Regularity of random elliptic operators with degenerate coefficients and applications to stochastic homogenization

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Abstract

We consider degenerate elliptic equations of second order in divergence form with a symmetric random coefficient field a. Extending the work of the first author, Fehrman, and Otto [Ann. Appl. Probab. **28** (2018), no. 3, 1379–1422], who established the large-scale $C^{1,\alpha}$ regularity of a-harmonic functions in a degenerate situation, we provide stretched exponential moments for the minimal radius r_* describing the minimal scale for this $C^{1,\alpha}$ regularity. As an application to stochastic homogenization, we partially generalize results by Gloria, Neukamm, and Otto [Anal. PDE **14** (2021), no. 8, 2497–2537] on the growth of the corrector, the decay of its gradient, and a quantitative two-scale expansion to the degenerate setting. On a technical level, we demand the ensemble of coefficient fields to be stationary and subject to a spectral gap inequality, and we impose moment bounds on a and a^{-1}. We also introduce the ellipticity radius r_e which encodes the minimal scale where these moments are close to their positive expectation value.

1 Introduction and main results

In these notes, we present some ideas to generalize results from stochastic homogenization of uniformly elliptic operators $-\nabla \cdot a\nabla$ to the case of degenerate and unbounded random coefficient fields a. The underlying random distribution is always assumed to be stationary and ergodic. To quantify the degeneracy and unboundedness, we impose moment bounds on the norm of $a(x)$ and its pointwise inverse $a(x)^{-1}$, $x \in \mathbb{R}^d$. A precise collection of our general assumptions is given in Definition 1.1.

For the sake of a simplified notation, we focus on scalar models where $a : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is a matrix field rather than a field of rank-4 tensors. But since we do not rely on results from scalar PDE theory like maximum principles, we believe that our methods also extend to systems provided that all arguments involving $|a|$ or $|a^{-1}|$ also apply to the respective generalizations μ and λ^{-1} as defined by the first author, Fehrman, and Otto [8]. As we shall explain in more detail below (see Remark 2.2), we currently have to restrict ourselves for technical reasons to symmetric matrix coefficient fields $a(x) = a(x)^T$; this issue might be resolved when directly working with the scalar quantities μ and λ^{-1}. However, this is beyond the scope of the current contribution and will be the subject of future work.

To some extent, this paper continues the studies of the first author, Fehrman, and Otto [8], where the large-scale $C^{1,\alpha}$ regularity and a first-order Liouville principle for a-harmonic functions were derived in the same setting. It is one of the goals of the present contribution to provide stretched exponential moments for the minimal radius r_*, which determines the minimal scale for the $C^{1,\alpha}$ regularity. Moreover, we provide quantitative estimates on the growth of the corrector and the decay of its gradient, and we derive a quantitative two-scale expansion in our degenerate setting.

The starting point of our analysis is the work of Gloria, Neukamm, and Otto [19] on the large-scale regularity of random elliptic operators. The main achievements of this publication are large-scale

Schauder and large-scale Calderón-Zygmund estimates valid on scales larger than the minimal radius r_*. Their approach is in turn motivated by the ideas of Avellaneda and Lin [5], who established a large-scale regularity theory for elliptic operators with periodic coefficients, hence, on the torus. This enabled the authors to apply compactness arguments which are generally not available. Previous preprints [17, 18] of [19] follow in some cases different strategies which can be equally valuable as they are sometimes better suited for an application in our situation. A key ingredient in all three versions are functional inequalities (e.g. spectral gap and logarithmic Sobolev inequalities), which allow to quantify certain aspects of the random ensemble in an advantageous manner. A comparison of various forms of functional inequalities and applications is given by Duerinckx and Gloria [15]. The basis for our results on the corrector in stochastic homogenization and the two-scale expansion is the contribution by Gloria, Neukamm, and Otto [20] on quantitative estimates in stochastic homogenization.

Definition 1.1 (Ensemble of coefficient fields)**.** *Let Ω be the space of symmetric coefficient fields $a : \mathbb{R}^d \to \mathbb{R}^{d \times d}$, $d \geq 2$, and let $\langle \cdot \rangle$ denote an ensemble of coefficient fields a, i.e. a probability measure on Ω, which we assume to be*

- ◼ stationary*, i.e. the probability distributions of a and $a(x + \cdot)$ coincide for all $x \in \mathbb{R}^d$,*

- ◼ ergodic*, i.e. every translation invariant random variable is almost surely constant.*

For any $a \in \Omega$, we define the (space-dependent) quantities

$$\lambda := \left| a^{-1} \right|^{-1} \quad \text{and} \quad \mu := |a|.$$

We suppose $\lambda(x), \mu(x) \in (0, \infty)$ for a.e. $x \in \mathbb{R}^d$, and that $p, q \in (1, \infty)$ exist satisfying

$$\left\langle \mu^p \right\rangle^{\frac{1}{p}} + \left\langle \lambda^{-q} \right\rangle^{\frac{1}{q}} =: K < \infty \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} < \frac{2}{d}, \tag{1.1}$$

where (1.1) is independent of $x \in \mathbb{R}^d$ due to the stationarity of the ensemble $\langle \cdot \rangle$.

The concept of imposing stochastic moment bounds on the coefficient field $a(x)$ instead of assuming uniform ellipticity was successfully applied in a similar context by Chiarini and Deuschel to prove an invariance principle for symmetric diffusion processes on $\mathbb{R}^d$ [13]. In the context of homogenization, condition (1.1) was first imposed by Andres, Deuschel, and Slowik [2] for an ergodic random conductance model and later also used in a time-dependent ergodic version thereof [14]. In our situation, the purpose of (1.1) is to guarantee the sublinearity of the corrector (cf. Remark 1.5) and to allow for specific Sobolev embeddings (e.g. in Lemma 2.3). Only recently, the first author and Schäffner [10] showed that the relaxed version

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{d - 1} \tag{1.2}$$

guarantees local boundedness and the existence of a Harnack inequality for solutions to linear, nonuniformly elliptic equations. The same result was already proven by Trudinger [25] under the more restrictive version in (1.1). Condition (1.2) is, in addition, optimal in the sense that local boundedness is generally not available if the right hand side is replaced by $\frac{2}{d-1} + \varepsilon$ for any $\varepsilon > 0$; we refer to the references in [10] for further details. As an application to stochastic homogenization, the authors show that the pointwise sublinearity of the corrector, which was proven by Chiarini and Deuschel [13] in a similar framework assuming (1.1), also holds under condition (1.2).

Stochastic moment bounds of the type (1.1) appeared recently also in studies on the regularity properties of non-uniformly parabolic operators; see [1, 11] and the references therein.

Remark 1.2. *Birkhoff's ergodic theorem (see e.g. [21]) guarantees that for a.e. coefficient field a, we have*

$$\lim_{r \to \infty} \left(\fint_{B_r(0)} \mu^p \right)^{\frac{1}{p}} + \left(\fint_{B_r(0)} \lambda^{-q} \right)^{\frac{1}{q}} = K. \tag{1.3}$$

As a consequence of (1.3), we may define the *ellipticity radius r_e*, which determines the minimal scale on which the system behaves approximately elliptic, as follows.

Definition 1.3. *We define the* ellipticity radius *$r_e \geq 1$ as the random variable*

$$r_e := \inf \left\{ r \geq 1 \;\middle|\; \forall \rho > r : \left(\fint_{B_\rho(0)} \mu^p \right)^{\frac{1}{p}} + \left(\fint_{B_\rho(0)} \lambda^{-q} \right)^{\frac{1}{q}} \leq 4K \right\},$$

where K is defined in (1.1).

We subsequently recall standard notions in stochastic homogenization including the extended corrector $(\phi, \sigma) = ((\phi_i)_i, (\sigma_{ijk})_{ijk})$ and the homogenized field a_{hom}. Existence and uniqueness of the extended corrector will be discussed afterwards.

Definition 1.4 (Definition of the extended corrector (ϕ, σ)). *In the situation of Definition 1.1 and for given $\xi \in \mathbb{R}^d$, one calls the sublinear solution ϕ_ξ of $-\nabla \cdot a(\xi + \nabla \phi_\xi) = 0$ on $\mathbb{R}^d$ the corrector associated to ξ. Specifically for ξ being a canonical basis vector, one considers the corrector ϕ_i, $1 \leq i \leq d$, being a solution to*

$$-\nabla \cdot q_i = 0, \qquad q_i := a(e_i + \nabla \phi_i). \tag{1.4}$$

The vector q_i is called the ith component of the flux *and one introduces the* flux correction *σ_{ijk}, $1 \leq i, j, k \leq d$, as a vector-valued potential solving*

$$\nabla \cdot \sigma_i = q_i - \langle q_i \rangle, \qquad -\Delta \sigma_i = \nabla \times q_i := (\partial_j q_{ik} - \partial_k q_{ij})_{jk}. \tag{1.5}$$

Finally, one defines the homogenized field *a_{hom} via $a_{\mathrm{hom}} e_i := \langle q_i \rangle$.*

A question which typically arises in this context, is concerned with the so-called Liouville principle. For example, given a subquadratic solution u of $-\nabla \cdot a\nabla u = 0$ on $\mathbb{R}^d$, can one prove that $u = c + \xi \cdot x + \phi_\xi(x)$ for some $\xi \in \mathbb{R}^d$? For the present setting, the first author, Fehrman, and Otto [8] have shown that such a Liouville property does hold (cf. Remark 1.5). The interest in such a principle also lies in its close relation to Schauder estimates, which has been highlighted by Simon [24]. Moreover, Liouville properties have been established in many situations including stationary and ergodic degenerate systems [12], higher-order Liouville principles [16], and Liouville theorems for uniformly parabolic systems in a random setting supposing stationarity and ergodicity [7].

The existence of an extended corrector (ϕ, σ) as in Definition 1.4 directly follows from [8, Lemma 1], while its uniqueness is an immediate consequence of the sublinearity of (ϕ, σ) [8, Lemma 2] and a related Liouville principle [8, Theorem 1]. For the sake of completeness, we recall these results in Remark 1.5 below. Identity (1.6) shows that the extended corrector (ϕ, σ) is sublinear w.r.t. ρ. While (1.6) gives only a qualitative statement, we will prove a quantified version thereof in Corollary 2.4, which will serve as an important tool in the latter part of this paper.

Remark 1.5 (Properties of the extended corrector (ϕ, σ) [8, Lemmas 1,2, Theorem 1], [13, Proposition 4.1]). *Under the hypotheses of Definition 1.1, there exist a constant $C > 0$ and random tensor fields ϕ_i and σ_{ijk}, $1 \le i, j, k \le d$, satisfying (1.4)–(1.5) and the skew-symmetry $\sigma_{ijk} = -\sigma_{ikj}$, while the gradient fields are stationary, of vanishing expectation*

$$\langle \nabla \phi_i \rangle = \langle \nabla \sigma_{ijk} \rangle = 0,$$

and having bounded moments

$$\sum_{i=1}^{d} \langle \nabla \phi_i \cdot a \nabla \phi_i \rangle + \sum_{i=1}^{d} \left\langle |\nabla \phi_i|^{\frac{2q}{q+1}} \right\rangle^{\frac{q+1}{2q}} + \sum_{i,j,k=1}^{d} \left\langle |\nabla \sigma_{ijk}|^{\frac{2p}{p+1}} \right\rangle^{\frac{p+1}{2p}} \le CK$$

where K is the constant from Definition 1.1. In addition, (ϕ, σ) is sublinear in the sense

$$\lim_{\rho \to \infty} \max \left\{ \frac{1}{\rho} \left(\fint_{B_\rho} \left| \phi - \fint_{B_\rho} \phi \right|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{2p}}, \frac{1}{\rho} \left(\fint_{B_\rho} \left| \sigma - \fint_{B_\rho} \sigma \right|^{\frac{2q}{q-1}} \right)^{\frac{q-1}{2q}} \right\} = 0, \qquad (1.6)$$

and a.e. coefficient field a satisfies the following Liouville principle: Any solution $u \in H_a^1(\mathbb{R}^d)$ to $-\nabla \cdot a \nabla a = 0$ in $\mathbb{R}^d$ subject to

$$\lim_{R \to \infty} R^{-(1+\alpha)} \left(\fint_{B_R} |u|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{2p}} = 0$$

for some $\alpha \in (0, 1)$ admits the representation $u(x) = c + \xi \cdot x + \phi_\xi(x)$ for some $c \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$. Finally, the homogenized field a_{hom} is uniformly elliptic.

The following lemma is basically a consequence of the collected results in Remark 1.5 and [8, Theorem 2]. The claimed mean-value property in (1.8) is a direct consequence of (1.7) which is typically referred to as a large-scale $C^{1,\alpha}$ regularity estimate. The first large-scale regularity result for a uniformly elliptic, scalar equation was obtained by Marahrens and Otto [22], where the ergodicity of the random ensemble was encoded by means of a logarithmic Sobolev inequality. For elliptic systems with stationary and coercive coefficients, the first author and Otto [9] derived moment bounds on the corrector gradient by employing either a logarithmic Sobolev inequality or a spectral gap estimate. We also mention the large-scale regularity theory for scalar equations in a random environment developed by Armstrong and Smart [4]. A crucial ingredient of their approach is the assumption of a finite range of dependence for the symmetric coefficient field. More recently, large-scale regularity results have also been shown for the random conductance model by Armstrong and Dario [3]. They prove that the corresponding solutions on supercritical percolation clusters are close to harmonic functions on large scales which admit stretched exponential moments. For similar models subject to long-range correlations and decoupling inequalities, Sapozhnikov [23] generalized several results, such as heat kernel bounds and parabolic Harnack inequalities, which have already been known for the Bernoulli percolation.

Lemma 1.6 (Large-scale $C^{1,\alpha}$ regularity and a mean-value property for a-harmonic functions). *For any $\alpha \in (0, 1)$ and $K > 0$, there exist constants C_0, C_1, and C_2 such that for all positive radii $r < R$ and $p, q \in (1, \infty)$ satisfying $\frac{1}{p} + \frac{1}{q} \le \frac{2}{d}$ the following holds: If*

$$\left(\fint_{B_\rho} \mu^p \right)^{\frac{1}{p}} + \left(\fint_{B_\rho} \lambda^{-q} \right)^{\frac{1}{q}} \le 2K$$

and

$$\max\left\{\frac{1}{\rho}\Big(\fint_{B_\rho}\Big|\phi-\fint_{B_\rho}\phi\Big|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}}, \frac{1}{\rho}\Big(\fint_{B_\rho}\Big|\sigma-\fint_{B_\rho}\sigma\Big|^{\frac{2q}{q-1}}\Big)^{\frac{q-1}{2q}}\right\}\leq\frac{1}{C_0}$$

for all $\rho \in [r, R]$, *then any solution* $u \in H_a^1(B_R)$ *of* $-\nabla \cdot a\nabla u = 0$ *fulfills the excess-decay*

$$\mathrm{Exc}(r) \leq C_1\Big(\frac{r}{R}\Big)^{2\alpha}\mathrm{Exc}(R), \tag{1.7}$$

where the excess $\mathrm{Exc}(\rho)$ *is defined as*

$$\mathrm{Exc}(\rho) := \inf_{\xi \in \mathbb{R}^d}\fint_{B_\rho}\big|\nabla u - (\xi + \nabla\phi_\xi)\big|_a^2,$$

while ∇u *satisfies the mean-value property*

$$\fint_{B_r}|\nabla u|^2 \leq C_2\fint_{B_R}|\nabla u|^2. \tag{1.8}$$

We now introduce the *minimal radius* r_* which quantifies the minimal scale on which the (extended) corrector (ϕ, σ) grows only sublinearly. For technical reasons, we do not only demand $r_* \geq r_e$ but even $r_* \geq M_0 r_e$ for a specific constant $M_0 \geq 1$ detailed below.

