Sharp deviation bounds and concentration phenomenon for the squared norm of a sub-gaussian vector

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Abstract

Let \mathbf{X} be a Gaussian zero mean vector with $\operatorname{Var}(\mathbf{X}) = B$. Then $\|\mathbf{X}\|^2$ well concentrates around its expectation $\mathbf{p} = \mathbb{E} \|\mathbf{X}\|^2 = \operatorname{tr} B$ provided that the latter is sufficiently large. Namely, $\mathbb{P}(\|\mathbf{X}\|^2 - \operatorname{tr} B > 2\sqrt{\operatorname{x}\operatorname{tr}(B^2)} + 2\|B\|\mathbf{x}) \leq e^{-\mathbf{x}}$ and $\mathbb{P}(\|\mathbf{X}\|^2 - \operatorname{tr} B < -2\sqrt{\operatorname{x}\operatorname{tr}(B^2)}) \leq e^{-\mathbf{x}}$; see Laurent and Massart (2000). This note provides an extension of these bounds to the case of a sub-gaussian vector \mathbf{X} . The results are based on the recent advances in Laplace approximation from Spokoiny (2022).

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1 Introduction

Let X be a zero mean Gaussian vector in \mathbb{R}^p for p large. Denote $B = \operatorname{Var}(X)$. Then for the squared norm $||X||^2$, it holds $\mathbb{E}||X||^2 = \operatorname{tr} B$, $\operatorname{Var}(||X||^2) = \operatorname{tr}(B^2)$, and this random variable concentrates around its expectation $\operatorname{tr} B$ in the sense that for any $\mathbf{x} > 0$

$$\mathbb{P}\left(\|\boldsymbol{X}\|^{2} - \operatorname{tr} B > 2\sqrt{\operatorname{x}\operatorname{tr}(B^{2})} + 2\|B\|\mathbf{x}\right) \leq e^{-\mathbf{x}}, \\
\mathbb{P}\left(\|\boldsymbol{X}\|^{2} - \operatorname{tr} B < -2\sqrt{\operatorname{x}\operatorname{tr}(B^{2})}\right) \leq e^{-\mathbf{x}};$$
(1.1)

see e.g. Laurent and Massart (2000). The upper bound here can easily be extended to the sub-gaussian case; see e.g. Hsu et al. (2012) or Section 2.1 later. Rudelson and Vershynin

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(2013) described the effect of sun-gaussian concentration and established deviation bounds for the centered quadratic form $||\mathbf{X}||^2 - \mathbf{E} ||\mathbf{X}||^2$ by extending the Hanson-Wright inequality (see Hanson and Wright (1971)). In the recent years, a number of new results were obtained in this direction. We refer to Klochkov and Zhivotovskiy (2020) for an extensive overview and advanced results on Hanson-Wright type concentration inequalities. This note aims at extending the concentration result from (1.1) to a non-gaussian case under possibly mild conditions. Namely, we establish a version of the upper bound in (1.1) using local smoothness of the moment generating function $\mathbf{E}e^{\langle u, \mathbf{X} \rangle}$ and the recent advances in Laplace approximation from Spokoiny (2022). The lower bound is obtained by similar arguments applied to the characteristic function $\mathbf{E}e^{i\langle u, \mathbf{X} \rangle}$.

The paper is organized as follows. Section 2.1 provides a simple but rough upper bound under sub-gaussian condition on X. The main results about concentration of $||X||^2$ are collected in Section 2.2. Section 2.3 specifies the results to the case when Xis a normalized sum of independent random vectors. In Section 2.4 we extend the upper bound to a sub-exponential case. Some useful technical facts about Gaussian quadratic forms are collected in the Appendix A and Appendix B.

2 Deviation bounds for non-Gaussian quadratic forms

This section collects some probability bounds for non-Gaussian quadratic forms starting from the sub-gaussian case. Then we extend the result to the case of exponential tails.

2.1 Sub-gaussian upper bound

Let $\boldsymbol{\xi}$ be a random vector in \mathbb{R}^p , $p \leq \infty$ satisfying $\mathbb{E}\boldsymbol{\xi} = 0$. We suppose that there exists an operator \mathbb{V} in \mathbb{R}^p such that

$$\log \mathbb{E} \exp\left(\langle \boldsymbol{u}, \mathbb{V}^{-1}\boldsymbol{\xi}\rangle\right) \leq \frac{\|\boldsymbol{u}\|^2}{2}, \qquad \boldsymbol{u} \in \mathbb{R}^p.$$
(2.1)

In the Gaussian case, one obviously takes $\mathbb{V}^2 = \operatorname{Var}(\boldsymbol{\xi})$. In general, $\mathbb{V}^2 \geq \operatorname{Var}(\boldsymbol{\xi})$. We consider a quadratic form $\|Q\boldsymbol{\xi}\|^2$, where $\boldsymbol{\xi}$ satisfies (2.1) and Q is a given linear operator $\mathbb{R}^p \to \mathbb{R}^q$ such that $B = Q \mathbb{V}^2 Q^\top$ is a trace operator. Denote

$$\mathbf{p} = \operatorname{tr}(B), \qquad \mathbf{v}^2 \stackrel{\text{def}}{=} \operatorname{tr}(B^2).$$

We show that under (2.1), the quadratic form $\|Q\boldsymbol{\xi}\|^2$ follows the same upper deviation bound $\mathbb{P}(\|Q\boldsymbol{\xi}\|^2 \ge z^2(B,\mathbf{x})) \le e^{-\mathbf{x}}$ with $z^2(B,\mathbf{x})$ from (B.3) as in the Gaussian case. Similar results can be found e.g. in Hsu et al. (2012). We present an independent proof for reference convenience.

Theorem 2.1. Suppose (2.1). With $B = Q \mathbb{V}^2 Q^{\top}$, it holds for any $\mu < 1/\|B\|$

$$\mathbb{E}\exp\left(\frac{\mu}{2}\|Q\boldsymbol{\xi}\|^2\right) \le \exp\left(\frac{\mu^2\operatorname{tr}(B^2)}{4(1-\|B\|\mu)} + \frac{\mu\operatorname{tr}(B)}{2}\right)$$

and for any x > 0

$$\mathbb{P}\left(\|Q\boldsymbol{\xi}\|^2 > \mathbf{p} + 2\mathbf{v}\sqrt{\mathbf{x}} + 2\mathbf{x}\right) \le e^{-\mathbf{x}}.$$
(2.2)

The bounds (B.9) through (B.10) of Theorem B.5 continue to apply as well.

Proof. Normalization by ||B|| reduces the proof to ||B|| = 1. For $\mu \in (0,1)$, we use

$$\mathbb{E}\exp\left(\mu\|Q\boldsymbol{\xi}\|^{2}/2\right) = \mathbb{E}\mathbb{E}_{\boldsymbol{\gamma}}\exp\left(\mu^{1/2}\langle \mathbb{W}Q^{\top}\boldsymbol{\gamma}, \mathbb{W}^{-1}\boldsymbol{\xi}\rangle\right),\tag{2.3}$$

where γ is standard Gaussian under \mathbb{E}_{γ} independent on $\boldsymbol{\xi}$. Application of Fubini's theorem, (2.1), and (B.5) yields

$$\mathbb{E}\exp\left(\frac{\mu}{2}\|Q\boldsymbol{\xi}\|^2\right) \le \mathbb{E}_{\boldsymbol{\gamma}}\exp\left(\frac{\mu}{2}\|\boldsymbol{W}Q^{\top}\boldsymbol{\gamma}\|^2\right) \le \exp\left(\frac{\mu^2\operatorname{tr}(B^2)}{4(1-\mu)} + \frac{\mu\operatorname{tr}(B)}{2}\right).$$

Further we proceed as in the Gaussian case.

The bound (2.2) looks identical to the Gaussian case, however, there is an essential difference: $\mathbf{p} = \operatorname{tr}(B)$ can be much larger than $\mathbf{E} || Q \boldsymbol{\xi} ||^2 = Q^{\top} \operatorname{Var}(\boldsymbol{\xi}) Q$. For supporting the concentration phenomenon of $|| Q \boldsymbol{\xi} ||^2$ around its expectation $\mathbf{E} || Q \boldsymbol{\xi} ||^2 = \operatorname{tr} \{Q^{\top} \operatorname{Var}(\boldsymbol{\xi}) Q\}$, the result from (2.2) is not accurate enough. Rudelson and Vershynin (2013) established deviation bounds for the centered quadratic form $|| Q \boldsymbol{\xi} ||^2 - \mathbf{E} || Q \boldsymbol{\xi} ||^2$ by applying Hanson-Wright inequality (see Hanson and Wright (1971)) to its absolute value. The next section presents some sufficient conditions for obtaining sharp Gaussian-like deviation bounds.

2.2 Sharp deviation bounds for the norm of a sub-gaussian vector

Let $\boldsymbol{\xi}$ be a centered random vector in \mathbb{R}^p with sub-gaussian tails. We study concentration effect of the squared norm $\|Q\boldsymbol{X}\|^2$ for a linear mapping Q and for $\boldsymbol{X} = \boldsymbol{V}^{-1}\boldsymbol{\xi}$ being the standardized version of $\boldsymbol{\xi}$, where $\boldsymbol{V}^2 = \operatorname{Var}(\boldsymbol{\xi})$. More generally, we allow $\boldsymbol{V}^2 \geq \operatorname{Var}(\boldsymbol{\xi})$ yielding $\operatorname{Var}(\boldsymbol{X}) \leq I_p$ to incorporate the case when $\operatorname{Var}(\boldsymbol{\xi})$ is ill-posed. Later we assume the following condition.

