \zeta[-\partial_t r - \Delta(\kappa_1 r - \kappa_2 (1 \circledast r)) - bq] \\
&\quad + \int_\Omega [\xi(T) p(T) + \partial_t \zeta(T) r(T) + \lambda \xi(T) r(T)] - \int_Q hr.
\end{aligned}
$$

Using the adjoint system (4.33)–(4.37) and the associated final conditions, and integrating by parts as well, we infer that

$$
\begin{aligned}
&\int_Q r(u - u^*) = \int_Q hr \\
&= \int_Q \xi\, \alpha_1(\varphi^* - \varphi_Q) + \int_Q \partial_t \zeta \,[\alpha_3 (1 \circledast (w^* - w_Q) + \alpha_4 (w^*(T) - w_\Omega) + \alpha_5 (\partial_t w^* - w_Q')] \\
&\quad + \int_\Omega \xi(T)\,[\alpha_2(\varphi^*(T) - \varphi_\Omega) - \lambda \alpha_6 (\partial_t w^*(T) - w_\Omega')] \\
&\quad + \int_\Omega \partial_t \zeta(T)\, \alpha_6(\partial_t w^*(T) - w_\Omega') + \int_\Omega \lambda \xi(T)\, \alpha_6 (\partial_t w^*(T) - w_\Omega') \\
&= \alpha_1 \int_Q (\varphi^* - \varphi_Q)\xi + \alpha_2 \int_\Omega (\varphi^*(T) - \varphi_\Omega)\xi(T) + \alpha_3 \int_Q (w^* - w_Q)\zeta \\
&\quad + \alpha_4 \int_\Omega (w^*(T) - w_\Omega)\zeta(T) + \alpha_5 \int_Q (\partial_t w^* - w_Q')\partial_t \zeta + \alpha_6 \int_\Omega (\partial_t w^*(T) - w_\Omega')\partial_t \zeta(T),
\end{aligned}
$$

so that (4.41) entails (4.40), and this concludes the proof. □

Corollary 4.8. *Suppose the assumptions of Theorem 4.7 are fulfilled, and let u^* be an optimal control with associated state $(\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$ and adjoint (p, q, r). Then, whenever $\nu > 0$, u^* is the L^2-orthogonal projection of $-\frac{1}{\nu} r$ onto $\mathcal{U}_{\rm ad}$. Besides, we have the pointwise characterization of the optimal control u^* as*

$$
u^*(x,t) = \max\Big\{u_{\min}(x,t), \min\{u_{\max}(x,t), -\frac{1}{\nu}\, r(x,t)\}\Big\} \quad \textit{for } a.a.\ (x,t) \in Q.
$$

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