Definition 1.7 (Minimal radius)**.** *In the situation of Definition 1.1, we define the* minimal radius *as the random variable* $r_* \geq M_0 r_e$ *given in the form*

$$r_* := \inf\left\{r \geq M_0 r_e \,\Big|\, \forall \rho > r : \max\left\{\frac{1}{\rho}\Big(\fint_{B_\rho}\Big|\phi-\fint_{B_\rho}\phi\Big|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}}, \right.\right.$$
$$\left.\left.\frac{1}{\rho}\Big(\fint_{B_\rho}\Big|\sigma-\fint_{B_\rho}\sigma\Big|^{\frac{2q}{q-1}}\Big)^{\frac{q-1}{2q}}\right\}\leq\frac{1}{C_0}\right\},$$

where C_0 *is the constant from Lemma 1.6, while* $M_0 \geq 1$ *is defined in* (2.30)*.*

Concerning the possible degeneracy and unboundedness of the coefficients $a(x)$, we mention that it is obviously not possible to perform estimates like $c|v|^2 \leq v \cdot av \leq C|v|^2$ for $v \in \mathbb{R}^d$ with uniform constants $C \geq c > 0$. It is, therefore, advantageous to introduce a separate notation for such quadratic forms and also for matrix products $a^{\frac{1}{2}}Ma^{\frac{1}{2}}$ with some $M \in \mathbb{R}^{d \times d}$.

Notation 1.8. *For* $a \in \mathbb{R}^{d \times d}$, $M \in \mathbb{R}^{d \times d}$, *and* $v \in \mathbb{R}^d$, *we set* $|v|_a^2 := v \cdot av$ *and* $|M|_a := \big|a^{\frac{1}{2}}Ma^{\frac{1}{2}}\big|$ *where* $|\cdot|$ *denotes the spectral norm on* $\mathbb{R}^{d \times d}$.

The following spectral gap estimate (1.10) is our main stochastic assumption on the underlying random environment. A very similar condition involving a coarsening partition $\{D\}$ of $\mathbb{R}^d$ was used in [17]. Alternatively, one can employ multiscale functional inequalities to describe the random ensemble (see Remark 1.10). A detailed exposition of these ideas is given by Duerinckx and Gloria in [15]. We point out that the more elementary spectral gap condition (1.10) is sufficient for the present study, where we deduce stretched exponential moments for r_* with a typically small exponent ε. Nevertheless, we believe that stronger stretched exponential bounds on r_* in the spirit of Gloria, Neukamm, and Otto [19] could be obtained also in the degenerate setting by relying on multiscale functional inequalities.

Definition 1.9 (Spectral gap inequality)**.** *Let the hypotheses of Definition 1.1 hold and assume that a partition $\{D\}$ of $\mathbb{R}^d$ and an exponent $\beta \in [0, 1)$ exist such that*

$$\operatorname{diam} D \leq (\operatorname{dist} D + 1)^\beta \leq C(d) \operatorname{inn diam} D, \tag{1.9}$$

where $\operatorname{inn diam} D := 2 \sup \{r \geq 0 \mid \exists\, x \in D : B_r(x) \subset D\}$ *denotes the* inner diameter *of $D \subset \mathbb{R}^d$.*

We say that a random field a satisfies the spectral gap *(or* Poincaré*) inequality, if there exists a constant $\kappa \in (0, 1]$ such that*

$$\left\langle \big(X(a) - \langle X(a)\rangle\big)^2 \right\rangle \leq \frac{1}{\kappa} \left\langle \sum_D \Big(\int_D \Big| \frac{\partial X(a)}{\partial a} \Big|_a \Big)^2 \right\rangle \tag{1.10}$$

for all $\sigma(a)$-measurable random variables $X(a)$, where we recall that

$$\int_D \Big| \frac{\partial X(a)}{\partial a} \Big|_a = \sup_{\|b\|_{L^\infty(D)}=1} \operatorname{ess} \int_D b : a^{\frac{1}{2}} \frac{\partial X(a)}{\partial a} a^{\frac{1}{2}} = \sup_{\|b\|_{L^\infty(D)}=1} \operatorname{ess} \lim_{t \to 0} \frac{X\big(a + t a^{\frac{1}{2}} b a^{\frac{1}{2}}\big) - X(a)}{t}.$$

An upgraded version of the standard spectral gap estimate (1.10) to higher order moments will be provided in Lemma 2.1, which will serve as a useful tool in various situations subsequently in this paper.

Remark 1.10. *Instead of* (1.10)*, one can also use a multiscale spectral gap inequality*

$$\left\langle \big(X(a) - \langle X(a)\rangle\big)^2 \right\rangle \leq \left\langle \int_0^\infty \int_{\mathbb{R}^d} \Big(\int_{B_\ell(x)} \Big| \frac{\partial X(a)}{\partial a} \Big| \Big)^2 \mathrm{d}x \, \frac{\pi(\ell)}{(\ell+1)^d} \mathrm{d}\ell \right\rangle \tag{1.11}$$

or (multiscale) logarithmic Sobolev inequalities for quantifying the ergodicity of the ensemble $\langle \cdot \rangle$. The weight function $\pi : [0, \infty) \to [0, \infty)$ in (1.11) is generally assumed to be integrable. Integrable correlations $\operatorname{Cov}(a(x); a(0))$ can be modeled with weights decaying like $\pi(\ell) \sim (\ell+1)^{-1-\alpha}$ for $\alpha > d$. We refer to the work of Duerinckx and Gloria [15] for further details.

The connection between the ellipticity radius r_e and the stochastic integrability of the underlying coefficient field a satisfying the spectral gap inequality (1.10) is clarified in the following lemma. We shall basically prove that stretched exponential moment bounds on averages of $|a|^p$ and $|a^{-1}|^q$ carry over to r_e.

Lemma 1.11 (Stretched exponential moments for r_e)**.** *Assume that an ensemble of coefficient fields $a \in \Omega$ is given according to Definition 1.1, which satisfies the spectral gap estimate (1.10) along with $\beta \in [0, 1)$ subject to (1.9). In case that $\int_{B_1} \lambda^{-q}$ and $\int_{B_1} \mu^p$ possess stretched exponential moments*

$$\max \left\{ \Big\langle \exp\Big(\frac{1}{C} \Big(\int_{B_1} \lambda^{-q} \Big)^\alpha \Big) \Big\rangle, \Big\langle \exp\Big(\frac{1}{C} \Big(\int_{B_1} \mu^p \Big)^\alpha \Big) \Big\rangle \right\} < 2 \tag{1.12}$$

for some constants $\alpha > 0$ and $C > 0$, then the ellipticity radius r_e from Definition 1.3 is subject to

$$\Big\langle \exp\Big(\frac{1}{C} r_e^{\frac{\alpha}{\alpha+1} \frac{d}{2}(1-\beta)} \Big) \Big\rangle < 2 \tag{1.13}$$

with the same parameter $\alpha > 0$ but a possibly different constant $C > 0$.

We are now in a position to show that the minimal radius r_* introduced in Definition 1.7 possesses stretched exponential moments by adapting the line of arguments from Gloria, Neukamm, and Otto [17]. In contrast to the final version [19] of the aforementioned preprint, optimal stochastic integrability is not achieved in [17]. The main tool which allows the authors to improve the stochastic integrability of r_* is a modified extended corrector (ϕ_T, σ_T) living on the length scale $\sqrt{T}$ and arising from a "massive approximation". As we are currently not able to adapt this approach to our situation, we resort to the more elementary approach in the preprint [17] where the same bound on r_* as in (1.14) is obtained.

Theorem 1.12 (Stretched exponential moments for r_*)**.** *Suppose that the hypotheses of Definition 1.1 on the ensemble of coefficient fields $a \in \Omega$ satisfying the spectral gap inequality* (1.10) *hold together with $\beta \in [0,1)$ subject to* (1.9). *Moreover, assume that $\int_{B_1} \lambda^{-q}$ and $\int_{B_1} \mu^p$ allow for the stretched exponential moments in* (1.12) *with $\alpha := \frac{\varepsilon}{1-\varepsilon}$ where $\varepsilon \in (0,1)$ is the hole-filling exponent from Proposition 1.13.*

Then, the minimal radius r_* as defined in Definition 1.7 fulfills

$$\left\langle \exp\left(\frac{1}{C} r_*^{\varepsilon \frac{d}{2}(1-\beta)}\right)\right\rangle < 2 \tag{1.14}$$

for a sufficiently large constant $C > 0$.

An essential part of the proof of Theorem 1.12 is concerned with the sensitivity analysis quantifying the dependence of $\nabla(\phi, \sigma)$ on the coefficient field a. At the end, we need to control averages of $\nabla(\phi, \sigma)$ on balls around the origin, but we shall give a slightly more general statement below. As above, the (ϕ_T, σ_T)-regularization currently prevents us from proceeding as in [19]. However, we could prove an analogue of the statement in the intermediate version [18], but as Theorem 1.12 is already posed in the language of [17], it suffices to generalize the sensitivity result in [17] to our setting.

Proposition 1.13 (Sensitivity estimate for average integrals)**.** *Let the assumptions of Definition 1.1 on the ensemble of coefficient fields $a \in \Omega$ be in place, and let a partition $\{D\}$ of $\mathbb{R}^d$ and $\beta \in [0,1)$ be given according to* (1.9). *Consider the linear functional*

$$F\psi = \int g \cdot \psi$$

acting on vector fields $\psi : \mathbb{R}^d \to \mathbb{R}^d$, where $g : \mathbb{R}^d \to \mathbb{R}^d$ is supported in B_r for some radius $r \geq r_e$.

Then, there exist a hole-filling exponent $\varepsilon = \varepsilon(d, K) \in (0,1)$ and a constant $C = C(d, K) > 1$ such that for any g as above satisfying

$$\left(\fint_{B_r} |g|^2_{a^{-1}}\right)^{\frac{1}{2}} \leq r^{-d}, \tag{1.15}$$

the following bound on the functional derivative of F holds:

$$\sum_D \left(\int_D \left|\frac{\partial F \nabla \phi}{\partial a}\right|_a\right)^2 + \sum_D \left(\int_D \left|\frac{\partial F \nabla \sigma_{jk}}{\partial a}\right|_a\right)^2 \leq C\left(\frac{(r + r_*)^{1-\varepsilon(1-\beta)}}{r}\right)^d. \tag{1.16}$$

We now employ the above results on the existence of stretched exponential moments for the minimal radius r_* (cf. Theorem 1.12) and the sensitivity estimate for (extended) corrector gradients (cf. Proposition 1.13) to derive quantitative estimates on the decay of the corrector gradient $\nabla(\phi, \sigma)$ and the growth of the corrector itself. Due to the relatively weak (ε-dependent) stretched exponential moments available for r_* (compared to [19] and its preprint [18]), the subsequent results also involve a dependence on ε.

Theorem 1.14 (Decay of the corrector gradient and growth of the corrector). *Assume that the ensemble of coefficient fields $a \in \Omega$ fulfills the assumptions of Definition 1.1 and satisfies the spectral gap estimate (1.10) along with $\beta \in [0, 1)$ subject to (1.9). Let $\varepsilon \in (0, 1)$ denote the constant from Proposition 1.13.*

Then, there exists a stationary random field $\mathcal{C}(x)$ with stretched exponential moments

$$\left\langle \exp\left(\frac{1}{C} \mathcal{C}^{\varepsilon(1-\beta)}\right)\right\rangle < 2 \tag{1.17}$$

for a sufficiently large constant $C > 0$ such that the following assertions hold:

1. *If $m : \mathbb{R}^d \to \mathbb{R}^d$ is bounded and supported in B_r, $r \geq 1$, $\fint_{B_r} |m|^2 = 1$, and assumption (1.12) holds true with $\alpha := \frac{\varepsilon}{1-\varepsilon}$, then, for all $x \in \mathbb{R}^d$,*

$$\left| \fint_{B_r} \nabla(\phi, \sigma)(x + y) \cdot m(y)\, dy \right| \leq \mathcal{C}(x) r^{-\frac{\varepsilon}{2} d(1-\beta)}.$$

2. *If $\varepsilon \in \left(0, \frac{\alpha}{\alpha+1} - \frac{1}{\min\{p,q\}}\right]$ and (1.12) holds for some $\alpha > \frac{1}{\min\{p,q\}-1}$, then the correctors (ϕ, σ) fulfill*

$$\left(\fint_{B_1(x)} |\phi|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}} + \left(\fint_{B_1(x)} |\sigma|^{\frac{2q}{q-1}}\right)^{\frac{q-1}{2q}} \lesssim \left| \fint_{B_1} (\phi, \sigma) \right| + \mathcal{C}(x)\mu(|x|)$$

together with

$$\mu(r) := \begin{cases} 1, & 0 \leq \beta < 1 - \frac{2}{\varepsilon d}, \\ \log(2 + r), & \beta = 1 - \frac{2}{\varepsilon d}, \\ r^{\frac{\varepsilon d}{2}\left(\frac{2}{\varepsilon d} - 1 + \beta\right)}, & \beta > 1 - \frac{2}{\varepsilon d}. \end{cases} \tag{1.18}$$

Our last result gives a quantitative estimate for a two-scale expansion. It is mainly a consequence of Theorem 1.14 on the growth of the corrector and the stochastic integrability of the random field $\mathcal{C}$ in (1.17). We formulate the statement in the same spirit as in [20]; in particular, we employ the same averaging procedure over small balls for reasons of generality (even though this might not be necessary in many cases). But in contrast to [20], we again encounter the small parameter ε (coming from Theorem 1.14), and we also get an additional term on the right hand side of (1.19) (which can be (formally) absorbed in the other term on the right hand side in the limit $q \to \infty$).

Corollary 1.15 (Quantitative two-scale expansion). *Suppose that the ensemble of coefficient fields $a \in \Omega$ meets the requirements of Definition 1.1 and fulfills the spectral gap estimate (1.10). Moreover, let (1.9) hold with $\beta \in [0, 1)$ and suppose that assumption (1.12) with $\alpha > \frac{1}{\min\{p,q\}-1}$ is in place, while the hole-filling exponent from Proposition 1.13 is restricted to $\varepsilon \in \left(0, \frac{\alpha}{\alpha+1} - \frac{1}{\min\{p,q\}}\right]$. For $R \geq r_e$ and $\delta > 0$, let $g \in W^{1, \frac{2q}{q-1}}(\mathbb{R}^d)$ be supported in B_R, and let u_δ and u_{hom} denote the solutions to*

$$-\nabla \cdot a\left(\tfrac{\cdot}{\delta}\right)\nabla u_\delta = \nabla \cdot g, \qquad -\nabla \cdot a_{\text{hom}} \nabla u_{\text{hom}} = \nabla \cdot g,$$

while the error z_δ in the two-scale expansion and the small-scale average $u_{\text{hom},\delta}$ are defined by

$$z_\delta := u_\delta - \left(u_{\text{hom},\delta} + \delta \phi_i\left(\tfrac{\cdot}{\delta}\right) \partial_i u_{\text{hom},\delta}\right), \qquad u_{\text{hom},\delta}(x) := \fint_{B_\delta(x)} u_{\text{hom}}.$$

We then have

$$\left(\int |\nabla z_\delta|_a^2 \right)^{\frac{1}{2}} \lesssim \delta^{1+\frac{d}{2q}} \left(\int |\nabla g|^{\frac{2q}{q-1}} \right)^{\frac{q-1}{2q}} + \mathcal{C}_{\delta,g} \delta \mu(\delta^{-1}) \left(\int \mu(|x|)^2 |\nabla g|^2 \right)^{\frac{1}{2}} \quad (1.19)$$

where $\mu(r)$ *is defined in* (1.18) *and where the random field* $\mathcal{C}_{\delta,g}$ *satisfies*

$$\left\langle \exp \left(\frac{1}{C} \mathcal{C}_{\delta,g}^{\left(1+\frac{\alpha+1}{\alpha}\frac{\varepsilon}{\min\{p,q\}}\right)^{-1}\varepsilon(1-\beta)} \right) \right\rangle < 2$$

for a sufficiently large constant $C > 0$ *independent of* δ, g, p, *and* q.

2 Large-scale $C^{1,\alpha}$ regularity quantified by the minimal radius r_*

2.1 Proof of Lemma 1.6: A mean-value property for a-harmonic functions

Proof of Lemma 1.6. We divide the proof into two steps. First, we derive a non-degeneracy property for $\xi + \nabla\phi_\xi$ with $\xi \in \mathbb{R}^d$, while the desired mean-value property is proven as a consequence in the second step.