4 Sharp deviation bounds for the squared norm of a sub-gaussian vector

(X) A random vector $X \in \mathbb{R}^p$ satisfies $\mathbb{E}X = 0$, $\operatorname{Var}(X) \leq I_p$. The function $\phi(u) \stackrel{\text{def}}{=} \log \mathbb{E} e^{\langle u, X \rangle}$ is finite and fulfills for some C_{ϕ}

$$\phi(\boldsymbol{u}) \stackrel{\text{def}}{=} \log \mathbb{E} \mathrm{e}^{\langle \boldsymbol{u}, \boldsymbol{X} \rangle} \le \frac{\mathsf{C}_{\phi} \|\boldsymbol{u}\|^2}{2}, \qquad \boldsymbol{u} \in \mathbb{R}^p.$$
(2.4)

The constant C_{ϕ} can be quite large, it does not show up in the leading term of the obtained bound. Also we will only use this condition for $||u|| \ge g$ for some sufficiently large g.

Given a linear mapping $Q: \mathbb{R}^p \to \mathbb{R}^q$ with $||Q|| \leq 1$, we expect the quadratic form $||QX||^2$ behaves nearly as X were a Gaussian vector. In that case for $\mu < 1$

$$\boldsymbol{E} \exp(\mu \| Q \boldsymbol{X} \|^2 / 2) = \det(I_p - \mu B)^{-1/2}, \qquad B \stackrel{\text{def}}{=} Q^\top \operatorname{Var}(\boldsymbol{X}) Q$$

Our results provide a bound on $|\mathbf{E} \exp(\mu || Q \mathbf{X} ||^2/2) - \det(\mathbf{I}_p - \mu B)^{-1/2}|$ for \mathbf{X} non-Gaussian but (\mathbf{X}) is fulfilled. Define

$$\mathbf{p} \stackrel{\text{def}}{=} \mathbf{E} \|Q\mathbf{X}\|^2 = \operatorname{tr}\{Q^\top \operatorname{Var}(\mathbf{X})Q\} = \operatorname{tr} B,$$
$$\mathbf{p}_Q \stackrel{\text{def}}{=} \mathbf{E} \|Q\boldsymbol{\gamma}\|^2 = \operatorname{tr}(Q^\top Q).$$

Fix g and define for $u \in \mathbb{R}^p$ with $||u|| \leq g$ a measure \mathbb{E}_u by

$$\boldsymbol{E}_{\boldsymbol{u}} \eta \stackrel{\text{def}}{=} \frac{\boldsymbol{E}(\eta \, \mathrm{e}^{\langle \boldsymbol{u}, \boldsymbol{X} \rangle})}{\boldsymbol{E} \mathrm{e}^{\langle \boldsymbol{u}, \boldsymbol{X} \rangle}}.$$
(2.5)

Also define

$$\tau_{3} \stackrel{\text{def}}{=} \sup_{\|\boldsymbol{u}\| \leq \mathbf{g}} \frac{1}{\|\boldsymbol{u}\|^{3}} \left| \boldsymbol{\mathcal{E}}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{X} - \boldsymbol{\mathcal{E}}_{\boldsymbol{u}} \boldsymbol{X} \rangle^{3} \right|,$$

$$\tau_{4} \stackrel{\text{def}}{=} \sup_{\|\boldsymbol{u}\| \leq \mathbf{g}} \frac{1}{\|\boldsymbol{u}\|^{4}} \left| \boldsymbol{\mathcal{E}}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{X} - \boldsymbol{\mathcal{E}}_{\boldsymbol{u}} \boldsymbol{X} \rangle^{4} - 3 \left\{ \boldsymbol{\mathcal{E}}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{X} - \boldsymbol{\mathcal{E}}_{\boldsymbol{u}} \boldsymbol{X} \rangle^{2} \right\}^{2} \right|.$$
(2.6)

These quantities are typically not only finite but also very small. Indeed, for X Gaussian they just vanish. If X is a normalized sum of n i.i.d. centred random vectors ξ_i then $\tau_m \simeq n^{-m/2+1}$; see Section 2.3.

Theorem 2.2. Fix a linear mapping $Q: \mathbb{R}^p \to \mathbb{R}^q$ s.t. ||Q|| = 1. Let a random vector $X \in \mathbb{R}^p$ satisfy $\mathbb{E}X = 0$, $\operatorname{Var}(X) \leq I_p$, and (X). Let also τ_3 and τ_4 be given by (2.6) and g be fixed to ensure $\omega \stackrel{\text{def}}{=} \operatorname{g} \tau_3/2 \leq 1/3$. Consider $\mu > 0$ satisfying

$$C_{\phi} \mu \le 1/3, \qquad \mu^{-1} g^2 \ge 9 C_{\phi} p_Q,$$
 (2.7)

with $\mathbf{p}_Q = \operatorname{tr}(Q^\top Q)$. Further, define

$$\mathbf{x}_{\mu} \stackrel{\text{def}}{=} \frac{1}{4} \left(\sqrt{\mathsf{C}_{\phi}^{-1} \mu^{-1} \mathsf{g}^2} - \sqrt{\mathsf{p}_Q} \right)^2,$$

$$\epsilon_{\mu} \stackrel{\text{def}}{=} \mathsf{C}_{\phi} \, \mu + \mathsf{C}_{\phi} \, \mu \sqrt{\mathsf{p}_Q / \mathsf{x}_{\mu}} \,.$$
(2.8)

It holds with $B = Q^{\top} \operatorname{Var}(\boldsymbol{X}) Q$

$$\left| \mathbf{E} \exp(\mu \| Q \mathbf{X} \|^2 / 2) - \det(I_p - \mu B)^{-1/2} \right| \le \Delta_\mu \det(I_p - \mu B)^{-1/2}, \qquad (2.9)$$

where, with some small quantities \diamondsuit_4 and ρ_{μ} given below,

$$\Delta_{\mu} \leq \diamondsuit_4 + \rho_{\mu} + \frac{1}{1 - \epsilon_{\mu}} \exp\{\mathsf{C}_{\phi} \,\mu\,\mathsf{p}_Q/2 - (1 - \epsilon_{\mu})\mathsf{x}_{\mu}\}.$$

$$(2.10)$$

Remark 2.1. Conditions (2.7) imply $\mathbf{x}_{\mu} \geq \mathbf{p}_{Q}$ and $\epsilon_{\mu} \leq 2/3$. If $\mu^{-1}\mathbf{g}^{2} \gg \mathbf{p}_{Q}$ then $\mathbf{x}_{\mu} \gg \mathbf{p}_{Q}$, and hence the last term in (2.10) is quite small. The same holds for the value ρ_{μ} ; see later in the proof.

Proof. We use representation (2.3) and Fubini theorem: with $\mathbb{E}_{\gamma} = \mathbb{E}_{\gamma \sim \mathcal{N}(0,I)}$

$$\mathbb{E}\exp\left(\mu\|Q\mathbf{X}\|^{2}/2\right) = \mathbb{E}\mathbb{E}_{\boldsymbol{\gamma}}\exp\left(\mu^{1/2}\langle Q^{\top}\boldsymbol{\gamma}, \mathbf{X}\rangle\right) = \mathbb{E}_{\boldsymbol{\gamma}}\exp\phi(\mu^{1/2}Q^{\top}\boldsymbol{\gamma}).$$
(2.11)

Further,

$$\mathbf{E}_{\boldsymbol{\gamma}} \exp \phi(\mu^{1/2} Q^{\top} \boldsymbol{\gamma}) = \mathbf{E}_{\boldsymbol{\gamma}} \exp \phi(\mu^{1/2} Q^{\top} \boldsymbol{\gamma}) \operatorname{I\!I}(\|\mu^{1/2} Q^{\top} \boldsymbol{\gamma}\| \le \mathsf{g})
+ \mathbf{E}_{\boldsymbol{\gamma}} \exp \phi(\mu^{1/2} Q^{\top} \boldsymbol{\gamma}) \operatorname{I\!I}(\|\mu^{1/2} Q^{\top} \boldsymbol{\gamma}\| > \mathsf{g}).$$
(2.12)

Each summand here will be bounded separately starting from the second one. By (2.4) and (B.11) of Theorem B.5, it holds under the condition $\epsilon_{\mu} < 1$ for ϵ_{μ} from (2.8)

Now we check that $\phi(u)$ satisfies (\mathcal{T}_3) and (\mathcal{T}_4) : for any $||u|| \leq g$

$$\begin{aligned} |\delta_{3}(\boldsymbol{u})| &\stackrel{\text{def}}{=} \left| \phi(\boldsymbol{u}) - \frac{1}{2} \langle \phi''(0), \boldsymbol{u}^{\otimes 2} \rangle \right| &\leq \frac{\tau_{3}}{6} \|\boldsymbol{u}\|^{3} ,\\ |\delta_{4}(\boldsymbol{u})| &\stackrel{\text{def}}{=} \left| \phi(\boldsymbol{u}) - \frac{1}{2} \langle \phi''(0), \boldsymbol{u}^{\otimes 2} \rangle - \frac{1}{6} \langle \phi'''(0), \boldsymbol{u}^{\otimes 3} \rangle \right| &\leq \frac{\tau_{4}}{24} \|\boldsymbol{u}\|^{4} . \end{aligned}$$

$$(2.14)$$

Consider first the univariate case. Let a r.v. X satisfy $\mathbb{E}X = 0$ and $\mathbb{E}X^2 \leq \sigma^2$. Define for any $t \in [0, g]$ a measure \mathbb{P}_t s.t. for any r.v. η

$$\mathbb{E}_t \eta \stackrel{\text{def}}{=} \frac{\mathbb{E}(\eta \, \mathrm{e}^{tX})}{\mathbb{E} \mathrm{e}^{tX}} \,.$$

Consider $\phi(t) \stackrel{\text{def}}{=} \log \mathbb{E} e^{tX}$ as a function of $t \in [0, \lambda]$. It is well defined and satisfies $\phi(0) = \phi'(0) = 0$, $\phi''(0) = \mathbb{E} X^2 \leq \sigma^2$,

$$\phi'(t) = \mathbf{E}_t X,$$

$$\phi''(t) = \mathbf{E}_t (X - \mathbf{E}_t X)^2,$$

$$\phi'''(t) = \mathbf{E}_t (X - \mathbf{E}_t X)^3,$$

$$\phi^{(4)}(t) = \mathbf{E}_t (X - \mathbf{E}_t X)^4 - 3 \{\mathbf{E}_t (X - \mathbf{E}_t X)^2\}^2.$$

Therefore, conditions (\mathcal{T}_3) and (\mathcal{T}_4) follow from (2.6). The multivariate case can be reduced to the univariate one by fixing a direction $u \in \mathbb{R}^p$ and considering the function $\phi(tu)$ of t.

Next consider the first term in the right hand-side of (2.12). Define $\mathcal{U} = \{ \boldsymbol{u} \colon \| \boldsymbol{\mu}^{1/2} Q^{\top} \boldsymbol{u} \| \leq g \}$. Then with $C_p = (2\pi)^{-p/2}$

$$\mathbb{E}_{\boldsymbol{\gamma}} \exp \phi(\mu^{1/2} Q^{\top} \boldsymbol{\gamma}) \, \mathbb{I}(\|\mu^{1/2} Q^{\top} \boldsymbol{\gamma}\| \leq g) = \mathtt{C}_p \int_{\mathcal{U}} \mathrm{e}^{f_{\mu}(\boldsymbol{u})} \, d\boldsymbol{u} \,,$$

where

$$f_{\mu}(\boldsymbol{u}) = \phi(\mu^{1/2} Q^{\top} \boldsymbol{u}) - \|\boldsymbol{u}\|^2/2$$

so that $f_{\mu}(0) = 0$, $\nabla f_{\mu}(0) = 0$. Also define

$$D^2_{\mu} \stackrel{\text{def}}{=} -\nabla^2 f_{\mu}(0) = -\mu Q^{\top} \operatorname{Var}(\boldsymbol{X}) Q + I_p = I_p - \mu B.$$

The function $f_{\mu}(\boldsymbol{u})$ inherits smoothness properties of $\phi(\mu^{1/2}Q^{\top}\boldsymbol{u})$. In particular,

$$\left|f_{\mu}(\boldsymbol{u}) - \frac{1}{2} \|D_{\mu}\boldsymbol{u}\|^{2}\right| \leq \frac{\tau_{3}}{6} \|\mu^{1/2}Q\boldsymbol{u}\|^{3}.$$

We apply Proposition A8 from Spokoiny (2022) to $f_{\mu}(\boldsymbol{u})$ yielding

$$\left| \frac{\int_{\mathcal{U}} e^{f_{\mu}(\boldsymbol{u})} d\boldsymbol{u} - \int_{\mathcal{U}} e^{-\|D_{\mu}\boldsymbol{u}\|^{2}/2} d\boldsymbol{u}}{\int e^{-\|D_{\mu}\boldsymbol{u}\|^{2}/2} d\boldsymbol{u}} \right| \leq \diamondsuit_{4},$$

$$\diamondsuit_{4} = \frac{1}{16(1-\omega)^{2}} \left\{ \tau_{3}^{2} \left(\mathbf{p}_{\mu} + 2\alpha_{\mu} \right)^{3} + 2\tau_{4} (\mathbf{p}_{\mu} + \alpha_{\mu})^{2} \right\}, \qquad (2.15)$$

with $\omega = g \tau_3/2 \le 1/3$ and

$$\mathbf{p}_{\mu} \stackrel{\text{def}}{=} \operatorname{tr} \{ D_{\mu}^{-2}(\mu Q^{\top} Q) \},$$
$$\alpha_{\mu} \stackrel{\text{def}}{=} \| D_{\mu}^{-1}(\mu Q^{\top} Q) D_{\mu}^{-1} \|$$

Note that $||B|| \leq 1$ implies with $\mathbf{p}_Q = \operatorname{tr}(Q^\top Q)$

$$\mathtt{p}_{\mu} \leq rac{\mu}{1-\mu} \mathtt{p}_Q, \qquad lpha_{\mu} \leq rac{\mu}{1-\mu},$$

and

$$\diamond_4 \le \frac{1}{16(1-\omega)^2} \Big\{ \frac{\tau_3^2 \,\mu^3 (\mathbf{p}_Q + 2)^3}{(1-\mu)^3} + \frac{2\tau_4 \mu^2 (\mathbf{p}_Q + 1)^2}{(1-\mu)^2} \Big\}.$$
(2.16)

Furthermore, it holds

$$\rho_{\mu} \stackrel{\text{def}}{=} 1 - \frac{\int_{\mathcal{U}} \mathrm{e}^{-\|D_{\mu}\boldsymbol{u}\|^{2}/2} \, d\boldsymbol{u}}{\int \mathrm{e}^{-\|D_{\mu}\boldsymbol{u}\|^{2}/2} \, d\boldsymbol{u}} = \mathbb{P}\left(\|\boldsymbol{\mu}^{1/2}\boldsymbol{Q}^{\top}\boldsymbol{D}_{\mu}^{-1}\boldsymbol{\gamma}\| > \mathsf{g}\right)$$
$$\leq \mathbb{P}\left(\|\boldsymbol{Q}^{\top}\boldsymbol{\gamma}\|^{2} > (1-\mu)\boldsymbol{\mu}^{-1}\mathsf{g}^{2}\right), \tag{2.17}$$

and the latter value is small provided $\mu^{-1}g^2 \gg p_Q$. This and (2.15) yield the bound

$$\left| \frac{\int_{\mathcal{U}} \mathrm{e}^{f_{\mu}(\boldsymbol{u})} \, d\boldsymbol{u}}{\int \mathrm{e}^{-\|D_{\mu}\boldsymbol{u}\|^{2}/2} \, d\boldsymbol{u}} - 1 \right| \le \diamondsuit_{4} + \rho_{\mu} \,. \tag{2.18}$$

It remains to note that

$$C_p \int e^{-\|D_{\mu}u\|^2/2} du = \frac{1}{\det D_{\mu}} = \det(I_p - \mu B)^{-1/2}$$

and (2.9) follows from (2.13) and (2.18) in view of $\det(I_p - \mu B) \leq 1$.

Upper deviation bounds for $||QX||^2$ can now be derived as in the Gaussian case by applying (2.9) with a proper choice of μ . This leads to a surprisingly sharp bound on the upper deviation probability which almost repeats bound (B.8) for X Gaussian.

Corollary 2.3. Let $B = Q^{\top} \operatorname{Var}(\mathbf{X})Q$. With $\mathbf{x} > 0$ fixed, define $\mu = \mu(\mathbf{x})$ by $\mu^{-1} = 1 + \sqrt{\operatorname{tr}(B^2)/(4\mathbf{x})}$. Assume the condition of Theorem 2.2 for this choice of μ . Then

$$\mathbb{P}(\|Q\mathbf{X}\| > z(B,\mathbf{x})) = \mathbb{P}(\|Q\mathbf{X}\|^2 > \operatorname{tr} B + 2\sqrt{\operatorname{x}\operatorname{tr}(B^2)} + 2\mathbf{x}) \le (1 + \Delta_{\mu})e^{-\mathbf{x}}.$$

Remark 2.2. Theorem 2.2 requires μ to be a small number to ensure (2.7). Alternatively, we need $\operatorname{tr}(B^2) \gg \mathbf{x}$. This is an important message: concentration of the squared

norm $||QX||^2$ is only possible in high dimension when $\operatorname{tr}(B^2)$ is sufficiently large. In typical situations it holds $\operatorname{tr} B \approx \operatorname{tr}(Q^{\top}Q)$ and also $\operatorname{tr}(B^2) \asymp \operatorname{tr} B \approx p_Q$. Then the effective trace of $Q^{\top}Q$ should be large. The choice of μ by $\mu^{-1} = 1 + \sqrt{\operatorname{tr}(B^2)/(4\mathbf{x})}$ leads to $\mu \asymp \sqrt{\mathbf{x} \mathbf{p}_Q}$. This helps to evaluate the term \diamondsuit_4 from (2.16) in the bound (2.10). Namely, (2.16) yields

$$\diamondsuit_4 \lesssim au_3^2 \, \mathrm{x}^{3/2} \, \mathrm{p}_Q^{3/2} + au_4 \, \mathrm{x} \, \mathrm{p}_Q$$

This value is small provided $\tau_3^2 \ll \mathbf{p}_Q^{-3/2}$ and $\tau_4 \ll \mathbf{p}_Q^{-1}$.

For getting the bound on the lower deviation probability, we need an analog of (2.9) for μ negative. Representation (2.11) reads as

$$\boldsymbol{E} e^{-\mu \|Q\boldsymbol{X}\|^2/2} = \boldsymbol{E} \boldsymbol{E}_{\boldsymbol{\gamma}} e^{i\sqrt{\mu}\langle Q^\top \boldsymbol{\gamma}, \boldsymbol{X} \rangle} = \boldsymbol{E}_{\boldsymbol{\gamma}} \boldsymbol{E} e^{i\sqrt{\mu}\langle Q^\top \boldsymbol{\gamma}, \boldsymbol{X} \rangle}$$
(2.19)

with $\mathbf{i} = \sqrt{-1}$. Our technique requires that the characteristic function $\mathbb{E} \exp(\mathbf{i} \langle u, X \rangle)$ does not vanish. This allows to define

$$\mathsf{f}(\boldsymbol{u}) \stackrel{\mathrm{def}}{=} \log E \,\mathrm{e}^{\mathrm{i} \langle \boldsymbol{u}, \boldsymbol{X}
angle}$$
 .

Later we assume that the function f(u) satisfies the condition similar to (X).