Step 1. Excess decay and non-degeneracy. Under the hypotheses of the lemma, we may apply [8, Theorem 2] to establish (1.7). Note that we subsequently use (1.7) with the choice $\alpha := \frac{1}{2}$. Following [19], we shall first prove a non-degeneracy condition for the correctors ϕ_ξ in the sense

$$c|\xi|^2 \le \fint_{B_r} |\xi + \nabla\phi_\xi|_a^2 \le C|\xi|^2 \quad (2.1)$$

for all $r \ge r_*$ and $\xi \in \mathbb{R}^d$ where $0 < c < C$ are independent of r and ξ. For the lower bound, we first recall the elementary bound

$$\left(\fint_{B_r} |\xi + \nabla\phi_\xi|^{\frac{2q}{q+1}} \right)^{\frac{q+1}{q}} \le \left(\fint_{B_r} \lambda^{-q} \right)^{\frac{1}{q}} \fint_{B_r} |\xi + \nabla\phi_\xi|_a^2.$$

Together with Poincaré's inequality, we derive

$$\left(\fint_{B_r} |\xi + \nabla\phi_\xi|_a^2 \right)^{\frac{1}{2}} \gtrsim \left(\fint_{B_r} |\xi + \nabla\phi_\xi|^{\frac{2q}{q+1}} \right)^{\frac{q+1}{2q}} \gtrsim \frac{1}{r} \left(\fint_{B_r} |\xi \cdot x + \phi_\xi - \fint_{B_r} \phi_\xi|^{\frac{2q}{q+1}} \right)^{\frac{q+1}{2q}}.$$

The triangle inequality, Jensen's inequality, and the sublinear growth of the corrector now yield

$$\left(\fint_{B_r} |\xi + \nabla\phi_\xi|_a^2 \right)^{\frac{1}{2}} \gtrsim \frac{1}{r} \left(\fint_{B_r} |\xi \cdot x|^{\frac{2q}{q+1}} \right)^{\frac{q+1}{2q}} - \frac{1}{r} \left(\fint_{B_r} \left| \phi_\xi - \fint_{B_r} \phi_\xi \right|^{\frac{2q}{q+1}} \right)^{\frac{q+1}{2q}}$$

$$\ge \frac{|\xi|}{r} \left(\fint_{B_r} |x|^{\frac{2q}{q+1}} \right)^{\frac{q+1}{2q}} - \frac{|\xi|}{r} \left(\fint_{B_r} \left| \phi - \fint_{B_r} \phi \right|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{2p}} \gtrsim |\xi| - \frac{1}{C_0}|\xi|$$

taking the scaling of ϕ_ξ and the notation $\phi = (\phi_i)_i$ into account. Choosing the constant $C_0 > 0$ sufficiently large, one arrives at the desired lower bound in (2.1). Similarly, the upper bound is a consequence of the Caccioppoli estimate (carried out e.g. in [8, Lemma 3])

$$\int_{B_r} |\xi + \nabla\phi_\xi|_a^2 \le \frac{4}{r^2} \left(\int_{B_{2r}} \mu^p \right)^{\frac{1}{p}} \left(\int_{B_{2r}} \left| \xi \cdot x + \phi_\xi - \fint_{B_{2r}} \phi_\xi \right|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{p}}.$$

By the same reasoning as above, we obtain

$$\Big(\fint_{B_r} |\xi + \nabla\phi_\xi|_a^2\Big)^{\frac{1}{2}} \lesssim \frac{|\xi|}{r}\Big(\fint_{B_{2r}} |x|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}} + \frac{|\xi|}{r}\Big(\fint_{B_{2r}} \Big|\phi - \fint_{B_{2r}}\phi\Big|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}} \lesssim |\xi| + \frac{1}{C_0}|\xi|.$$

The claimed bound (2.1) now follows.

Step 2. Mean-value property. The ideas of [19] also apply to our situation, but we present the main steps for completeness. The lower bound in (2.1) ensures for any $\rho \in [r_*, R]$ the existence of a unique $\xi_\rho \in \mathbb{R}^d$ such that

$$\mathrm{Exc}(\rho) = \fint_{B_\rho} \big|\nabla u - (\xi_\rho + \nabla\phi_{\xi_\rho})\big|_a^2. \tag{2.2}$$

For radii $\rho, \rho' \in [r_*, R]$ satisfying $0 < \rho' - \rho \le \rho$, we deduce by virtue of (2.1), the linearity of $\xi \mapsto \phi_\xi$, and the triangle inequality

$$|\xi_\rho - \xi_{\rho'}|^2 \lesssim \fint_{B_\rho} \big|\xi_\rho - \xi_{\rho'} + \nabla\phi_{\xi_\rho - \xi_{\rho'}}\big|_a^2 \lesssim \fint_{B_\rho} \big|\nabla u - (\xi_\rho + \nabla\phi_{\xi_\rho})\big|_a^2 + \fint_{B_\rho} \big|\nabla u - (\xi_{\rho'} + \nabla\phi_{\xi_{\rho'}})\big|_a^2.$$

Due to the minimality property (2.2) of ξ_ρ and $\rho < \rho' \le 2\rho$, this entails

$$|\xi_\rho - \xi_{\rho'}|^2 \lesssim \fint_{B_\rho} \big|\nabla u - (\xi_{\rho'} + \nabla\phi_{\xi_{\rho'}})\big|_a^2 \lesssim \mathrm{Exc}(\rho'). \tag{2.3}$$

For arbitrary $R \ge r \ge r_*$, we let $N \in \mathbb{N}$ be the integer such that $2^{-(N+1)}R < r \le 2^{-N}R$, which allows us to use (2.3) and (1.7) (with $\alpha = \frac{1}{2}$) to estimate

$$|\xi_r - \xi_R|^2 \le \Big(\sum_{n=0}^{N} |\xi_{2^{-(n+1)}R} - \xi_{2^{-n}R}|\Big)^2 \lesssim \Big(\sum_{n=0}^{N} 2^{-\frac{n}{2}}\sqrt{\mathrm{Exc}(R)}\Big)^2 \lesssim \mathrm{Exc}(R). \tag{2.4}$$

By means of (2.2), (2.1), and (2.4), we thus get

$$\fint_{B_r} |\nabla u|_a^2 \lesssim \mathrm{Exc}(r) + |\xi_r|^2 \le \mathrm{Exc}(r) + \mathrm{Exc}(R) + |\xi_R|^2.$$

Moreover, (1.7) and the definition of the excess ensure $\mathrm{Exc}(r) \lesssim \mathrm{Exc}(R) \le \fint_{B_R} |\nabla u|_a^2$, while

$$|\xi_R|^2 \lesssim \fint_{B_R} \big|\xi_R + \nabla\phi_{\xi_R}\big|_a^2 \lesssim \mathrm{Exc}(R) + \fint_{B_R} |\nabla u|_a^2$$

is a result of (2.1) and (2.2). This concludes the argument. $\qquad\square$

2.2   Proof of Lemma 1.11: Stretched exponential moments for r_e

Lemma 2.1 (Pth power spectral gap estimate). *Let the ensemble of coefficient fields $a \in \Omega$ satisfy the assumptions in Definition 1.1 and the spectral gap estimate* (1.10) *with an arbitrary partition $\{D\}$ of $\mathbb{R}^d$. Then, there exists a constant $C > 0$ such that*

$$\big\langle (\zeta - \langle\zeta\rangle)^{2P}\big\rangle^{\frac{1}{P}} \le \frac{CP^2}{\kappa}\Big\langle\Big(\sum_D \Big(\int_D \big|\frac{\partial\zeta}{\partial a}\big|_a\Big)^2\Big)^P\Big\rangle^{\frac{1}{P}} \tag{2.5}$$

for any random variable ζ and all $P \in \mathbb{N}$, $P \ge 2$.

Proof. The arguments are basically the same as in [17] but adapted to our degenerate setting. Applying the spectral gap estimate (1.10) to ζ^P, we first derive

$$\langle \zeta^{2P} \rangle \leq \langle \zeta^P \rangle^2 + \frac{1}{\kappa} \Big\langle \sum_D \Big(\int_D \big| \frac{\partial \zeta^P}{\partial a} \big|_a \Big)^2 \Big\rangle.$$

Elementary calculus guarantees that

$$\sum_D \Big(\int_D \big| \frac{\partial \zeta^P}{\partial a} \big|_a \Big)^2 = P^2 \zeta^{2(P-1)} \sum_D \Big(\int_D \big| \frac{\partial \zeta}{\partial a} \big|_a \Big)^2,$$

while Hölder's inequality on the level of the probability measure $\langle \cdot \rangle$ yields

$$\Big\langle \sum_D \Big(\int_D \big| \frac{\partial \zeta^P}{\partial a} \big|_a \Big)^2 \Big\rangle \leq P^2 \langle \zeta^{2P} \rangle^{1-\frac{1}{P}} \Big\langle \Big(\sum_D \Big(\int_D \big| \frac{\partial \zeta}{\partial a} \big|_a \Big)^2 \Big)^P \Big\rangle^{\frac{1}{P}}.$$

Young's inequality now allows to get rid of $\langle \zeta^{2P} \rangle$ on the right hand side and to derive

$$\langle \zeta^{2P} \rangle \leq C \langle \zeta^P \rangle^2 + \Big(\frac{CP^2}{\kappa} \Big)^P \Big\langle \Big(\sum_D \Big(\int_D \big| \frac{\partial \zeta}{\partial a} \big|_a \Big)^2 \Big)^P \Big\rangle \tag{2.6}$$

with some constant $C > 0$. We now argue how to replace $\langle \zeta^P \rangle^2$ by $\langle \zeta^2 \rangle^P$ on the right hand side. To this end, one writes $\zeta^P = \zeta^{P\frac{P-2}{P-1}} \zeta^{P\frac{1}{P-1}}$ and applies Hölder's inequality with exponents $2\frac{P-1}{P-2}$ and $2\frac{P-1}{P}$ followed by Young's inequality leading to

$$\langle \zeta^P \rangle^2 \leq \langle \zeta^{2P} \rangle^{\frac{P-2}{P-1}} \langle \zeta^2 \rangle^{\frac{P}{P-1}} \leq \frac{1}{C} \langle \zeta^{2P} \rangle + C^{P-2} \langle \zeta^2 \rangle^P$$

with another constant $C > 0$. Using again the original spectral gap inequality and noting that it suffices to prove (2.5) for the case $\langle \zeta \rangle = 0$, we further obtain

$$\langle \zeta^2 \rangle^P \leq \frac{1}{\kappa^P} \Big\langle \sum_D \Big(\int_D \big| \frac{\partial \zeta}{\partial a} \big|_a \Big)^2 \Big\rangle^P.$$

The proof is finished taking Jensen's inequality $\langle (\cdot) \rangle^P \leq \langle (\cdot)^P \rangle$ into account. □

Proof of Lemma 1.11. We divide the proof into two steps.

Step 1. Exponential concentration for $\fint_{B_R} \mu^p$ and $\fint_{B_R} \lambda^{-q}$. We start by recalling the upgraded spectral gap estimate from (2.5) and by applying it to $X(a) := \fint_{B_R} \mu^p$ for some arbitrary $R \geq 1$. The same arguments are also applicable to $\fint_{B_R} \lambda^{-q}$. This yields

$$\Big\langle \big(X(a) - \langle X(a) \rangle \big)^{2r} \Big\rangle^{\frac{1}{2r}} \leq Cr \Big\langle \Big(\sum_D \Big(\int_D \big| \frac{\partial X(a)}{\partial a} \big|_a \Big)^2 \Big)^r \Big\rangle^{\frac{1}{2r}} \tag{2.7}$$

for all $r \in \mathbb{N}, r \geq 2$. In a similar setting, exponential concentration and stretched exponential moments were shown in [15, Proposition 1.10] for arbitrary random variables $X(a)$ by assuming a deterministic bound of the form

$$\sum_D \Big(\int_D \big| \frac{\partial X(a)}{\partial a} \big|_a \Big)^2 \leq \overline{C}$$

(albeit employing a multiscale spectral gap inequality). Such a deterministic bound cannot be expected in our situation, instead we shall prove that

$$\left\langle \left(\sum_D \left(\fint_D \left| \frac{\partial X(a)}{\partial a} \right|_a \right)^2 \right)^r \right\rangle^{\frac{1}{2r}} \lesssim r^{\frac{1}{\alpha}} R^{-\frac{d}{2}(1-\beta)} \tag{2.8}$$

holds true where $\lesssim$ means $\leq$ up to the prescribed parameters d, p, q, and K. To this end, we calculate

$$\sum_D \left(\fint_D \left| \frac{\partial X(a)}{\partial a} \right|_a \right)^2 = \sum_D \left(\operatorname*{sup\,ess}_{\|b\|_{L^\infty(D)}=1} \fint_D b : a^{\frac{1}{2}} \frac{\partial X(a)}{\partial a} a^{\frac{1}{2}} \right)^2$$

$$= \sum_D \left(\operatorname*{sup\,ess}_{\|b\|_{L^\infty(D)}=1} \lim_{t \to 0} \frac{X\left(a + ta^{\frac{1}{2}}ba^{\frac{1}{2}}\right) - X(a)}{t} \right)^2$$

and

$$\frac{X\left(a + ta^{\frac{1}{2}}ba^{\frac{1}{2}}\right) - X(a)}{t} = \frac{1}{t} \fint_{B_R} \left(\left| a + ta^{\frac{1}{2}}ba^{\frac{1}{2}} \right|^p - |a|^p \right)$$

$$= p \fint_{B_R} \left((1-\theta)|a| + \theta \left| a + ta^{\frac{1}{2}}ba^{\frac{1}{2}} \right| \right)^{p-1} \left| a + \vartheta ta^{\frac{1}{2}}ba^{\frac{1}{2}} \right|' : a^{\frac{1}{2}}ba^{\frac{1}{2}}$$

$$\lesssim \fint_{B_R} |a|^{p-1} \left(1 - \theta + \theta |I + tb| \right)^{p-1} \left| a^{\frac{1}{2}}ba^{\frac{1}{2}} \right|$$

$$\lesssim R^{-d} \int_{B_R \cap D} \mu^p$$

where we use the identity $|A^2| = |A|^2$ for the spectral norm of any symmetric matrix $A \in \mathbb{R}^{d \times d}$ and the uniform boundedness $\left| |A|' : B \right| \lesssim |B|$ of the derivative of the spectral norm for any $A, B \in \mathbb{R}^{d \times d}$, while we assume w.l.o.g. that $\frac{1}{t} \left(X\left(a + ta^{\frac{1}{2}}ba^{\frac{1}{2}}\right) - X(a) \right)$ is positive for $t > 0$ sufficiently small. Note that we further applied the elementary mean value theorem with some $\theta, \vartheta \in [0, 1]$, the boundedness of b, and the fact that b vanishes outside of D. Moreover, every instance of a and b inside an integral refers to $a(x)$ and $b(x)$, respectively. As a consequence,

$$\left\langle \left(\sum_D \left(\fint_D \left| \frac{\partial X(a)}{\partial a} \right|_a \right)^2 \right)^r \right\rangle^{\frac{1}{2r}} \lesssim R^{-d} \left\langle \left(\sum_{D \cap B_R \neq \emptyset} \left(\int_{B_R \cap D} \mu^p \right)^2 \right)^r \right\rangle^{\frac{1}{2r}}.$$

By recalling (1.9), we notice that the number of subdomains D obeying $D \cap B_R \neq \emptyset$ equals (up to fixed constants)

$$\int_0^R \left(\frac{l}{(l+1)^\beta} \right)^{d-1} \frac{dl}{(l+1)^\beta} \lesssim \int_0^R l^{(1-\beta)(d-1)-\beta} dl \lesssim R^{d(1-\beta)}.$$

Jensen's inequality then leads to

$$\left\langle \left(\sum_D \left(\fint_D \left| \frac{\partial X(a)}{\partial a} \right|_a \right)^2 \right)^r \right\rangle^{\frac{1}{2r}} \lesssim R^{-\frac{d}{2}(1+\beta)} \left\langle R^{-d(1-\beta)} \sum_{D \cap B_R \neq \emptyset} \left(\int_D \mu^p \right)^{2r} \right\rangle^{\frac{1}{2r}}.$$