(iX) For some fixed g and C_f , the function $f(u) = \log \mathbb{E} e^{i\langle u, X \rangle}$ satisfies

$$|\mathsf{f}(\boldsymbol{u})| = |\log I\!\!E \, \mathrm{e}^{\mathrm{i} \langle \boldsymbol{u}, \boldsymbol{X} \rangle}| \le \mathsf{C}_\mathsf{f} \,, \qquad \|\boldsymbol{u}\| \le \mathsf{g} \,.$$

Note that this condition can easily be ensured by replacing X with $X + \alpha \gamma$ for any positive α and $\gamma \sim \mathcal{N}(0, I_p)$. The constant C_f is unimportant, it does not show up in our results. It, however, enables us to define similarly to (2.6)

$$\tau_{3} \stackrel{\text{def}}{=} \sup_{\|\boldsymbol{u}\| \leq g} \frac{1}{\|\boldsymbol{u}\|^{3}} \left| \boldsymbol{\mathcal{E}}_{\boldsymbol{i}\boldsymbol{u}} \langle \boldsymbol{i}\boldsymbol{u}, \boldsymbol{X} - \boldsymbol{\mathcal{E}}_{\boldsymbol{i}\boldsymbol{u}} \boldsymbol{X} \rangle^{3} \right|,$$

$$\tau_{4} \stackrel{\text{def}}{=} \sup_{\|\boldsymbol{u}\| \leq g} \frac{1}{\|\boldsymbol{u}\|^{4}} \left| \boldsymbol{\mathcal{E}}_{\boldsymbol{i}\boldsymbol{u}} \langle \boldsymbol{i}\boldsymbol{u}, \boldsymbol{X} - \boldsymbol{\mathcal{E}}_{\boldsymbol{i}\boldsymbol{u}} \boldsymbol{X} \rangle^{4} - 3 \left\{ \boldsymbol{\mathcal{E}}_{\boldsymbol{i}\boldsymbol{u}} \langle \boldsymbol{i}\boldsymbol{u}, \boldsymbol{X} - \boldsymbol{\mathcal{E}}_{\boldsymbol{i}\boldsymbol{u}} \boldsymbol{X} \rangle^{2} \right\}^{2} \right|.$$

$$(2.20)$$

Theorem 2.4. Let ||Q|| = 1, $p_Q = tr(Q^{\top}Q)$. Let X satisfy $\mathbb{E}X = 0$, $Var(X) \leq I_p$, and (iX) for a fixed g. Let also τ_3 and τ_4 be given by (2.20) and $\omega \stackrel{\text{def}}{=} g \tau_3/2 \leq 1/3$. For any $\mu > 0$ s.t. $\mu^{-1}g^2 \geq 4p_Q$, it holds with $B = Q^{\top} Var(X)Q$

$$\mathbb{E}e^{-\mu \|Q\mathbf{X}\|^{2}/2} - \det(I_{p} + \mu B)^{-1/2} | \leq (\diamondsuit_{4} + \rho_{\mu}) \det(I_{p} + \mu B)^{-1/2} + \rho_{\mu};$$

$$\rho_{\mu} \leq \mathbb{P}_{\gamma} (\|Q\gamma\|^{2} \geq \mu^{-1}g^{2}) \leq \frac{1}{4} (\sqrt{\mu^{-1}g^{2}} - \sqrt{p_{Q}})^{2}.$$
 (2.21)

Proof. We follow the line of the proof of Theorem 2.2 replacing everywhere $\phi(\boldsymbol{u})$ with $f(\boldsymbol{u})$. In particular, we start with representation (2.19) and apply

$$\begin{split} \mathbf{E} \, \mathrm{e}^{-\mu \|Q\mathbf{X}\|^{2}/2} &= \mathbf{E}_{\gamma} \mathrm{e}^{\mathrm{f}(\sqrt{\mu} \, Q^{\top} \gamma)} \\ &= \mathbf{E}_{\gamma} \mathrm{e}^{\mathrm{f}(\sqrt{\mu} \, Q^{\top} \gamma)} \, \mathrm{I\!I}(\|\sqrt{\mu} \, Q^{\top} \gamma\| \leq \mathrm{g}) + \mathbf{E}_{\gamma} \mathrm{e}^{\mathrm{f}(\sqrt{\mu} \, Q^{\top} \gamma)} \, \mathrm{I\!I}(\|\sqrt{\mu} \, Q^{\top} \gamma\| > \mathrm{g}). \end{split}$$

It holds

$$\mathsf{f}(0) = 0, \quad \nabla \mathsf{f}(0) = 0, \quad -\nabla^2 \mathsf{f}(0) = \operatorname{Var}(\boldsymbol{X}) \le I_p.$$

Moreover, smoothness conditions (2.14) are automatically fulfilled for f(u) with the same τ_3 and τ_4 . The most important observation for the proof is that the bound (2.18) continues to apply for $\mu < 0$ and

$$f_{\mu}(\boldsymbol{u}) = \mathsf{f}(\sqrt{\mu} Q^{\top} \boldsymbol{u}) - \|\boldsymbol{u}\|^2 / 2,$$

with \diamondsuit_4 from (2.15) and

$$D_{\mu}^{2} \stackrel{\text{def}}{=} -\nabla^{2} f_{\mu}(0) = \mu Q^{\top} \operatorname{Var}(\boldsymbol{X}) Q + I_{p} = I_{p} + \mu B,$$

$$p_{\mu} \stackrel{\text{def}}{=} \operatorname{tr} \left\{ D_{\mu}^{-2}(\mu Q^{\top} Q) \right\} \leq \frac{\mu}{1+\mu} \operatorname{tr}(Q^{\top} Q) \leq \mu p_{Q},$$

$$\alpha_{\mu} \stackrel{\text{def}}{=} \| D_{\mu}^{-1}(\mu Q^{\top} Q) D_{\mu}^{-1} \| \leq \frac{\mu}{1+\mu},$$

and $\rho_{\mu} \leq \mathbb{P}(\|Q\boldsymbol{\gamma}\|^2 \geq \mu^{-1}g^2)$; cf. (2.17). This yields

$$\left| \mathbb{E}_{\boldsymbol{\gamma}} \mathrm{e}^{\mathrm{f}(\sqrt{\mu} Q^{\top} \boldsymbol{\gamma})} \, \mathrm{I\!I}(\|\sqrt{\mu} Q^{\top} \boldsymbol{\gamma}\| \leq \mathrm{g}) - \frac{1}{\det(I_p + \mu B)^{1/2}} \right| \leq \frac{\diamondsuit_4 + \rho_{\mu}}{\det(I_p + \mu B)^{1/2}}$$

Finally we use $|e^{f(u)}| \leq 1$ and thus,

$$\left| \mathbb{E}_{\boldsymbol{\gamma}} \mathrm{e}^{\mathrm{f}(\sqrt{\mu} Q^{\top} \boldsymbol{\gamma})} \, \mathrm{I\!I}(\|\sqrt{\mu} Q^{\top} \boldsymbol{\gamma}\| > \mathrm{g}) \right| \leq \mathbb{P}\left(\|\sqrt{\mu} Q^{\top} \boldsymbol{\gamma}\| > \mathrm{g} \right)$$

and (2.21) follows.

Corollary 2.5. With $\mathbf{x} > 0$ fixed, define $\mu = 2\mathbf{v}^{-1}\sqrt{\mathbf{x}}$ for $\mathbf{v}^2 = \operatorname{tr} B^2$. Assume the condition of Theorem 2.4 for this choice of μ . Then with $\rho_{\mu} = \mathbb{P}(||Q\gamma||^2 \ge \mu^{-1}g^2)$

$$\mathbb{P}\left(\|Q\mathbf{X}\|^2 < \operatorname{tr} B - 2\mathtt{v}\sqrt{\mathtt{x}}\right) \le (1 + \diamondsuit_4 + \rho_\mu) \mathrm{e}^{-\mathtt{x}} + \rho_\mu \exp(\mathtt{v}^{-1} \operatorname{tr} B\sqrt{\mathtt{x}} - 2\mathtt{x}).$$

Proof. By the exponential Chebyshev inequality and (2.21)

$$\mathbb{P}\left(\operatorname{tr} B - \|Q\mathbf{X}\|^{2} > 2\mathtt{v}\sqrt{\mathtt{x}}\right) \leq \exp(-\mu\,\mathtt{v}\sqrt{\mathtt{x}})\mathbb{E}\exp\left\{\mu\,\operatorname{tr} B/2 - \mu\|Q\mathbf{X}\|^{2}/2\right\} \\
\leq \exp(\mu\,\operatorname{tr} B/2 - \mu\,\mathtt{v}\sqrt{\mathtt{x}})\left\{(\diamondsuit_{4} + \rho_{\mu})\det(I_{p} + \mu B)^{-1/2} + \rho_{\mu}\right\}.$$

It remains to note that by $x - \log(1+x) \le x^2/2$ and $\mu = 2\mathbf{v}^{-1}\sqrt{\mathbf{x}}$, it holds

$$-\mu \, v \sqrt{x} + \mu \, \mathrm{tr} \, B/2 + \log \det(I_p + \mu B)^{-1/2} \le -\mu \, v \sqrt{x} + \mu^2 v^2/4 = -x$$

and also $\mu \operatorname{tr} B/2 - \mu \operatorname{v} \sqrt{\operatorname{x}} = \operatorname{v}^{-1} \operatorname{tr} B \sqrt{\operatorname{x}} - 2\operatorname{x}$.

Remark 2.3. The statement of Corollary 2.5 is meaningful and informative if $\mu^{-1} \mathbf{g}^2 \gg \mathbf{p}_Q$. If $\mathbf{v}^2 = \operatorname{tr} B^2 \asymp \operatorname{tr} B \asymp \mathbf{p}_Q$, it suffices to ensure $\mathbf{g}^2 \gg \mathbf{p}_Q^{1/2}$.

2.3 Sum of i.i.d. random vectors

Here we specify the obtained results to the case when $\mathbf{X} = n^{-1/2} \sum_{i=1}^{n} \boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_i$ are i.i.d. in \mathbb{R}^p with $\mathbb{E}\boldsymbol{\xi}_i = 0$ and $\operatorname{Var}(\boldsymbol{\xi}_i) \leq I_p$. In fact, the i.i.d. structure of the $\boldsymbol{\xi}_i$'s is not used, it suffices to check that all the moment conditions later on are satisfied uniformly over $i \leq n$. However, the formulation slightly simplifies in the i.i.d case. Let some $Q: \mathbb{R}^p \to \mathbb{R}^q$ be fixed with ||Q|| = 1. It holds

$$\mathbf{p} = \mathbf{I} \mathbf{E} \| Q \mathbf{X} \|^2 = \operatorname{tr} B, \qquad B = Q^{\top} \Sigma Q.$$

Also define $\mathbf{p}_Q = Q^\top Q$. We study the concentration phenomenon for $||Q\mathbf{X}||^2$ under two basic. Later we assume that $\mathbf{p} \approx \mathbf{p}_Q$ is a large number and $\mathbf{v}^2 = \operatorname{tr}(B^2) \approx \mathbf{p} \approx \mathbf{p}_Q$. The goal is to apply Corollary 2.3 and Corollary 2.5 claiming that $||Q\mathbf{X}||^2 - \mathbf{p}$ can be sandwiched between $-2\mathbf{v}\sqrt{\mathbf{x}}$ and $2\mathbf{v}\sqrt{\mathbf{x}} + 2\mathbf{x}$ with probability at least $1 - 2e^{-\mathbf{x}}$. The major required condition is sub-gaussian behavior of $\boldsymbol{\xi}_1$. The whole list is given here.

- $(\boldsymbol{\xi}_1)$ A random vector $\boldsymbol{\xi}_1 \in \mathbb{R}^p$ satisfies $\mathbb{E}\boldsymbol{\xi}_1 = 0$, $\operatorname{Var}(\boldsymbol{\xi}_1) \leq I_p$. Also
 - 1. The function $\phi_1(\boldsymbol{u}) \stackrel{\text{def}}{=} \log \mathbb{E} e^{\langle \boldsymbol{u}, \boldsymbol{\xi}_1 \rangle}$ is finite and fulfills for some C_{ϕ}

$$\phi_1(oldsymbol{u}) \stackrel{ ext{def}}{=} \log I\!\!\!E \mathrm{e}^{\langleoldsymbol{u},oldsymbol{\xi}_1
angle} \leq rac{\mathsf{C}_\phi \|oldsymbol{u}\|^2}{2}\,, \qquad oldsymbol{u} \in I\!\!\!R^p$$

2. For some $\rho > 0$ and some constants c_3 , c_4 , it holds with \mathbb{E}_u from (2.5)

$$\begin{split} \sup_{\|\boldsymbol{u}\| \leq \varrho} \frac{1}{\|\boldsymbol{u}\|^3} \Big| \boldsymbol{\mathbb{E}}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{\xi}_1 - \boldsymbol{\mathbb{E}}_{\boldsymbol{u}} \boldsymbol{\xi}_1 \rangle^3 \Big| \leq \mathsf{c}_3 \,, \\ \sup_{\|\boldsymbol{u}\| \leq \varrho} \frac{1}{\|\boldsymbol{u}\|^4} \Big| \boldsymbol{\mathbb{E}}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{\xi}_1 - \boldsymbol{\mathbb{E}}_{\boldsymbol{u}} \boldsymbol{\xi}_1 \rangle^4 - 3 \Big\{ \boldsymbol{\mathbb{E}}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{\xi}_1 - \boldsymbol{\mathbb{E}}_{\boldsymbol{u}} \boldsymbol{\xi}_1 \rangle^2 \Big\}^2 \Big| \leq \mathsf{c}_4 \,. \end{split}$$

3. The function $\log \mathbb{E} e^{i \langle \boldsymbol{u}, \boldsymbol{\xi}_1 \rangle}$ is well defined and

$$\begin{split} \sup_{\|\boldsymbol{u}\| \leq \varrho} \frac{1}{\|\boldsymbol{u}\|^3} \Big| \boldsymbol{\mathcal{E}}_{\boldsymbol{i}\boldsymbol{u}} \langle \boldsymbol{i}\boldsymbol{u}, \boldsymbol{\xi}_1 - \boldsymbol{\mathcal{E}}_{\boldsymbol{i}\boldsymbol{u}} \boldsymbol{\xi}_1 \rangle^3 \Big| \leq \mathsf{c}_3 \,, \\ \sup_{\|\boldsymbol{u}\| \leq \varrho} \frac{1}{\|\boldsymbol{u}\|^4} \Big| \boldsymbol{\mathcal{E}}_{\boldsymbol{i}\boldsymbol{u}} \langle \boldsymbol{i}\boldsymbol{u}, \boldsymbol{\xi}_1 - \boldsymbol{\mathcal{E}}_{\boldsymbol{i}\boldsymbol{u}} \boldsymbol{\xi}_1 \rangle^4 - 3 \big\{ \boldsymbol{\mathcal{E}}_{\boldsymbol{i}\boldsymbol{u}} \langle \boldsymbol{i}\boldsymbol{u}, \boldsymbol{\xi}_1 - \boldsymbol{\mathcal{E}}_{\boldsymbol{i}\boldsymbol{u}} \boldsymbol{\xi}_1 \rangle^2 \big\}^2 \Big| \leq \mathsf{c}_4 \,. \end{split}$$

We are now well prepared to state the result for the i.i.d. case. Apart (ξ_1) , we need p_Q to be sufficiently large to ensure the condition $C_{\phi} \mu \leq 1/3$; see (2.7). Also we require n to be large enough for the relation $p_Q^{3/2} \ll n$, where $a \ll b$ means that $a/b \leq c$ for some small absolute constant c. Similarly $a \lesssim b$ means $a/b \leq C$ for an absolute constant C.

Theorem 2.6. Let $\mathbf{X} = n^{-1/2} \sum_{i=1}^{n} \boldsymbol{\xi}_i$, $\boldsymbol{\xi}_i$ are *i.i.d.* in \mathbb{R}^p with $\mathbb{E} \boldsymbol{\xi}_1 = 0$ and $\operatorname{Var}(\boldsymbol{\xi}_1) \leq I_p$. For a fixed \mathbf{x} , assume $(\boldsymbol{\xi}_1)$ with $n \varrho^2 \gg \mathbf{x} \mathbf{p}_Q$. Let also $\mathbf{p}_Q \gg C_{\varphi}^2 \mathbf{x}$ and $n \gg \mathbf{p}_Q^{3/2}$. Then

$$\begin{split} \mathbb{P}\big(\|Q\mathbf{X}\|^2 > \operatorname{tr} B + 2\sqrt{\operatorname{x}\operatorname{tr}(B^2)} + 2\operatorname{x}\big) &\leq (1 + \Delta_{\mu})\mathrm{e}^{-\operatorname{x}}, \\ \mathbb{P}\big(\|Q\mathbf{X}\|^2 < \operatorname{tr} B - 2\operatorname{v}\sqrt{\operatorname{x}}\big) &\leq (1 + \Delta_{\mu})\mathrm{e}^{-\operatorname{x}}, \end{split}$$

with

$$\Delta_{\mu} \lesssim \frac{\mathbf{x}^{3/2} \mathbf{p}_Q^{3/2}}{n} \,.$$

Proof. The definition and i.i.d structure of the ξ_i 's yield

$$\phi(\boldsymbol{u}) = \log \boldsymbol{E} \mathrm{e}^{\langle \boldsymbol{X}, \boldsymbol{u} \rangle} = n \phi_1(n^{-1/2} \boldsymbol{u}).$$

Moreover, for any u

$$egin{aligned} & E_{oldsymbol{u}}\langleoldsymbol{u},oldsymbol{X}-E_{oldsymbol{u}}oldsymbol{X}
angle^2 &= E_{oldsymbol{u}}\langleoldsymbol{u},oldsymbol{\xi}_1-E_{oldsymbol{u}}oldsymbol{\xi}_1
angle^2, \ & E_{oldsymbol{u}}\langleoldsymbol{u},oldsymbol{X}-E_{oldsymbol{u}}oldsymbol{X}
angle^3 &= n^{-1/2}E_{oldsymbol{u}}\langleoldsymbol{u},oldsymbol{\xi}_1-E_{oldsymbol{u}}oldsymbol{\xi}_1
angle^3, \end{aligned}$$

and

$$\mathbb{E}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{X} - \mathbb{E}_{\boldsymbol{u}} \boldsymbol{X} \rangle^{4} - 3 \big\{ \mathbb{E}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{X} - \mathbb{E}_{\boldsymbol{u}} \boldsymbol{X} \rangle^{2} \big\}^{2} \\
 = n^{-1} \mathbb{E}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{\xi}_{1} - \mathbb{E}_{\boldsymbol{u}} \boldsymbol{\xi}_{1} \rangle^{4} - 3n^{-1} \big\{ \mathbb{E}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{\xi}_{1} - \mathbb{E}_{\boldsymbol{u}} \boldsymbol{\xi}_{1} \rangle^{2} \big\}^{2}.$$

This implies (2.6) for any ${\tt g}$ with ${\tt g}/\sqrt{n} \leq \varrho$ and

$$au_3 \le n^{-1/2} c_3, \qquad au_4 \le n^{-1} c_4.$$

12 Sharp deviation bounds for the squared norm of a sub-gaussian vector

Moreover, the quantity \diamond_4 from (2.16) satisfies

$$\diamondsuit_4 \lesssim rac{\mathtt{x}^{3/2} \mathtt{p}_Q^{3/2}}{n},$$

while the other terms like ρ_{μ} in the definition (2.10) of Δ_{μ} are exponentially small. Now the upper bound follows from Corollary 2.3. Similar arguments can be used for checking the lower bound by Corollary 2.5.