Likewise, any subdomain D can be covered by at most $|D| \lesssim R^{\beta d}$ unit balls $B_1(x_k)$ with appropriate $x_k \in D$, hence, applying Jensen's inequality once more results in

$$\left\langle \left(\sum_D \left(\fint_D \left| \frac{\partial X(a)}{\partial a} \right|_a \right)^2 \right)^r \right\rangle^{\frac{1}{2r}} \lesssim R^{-\frac{d}{2}(1-\beta)} \left\langle R^{-d(1-\beta)} \sum_{D \cap B_R \neq \emptyset} R^{-\beta d} \sum_k \left(\int_{B_1(x_k)} \mu^p \right)^{2r} \right\rangle^{\frac{1}{2r}}.$$

Pulling the expectation inside and using the stationarity of the underlying ensemble, we deduce

$$\Big\langle \Big(\sum_D \Big(\int_D \Big| \frac{\partial X(a)}{\partial a} \Big|_a \Big)^2 \Big)^r \Big\rangle^{\frac{1}{2r}} \lesssim R^{-\frac{d}{2}(1-\beta)} \Big\langle \Big(\int_{B_1} \mu^p \Big)^{2r} \Big\rangle^{\frac{1}{2r}}.$$

Owing to (1.12) and Lemma A.1, we know that $\big\langle (\int_{B_1} \mu^p)^{2r} \big\rangle^{\frac{1}{2r}} \lesssim r^{\frac{1}{\alpha}}$, which gives rise to (2.8). Together with (2.7), this results in

$$\Big\langle \big(X(a) - \langle X(a) \rangle \big)^{2r} \Big\rangle^{\frac{1}{2r}} \lesssim r^{\frac{\alpha+1}{\alpha}} R^{-\frac{d}{2}(1-\beta)}.$$

An elementary argument shows that $\big\langle \big(X(a) - \langle X(a) \rangle \big)^{\frac{\alpha}{\alpha+1}r} \big\rangle^{\frac{1}{r}} \lesssim r R^{-\frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)}$, which by Lemma A.1 entails

$$\Big\langle \exp \Big(\frac{1}{C} R^{\frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)} \big(X(a) - \langle X(a) \rangle \big)^{\frac{\alpha}{\alpha+1}} \Big) \Big\rangle < 2$$

for a sufficiently large constant $C > 0$ depending only on fixed model parameters. Therefore,

$$\Big\langle I\Big(\fint_{B_R} \mu^p - \langle \mu^p \rangle > \delta \Big) \Big\rangle \lesssim \exp \Big(-\frac{1}{C} \delta^{\frac{\alpha}{\alpha+1}} R^{\frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)} \Big). \tag{2.9}$$

Step 2. Stretched exponential moments for r_e. For any $r_0 > 1$ we now estimate the probability of the event $r_e > r_0$ as follows:

$$\big\langle I(r_e > r_0) \big\rangle \leq \Big\langle I\Big(\exists r > r_0 \ : \ \fint_{B_r} \mu^p > (2K)^p \ \vee \ \fint_{B_r} \lambda^{-q} > (2K)^q \Big) \Big\rangle$$

$$\leq \Big\langle I\Big(\exists n \geq n_0 \ : \ \fint_{B_{b^n}} \mu^p > \frac{(2K)^p}{b^d} \ \vee \ \fint_{B_{b^n}} \lambda^{-q} > \frac{(2K)^q}{b^d} \Big) \Big\rangle$$

where $b = b(d, p, q, K) \in (1, 2)$ is a constant specified below and $n_0, n \in \mathbb{N}$ satisfy $b^{n_0-1} < r_0 \leq b^{n_0}$ and $b^{n-1} < r \leq b^n$. Notice that $\fint_{B_{b^n}} \mu^p > (\frac{r}{b^n})^d (2K)^p > b^{-d}(2K)^p$ according to the assumption in the first line and that an analogous estimate holds for λ^{-q}. The previous estimate is continued via

$$\big\langle I(r_e > r_0) \big\rangle \leq \sum_{n=n_0}^{\infty} \Big\langle I\Big(\fint_{B_{b^n}} \mu^p > \frac{(2K)^p}{b^d} \Big) \Big\rangle + \sum_{n=n_0}^{\infty} \Big\langle I\Big(\fint_{B_{b^n}} \lambda^{-q} > \frac{(2K)^q}{b^d} \Big) \Big\rangle.$$

We are hence in a position to employ (2.9) after choosing $b \in (1, 2)$ sufficiently close to 1 in order to guarantee that the lower bounds inside the indicator functions subsequently stay positive. Besides, we only provide the argument for the term involving μ, while the same reasoning also applies to the other term. This yields

$$\sum_{n=n_0}^{\infty} \Big\langle I\Big(\fint_{B_{b^n}} \mu^p > \frac{(2K)^p}{b^d} \Big) \Big\rangle \leq \sum_{n=n_0}^{\infty} \Big\langle I\Big(\fint_{B_{b^n}} \mu^p - \langle \mu^p \rangle > \frac{(2K)^p}{b^d} - K^p \Big) \Big\rangle$$

$$\leq \sum_{n=n_0}^{\infty} \exp \Big(-\frac{1}{C} \Big(\frac{(2K)^p}{b^d} - K^p \Big)^{\frac{\alpha}{\alpha+1}} (b^n)^{\frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)} \Big)$$

$$\leq C_0 \exp \Big(-c_1 r_0^{\frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)} \Big)$$

together with (large) constants $\overline{C}, C_0 \geq 1$, and a (small) constant $c_1 \in (0,1)$. For the third inequality above, we pull out the factor $\exp\left(-c_1(b^{n_0})^{\frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)}\right)$ and estimate the remaining sum using the crude bound $(b^n)^{\frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)} - (b^{n_0})^{\frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)} \gtrsim \log(b)(n-n_0)$. We can now derive moment bounds of order $k \geq 1$ via

$$\langle r_e^k \rangle = \left\langle \int_0^\infty I(r_e > r)\frac{\mathrm{d}}{\mathrm{d}r}r^k \mathrm{d}r \right\rangle \leq \int_0^\infty C_0 \exp\left(-c_1 r^{\frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)}\right)\frac{\mathrm{d}}{\mathrm{d}r}r^k \mathrm{d}r$$

$$= \int_0^\infty C_0 c_1 \frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)r^{\frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)-1}\exp\left(-c_1 r^{\frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)}\right)r^k \mathrm{d}r$$

By means of the substitution $t = c_1 r^{\frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)}$, the last expression rewrites as

$$\langle r_e^k \rangle = \int_0^\infty C_0 e^{-t}\left(\frac{t}{c_1}\right)^{\frac{\alpha+1}{\alpha}\frac{2k}{d(1-\beta)}}\mathrm{d}t = C_0 c_1^{-\frac{\alpha+1}{\alpha}\frac{2k}{d(1-\beta)}}\Gamma\left(\frac{\alpha+1}{\alpha}\frac{2k}{d(1-\beta)}+1\right).$$

The stretched exponential bound (1.13) for r_e of order $\frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)$ now immediately follows as

$$\left\langle \exp\left(\frac{1}{C}r_e^{\frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)}\right)\right\rangle \leq 1 + \sum_{k=1}^\infty \frac{\left\langle r_e^{\frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)k}\right\rangle}{C^k k!} \leq 1 + C_0 \sum_{k=1}^\infty \frac{\Gamma(k+1)}{C^k c_1^k k!} < 2 \qquad (2.10)$$

for $C > 0$ large enough. $\qquad\qquad\square$

2.3 Proof of Proposition 1.13: A sensitivity estimate for average integrals

Remark 2.2. *The reason for demanding symmetric coefficient fields $a(x) = a(x)^T$ is mainly related to the subsequent proof of Proposition 1.13, which does not seem to generalize to the case of non-symmetric a. In particular, the arguments in (2.18) and (2.19) heavily rely on the symmetry of a. In (2.19), we smuggle in $a^{-\frac{1}{2}}$ and $a^{\frac{1}{2}}$ leading to $|g|_{a^{-1}}$ and $|\psi|_a$ after applying Hölder's inequality. In the absence of symmetry, the terms which we insert should still cancel and be of the order $-\frac{1}{2}$ and $\frac{1}{2}$ w.r.t. a. Owing to (2.18) and the fact that the matrix a should partially cancel within the norm $|\cdot|_{a^{-1}}$, we see that the definition of $|\cdot|_{a^{-1}}$ has to be of the form $|g|_{a^{-1}} := g^T(a^{\frac{1}{2}})^{-T}(a^{\frac{1}{2}})^{-1}g$. Since (2.18) shall be controlled in terms of $|\cdot|_a$ in the subsequent estimate, $|\cdot|_a$ needs to be defined as $|\psi|_a := \psi^T(a^{\frac{1}{2}})^T a^{\frac{1}{2}}\psi$. But the structure of $|\cdot|_{a^{-1}}$ and $|\cdot|_a$ now prevents us from proceeding as in (2.19) unless a is symmetric.*

Proof of Proposition 1.13. We follow a strategy similar to the one in [17], which separates the proof into several steps. Throughout the proof, we will use the notation

$$F\nabla(\phi,\sigma) := \int \tilde{g}\cdot\nabla\phi + \int \bar{g}\cdot\nabla\sigma$$

for compactly supported $g = (\tilde{g}, \bar{g})$.

Step 1. Energy estimates for $r \geq r_*$. We start by noting that

$$\fint_{B_R} |\nabla\phi_i + e_i|_a^2 \lesssim 1 \qquad\qquad (2.11)$$

for all $R \geq r$, where the constant on the right hand side only depends on d and K. This follows from a Caccioppoli estimate as stated in [8, Lemma 3] since $r_* \geq r_e$ in particular ensures $R \geq r_e$. The definition of the minimal radius r_* in Definition 1.7 then allows for a constant upper bound.

As in [17], we now claim that for all $\gamma \in (0, d)$ and any decaying functions u and g related via

$$-\nabla \cdot a \nabla u = \nabla \cdot g, \tag{2.12}$$

we have

$$\fint_{B_r} |\nabla u|_a^2 \lesssim \int \left(\frac{|x|}{r} + 1 \right)^{-\gamma} |g|_{a^{-1}}^2 \tag{2.13}$$

with generic constants only depending on d and γ in this paragraph. Restricting oneself by scaling to the case $r = 1$, one is left to prove

$$\left(\fint_{B_1} |\nabla u|_a^2 \right)^{\frac{1}{2}} \lesssim \left(\fint_{B_1} |g|_{a^{-1}}^2 \right)^{\frac{1}{2}} + \sum_{n=1}^{\infty} \left(\frac{1}{(2^n)^d} \int_{2^{n-1} < |x| < 2^n} |g|_{a^{-1}}^2 \right)^{\frac{1}{2}}$$

taking the following elementary estimate into account:

$$\left(\fint_{B_1} |g|_{a^{-1}}^2 \right)^{\frac{1}{2}} + \sum_{n=1}^{\infty} \left(\frac{1}{(2^n)^d} \int_{2^{n-1} < |x| < 2^n} |g|_{a^{-1}}^2 \right)^{\frac{1}{2}}$$

$$\lesssim \left(\fint_{B_1} \left(|x| + 1 \right)^{-\gamma} |g|_{a^{-1}}^2 \right)^{\frac{1}{2}} + \sum_{n=1}^{\infty} \left(2^n \right)^{\frac{\gamma - d}{2}} \left(\int_{2^{n-1} < |x| < 2^n} \left(|x| + 1 \right)^{-\gamma} |g|_{a^{-1}}^2 \right)^{\frac{1}{2}}$$

$$\lesssim \int \left(|x| + 1 \right)^{-\gamma} |g|_{a^{-1}}^2.$$

As a result of the unique solvability of (2.12) in the class of decaying solutions, we may assume that g is either supported in B_1 or in $B_{2^n} \backslash B_{2^{n-1}}$ for some $n \in \mathbb{N}$. In the first case, we employ the energy estimate for (2.12) to derive $\fint_{B_1} |\nabla u|_a^2 \leq \int |\nabla u|_a^2 \lesssim \fint_{B_1} |g|_{a^{-1}}^2$. If $\operatorname{supp} g \subset B_{2^n} \backslash B_{2^{n-1}}$, we additionally use the mean-value property from Lemma 1.6 to deduce $\fint_{B_1} |\nabla u|_a^2 \lesssim (2^{n-1})^{-d} \int_{B_{2^{n-1}}} |\nabla u|_a^2 \lesssim (2^n)^{-d} \int_{2^{n-1} < |x| < 2^n} |g|_{a^{-1}}^2$.

The generalization of the previous estimate (2.13) provided in [17] also holds in our situation. For any $0 < \gamma' < \gamma < d$ and functions u and g subject to (2.12), one has

$$\fint_{B_r} \left(\frac{|x|}{r} + 1 \right)^{-\gamma} |\nabla u|_a^2 \lesssim \int \left(\frac{|x|}{r} + 1 \right)^{-\gamma'} |g|_{a^{-1}}^2 \tag{2.14}$$

where here the generic constants only depend on d, γ, and γ'. As above, we restrict ourselves to $r = 1$. By using (2.13) with $r := 2\rho$ therein for some $\rho \geq 1$, we find

$$\fint_{\rho \leq |x| \leq 2\rho} |\nabla u|_a^2 \lesssim \int \left(\frac{|x|}{2\rho} + 1 \right)^{-\gamma'} |g|_{a^{-1}}^2 \lesssim \rho^{\gamma'} \int \left(|x| + 1 \right)^{-\gamma'} |g|_{a^{-1}}^2.$$

Multiplying with $\rho^{-\gamma}$, we obtain $\fint_{\rho \leq |x| < 2\rho} |x|^{-\gamma} |\nabla u|_a^2 \lesssim \rho^{\gamma' - \gamma} \int (|x| + 1)^{-\gamma'} |g|_{a^{-1}}^2$. Setting $\rho := 2^n$, recalling $\gamma' < \gamma$, and taking the sum over $n \in \mathbb{N}$, we arrive at

$$\fint_{|x| \geq 1} \left(|x| + 1 \right)^{-\gamma} |\nabla u|_a^2 \lesssim \int \left(|x| + 1 \right)^{-\gamma'} |g|_{a^{-1}}^2.$$

The case $|x| < 1$ is treated by the previous estimate (2.13) for $r = 1$:

$$\int_{|x|<1} (|x|+1)^{-\gamma} |\nabla u|_a^2 \leq \int_{|x|<1} |\nabla u|_a^2 \lesssim \int (|x|+1)^{-\gamma'} |g|_{a^{-1}}^2.$$

Step 2. Energy estimate for all $r \geq r_e$. We claim that any a-harmonic function u on $\mathbb{R}^d$, i.e. any solution to

$$-\nabla \cdot a \nabla u = 0, \tag{2.15}$$

satisfies

$$\fint_{B_r(x)} |\nabla u|_a^2 \lesssim \left(\frac{r}{R}\right)^{\varepsilon d} \fint_{B_R(x)} |\nabla u|_a^2 \tag{2.16}$$

for some $\varepsilon > 0$ and for all $R \geq r \geq r_e$, where generic constants here and in the subsequent proof only depend on d and K.