2.4 Light exponential tails

Now we turn to the main case of light exponential tails of $\boldsymbol{\xi}$. Namely, we suppose that $\boldsymbol{E}\boldsymbol{\xi} = 0$ and for some fixed $\mathbf{g} > 0$

$$\phi(\boldsymbol{u}) \stackrel{\text{def}}{=} \log \boldsymbol{E} \exp\left(\langle \boldsymbol{u}, \boldsymbol{W}^{-1}\boldsymbol{\xi}\rangle\right) \leq \frac{\|\boldsymbol{u}\|^2}{2}, \qquad \boldsymbol{u} \in \boldsymbol{\mathbb{R}}^p, \, \|\boldsymbol{u}\| \leq \mathsf{g}, \tag{2.22}$$

for some self-adjoint operator \mathbb{V} in \mathbb{R}^p , $\mathbb{V} \ge I_p$. In fact, it suffices to assume that

$$\sup_{\|\boldsymbol{u}\| \leq g} \boldsymbol{E} \exp(\langle \boldsymbol{u}, \boldsymbol{V}^{-1} \boldsymbol{\xi} \rangle) \leq C.$$
(2.23)

The quantity C can be very large but it is not important and does not enter in the established bounds. In fact, condition (2.23) implies an analog of (2.22) for a g < g: by (2.14)

$$\phi(m{u}) \leq rac{\|m{u}\|^2}{2} + rac{ au_3 \|m{u}\|^3}{6} \leq rac{\|m{u}\|^2}{2} \Big(1 + rac{ au_3 \mathbf{g}}{3}\Big), \qquad \|m{u}\| \leq \mathbf{g},$$

for a small value τ_3 . Moreover, reducing g allows to take \mathbb{V}^2 equal or close to $\operatorname{Var}(\boldsymbol{\xi})$.

Now we continue with a vector $\boldsymbol{\xi}$ satisfying (2.22). As previously, the goal is to establish possibly sharp deviation bounds on $\|Q\boldsymbol{\xi}\|^2$ for a given linear mapping $Q: \mathbb{R}^p \to \mathbb{R}^q$. Remind the notation $B = Q \mathbb{V}^2 Q^\top$. By normalization, one can easily reduce the study to the case $\|B\| = 1$. Let $\mathbf{p} = \operatorname{tr}(B)$, $\mathbf{v}^2 = \operatorname{tr}(B^2)$, and $\mu(\mathbf{x})$ be defined by $\mu(\mathbf{x}) = \left(1 + \frac{\mathbf{v}}{2\sqrt{\mathbf{x}}}\right)^{-1}$; see (B.4). Obviously $\mu(\mathbf{x})$ grows with \mathbf{x} . Define the value \mathbf{x}_c as the root of the equation

$$\frac{\mathsf{g} - \sqrt{\mathsf{p}\,\mu(\mathsf{x})}}{\mu(\mathsf{x})} = z(B,\mathsf{x}) + 1. \tag{2.24}$$

The left hand-side here decreases with \mathbf{x} , while the right hand-side is increasing in \mathbf{x} to infinity. Therefore, the solution exists and is unique. Also denote $\mu_c = \mu(\mathbf{x}_c)$ and

$$g_c = g - \sqrt{p\mu_c}, \qquad (2.25)$$

so that

$$\mathbf{g}_c/\mu_c = z(B, \mathbf{x}_c) + 1.$$

Theorem 2.7. Let (2.22) hold and let Q be such that $B = Q \mathbf{W}^2 Q^\top$ satisfies ||B|| = 1and $\mathbf{p} = \operatorname{tr}(B) < \infty$. Define \mathbf{x}_c by (2.24) and \mathbf{g}_c by (2.25), and suppose $\mathbf{g}_c \ge 1$. Then for any $\mathbf{x} > 0$

$$\mathbb{P}\left(\|Q\boldsymbol{\xi}\|^2 \ge z_c^2(B, \mathbf{x})\right) \le 2\mathrm{e}^{-\mathbf{x}} + \mathrm{e}^{-\mathbf{x}_c} \mathbb{I}(\mathbf{x} < \mathbf{x}_c) \le 3\mathrm{e}^{-\mathbf{x}},\tag{2.26}$$

where $z_c(B, \mathbf{x})$ is defined by

$$z_c(B,\mathbf{x}) \stackrel{\text{def}}{=} \begin{cases} \sqrt{\mathbf{p} + 2\mathbf{v}\,\mathbf{x}^{1/2} + 2\mathbf{x}} \,, & \mathbf{x} \le \mathbf{x}_c \,, \\ \mathbf{g}_c/\mu_c + 2(\mathbf{x} - \mathbf{x}_c)/\mathbf{g}_c \,, & \mathbf{x} > \mathbf{x}_c \,, \end{cases}$$
$$\leq \begin{cases} \sqrt{\mathbf{p}} + \sqrt{2\mathbf{x}} \,, & \mathbf{x} \le \mathbf{x}_c \,, \\ \mathbf{g}_c/\mu_c + 2(\mathbf{x} - \mathbf{x}_c)/\mathbf{g}_c \,, & \mathbf{x} > \mathbf{x}_c \,. \end{cases}$$

Moreover, if, given x, it holds

$$g \ge x^{1/2}/2 + (px/4)^{1/4},$$
 (2.27)

then

$$\mathbb{P}\left(\|Q\boldsymbol{\xi}\| \ge \sqrt{p} + \sqrt{2\mathbf{x}}\right) \le 3\mathrm{e}^{-\mathbf{x}}.$$
(2.28)

Remark 2.4. Depending on the value \mathbf{x} , we have two types of tail behavior of the quadratic form $||Q\boldsymbol{\xi}||^2$. For $\mathbf{x} \leq \mathbf{x}_c$, we have essentially the same deviation bounds as in the Gaussian case with the extra-factor two in the deviation probability. For $\mathbf{x} > \mathbf{x}_c$, we switch to the special regime driven by the exponential moment condition (2.22). Usually \mathbf{g}^2 is a large number (of order n in the i.i.d. setup) yielding \mathbf{x}_c also large, and the second term in (2.26) can be simply ignored. The function $z_c(B, \mathbf{x})$ is discontinuous at the point \mathbf{x}_c . Indeed, $z_c(B, \mathbf{x}) = z(B, \mathbf{x})$ for $\mathbf{x} < \mathbf{x}_c$, while by (2.24), it holds $\mathbf{g}_c/\mu_c = z(B, \mathbf{x}_c) + 1$. However, the jump at \mathbf{x}_c is at most one.

As a corollary, we state the result for the norm of $\xi\in\mathbb{R}^p$ corresponding to the case $\mathbb{V}^{-2}=Q=I_p$ and $p<\infty$. Then

$$\mathbf{p} = \mathbf{v}^2 = p.$$

14 Sharp deviation bounds for the squared norm of a sub-gaussian vector

Corollary 2.8. Let (2.22) hold with $\mathbb{V} = I_p$. Then for each x > 0

$$\mathbb{P}\left(\|\boldsymbol{\xi}\| \ge z_c(p, \mathbf{x})\right) \le 2\mathrm{e}^{-\mathbf{x}} + \mathrm{e}^{-\mathbf{x}_c} \, \mathrm{I\!I}(\mathbf{x} < \mathbf{x}_c),$$

where $z_c(p, \mathbf{x})$ is defined by

$$z_c(p,\mathbf{x}) \stackrel{\text{def}}{=} \begin{cases} \left(p + 2\sqrt{p\,\mathbf{x}} + 2\mathbf{x}\right)^{1/2}, & \mathbf{x} \le \mathbf{x}_c, \\ \mathbf{g}_c/\mu_c + 2\mathbf{g}_c^{-1}(\mathbf{x} - \mathbf{x}_c), & \mathbf{x} > \mathbf{x}_c. \end{cases}$$

If $\mathtt{g} \geq \mathtt{x}^{1/2}/2 + (p\mathtt{x}/4)^{1/4}$, then

$$\mathbb{P}\big(\|\boldsymbol{\xi}\| \ge z(p, \mathbf{x})\big) \le 3\mathrm{e}^{-\mathbf{x}}$$

Proof of Theorem 2.7. First we consider the most interesting case $\mathbf{x} \leq \mathbf{x}_c$. We expect to get Gaussian type deviation bounds for such \mathbf{x} . The main tool of the proof is the following lemma.

Lemma 2.9. Let $\mu \in (0,1)$ and $\mathfrak{z}(\mu) = \mathfrak{g}/\mu - \sqrt{\mathfrak{p}/\mu} > 0$. Then (2.22) implies

$$\mathbb{E}\exp\left(\mu\|Q\boldsymbol{\xi}\|^{2}/2\right)\mathbb{I}\left(\|\mathbb{V}Q^{\top}Q\boldsymbol{\xi}\|\leq\mathfrak{z}(\mu)\right)\leq 2\exp\left(\frac{\mu^{2}\mathsf{v}^{2}}{4(1-\mu)}+\frac{\mu\,\mathsf{p}}{2}\right).$$
(2.29)

Proof. Let us fix for a moment some $\boldsymbol{\xi} \in \mathbb{R}^p$ and $\mu < 1$ and define

$$a = \mathbf{V}^{-1} \boldsymbol{\xi}, \qquad \Sigma = \mu \mathbf{V} Q^{\top} Q \mathbf{V}.$$

Consider the Gaussian measure $\mathbb{P}_{a,\Sigma} = \mathcal{N}(a,\Sigma^{-1})$, and let $U \sim \mathcal{N}(0,\Sigma^{-1})$. By the Girsanov formula

and for any set $A \in \mathbb{R}^p$

$$\mathbb{P}_{\boldsymbol{a},\boldsymbol{\Sigma}}(A) = \mathbb{P}_{0,\boldsymbol{\Sigma}}(A - \boldsymbol{a}) = \mathbb{E}_{0,\boldsymbol{\Sigma}}\left[\exp\left\{\langle\boldsymbol{\Sigma}\boldsymbol{U},\boldsymbol{a}\rangle - \frac{1}{2}\langle\boldsymbol{\Sigma}\boldsymbol{a},\boldsymbol{a}\rangle\right\}\mathbb{I}(A)\right]$$

Now we select $A = \{ \boldsymbol{u} : \| \boldsymbol{\Sigma} \boldsymbol{u} \| \leq g \}$. Under $\mathbb{P}_{0,\boldsymbol{\Sigma}}$, one can represent $\boldsymbol{\Sigma} \boldsymbol{U} = \boldsymbol{\Sigma}^{1/2} \boldsymbol{\gamma}$ with a standard Gaussian $\boldsymbol{\gamma}$. Therefore,

$$\begin{split} I\!\!P_{0,\varSigma}(A-oldsymbol{a}) &= I\!\!P_{oldsymbol{\gamma}\sim\mathcal{N}(0,I)} ig(\|\varSigma^{1/2}(oldsymbol{\gamma}-\varSigma^{1/2}oldsymbol{a})\|\leq \mathsf{g} ig) \ &\geq I\!\!P_{oldsymbol{\gamma}\sim\mathcal{N}(0,I)} ig(\|\varSigma^{1/2}oldsymbol{\gamma}\|\leq \mathsf{g}-\|\varSigmaoldsymbol{a}\|ig). \end{split}$$