Applying the Caccioppoli estimate for solutions to (2.15) from [8, Lemma 3] entails

$$\fint_{B_{r_e}(x)} |\nabla u|_a^2 \lesssim r_e^{-2} \left(\fint_A |u - \bar{u}|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}$$

where we abbreviate $A := B_{2r_e}(x) \backslash B_{r_e}(x)$ and $\bar{u} := \fint_A u$. Since the condition $\frac{1}{p} + \frac{1}{q} \leq \frac{2}{d}$ guarantees the embedding $W^{1,\frac{2q}{q+1}}(A) \hookrightarrow L^{\frac{2p}{p-1}}(A)$, we infer from Sobolev's inequality (observing the correct scaling w.r.t. r_e) that

$$\fint_{B_{r_e}(x)} |\nabla u|_a^2 \lesssim r_e^{-2-d\frac{p-1}{p}} \left(\int_A |u - \bar{u}|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}$$
$$\lesssim r_e^{-2-d\frac{p-1}{p}} r_e^{-d(\frac{1}{p}+\frac{1}{q})} \left(\left(\int_A |u - \bar{u}|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{q}} + r_e^2 \left(\int_A |\nabla u|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{q}}\right)$$
$$\lesssim r_e^{-2-d\frac{p-1}{p}} r_e^{-d(\frac{1}{p}+\frac{1}{q})} r_e^2 \left(\int_A |\nabla u|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{q}}$$
$$= r_e^{-d\frac{q+1}{q}} \left(\int_A |\nabla u|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{q}}$$

where we employed the correctly scaled Poincaré inequality for the third estimate. Thanks to Hölder's inequality (cf. [8, Lemma 3]), we conclude that

$$\fint_{B_{r_e}(x)} |\nabla u|_a^2 \lesssim \left(\fint_A |\nabla u|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{q}} \lesssim \fint_A |\nabla u|_a^2.$$

Rewriting this estimate in terms of an explicit constant $C(d, K)$, we get

$$\fint_{B_{r_e}(x)} |\nabla u|_a^2 \leq C \fint_{B_{2r_e}(x) \backslash B_{r_e}(x)} |\nabla u|_a^2.$$

The bound in (2.16) now follows from a so-called hole-filling trick, which amounts to adding $C \fint_{B_{r_e}(x)} |\nabla u|_a^2$ to both sides. This results in

$$\fint_{B_{r_e}(x)} |\nabla u|_a^2 \leq \frac{C}{C+1} \fint_{B_{2r_e}(x)} |\nabla u|_a^2 \leq \left(\frac{C}{C+1}\right)^n \fint_{B_{2^n r_e}(x)} |\nabla u|_a^2$$

where the second bound simply follows by iteration. Defining $\varepsilon > 0$, $n \in \mathbb{N}$, and $N \in \mathbb{N}$ via

$$\frac{C}{C+1} = 2^{-\varepsilon d}, \qquad 2^{n-1} r_e \leq r < 2^n r_e, \qquad 2^{N-1} r_e \leq R < 2^N r_e,$$

we arrive at

$$\int_{B_r(x)} |\nabla u|_a^2 \leq \int_{B_{2^n r_e}(x)} |\nabla u|_a^2 \leq 2^{-\varepsilon d(N-1-n)} \int_{B_{2^{N-1} r_e}(x)} |\nabla u|_a^2 \leq 2^{2\varepsilon d} \left(\frac{r}{R}\right)^{\varepsilon d} \int_{B_R(x)} |\nabla u|_a^2$$

provided $N - 1 \geq n$. In case that $N = n$, (2.16) holds true by adapting the generic constant appropriately.

Step 3. Sensitivity estimate for all $r \geq r_*$. We proceed by following [17] and recall that the defining equations for the decaying functions ϕ and σ_{jk} (where we skip the index i for notational convenience) read

$$-\nabla \cdot a(\nabla \phi + e) = 0, \qquad -\Delta \sigma_{jk} = \partial_j q_k - \partial_k q_j, \qquad q = a(\nabla \phi + e). \tag{2.17}$$

Fixing an element D of the underlying partition of $\mathbb{R}^d$, we shall write a_D for a coefficient field which may differ from a only inside of D. The corresponding solutions to (2.17) for a replaced by a_D are then denoted by ϕ_D and σ_{jkD}. As a result, the differences $\phi - \phi_D$ and $\sigma - \sigma_{jkD}$ are subject to

$$-\nabla \cdot a\nabla(\phi - \phi_D) = \nabla \cdot (a - a_D)(\nabla \phi_D + e),$$
$$-\Delta(\sigma_{jk} - \sigma_{jkD}) = \partial_j\big(a(\nabla \phi + e) - a_D(\nabla \phi_D + e)\big)_k - \partial_k\big(a(\nabla \phi + e) - a_D(\nabla \phi_D + e)\big)_j.$$

Taking a linear combination with scalar coefficients $\{c_D\}_D$, we get

$$-\nabla \cdot a\nabla \sum_D c_D(\phi - \phi_D) = \nabla \cdot \sum_D c_D(a - a_D)(\nabla \phi_D + e),$$
$$-\Delta \sum_D c_D(\sigma_{jk} - \sigma_{jkD}) = \partial_j \sum_D c_D\big(a(\nabla \phi + e) - a_D(\nabla \phi_D + e)\big)_k$$
$$- \partial_k \sum_D c_D\big(a(\nabla \phi + e) - a_D(\nabla \phi_D + e)\big)_j.$$

With the help of estimate (2.14) in Step 1, we now derive

$$\int_{B_r} \Big|\nabla \sum_D c_D(\phi - \phi_D)\Big|_a^2 + \int_{B_r} \Big(\frac{|x|}{r} + 1\Big)^{-\gamma} \Big|\nabla \sum_D c_D(\phi - \phi_D)\Big|_a^2$$
$$\lesssim \int \Big(\frac{|x|}{r} + 1\Big)^{-\gamma'} \Big|\sum_D c_D(a - a_D)(\nabla \phi_D + e)\Big|_{a^{-1}}^2$$
$$\leq \int \Big(\frac{|x|}{r} + 1\Big)^{-\gamma'} \sum_D c_D^2 \Big|a^{-\frac{1}{2}}(a - a_D)a^{-\frac{1}{2}} a^{\frac{1}{2}}(\nabla \phi_D + e)\Big|^2$$
$$\leq \sum_D c_D^2 \sup_{x \in D} |a - a_D|_{a^{-1}}^2 \int_D \Big(\frac{|x|}{r} + 1\Big)^{-\gamma'} |\nabla \phi_D + e|_a^2.$$

Note that we crucially employed the inclusion $\operatorname{supp}(a - a_D) \subset D$ and the fact that the elements of the partition are disjoint when pulling the square inside the sum over all D. Moreover, all generic constants appearing in this part of the proof only depend on d, γ, and γ'. Next, we observe that

$$a(\nabla \phi + e) - a_D(\nabla \phi_D + e) = a\nabla(\phi - \phi_D) + (a - a_D)(\nabla \phi_D + e)$$

together with the previous estimates and $\gamma' \leq \gamma$ leads to

$$\int_{B_r} \Big| \sum_D c_D (\nabla \phi - \nabla \phi_D) \Big|_a^2 + \int_{B_r} \Big(\frac{|x|}{r} + 1 \Big)^{-\gamma} \Big| \sum_D c_D \big(a(\nabla \phi + e) - a_D(\nabla \phi_D + e) \big) \Big|_{a^{-1}}^2$$

$$\leq \int_{B_r} \Big| \sum_D c_D (\nabla \phi - \nabla \phi_D) \Big|_a^2 + \int_{B_r} \Big(\frac{|x|}{r} + 1 \Big)^{-\gamma} \Big| a \nabla \sum_D c_D (\phi - \phi_D) \Big|_{a^{-1}}^2$$

$$+ \int_{B_r} \Big(\frac{|x|}{r} + 1 \Big)^{-\gamma'} \Big| \sum_D c_D (a - a_D)(\nabla \phi_D + e) \Big|_{a^{-1}}^2 \qquad (2.18)$$

$$\lesssim \sum_D c_D^2 \sup_{x \in D} |a - a_D|_{a^{-1}}^2 \int_D \Big(\frac{|x|}{r} + 1 \Big)^{-\gamma'} |\nabla \phi_D + e|_a^2.$$

Similarly, we may apply (2.13) to find

$$\int_{B_r} \Big| \nabla \sum_D c_D (\sigma_{jk} - \sigma_{jkD}) \Big|_a^2 \lesssim \int_{B_r} \Big(\frac{|x|}{r} + 1 \Big)^{-\gamma} \Big| \sum_D c_D \big(a(\nabla \phi + e) - a_D(\nabla \phi_D + e) \big) \Big|_{a^{-1}}^2$$

$$\lesssim \sum_D c_D^2 \sup_{x \in D} |a - a_D|_{a^{-1}}^2 \int_D \Big(\frac{|x|}{r} + 1 \Big)^{-\gamma'} |\nabla \phi_D + e|_a^2.$$

Due to the definition of the functional

$$F\psi = \int g \cdot \psi \quad \text{with} \quad \Big(\fint_{B_r} |g|_{a^{-1}}^2 \Big)^{\frac{1}{2}} \leq r^{-d},$$

we have

$$|F\psi| = \Big| \int_{B_r} g^T a^{-\frac{1}{2}} a^{\frac{1}{2}} \psi \Big| \leq \Big(\int_{B_r} |a^{-\frac{1}{2}} g|^2 \Big)^{\frac{1}{2}} \Big(\int_{B_r} |a^{\frac{1}{2}} \psi|^2 \Big)^{\frac{1}{2}} \leq \Big(\fint_{B_r} |\psi|_a^2 \Big)^{\frac{1}{2}}. \qquad (2.19)$$

Consequently, we deduce that

$$r^d \Big| \sum_D c_D (F\nabla \phi - F\nabla \phi_D) \Big|^2 + r^d \Big| \sum_D c_D (F\nabla \sigma_{jk} - F\nabla \sigma_{jkD}) \Big|^2$$

$$\lesssim \sum_D c_D^2 \sup_{x \in D} |a - a_D|_{a^{-1}}^2 \int_D \Big(\frac{|x|}{r} + 1 \Big)^{-\gamma'} |\nabla \phi_D + e|_a^2.$$

Thus, an elementary l^2 duality argument guarantees the bound

$$r^d \sum_D |F\nabla \phi - F\nabla \phi_D|^2 + r^d \sum_D |F\nabla \sigma_{jk} - F\nabla \sigma_{jkD}|^2$$

$$\lesssim \sup_D \Big(\sup_{x \in D} |a - a_D|_{a^{-1}}^2 \int_D \Big(\frac{|x|}{r} + 1 \Big)^{-\gamma'} |\nabla \phi_D + e|_a^2 \Big).$$

We now specify $a_D := a + t a^{\frac{1}{2}} \delta a_D a^{\frac{1}{2}}$ for $t \in \mathbb{R}$, $|t| \ll 1$, and a perturbation δa_D being bounded and supported in D. In the limit $t \to 0$, the previous estimate becomes

$$r^d \sum_D \Big| \int_D \frac{\partial F\nabla \phi}{\partial a} : a^{\frac{1}{2}} \delta a_D a^{\frac{1}{2}} \Big|^2 + r^d \sum_D \Big| \int_D \frac{\partial F\nabla \sigma_{jk}}{\partial a} : a^{\frac{1}{2}} \delta a_D a^{\frac{1}{2}} \Big|^2$$

$$\lesssim \sup_D \sup_{x \in D} |\delta a_D|^2 \cdot \sup_D \int_D \Big(\frac{|x|}{r} + 1 \Big)^{-\gamma'} |\nabla \phi_D + e|_a^2.$$

Lemma A.3 now allows for an explicit estimate of the matrix-valued derivative of F in terms of specific matrix norms. But as all matrix norms on $\mathbb{R}^{d \times d}$ are equivalent, Lemma A.3 also holds for the spectral norm $|\cdot|$ up to an additional constant. Hence, we arrive at

$$r^d \sum_D \Big(\int_D \big| \frac{\partial F \nabla \phi}{\partial a} \big|_a \Big)^2 + r^d \sum_D \Big(\int_D \big| \frac{\partial F \nabla \sigma_{jk}}{\partial a} \big|_a \Big)^2$$

$$\lesssim \sup_D \int_D \Big(\frac{|x|}{r} + 1 \Big)^{-\gamma'} |\nabla \phi_D + e|_a^2 \lesssim \sup_D \Big(\frac{\operatorname{dist} D}{r} + 1 \Big)^{-\gamma'} \int_D |\nabla \phi_D + e|_a^2. \quad (2.20)$$

To derive the announced sensitivity estimate for $r \geq r_*$, we set $\rho := \operatorname{diam} D$ and choose $x \in D$ such that $R := \operatorname{dist} D = |x|$. The assumption $\rho \leq (R+1)^\beta$ with $\beta \in (0,1)$ on the coarseness of the partition as well as the hole-filling estimate (2.16) ensure that

$$\int_D |\nabla \phi_D + e|_a^2 \leq \int_{B_\rho(x)} |\nabla \phi_D + e|_a^2 \lesssim \Big(\frac{1}{R+1} \Big)^{\varepsilon d(1-\beta)} \int_{B_{R+1}(x)} |\nabla \phi_D + e|_a^2.$$

Applying (2.16) once more in case that $r_* > 2R + 1$, we obtain

$$\int_D |\nabla \phi_D + e|_a^2 \lesssim \Big(\frac{1}{R + r_* + 1} \Big)^{\varepsilon d(1-\beta)} \int_{B_{\max\{2R+1, r_*\}}} |\nabla \phi_D + e|_a^2.$$

While this estimate trivially follows from the previous one in case that $r_* \leq 2R + 1$, we treat the additional factor from the hole-filling estimate in case that $r_* > 2R+1$ via $\big(\frac{1}{R+1} \big)^{\varepsilon d(1-\beta)} \big(\frac{2R+1}{r_*} \big)^{\varepsilon d} \leq \big(\frac{1}{R+1} \big)^{\varepsilon d(1-\beta)} \big(\frac{2R+1}{r_*} \big)^{\varepsilon d(1-\beta)} \lesssim \big(\frac{1}{r_*} \big)^{\varepsilon d(1-\beta)} \lesssim \big(\frac{1}{R+r_*+1} \big)^{\varepsilon d(1-\beta)}$. The remaining integral is controlled via the Caccioppoli estimate (2.11), which results in

$$\int_D |\nabla \phi_D + e|_a^2 \lesssim \Big(\frac{1}{R + r_* + 1} \Big)^{\varepsilon d(1-\beta)} (R + r_* + 1)^d \leq (R+r)^{d(1-\varepsilon(1-\beta))}$$

recalling $r \geq r_* \geq 1$. Going back to (2.20) and defining $\gamma' := d(1 - \varepsilon(1-\beta)) < d$, we conclude that

$$r^d \sum_D \Big(\int_D \big| \frac{\partial F \nabla \phi}{\partial a} \big|_a \Big)^2 + r^d \sum_D \Big(\int_D \big| \frac{\partial F \nabla \sigma_{jk}}{\partial a} \big|_a \Big)^2 \lesssim r^{d(1-\varepsilon(1-\beta))}. \quad (2.21)$$

Step 4. Sensitivity estimate for all $r \geq r_e$. The range of radii $r \in [r_e, r_*)$ is covered by applying the result from the previous step to the functional

$$\tilde{F}\psi := \Big(\frac{r}{r_*} \Big)^{\frac{d}{2}} F\psi.$$

Since (1.15) holds true, we infer

$$\big| \tilde{F}\psi \big|^2 = \Big(\frac{r}{r_*} \Big)^d |F\psi|^2 \leq \Big(\frac{r}{r_*} \Big)^d \fint_{B_r} |\psi|_a^2 \leq \fint_{B_{r_*}} |\psi|_a^2.$$

This enables us to employ (2.21) for $\tilde{F}$ and $r = r_*$ therein resulting in

$$(r_*)^d \Big(\frac{r}{r_*} \Big)^d \sum_D \Big(\int_D \big| \frac{\partial F \nabla \phi}{\partial a} \big|_a \Big)^2 + (r_*)^d \Big(\frac{r}{r_*} \Big)^d \sum_D \Big(\int_D \big| \frac{\partial F \nabla \sigma_{jk}}{\partial a} \big|_a \Big)^2 \lesssim (r_*)^{d(1-\varepsilon(1-\beta))}.$$

$$(2.22)$$

Combining (2.21) and (2.22) establishes the announced estimate (1.16). □

2.4 Auxiliary results on the sublinear growth of the extended corrector (ϕ, σ)

Lemma 2.3. *For $0 < L < \rho < \infty$, $1 \le s < d$, $s \le S < \infty$ such that $\theta := d(\frac{1}{s} - \frac{1}{S}) \in [0,1)$, and $u \in W^{1,s}(B_r)$, we have*

$$
\frac{1}{\rho} \left(\fint_{B_\rho} \left| u - \fint_{B_\rho} u \right|^S \right)^{\frac{1}{S}} \le C(d) \left(\fint_{B_\rho} \left| \fint_{B_{L(x)}} \nabla u \right|^s \mathrm{d}x \right)^{\frac{1}{s}} + C(d) \left(\frac{L}{\rho} \right)^{1-\theta} \left(\fint_{B_{2\rho}} |\nabla u|^s \right)^{\frac{1}{s}}.
\tag{2.23}
$$