We now use that $\mathbb{P}_{\gamma \sim \mathcal{N}(0,I)} (\|\Sigma^{1/2} \gamma\|^2 \leq \operatorname{tr}(\Sigma)) \geq 1/2$ with $\operatorname{tr}(\Sigma) = \mu \operatorname{tr}(B) = \mu \operatorname{p}$. Therefore, the condition $\|\Sigma \boldsymbol{a}\| + \sqrt{\mu \operatorname{p}} \leq \operatorname{g}$ implies in view of $\langle \Sigma \boldsymbol{a}, \boldsymbol{a} \rangle = \mu \|Q \boldsymbol{\xi}\|^2$

$$1/2 \leq \mathbb{P}_{\boldsymbol{a}, \boldsymbol{\Sigma}}(A) = \mathbb{E}_{0, \boldsymbol{\Sigma}} \Big[\exp \Big\{ \langle \boldsymbol{\Sigma} \boldsymbol{U}, \mathbb{V}^{-1} \boldsymbol{\xi} \rangle - \mu \| Q \boldsymbol{\xi} \|^2 / 2 \Big\} \, \mathbb{I}(\| \boldsymbol{\Sigma} \boldsymbol{U} \| \leq \mathsf{g}) \Big]$$

or

We now take the expectation of the each side of this equation w.r.t. $\boldsymbol{\xi}$, change the integration order, and use (2.22) yielding

$$\begin{split} \boldsymbol{E} \exp\left(\mu \|Q\boldsymbol{\xi}\|^2/2\right) \, \boldsymbol{\mathbb{I}}\left(\|\boldsymbol{\Sigma} \, \boldsymbol{\mathbb{V}}^{-1} \boldsymbol{\xi}\| \leq \mathsf{g} - \sqrt{\mu \, \mathsf{p}}\right) &\leq 2\boldsymbol{\mathbb{E}}_{0,\boldsymbol{\Sigma}} \exp\left(\|\boldsymbol{\Sigma} \boldsymbol{U}\|^2/2\right) \\ &= 2\boldsymbol{\mathbb{E}}_{\boldsymbol{\gamma}\sim\mathcal{N}(0,I)} \exp\left(\|\boldsymbol{\Sigma}^{1/2} \boldsymbol{\gamma}\|^2/2\right) = 2 \det\left(I - \boldsymbol{\Sigma}\right)^{-1/2} = 2 \det\left(I - \mu B\right)^{-1/2}. \end{split}$$

We also use that for any $\mu > 0$

$$\log \det (I - \mu B)^{-1/2} - \frac{\mu \operatorname{tr}(B)}{2} \le \frac{\mu^2 \operatorname{tr}(B^2)}{4(1 - \mu)};$$

see (B.5), and the first statement follows in view of $\Sigma \mathbf{V}^{-1} \boldsymbol{\xi} = \mu \mathbf{V} Q^{\top} Q \boldsymbol{\xi}$.

The use of μ from (B.4) in (2.29) yields similarly to the proof of Theorem B.1

$$\mathbb{P}\Big(\|Q\boldsymbol{\xi}\|^2 > z^2(B,\mathbf{x}), \|\mathbb{V}Q^\top Q\boldsymbol{\xi}\| \le \mathfrak{z}(\mu)\Big) \le 2\mathrm{e}^{-\mathbf{x}}.$$
(2.30)

It remains to consider the probability of large deviation $\mathbb{P}(\|\mathbb{W}Q^{\top}Q\boldsymbol{\xi}\| > \mathfrak{z}(\mu))$.

Lemma 2.10. For any $\mathbf{x}_c > 0$ such that $z(B, \mathbf{x}_c) + 1 \leq \mathbf{g}_c/\mu_c$, it holds with $\mu_c = \{1 + \mathbf{v}/(2\sqrt{\mathbf{x}_c})\}^{-1}$ and $z_c = \mathfrak{z}(\mu_c) = \mathbf{g}/\mu_c - \sqrt{\mathbf{p}/\mu_c}$

$$\mathbb{P}(\|\mathbb{W}Q^{\top}Q\boldsymbol{\xi}\| > z_c) \le \mathbb{P}(\|Q\boldsymbol{\xi}\|^2 > z_c^2) \le e^{-\mathbf{x}_c}$$

Proof. Define

$$\Phi(\mu) \stackrel{\text{def}}{=} \frac{\mu^2 \mathbf{v}^2}{4(1-\mu)} + \frac{\mu \,\mathbf{p}}{2}$$

It follows due to (B.4) and (B.6) for any $\mu \leq \mu_c$

$$\Phi(\mu) \le \Phi(\mu_c) \le \frac{\mu_c z^2(B, \mathbf{x}_c)}{2} - \mathbf{x}_c,$$

where the right hand-side does not depend on μ . Denote $\eta = ||Q\boldsymbol{\xi}||$ and use that $||\boldsymbol{W}Q^{\top}Q\boldsymbol{\xi}|| \leq ||Q\boldsymbol{W}^{2}Q^{\top}||^{1/2}||Q\boldsymbol{\xi}|| \leq \eta$. Then by (2.29)

$$\mathbb{E}\exp(\mu\eta^2/2)\,\mathbb{I}\left(\eta \leq \mathfrak{z}(\mu)\right) \leq 2\exp\Phi(\mu) \leq 2\exp\Phi(\mu_c). \tag{2.31}$$

Define the inverse function $\mu(\mathfrak{z}) = \mathfrak{z}^{-1}(\mu)$. For any $\mathfrak{z} \ge z_c$, it follows from (2.31) with $\mu = \mu(\mathfrak{z})$

$$\mathbb{E}\exp\left\{\mu(\mathfrak{z})(\mathfrak{z}-1)^2/2\right\}\,\mathbb{I}\left(\mathfrak{z}-1\leq\eta\leq\mathfrak{z}\right)\leq 2\exp\Phi(\mu_c)$$

yielding

$$\mathbb{P}\left(\mathfrak{z}-1\leq\eta\leq\mathfrak{z}\right)\leq 2\exp\left(-\mu(\mathfrak{z})\left(\mathfrak{z}-1\right)^2/2+\Phi(\mu_c)\right)$$

and hence,

$$\mathbb{P}(\eta > \mathfrak{z}) \leq 2 \int_{\mathfrak{z}}^{\infty} \exp\{-\mu(z)(z-1)^2/2 + \Phi(\mu_c)\} dz.$$

Further, $\mu \mathfrak{z}(\mu) = g - \sqrt{p\mu}$ and

$$g_c = \mu_c z_c \le \mu \mathfrak{z}(\mu) \le g, \quad \mu \le \mu_c.$$

This implies the same bound for the inverse function:

$$\mathsf{g}_c \leq \mathfrak{z}\,\mu(\mathfrak{z}) \leq \mathsf{g}, \quad \mathfrak{z} \geq z_c$$

and for $\mathfrak{z} \geq 2$

$$\mathbb{P}(\eta > \mathfrak{z}) \leq 2 \int_{\mathfrak{z}}^{\infty} \exp\{-\mu(z)(z^2/2 - z) + \Phi(\mu_c)\}dz$$

$$\leq 2 \int_{\mathfrak{z}}^{\infty} \exp\{-\mathfrak{g}_c(z/2 - 1) + \Phi(\mu_c)\}dz$$

$$\leq \frac{4}{\mathfrak{g}_c} \exp\{-\mathfrak{g}_c(\mathfrak{z}/2 - 1) + \Phi(\mu_c)\}.$$
(2.32)

Conditions $\mathbf{g}_c z_c = \mu_c^{-1} \mathbf{g}_c^2 \ge \mu_c \{ z(B, \mathbf{x}_c) + 1 \}^2$ and $\mathbf{g}_c \ge 1$ ensure that $\mathbb{P}(\eta > z_c) \le e^{-\mathbf{x}_c}$.

Remind that \mathbf{x}_c is the largest \mathbf{x} -value ensuring the condition $\mathbf{g}_c \geq z(B, \mathbf{x}_c) + 1$. We also use that for $\mathbf{x} \leq \mathbf{x}_c$, it holds $\mathfrak{z}(\mu) \geq \mathfrak{z}(\mu_c) = z_c$. Therefore, by (2.30) and Lemma 2.10

$$\begin{split} \mathbb{P}\big(\|Q\boldsymbol{\xi}\|^2 \ge z^2(B,\mathbf{x})\big) &\leq \mathbb{P}\big(\|Q\boldsymbol{\xi}\|^2 \ge z^2(B,\mathbf{x}), \|\mathbb{W}Q^\top Q\boldsymbol{\xi}\| \le \mathfrak{z}(\mu)\big) + \mathbb{P}\big(\|Q\boldsymbol{\xi}\|^2 \ge z_c^2\big) \\ &\leq 2\mathrm{e}^{-\mathbf{x}} + \mathrm{e}^{-\mathbf{x}_c} \,. \end{split}$$

Finally we consider $\, {\tt x} > {\tt x}_c \, .$ Applying (2.32) yields by $\, \mathfrak{z} \geq z_c \,$

$$\begin{split} \mathbb{P}\left(\eta > \mathfrak{z}\right) &\leq \frac{2}{\mu_c \, z_c} \exp\left\{-\mu_c \, z_c^2/2 + \mathsf{g} + \mu_c \, z^2(B, \mathsf{x}_c)/2 - \mathsf{x}_c\right\} \exp\left\{-\mu_c \, z_c(\mathfrak{z} - z_c)/2\right\} \\ &\leq \mathrm{e}^{-\mathsf{x}_c} \exp\left\{-\mathsf{g}_c(\mathfrak{z} - z_c)/2\right\}. \end{split}$$

The choice \mathfrak{z} by

$$g_c(\mathfrak{z}-z_c)/2 = \mathbf{x} - \mathbf{x}_c$$

ensures the desired bound.