Proof. By scaling we can assume $\rho = 1$, in which case (2.23) reduces for some $L \le 1$ to

$$
\left(\fint_{B_1} \left| u - \fint_{B_1} u \right|^S \right)^{\frac{1}{S}} \le C(d) \left(\fint_{B_1} \left| \fint_{B_{L(x)}} \nabla u \right|^s \mathrm{d}x \right)^{\frac{1}{s}} + C(d) L^{1-\theta} \left(\fint_{B_2} |\nabla u|^s \right)^{\frac{1}{s}}.
\tag{2.24}
$$

To show this, we apply the triangle inequality to estimate the left-hand side by

$$
\left(\fint_{B_1} \left| u - \fint_{B_1} u \right|^S \right)^{\frac{1}{S}} \le \left(\fint_{B_1} \left| (u - u_L) - \fint_{B_1} (u - u_L) \right|^S \right)^{\frac{1}{S}} + \left(\fint_{B_1} \left| u_L - \fint_{B_1} u_L \right|^S \right)^{\frac{1}{S}},
$$

where $u_L(x) := \fint_{B_{L(x)}} u$. Combining Jensen's, Sobolev's, and Poincaré's inequalities (while using $\frac{1}{S} \ge \frac{1}{s} - \frac{1}{d}$), we get for the second term on the right-hand side

$$
\left(\fint_{B_1} \left| u_L - \fint_{B_1} u_L \right|^S \right)^{\frac{1}{S}} \le C(d) \left(\fint_{B_1} \left| u_L - \fint_{B_1} u_L \right|^{\frac{ds}{d-s}} \right)^{\frac{d-s}{ds}}
$$

$$
\le C(d) \left(\fint_{B_1} |\nabla(u_L)|^s \right)^{\frac{1}{s}} = C(d) \left(\fint_{B_1} |(\nabla u)_L|^s \right)^{\frac{1}{s}}.
$$

For the first term, we apply Hölder's inequality (with exponents $\frac{s}{s-S(1-\theta)}, \frac{s}{S(1-\theta)} \in (1, \infty)$ after splitting the integrand as $|\cdot|^S = |\cdot|^{\theta S} \cdot |\cdot|^{(1-\theta)S}$) and Jensen's inequality, followed in the next step by the above Sobolev inequality and the convolution estimate:

$$
\left(\fint_{B_1} \left| (u - u_L) - \fint_{B_1} (u - u_L) \right|^S \right)^{\frac{1}{S}}
$$

$$
\le C(d) \left(\fint_{B_1} \left| (u - u_L) - \fint_{B_1} (u - u_L) \right|^{\frac{ds}{d-s}} \right)^{\theta \frac{d-s}{ds}} \left(\fint_{B_1} |u - u_L|^s \right)^{\frac{1-\theta}{s}}
$$

$$
\le C(d) \left(\fint_{B_1} |\nabla(u - u_L)|^s \right)^{\frac{\theta}{s}} L^{1-\theta} \left(\fint_{B_2} |\nabla u|^s \right)^{\frac{1-\theta}{s}} \le C(d) L^{1-\theta} \left(\fint_{B_2} |\nabla u|^s \right)^{\frac{1}{s}},
$$

which proves (2.24). $\qquad\square$

Corollary 2.4. *For $0 < L < r$, $\theta := \frac{d}{2}(\frac{1}{p} + \frac{1}{q}) \in (0,1)$, and*

$$
\bar{K} := \sup_{R \ge r} \left(\fint_{B_R} |a|^p \right)^{\frac{1}{p}} + \left(\fint_{B_R} |a^{-1}|^q \right)^{\frac{1}{q}},
$$

the extended corrector (ϕ, σ) from Definition 1.4 satisfies

$$\frac{1}{r}\left(\fint_{B_r}\left|\phi - \fint_{B_r}\phi\right|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}} + \frac{1}{r}\left(\fint_{B_r}\left|\sigma - \fint_{B_r}\sigma\right|^{\frac{2q}{q-1}}\right)^{\frac{q-1}{2q}}$$

$$\lesssim \left(\fint_{B_r}\left|\fint_{B_{L(x)}}\nabla\phi\right|^{\frac{2q}{q+1}}\mathrm{d}x\right)^{\frac{q+1}{2q}} + \left(\fint_{B_r}\left|\fint_{B_{L(x)}}\nabla\sigma\right|^{\frac{2p}{p+1}}\mathrm{d}x\right)^{\frac{p+1}{2p}} \tag{2.25}$$

$$+ \bar{K}^{\frac{1}{2}}\left(\frac{L}{r}\right)^{1-\theta}\left(1 + \frac{1}{r}\left(\fint_{B_{8r}}\left|\phi - \fint_{B_{8r}}\phi\right|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{2p}} + \frac{1}{r}\left(\fint_{B_{8r}}\left|\sigma - \fint_{B_{8r}}\sigma\right|^{\frac{2q}{q-1}}\right)^{\frac{q-1}{2q}}\right).$$

Proof. We start with σ. Using Lemma 2.3 with $u = \sigma$, $S = \frac{2q}{q-1}$, and $s = \frac{2p}{p+1}$, we get that

$$\frac{1}{r}\left(\fint_{B_r}\left|\sigma - \fint_{B_r}\sigma\right|^{\frac{2q}{q-1}}\right)^{\frac{q-1}{2q}}$$

$$\leq C(d)\left(\fint_{B_r}\left|\fint_{B_{L(x)}}\nabla\sigma\right|^{\frac{2p}{p+1}}\mathrm{d}x\right)^{\frac{p+1}{2p}} + C(d)\left(\frac{L}{r}\right)^{1-\theta}\left(\fint_{B_{2r}}|\nabla\sigma|^{\frac{2p}{p+1}}\right)^{\frac{p+1}{2p}}, \tag{2.26}$$

with $\theta = \frac{d}{2}\left(\frac{1}{p} + \frac{1}{q}\right) < 1$. To estimate the second term on the right-hand side, we assume w.l.o.g. that $\fint_{B_{4r}}\sigma = 0$ and consider $\eta\sigma$ with a smooth cut-off function η for B_{2r} in B_{4r}, which then by $-\Delta\sigma_{ijk} = \partial_j q_{ik} - \partial_k q_{ij} =: \nabla \cdot \tilde{q}$ with $\tilde{q} = q_{ik}e_j - q_{ij}e_k$ satisfies

$$\Delta(\eta\sigma_{ijk}) = \nabla \cdot (2\sigma_{ijk}\nabla\eta - \eta\tilde{q}) + \nabla\eta \cdot \tilde{q} - \sigma_{ijk}\Delta\eta.$$

By a Calderón–Zygmund estimate for the Laplacian combined with Sobolev's and Jensen's inequalities, this implies

$$\left(\int_{B_{2r}}|\nabla\sigma_{ijk}|^{\frac{2p}{p+1}}\right)^{\frac{p+1}{2p}} \leq \left(\int_{\mathbb{R}^d}|\nabla(\eta\sigma_{ijk})|^{\frac{2p}{p+1}}\right)^{\frac{p+1}{2p}}$$

$$\lesssim \frac{1}{r}\left(\int_{B_{4r}}|\sigma_{ijk}|^{\frac{2p}{p+1}}\right)^{\frac{p+1}{2p}} + \left(\int_{B_{4r}}|\tilde{q}|^{\frac{2p}{p+1}}\right)^{\frac{p+1}{2p}}.$$

Using the definition of $\tilde{q}$ via $q_i = a(\nabla\phi_i + e_i)$ we see that by Hölder's inequality

$$\left(\fint_{B_{4r}}|\tilde{q}|^{\frac{2p}{p+1}}\right)^{\frac{p+1}{2p}} \leq \left(\fint_{B_{4r}}|a|^p\right)^{\frac{1}{2p}}\left(\fint_{B_{4r}}|\nabla\phi_i + e_i|_a^2\right)^{\frac{1}{2}},$$

which combined with the previous inequality (after taking averages on both sides) yields

$$\left(\fint_{B_{2r}}|\nabla\sigma_{ijk}|^{\frac{2p}{p+1}}\right)^{\frac{p+1}{2p}} \lesssim \frac{1}{r}\left(\fint_{B_{4r}}|\sigma_{ijk}|^{\frac{2p}{p+1}}\right)^{\frac{p+1}{2p}} + \bar{K}^{\frac{1}{2}}\left(\fint_{B_{4r}}|\nabla\phi_i + e_i|_a^2\right)^{\frac{1}{2}}.$$

Note that the same arguments apply if we replace σ by $\sigma - \fint_{B_{4r}}\sigma$. We then plug the previous estimate into (2.26) and use Jensen's inequality to obtain

$$\frac{1}{r}\left(\fint_{B_r}\left|\sigma - \fint_{B_r}\sigma\right|^{\frac{2q}{q-1}}\right)^{\frac{q-1}{2q}} \leq C(d)\left(\fint_{B_r}\left|\fint_{B_{L(x)}}\nabla\sigma\right|^{\frac{2p}{p+1}}\mathrm{d}x\right)^{\frac{p+1}{2p}} \tag{2.27}$$

$$+ C(d)\left(\frac{L}{r}\right)^{1-\theta}\frac{1}{r}\left(\fint_{B_{4r}}\left|\sigma - \fint_{B_{4r}}\sigma\right|^{\frac{2q}{q-1}}\right)^{\frac{q-1}{2q}} + C(d)\left(\frac{L}{r}\right)^{1-\theta}\bar{K}^{\frac{1}{2}}\left(\fint_{B_{4r}}|\nabla\phi + e|_a^2\right)^{\frac{1}{2}}.$$

To control the last term on the right-hand side, we observe that testing $-\nabla \cdot a\big(\nabla(\phi - \fint_{B_{8r}} \phi) + e\big) = 0$ with $\eta^2(\phi - \fint_{B_{8r}} \phi)$, where η is a cut-off function for B_{4r} in B_{8r}, entails

$$\fint_{B_{8r}} \Big|\nabla\big(\eta(\phi - \fint_{B_{8r}} \phi)\big)\Big|_a^2 \lesssim \fint_{B_{8r}} |e|_a^2 + \fint_{B_{8r}} \Big|\phi - \fint_{B_{8r}} \phi\Big|^2 |\nabla\eta|_a^2$$

$$\lesssim \bar{K}\left(1 + \frac{1}{r^2}\Big(\fint_{B_{8r}} \Big|\phi - \fint_{B_{8r}} \phi\Big|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{p}}\right), \qquad (2.28)$$

where we used Hölder's inequality together with $|\nabla\eta| \lesssim \frac{1}{r}$.

Using Lemma 2.3 with $u = \phi$, $S = \frac{2p}{p-1}$ and $s = \frac{2q}{q+1}$, we get that

$$\frac{1}{r}\Big(\fint_{B_r} \Big|\phi - \fint_{B_r} \phi\Big|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}}$$

$$\leq C(d)\Big(\fint_{B_r} \Big|\fint_{B_{L(x)}} \nabla\phi\Big|^{\frac{2q}{q+1}} \mathrm{d}x\Big)^{\frac{q+1}{2q}} + C(d)\Big(\frac{L}{r}\Big)^{1-\theta}\Big(\fint_{B_{2r}} |\nabla\phi|^{\frac{2q}{q+1}}\Big)^{\frac{q+1}{2q}}. \qquad (2.29)$$

By means of Hölder's inequality, we see that

$$\Big(\fint_{B_{2r}} \Big|\nabla\big(\phi - \fint_{B_{8r}} \phi\big)\Big|^{\frac{2q}{q+1}}\Big)^{\frac{q+1}{2q}} \lesssim \bar{K}^{\frac{1}{2}}\Big(\fint_{B_{8r}} \Big|\nabla\big(\eta(\phi - \fint_{B_{8r}} \phi)\big)\Big|_a^2\Big)^{\frac{1}{2}},$$

which we use in (2.29) to control the last integral. Finally, such modified (2.29) together with (2.28) and (2.27) yield (2.25), which completes the proof. $\qquad\qquad\square$

2.5 Proof of Theorem 1.12: Stretched exponential moments for r_*

Proof of Theorem 1.12. Step 1. Control of the minimal radius r_*. Let $p, q \in (1, \infty)$ be the integrability exponents of a and a^{-1} from Definition 1.1. We shall follow the lines of [17] to derive an estimate for the minimal radius r_*. To this end, we first assume that $M_0 r_e < r \leq r_*$ holds with some positive radius r and the positive constant M_0 from Definition 1.7 which is specified in (2.30) below. We introduce

$$X(r) := \max\left\{\frac{1}{r}\Big(\fint_{B_r} \Big|\phi - \fint_{B_r} \phi\Big|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}}, \frac{1}{r}\Big(\fint_{B_r} \Big|\sigma - \fint_{B_r} \sigma\Big|^{\frac{2q}{q-1}}\Big)^{\frac{q-1}{2q}}\right\}.$$

With another positive radius r' subject to $r' < r$, we may employ Corollary 2.4 to obtain

$$X(r) \lesssim \left(\Big(\frac{r'}{r}\Big)^{1-\theta}(1 + X(8r)) + \Big(\fint_{B_r} \Big|\fint_{B_{r'}(x)} \nabla\phi\Big|^{\frac{2p}{p+1}}\Big)^{\frac{p+1}{2p}} + \Big(\fint_{B_r} \Big|\fint_{B_{r'}(x)} \nabla\sigma\Big|^{\frac{2q}{q+1}}\Big)^{\frac{q+1}{2q}}\right)$$

with $\theta = \frac{d}{2}\big(\frac{1}{p} + \frac{1}{q}\big) < 1$. From the definition of r_* and the constant C_0 in Definition 1.7, we deduce that there exists some $\rho \geq r$ such that $X(\rho) \geq \frac{1}{C_0}$ while $X(\rho') \leq \frac{2}{C_0}$ for all $\rho' \geq \rho$. Note that we crucially employed the assumption $M_0 r_e < r_*$, which ensures that such a $\rho \geq r$ actually exists. We infer

$$X(\rho') \geq \Big(\frac{\rho}{\rho'}\Big)^{1+d\max\big\{\frac{p-1}{2p}, \frac{q-1}{2q}\big\}} X(\rho) \geq \Big(\frac{1}{2}\Big)^{1+d\max\big\{\frac{p-1}{2p}, \frac{q-1}{2q}\big\}} \frac{1}{C_0}, \qquad X(8\rho') \leq \frac{2}{C_0}$$

for all $\rho' \in (\rho, 2\rho)$. The choice $r := \rho'$ and $r' := \rho'' \in (0, \rho')$ entails

$$\frac{1}{C} \le \Big(\frac{\rho''}{\rho'}\Big)^{1-\theta} + \Big(\fint_{B_{\rho'}} \big| \fint_{B_{\rho''}(x)} \nabla\phi \big|^{\frac{2p}{p+1}}\Big)^{\frac{p+1}{2p}} + \Big(\fint_{B_{\rho'}} \big| \fint_{B_{\rho''}(x)} \nabla\sigma \big|^{\frac{2q}{q+1}}\Big)^{\frac{q+1}{2q}}$$

with a positive constant $C(d, p, q, K, C_0)$ which is in particular independent of M_0. One can now absorb the first term on the right-hand side by setting $\rho'' := \frac{\rho'}{M_0}$ with

$$M_0 := (2C)^{\frac{1}{1-\theta}} \tag{2.30}$$

leading to

$$\frac{1}{2C} \le \Big(\fint_{B_{\rho'}} \big| \fint_{B_{\frac{\rho'}{M_0}}(x)} \nabla\phi \big|^{\frac{2p}{p+1}}\Big)^{\frac{p+1}{2p}} + \Big(\fint_{B_{\rho'}} \big| \fint_{B_{\frac{\rho'}{M_0}}(x)} \nabla\sigma \big|^{\frac{2q}{q+1}}\Big)^{\frac{q+1}{2q}}.$$

Taking the power $2P$, $P \in \mathbb{N}$, applying Jensen's inequality, and integrating over $(\rho, 2\rho)$, we find

$$\frac{1}{4(4C)^{2P}} \le \int_\rho^{2\rho} \fint_{B_{\rho'}} \big| \fint_{B_{\frac{\rho'}{M_0}}(x)} \nabla(\phi, \sigma) \big|^{2P} \frac{1}{\rho'}.$$

Since the previous calculation holds for any configuration satisfying $M_0 r_e < r \le r_*$, we arrive at

$$\langle I(M_0 r_e < r \le r_*) \rangle \le C^P \int_{\frac{r}{M_0}}^\infty \Big\langle \big| \fint_{B_\rho} \nabla(\phi, \sigma) \big|^{2P} \Big\rangle \frac{1}{\rho} \tag{2.31}$$

extending the range of integration, renaming variables, and using the stationarity of $\fint_{B_\rho(x)} \nabla(\phi, \sigma)$.

Step 2. Control of the corrector gradient $\nabla(\phi, \sigma)$. We keep the assumptions and the notation from the previous step and consider some $\rho > \frac{r}{M_0}$. As a result of the vanishing expectation of $\nabla(\phi, \sigma)$, the spectral gap estimate (2.5) ensures the bound

$$\Big\langle \big| \fint_{B_\rho} \nabla(\phi, \sigma) \big|^{2P} \Big\rangle^{\frac{1}{P}} \le \frac{CP^2}{\kappa} \Big\langle \Big(\sum_D \Big(\int_D \big| \frac{\partial}{\partial a} \fint_{B_\rho} \nabla(\phi, \sigma) \big|_a \Big)^2 \Big)^P \Big\rangle^{\frac{1}{P}}.$$

Recalling the notation $F\nabla(\phi, \sigma) = \int_{\mathbb{R}^d} g\nabla(\phi, \sigma) = \fint_{B_\rho} \nabla(\phi, \sigma)$ with $g := |B_\rho|^{-1} \mathbb{1}_{B_\rho}$, which in particular fulfills the condition $\big(\fint_{B_\rho} |g|^2_{a^{-1}} \big)^{\frac{1}{2}} \lesssim \rho^{-d} \big(\fint_{B_\rho} |a^{-1}| \big)^{\frac{1}{2}} \lesssim \rho^{-d}$, Proposition 1.13 guarantees

$$\Big\langle \big| \fint_{B_\rho} \nabla(\phi, \sigma) \big|^{2P} \Big\rangle^{\frac{1}{P}} \lesssim P^2 \Big\langle \Big(\frac{(\rho + r_*)^{1-\varepsilon(1-\beta)}}{\rho} \Big)^{dP} \Big\rangle^{\frac{1}{P}}.$$

Here and in the remainder of this proof, all generic constants may depend on d, p, q, β, ε, κ, K, and C_0. Together with the estimate on the minimal radius in (2.31), we derive

$$\langle I(M_0 r_e < r \le r_*) \rangle^{\frac{1}{P}} \lesssim \Big(\int_{\frac{r}{M_0}}^\infty \Big\langle \big| \fint_{B_\rho} \nabla(\phi, \sigma) \big|^{2P} \Big\rangle \frac{1}{\rho} \Big)^{\frac{1}{P}}$$

$$\lesssim P^2 \Big(\int_{\frac{r}{M_0}}^\infty \Big\langle \Big(\frac{(\rho + r_*)^{1-\varepsilon(1-\beta)}}{\rho} \Big)^{dP} \Big\rangle \frac{1}{\rho} \Big)^{\frac{1}{P}}.$$

An evaluation of the integral over ρ after using the triangle inequality gives rise to

$$\langle I(M_0 r_e < r \le r_*)\rangle^{\frac{1}{P}} \lesssim P^2\bigg(\bigg(\frac{1}{\varepsilon dP(1-\beta)}\Big(\frac{r}{M_0}\Big)^{-\varepsilon dP(1-\beta)}\bigg)^{\frac{1}{P}}$$
$$+\Big(\frac{1}{dP}\Big(\frac{r}{M_0}\Big)^{-dP}\Big)^{\frac{1}{P}}\big\langle r_*^{dP(1-\varepsilon(1-\beta))}\big\rangle^{\frac{1}{P}}\bigg).$$

Keeping track only of the dependence of the constants on P, this expression simplifies to

$$\langle I(M_0 r_e < r \le r_*)\rangle^{\frac{1}{P}} \lesssim P^2\Big(\frac{1}{r^{\varepsilon d(1-\beta)}} + \frac{1}{r^d}\big\langle r_*^{dP(1-\varepsilon(1-\beta))}\big\rangle^{\frac{1}{P}}\Big). \tag{2.32}$$

To derive a bound on $\langle I(r_* \ge r)\rangle$, we first note that

$$\langle I(r_* \ge r)\rangle^{\frac{1}{P}} \le \langle I(M_0 r_e \ge r)\rangle^{\frac{1}{P}} + \langle I(M_0 r_e < r \le r_*)\rangle^{\frac{1}{P}}$$

by elementary arguments, where $\langle I(M_0 r_e \ge r)\rangle^{\frac{1}{P}} \lesssim \exp\big(-\frac{1}{CP}\big(\frac{r}{M_0}\big)^{\varepsilon\frac{d}{2}(1-\beta)}\big)$ due to (1.13) with $\alpha = \frac{\varepsilon}{1-\varepsilon}$. As a consequence of the scalar inequality $e^{-x} \le x^{-2}$ for any $x > 0$, we obtain

$$\exp\Big(-\frac{1}{CP}\Big(\frac{r}{M_0}\Big)^{\varepsilon\frac{d}{2}(1-\beta)}\Big) \le \frac{C^2 P^2 M_0^{\varepsilon d(1-\beta)}}{r^{\varepsilon d(1-\beta)}}.$$

Hence, we established the estimate

$$\langle I(r_* \ge r)\rangle^{\frac{1}{P}} \lesssim P^2\Big(\frac{1}{r^{\varepsilon d(1-\beta)}} + \frac{1}{r^d}\big\langle r_*^{dP(1-\varepsilon(1-\beta))}\big\rangle^{\frac{1}{P}}\Big). \tag{2.33}$$

Step 3. Buckling and exponential moments of r_*. We introduce $s := r^{\frac{\varepsilon}{2}d(1-\beta)}$, $s_* := r_*^{\frac{\varepsilon}{2}d(1-\beta)}$, $Q := \frac{1}{\varepsilon(1-\beta)} > 1$, and we replace $P \ge 1$ by $\frac{P}{2}$ (now with $P \ge 2$) in (2.33). The transformed estimate then reads

$$\langle I(s_* \ge s)\rangle^{\frac{1}{P}} \le \widetilde{C}P\Big(\frac{1}{s} + \frac{1}{s^Q}\big\langle (s_*^{Q-1})^P\big\rangle^{\frac{1}{P}}\Big) \tag{2.34}$$

with some constant $\widetilde{C} \ge 1$. The basic idea of buckling in this context is to assume that there exists a constant $\Lambda \ge 1$ such that

$$\big\langle I(s_* \ge s)\big\rangle \le \exp\Big(-\frac{s}{\Lambda}\Big) \tag{2.35}$$

holds true for all $s \ge \Lambda$. This assumption is in general only satisfied for $\min\{s_*, \delta^{-1}\}$ with $\delta > 0$ instead of s_*. Showing that (2.35) (with $\min\{s_*, \delta^{-1}\}$) entails $\Lambda \le C$ for some constant $C \ge 1$, which is in particular independent of δ, allows to transfer the uniform exponential decay to $\big\langle I(s_* \ge s)\big\rangle$.

Imposing (2.35), we calculate for any $P \ge 2$,

$$\langle s_*^P\rangle \le \int_0^\infty \big\langle I(s_* \ge s)\big\rangle\frac{\mathrm{d}}{\mathrm{d}s}s^P\,\mathrm{d}s \le \int_0^\infty \exp\Big(1 - \frac{s}{\Lambda}\Big)\frac{\mathrm{d}}{\mathrm{d}s}s^P\,\mathrm{d}s$$
$$= e\int_0^\infty \frac{1}{\Lambda}\exp\Big(-\frac{s}{\Lambda}\Big)s^P\,\mathrm{d}s = e\int_0^\infty e^{-t}(\Lambda t)^P\,\mathrm{d}t = e\Lambda^P P!.$$

We thus obtain $\langle s_*^P\rangle^{\frac{1}{P}} \le eP\Lambda$ for all $P \ge 2$ and infer from (2.34) the estimate

$$\langle I(s_* \ge s)\rangle^{\frac{1}{P}} \le \widetilde{C}P\Big(\frac{1}{s} + \frac{1}{s^Q}\big(eP(Q-1)\Lambda\big)^{Q-1}\Big) \le C_0\frac{P}{s}\Big(1 + \Big(\frac{P\Lambda}{s}\Big)^{Q-1}\Big)$$

with another constant $C_0 \geq 1$. By defining $\Lambda' := 2C_0\Lambda^{\frac{Q-1}{Q}} \geq 1$, we have

$$\langle I(s_* \geq s)\rangle^{\frac{1}{P}} \leq C_0 \frac{P}{s}\left(\frac{\Lambda'}{2C_0} + \left(\frac{\Lambda}{2C_0\Lambda^{\frac{Q-1}{Q}}}\right)^{Q-1}\right) \leq \frac{P}{s}\Lambda'$$

provided $s \geq P\Lambda'$. The best upper bound for $\langle I(s_* \geq s)\rangle \leq \left(\frac{P}{s}\Lambda'\right)^P$ can be found by optimizing the right-hand side in P leading to $P = \frac{s}{e\Lambda'}$. In order to guarantee $P \geq 2$ (due to the variable transformation at the beginning of this step), we have to demand $s \geq 2e\Lambda'$. Note that $s \geq P\Lambda'$ is trivially satisfied. Consequently,

$$\langle I(s_* \geq s)\rangle \leq e^{-P} \leq \exp\left(-\frac{s}{2e\Lambda'}\right)$$

for all $s \geq 2e\Lambda'$. This shows that (2.35) also holds true with $2e\Lambda'$ instead of Λ. Taking the best constant Λ in (2.35) ensures $\Lambda \leq 2e\Lambda' = 4C_0 e\Lambda^{\frac{Q-1}{Q}}$ and, finally, $\Lambda \leq (4C_0 e)^Q$.

Moments of s_* of order $k \in \mathbb{N}$ are now derived as above via

$$\langle s_*^k\rangle \leq \int_0^\infty \langle I(s_* \geq s)\rangle \frac{\mathrm{d}}{\mathrm{d}s}s^k\,\mathrm{d}s \leq \int_0^\infty \exp\left(-\frac{s}{\Lambda}\right)\frac{\mathrm{d}}{\mathrm{d}s}s^k\,\mathrm{d}s$$
$$= \int_0^\infty \frac{1}{\Lambda}\exp\left(-\frac{s}{\Lambda}\right)s^k\,\mathrm{d}s = \int_0^\infty e^{-t}(\Lambda t)^k\,\mathrm{d}t = \Lambda^k k!.$$

As an immediate consequence, we obtain stretched exponential moments of r_* in the sense that

$$\left\langle \exp\left(\frac{1}{C}r_*^{\frac{\varepsilon}{2}d(1-\beta)}\right)\right\rangle = 1 + \sum_{k=1}^\infty \frac{1}{k!}\frac{\langle s_*^k\rangle}{C^k} \leq 1 + \sum_{k=1}^\infty \frac{\Lambda^k}{C^k} < 2$$

for a sufficiently large constant $C > 0$. $\qquad\square$

3 Two applications to stochastic homogenization

3.1 Proof of Theorem 1.14: Decay and growth properties of the corrector

Proof of Theorem 1.14. We define the random variable

$$F(x) := \fint_{B_r} \nabla(\phi, \sigma)(x + y) \cdot m(y)\,dy$$

which is stationary and satisfying $\langle F\rangle = 0$ thanks to the according properties of $\nabla(\phi, \sigma)$. From the P-spectral gap inequality (2.5) and the sensitivity estimate (1.16), we thus derive

$$\langle|F|^{2P}\rangle^{\frac{1}{2P}} \lesssim P\left\langle\left(\sum_D\left(\int_D\left|\frac{\partial F}{\partial a}\right|_a\right)^2\right)^P\right\rangle^{\frac{1}{2P}} \lesssim P\left\langle\left(\frac{(r + r_*)^{1-\varepsilon(1-\beta)}}{r}\right)^{Pd}\right\rangle^{\frac{1}{2P}}$$

for any $P \in \mathbb{N}$, $P \geq 2$. Taking $r \geq 1$ and $r_* \geq 1$ into account, this simplifies to

$$\langle|F|^{2P}\rangle^{\frac{1}{2P}} \lesssim Pr^{-\frac{\varepsilon}{2}d(1-\beta)}\left\langle r_*^{(1-\varepsilon(1-\beta))Pd}\right\rangle^{\frac{1}{2P}}. \tag{3.1}$$

The first assertion of the theorem is a consequence of establishing exponentially stretched moments of order $\varepsilon(1-\beta)$ for

$$\mathcal{C}(x) := r^{\frac{\varepsilon}{2}d(1-\beta)}\big|F(x)\big|.$$

To this end, we choose $Q \geq 1$ and let $P \in \mathbb{N}$ be the integer such that $Q \leq P < Q+1$. Estimate (3.1) then entails

$$\big\langle |\mathcal{C}|^{2Q}\big\rangle^{\frac{1}{2Q}} \lesssim Q\big\langle r_*^{(1-\varepsilon(1-\beta))(Q+1)d}\big\rangle^{\frac{1}{2(Q+1)}}.$$

Replacing Q by $\frac{\mu Q}{2}$ for some $\mu > 0$ and demanding $Q \geq \frac{4}{\mu}$ to guarantee $\frac{\mu Q}{2} \geq 2$, we obtain

$$\big\langle |\mathcal{C}|^{\mu Q}\big\rangle^{\frac{1}{Q}} \lesssim Q^\mu\big\langle r_*^{(1-\varepsilon(1-\beta))(\frac{\mu Q}{2}+1)d}\big\rangle^{\frac{\mu}{2(\frac{\mu Q}{2}+1)}} \leq Q^\mu\big\langle r_*^{(1-\varepsilon(1-\beta))\mu Q d}\big\rangle^{\frac{1}{2Q}}.$$

Next, we infer from Theorem 1.12 and Lemma A.1 the bound

$$\Big\langle r_*^{(1-\varepsilon(1-\beta))\mu Q d}\Big\rangle \leq \Big(C\frac{(1-\varepsilon(1-\beta))2\mu Q}{\varepsilon(1-\beta)}\Big)^{\frac{(1-\varepsilon(1-\beta))2\mu Q}{\varepsilon(1-\beta)}}$$

for some constant $C \geq 1$, which results in

$$\big\langle |\mathcal{C}|^{\mu Q}\big\rangle^{\frac{1}{Q}} \lesssim Q^{\mu\big(1+\frac{1-\varepsilon(1-\beta)}{\varepsilon(1-\beta)}\big)} = Q^{\frac{\mu}{\varepsilon(1-\beta)}} = Q$$

for $\mu := \varepsilon(1-\beta)$. Applying again Lemma A.1 concludes the argument.

Concerning the growth of the correctors (ϕ,σ), we start by employing Sobolev's inequality related to the embedding $W^{1,\frac{2q}{q+1}}(B(x)) \hookrightarrow L^{\frac{2p}{p-1}}(B(x))$ on the unit ball $B(x) := B_1(x)$, which holds by the condition $\frac{1}{p}+\frac{1}{q} \leq \frac{2}{d}$. We infer

$$\Big(\fint_{B(x)} |\phi|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}} \lesssim \Big(\fint_{B(x)} |\phi|^{\frac{2q}{q+1}}\Big)^{\frac{q+1}{2q}} + \Big(\fint_{B(x)} |\nabla\phi|^{\frac{2q}{q+1}}\Big)^{\frac{q+1}{2q}}.$$

Poincaré's inequality and an elementary estimate involving the ellipticity radius $r_e(x)$ give rise to

$$\Big(\fint_{B(x)} |\phi|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}} \lesssim \Big|\fint_{B(x)} \phi\Big| + r_e(x)^{d\frac{q+1}{2q}}\Big(\fint_{B_{r_e(x)}(x)} |\nabla\phi|^{\frac{2q}{q+1}}\Big)^{\frac{q+1}{2q}}.$$

Hölder's inequality followed by a hole-filling argument then yields

$$\Big(\fint_{B(x)} |\phi|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}} \lesssim \Big|\fint_{B(x)} \phi\Big| + r_e(x)^{d\frac{q+1}{2q}}\Big(\frac{r_*(x)}{r_e(x)}\Big)^{\frac{d}{2}(1-\varepsilon)}\Big(\fint_{B_{r_*(x)}(x)} |\nabla\phi|_a^2\Big)^{\frac{1}{2}}.$$

Owing to a Caccioppoli estimate of the last term above and the definition of the minimal radius $r_*(x)$ from Definition 1.7, we deduce

$$\Big(\fint_{B(x)} |\phi|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}} \lesssim \Big|\fint_{B(x)} \phi\Big| + r_e(x)^{d\frac{q+1}{2q}}\Big(\frac{r_*(x)}{r_e(x)}\Big)^{\frac{d}{2}(1-\varepsilon)}\Big(\fint_{B_{r_*(x)}(x)} \Big|\phi - \fint_{B_{r_*(x)}(x)}\phi\Big|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{2p}}$$

$$\lesssim \Big|\fint_{B(x)} \phi\Big| + r_*(x)^{\frac{d}{2}(1-\varepsilon)}r_e(x)^{\frac{d}{2}(\frac{1}{q}+\varepsilon)}.$$

As in [18], we proceed by calculating

$$\fint_{B_t(x)} (\phi(y)-\phi(x))\,dy = \int_0^t \fint_{B_r(x)} \nabla\phi(y)\cdot\frac{y-x}{r}\,dy\,dr$$

for any $t \geq 1$. This identity easily follows from the integral mean value theorem. Hence,

$$\partial_t \fint_{B_t(x)} \phi = \fint_{B_t(x)} \nabla \phi(y) \cdot \frac{y - x}{t} \, dy.$$

Moreover, we have $\fint_{B_t(x)} \left| \frac{y-x}{t} \right|^2 dy \simeq 1$, which allows us to employ the first part of this theorem ensuring that

$$\left| \partial_t \fint_{B_t(x)} \phi \right| \leq \mathcal{C}(x) t^{-\frac{\varepsilon}{2} d(1-\beta)}$$

and, consequently,

$$\left| \fint_{B_R(x)} \phi - \fint_{B(x)} \phi \right| \leq \int_1^R \left| \partial_t \fint_{B_t(x)} \phi \right| \leq \mathcal{C}(x) \int_1^R t^{-\frac{\varepsilon}{2} d(1-\beta)} \lesssim \mathcal{C}(x) \mu(R)$$

recalling the definition of $\mu(r)$ from (1.18). In a similar fashion, we find that

$$\fint_{B_R(x)} \phi - \fint_{B_R} \phi = \fint_{B_R} \left(\phi(x+y) - \phi(y) \right) dy$$
$$= \fint_{B_R} \int_0^1 \nabla \phi(y + tx) \cdot x \, dt \, dy = |x| \int_0^1 \fint_{B_R} \nabla \phi(y + tx) \cdot \frac{x}{|x|} \, dt \, dy.$$

Since $\fint_{B_R} \left| \frac{x}{|x|} \right|^2 = 1$, we conclude as above that

$$\left| \fint_{B_R(x)} \phi - \fint_{B_R} \phi \right| \leq R^{-\frac{\varepsilon}{2} d(1-\beta)} |x| \int_0^1 \mathcal{C}(tx) \, dt.$$

We may now apply the previous estimates to the right-hand side of the following inequality,

$$\left| \fint_{B(x)} \phi \right| \leq \left| \fint_B \phi \right| + \left| \fint_B \phi - \fint_{B_R} \phi \right| + \left| \fint_{B_R} \phi - \fint_{B_R(x)} \phi \right| + \left| \fint_{B_R(x)} \phi - \fint_{B(x)} \phi \right|,$$

leading to

$$\left| \fint_{B(x)} \phi \right| \leq \left| \fint_B \phi \right| + \left(\mathcal{C}(0) + \mathcal{C}(x) + \int_0^1 \mathcal{C}(tx) \, dt \right) \mu(|x|)$$

when choosing $R := |x|$. Together, we have

$$\left(\fint_{B(x)} |\phi|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{2p}} \lesssim r_*(x)^{\frac{d}{2}} + r_e(x)^{\frac{d}{2}(\frac{1}{\varepsilon q}+1)} + \left| \fint_B \phi \right| + \left(\mathcal{C}(0) + \mathcal{C}(x) + \int_0^1 \mathcal{C}(tx) \, dt \right) \mu(|x|).$$

Observe that $r_*(x)^{\frac{d}{2}}$ is stochastically integrable with stretched exponential moment $\varepsilon(1-\beta)$ as $\left\langle \exp\left(\frac{1}{C} r_*^{\frac{\varepsilon}{2} d(1-\beta)} \right) \right\rangle < 2$ due to (1.14) and that $r_e(x)^{\frac{d}{2}(\frac{1}{\varepsilon q}+1)}$ is also integrable with the same moment $\varepsilon(1-\beta)$ since $\left(\frac{1}{\varepsilon q} + 1 \right)\varepsilon \leq \frac{\alpha}{\alpha+1}$ and $\left\langle \exp\left(\frac{1}{C} r_e^{\frac{\alpha}{\alpha+1} \frac{d}{2}(1-\beta)} \right) \right\rangle < 2$ due to (1.13). This closes the proof for the second part of the theorem as the same arguments also apply to σ by basically exchanging p and q. $\qquad\square$

3.2 Proof of Corollary 1.15: A quantitative two-scale expansion

Proof of Corollary 1.15. We follow the strategy presented in [18, 20] and first notice that by a scaling argument it is sufficient to prove the claim for $\delta = 1$. Next, we derive the following equation for z where $(\cdot)_1$ denotes averaging over a ball of radius 1:

$$-\nabla \cdot a\nabla z = \nabla \cdot (g - g_1 + (a\phi_i - \sigma_i)\nabla\partial_i u_{\text{hom},1}). \tag{3.2}$$

The calculation is carried out in [20] but we recall the main steps of the proof for completeness. From the representation $z := u - \big(u_{\text{hom},1} + \phi_i \partial_i u_{\text{hom},1}\big)$, we obtain

$$a\nabla z = a\nabla u - \partial_i u_{\text{hom},1} a(\nabla\phi_i + e_i) - a\phi_i \nabla\partial_i u_{\text{hom},1}$$

and

$$-\nabla \cdot a\nabla z = \nabla \cdot (g - g_1) + \nabla \cdot (a\phi_i \nabla\partial_i u_{\text{hom},1}) + \nabla\partial_i u_{\text{hom},1} \cdot (\nabla \cdot \sigma_i).$$

The claim is now a result of the skew-symmetry of σ_i, more precisely of

$$\nabla\partial_i u_{\text{hom},1} \cdot (\nabla \cdot \sigma_i) = -\nabla \cdot (\sigma_i \nabla\partial_i u_{\text{hom},1}).$$

Testing (3.2) with z entails

$$\int |\nabla z|_a^2 \lesssim \int |g - g_1|_{a^{-1}}^2 + \int |\phi_i \nabla\partial_i u_{\text{hom},1}|_a^2 + \int |\sigma_i \nabla\partial_i u_{\text{hom},1}|_{a^{-1}}^2.$$

For the first term on the right-hand side we aim to apply Poincaré's inequality noting that $g - g_1 \in W^{1,\frac{2q}{q-1}}(\mathbb{R}^d)$ is supported in B_{R+1}. Up to several constants depending in particular on R, we get

$$\int |g - g_1|_{a^{-1}}^2 \lesssim \Big(\fint_{B_{R+1}} |a^{-1}|^q\Big)^{\frac{1}{q}} \Big(\fint_{B_{R+1}} |\nabla g - \mathbb{1} * \nabla g|^{\frac{2q}{q-1}}\Big)^{\frac{q-1}{q}} \lesssim \Big(\int |\nabla g|^{\frac{2q}{q-1}}\Big)^{\frac{q-1}{q}}.$$

For the second term, we note that $\nabla\partial_i u_{\text{hom},1} = \frac{1}{|B|}\mathbb{1}_B * \nabla\partial_i u_{\text{hom}}$ (with $B := B_1(0)$) leading together with Jensen's inequality to

$$\int |\phi_i \nabla\partial_i u_{\text{hom},1}|_a^2 \lesssim \int |a||\phi_i|^2 \Big|\fint_{B(x)} \nabla\partial_i u_{\text{hom}}\Big|^2 \lesssim \int |a||\phi|^2 \, \mathbb{1}_B * |\nabla^2 u_{\text{hom}}|^2.$$

Proceeding with Hölder's inequality, we deduce

$$\int |\phi_i \nabla\partial_i u_{\text{hom},1}|_a^2 \lesssim \int \Big(\fint_{B(x)} |a||\phi|^2\Big)|\nabla^2 u_{\text{hom}}|^2$$

$$\leq \int \Big(\fint_{B(x)} |a|^p\Big)^{\frac{1}{p}}\Big(\fint_{B(x)} |\phi|^{\frac{2p}{p-1}}\Big)^{\frac{p-1}{p}} |\nabla^2 u_{\text{hom}}|^2.$$

Theorem 1.14 and $\fint_B \phi = 0$, therefore, result in

$$\int |\phi_i \nabla\partial_i u_{\text{hom},1}|_a^2 \lesssim \int 4Kr_e(x)^{\frac{d}{p}}\mathcal{C}(x)^2\mu(|x|)^2|\nabla^2 u_{\text{hom}}|^2 \lesssim \mathcal{C}_{g,p}^2 \int \mu(|x|)^2|\nabla g|^2$$

where we introduced the random field

$$\mathcal{C}_{g,p}^2 := \frac{\int \mathcal{C}(x)^2 r_e(x)^{\frac{d}{p}}\mu(|x|)^2|\nabla^2 u_{\text{hom}}|^2}{\int \mu(|x|)^2|\nabla g|^2}.$$

Employing the stationarity of $\mathcal{C}(x)$ and $r_e(x)$ as well as a weighted L^2 estimate for $\nabla^2 u_{\mathrm{hom}}$ (recall that $\mu(\cdot)$ as defined in (1.18) is a Muckenhoupt weight), we infer the bound

$$\langle \mathcal{C}^r_{g,p} \rangle \leq \left(\frac{\int \langle \mathcal{C}(x)^r r_e(x)^{\frac{dr}{2p}} \rangle^{\frac{2}{r}} \mu(|x|)^2 |\nabla^2 u_{\mathrm{hom}}|^2}{\int \mu(|x|)^2 |\nabla g|^2} \right)^{\frac{r}{2}} \lesssim \left\langle \mathcal{C}^r r_e^{\frac{dr}{2p}} \right\rangle$$

for any $r \geq 2$. Now let $r \geq \frac{2}{\varepsilon(1-\beta)}\left(1 + \frac{\alpha+1}{\alpha}\frac{\varepsilon}{p}\right)$ and observe that the previous estimate gives rise to

$$\left\langle \mathcal{C}^{(1+\frac{\alpha+1}{\alpha}\frac{\varepsilon}{p})^{-1}\varepsilon(1-\beta)r}_{g,p} \right\rangle \lesssim \left\langle \mathcal{C}^{(1+\frac{\alpha+1}{\alpha}\frac{\varepsilon}{p})^{-1}\varepsilon(1-\beta)r} r_e^{\frac{d}{2p}(1+\frac{\alpha+1}{\alpha}\frac{\varepsilon}{p})^{-1}\varepsilon(1-\beta)r} \right\rangle$$

$$\leq \left\langle \mathcal{C}^{\varepsilon(1-\beta)r} \right\rangle^{\frac{1}{1+\frac{\alpha+1}{\alpha}\frac{\varepsilon}{p}}} \left\langle r_e^{\frac{\alpha}{\alpha+1}\frac{d}{2}(1-\beta)r} \right\rangle^{\frac{\frac{\alpha+1}{\alpha}\frac{\varepsilon}{p}}{1+\frac{\alpha+1}{\alpha}\frac{\varepsilon}{p}}} \lesssim (cr)^r$$

using Hölder's inequality, the stochastic integrability of $\mathcal{C}$ and r_e from Theorem 1.14 and Lemma 1.11, respectively, and Lemma A.1. This ensures the announced stretched exponential moment bounds for $\mathcal{C}_{g,p}$. The same arguments also show that

$$\int |\sigma_i \nabla \partial_i u_{\mathrm{hom},1}|^2_{a^{-1}} \lesssim \mathcal{C}^2_{g,q} \int \mu(|x|)^2 |\nabla g|^2$$

together with the random field

$$\mathcal{C}^2_{g,q} := \frac{\int \mathcal{C}(x)^2 r_e(x)^{\frac{d}{q}} \mu(|x|)^2 |\nabla^2 u_{\mathrm{hom}}|^2}{\int \mu(|x|)^2 |\nabla g|^2}$$

allowing for the same stochastic integrability as $\mathcal{C}_{g,p}$ up to replacing p by q. $\qquad\square$

A Some auxiliary tools

Lemma A.1 (see e.g. [18, Lemma 6]). *The following statements on a nonnegative random variable F are equivalent.*

    1 *There exists a constant $C \geq 1$ such that*

$$\left\langle \exp\left(\frac{1}{C}F\right) \right\rangle < 2.$$

    2 *There exists some $p_0 \in \mathbb{N}$ and a constant $C \geq 1$ such that*

$$\left\langle F^p \right\rangle^{\frac{1}{p}} \leq Cp$$

    for all $p \in \mathbb{N}$, $p \geq p_0$.

Notation A.2. *For a matrix $M \in \mathbb{R}^{d \times d}$, we write*

$$|M|_{F1} := \sum_{i=1}^{d}\sum_{j=1}^{d}|M_{ij}|, \qquad |M|_{F\infty} := \sup_{i \leq d}\sup_{j \leq d}|M_{ij}|.$$

Lemma A.3. *Let* $M : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ *and suppose that*

$$\sum_D \left(\int_D M : a^{\frac{1}{2}} N a^{\frac{1}{2}} \right)^2 \leq c^2 \sup_D \||N|_{F\infty}\|^2_{L^\infty(D)}$$

for some $c > 0$ *and all bounded* $N : \mathbb{R}^d \to \mathbb{R}^{d \times d}$. *Then, we have*

$$\sum_D \left\||a^{\frac{1}{2}} M a^{\frac{1}{2}}|_{F1}\right\|^2_{L^1(D)} \leq c^2.$$

Proof. We first observe that

$$\sum_D \||M|_{F1}\|^2_{L^1(D)} = \sup_{\substack{N \in \mathbb{R}^{d \times d} \\ N \text{ bounded}}} \frac{\sum_D \left(\int_D M : N \right)^2}{\sup_D \||N|_{F\infty}\|^2_{L^\infty(D)}}$$

for all matrix-valued functions $M, N \in \mathbb{R}^{d \times d}$ provided that $N_{ij}(\cdot)$ is bounded for all i and j. This can be verified by elementary arguments from linear algebra. The previous identity, in particular, implies that the bound $\sum_D \||M|_{F1}\|^2_{L^1(D)} \leq c^2$ holds, if

$$\sum_D \left(\int_D M : N \right)^2 \leq c^2 \sup_D \||N|_{F\infty}\|^2_{L^\infty(D)}$$

for some $c > 0$ and all bounded $N : \mathbb{R}^d \to \mathbb{R}^{d \times d}$. Replacing M by $a^{\frac{1}{2}} M a^{\frac{1}{2}}$ and observing that $a^{\frac{1}{2}} M a^{\frac{1}{2}} : N = M : a^{\frac{1}{2}} N a^{\frac{1}{2}}$ allows to conclude. $\qquad\square$

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