Now, for a prescribed **x**, we evaluate the minimal value **g** ensuring the bound (2.26) with $\mathbf{x}_c \geq \mathbf{x}$. For simplicity we apply the sub-optimal choice $\mu(\mathbf{x}) = (1 + 2\sqrt{p/x})^{-1}$; see Remark B.3. Then for any $\mathbf{x} \geq 1$

$$\begin{split} \mu(\mathbf{x}) \left\{ z(B,\mathbf{x}) + 1 \right\} &\leq \frac{\sqrt{\mathbf{x}}}{\sqrt{\mathbf{x}} + 2\sqrt{\mathbf{p}}} \left(\sqrt{\mathbf{p} + 2(\mathbf{x}\mathbf{p})^{1/2} + 2\mathbf{x}} + 1 \right), \\ \mathbf{p} \, \mu(\mathbf{x}) &= \frac{\sqrt{\mathbf{x}} \, \mathbf{p}}{\sqrt{\mathbf{x}} + 2\sqrt{\mathbf{p}}} \,. \end{split}$$

It is now straightforward to check that

$$\mu(\mathbf{x})\left\{z(B,\mathbf{x})+1\right\}+\sqrt{\mathbf{p}\,\mu(\mathbf{x})} \le \sqrt{\mathbf{x}}/2+(\mathbf{x}\,\mathbf{p}/4)^{1/4}.$$

Therefore, if (2.27) holds for the given \mathbf{x} , then (2.24) is fulfilled with $\mathbf{x}_c \geq \mathbf{x}$ yielding (2.28).

18 Sharp deviation bounds for the squared norm of a sub-gaussian vector

A Moments of a Gaussian quadratic form

Let γ be standard normal in \mathbb{R}^p for $p \leq \infty$. Given a self-adjoint trace operator B, consider a quadratic form $\langle B\gamma, \gamma \rangle$.

Lemma A.1. It holds

$$E\langle B\gamma, \gamma \rangle = \operatorname{tr} B,$$

 $\operatorname{Var}\langle B\gamma, \gamma \rangle = 2 \operatorname{tr} B^{2}.$

Moreover,

$$\begin{split} \boldsymbol{E} & \left(\left\langle B\boldsymbol{\gamma},\boldsymbol{\gamma} \right\rangle - \operatorname{tr} B \right)^2 = 2 \operatorname{tr} B^2, \\ \boldsymbol{E} & \left(\left\langle B\boldsymbol{\gamma},\boldsymbol{\gamma} \right\rangle - \operatorname{tr} B \right)^3 = 8 \operatorname{tr} B^3, \\ \boldsymbol{E} & \left(\left\langle B\boldsymbol{\gamma},\boldsymbol{\gamma} \right\rangle - \operatorname{tr} B \right)^4 = 48 \operatorname{tr} B^4 + 12 (\operatorname{tr} B^2)^2, \end{split}$$

and

$$\begin{split} E \langle B\gamma, \gamma \rangle^2 &= (\operatorname{tr} B)^2 + 2 \operatorname{tr} B^2, \\ E \langle B\gamma, \gamma \rangle^3 &= (\operatorname{tr} B)^3 + 6 \operatorname{tr} B \operatorname{tr} B^2 + 8 \operatorname{tr} B^3, \\ E \langle B\gamma, \gamma \rangle^4 &= (\operatorname{tr} B)^4 + 12 (\operatorname{tr} B)^2 \operatorname{tr} B^2 + 32 (\operatorname{tr} B) \operatorname{tr} B^3 + 48 \operatorname{tr} B^4 + 12 (\operatorname{tr} B^2)^2, \\ \operatorname{Var} \langle B\gamma, \gamma \rangle^2 &= 8 (\operatorname{tr} B)^2 \operatorname{tr} B^2 + 32 (\operatorname{tr} B) \operatorname{tr} B^3 + 48 \operatorname{tr} B^4 + 8 (\operatorname{tr} B^2)^2. \end{split}$$

Moreover, if $B \leq I_p$ and $\mathbf{p} = \operatorname{tr} B$, then $\operatorname{tr} B^m \leq \mathbf{p} \|B\|^{m-1}$ for $m \geq 1$ and

$$\begin{split} E \langle B\gamma, \gamma \rangle^2 &\leq p^2 + 2p \|B\| &\leq (p + \|B\|)^2, \\ E \langle B\gamma, \gamma \rangle^3 &\leq p^3 + 6p^2 \|B\| + 8p \|B\|^2 &\leq (p + 2\|B\|)^3, \\ E \langle B\gamma, \gamma \rangle^4 &\leq p^4 + 12p^3 \|B\| + 44p^2 \|B\|^2 + 48p \|B\|^3 &\leq (p + 3\|B\|)^4, \\ Var \langle B\gamma, \gamma \rangle^2 &\leq 8p^3 + 40p^2 \|B\| + 48p \|B\|^2. \end{split}$$

Proof. Let $\chi = \gamma^2 - 1$ for γ standard normal. Then $\mathbf{E}\chi = 0$, $\mathbf{E}\chi^2 = 2$, $\mathbf{E}\chi^3 = 8$, $\mathbf{E}\chi^4 = 60$. Without loss of generality assume *B* diagonal: $B = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_p)$. Then

$$\xi \stackrel{\text{def}}{=} \langle B\boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle - \operatorname{tr} B = \sum_{j=1}^{p} \lambda_j (\gamma_j^2 - 1),$$

where γ_j are i.i.d. standard normal. This easily yields

$$\begin{split} \mathbf{E}\xi^2 &= \sum_{j=1}^p \lambda_j^2 \mathbf{E}(\gamma_j^2 - 1)^2 = \mathbf{E}\chi^2 \operatorname{tr} B^2 = 2\operatorname{tr} B^2, \\ \mathbf{E}\xi^3 &= \sum_{j=1}^p \lambda_j^3 \mathbf{E}(\gamma_j^2 - 1)^3 = \mathbf{E}\chi^3 \operatorname{tr} B^3 = 8\operatorname{tr} B^3, \\ \mathbf{E}\xi^4 &= \sum_{j=1}^p \lambda_j^4 (\gamma_j^2 - 1)^4 + \sum_{i \neq j} \lambda_i^2 \lambda_j^2 \mathbf{E}(\gamma_i^2 - 1)^2 \mathbf{E}(\gamma_j^2 - 1)^2 \\ &= \left(\mathbf{E}\chi^4 - 3(\mathbf{E}\chi^2)^2\right) \operatorname{tr} B^4 + 3(\mathbf{E}\chi^2 \operatorname{tr} B^2)^2 = 48\operatorname{tr} B^4 + 12(\operatorname{tr} B^2)^2, \end{split}$$

ensuring

$$\begin{split} \mathbf{E} \langle B\boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle^2 &= \left(\mathbf{E} \langle B\boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle \right)^2 + \mathbf{E} \xi^2 = (\operatorname{tr} B)^2 + 2 \operatorname{tr} B^2, \\ \mathbf{E} \langle B\boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle^3 &= \mathbf{E} \left(\xi + \operatorname{tr} B \right)^3 = (\operatorname{tr} B)^3 + \mathbf{E} \xi^3 + 3 \operatorname{tr} B \ \mathbf{E} \xi^2 \\ &= (\operatorname{tr} B)^3 + 6 \operatorname{tr} B \ \operatorname{tr} B^2 + 8 \operatorname{tr} B^3, \end{split}$$

and

$$\begin{aligned} \operatorname{Var} \langle B\gamma, \gamma \rangle^2 &= \mathbb{E} \big(\xi + \operatorname{tr} B \big)^4 - \big(\mathbb{E} \langle B\gamma, \gamma \rangle \big)^2 \\ &= \big(\operatorname{tr} B \big)^4 + 6 (\operatorname{tr} B)^2 \mathbb{E} \xi^2 + 4 \operatorname{tr} B \mathbb{E} \xi^3 + \mathbb{E} \xi^4 - \big((\operatorname{tr} B)^2 + 2 \operatorname{tr} B^2 \big)^2 \\ &= 8 (\operatorname{tr} B)^2 \operatorname{tr} B^2 + 32 (\operatorname{tr} B) \operatorname{tr} B^3 + 48 \operatorname{tr} B^4 + 8 (\operatorname{tr} B^2)^2. \end{aligned}$$

This implies the results of the lemma.

Now we compute the exponential moments of centered and non-centered quadratic forms.

Lemma A.2. Let $||B||_{op} = \lambda$ and $\gamma \sim \mathcal{N}(0, I_p)$. Then for any $\mu \in (0, \lambda^{-1})$,

$$\mathbb{E}\exp\left\{\frac{\mu}{2}(\langle B\boldsymbol{\gamma},\boldsymbol{\gamma}\rangle-\boldsymbol{p})\right\} = \det(I-\mu B)^{-1/2}.$$

Moreover, with $\mathbf{p} = \operatorname{tr} B$ and $\mathbf{v}^2 = \operatorname{tr} B^2$

$$\log \mathbb{E} \exp\left\{\frac{\mu}{2} \left(\langle B\gamma, \gamma \rangle - p \right) \right\} \le \frac{\mu^2 v^2}{4(1 - \lambda \mu)}.$$
(A.1)

If B is positive semidefinite, $\lambda_j \ge 0$, then

$$\log \mathbb{E} \exp\left\{-\frac{\mu}{2} \left(\langle B\gamma, \gamma \rangle - p\right)\right\} \le \frac{\mu^2 v^2}{4}.$$
(A.2)

Proof. W.l.o.g. assume $\lambda = 1$. Let λ_j be the eigenvalues of B, $|\lambda_j| \leq 1$. By an orthogonal transform, one can reduce the statement to the case of a diagonal matrix $B = \operatorname{diag}(\lambda_j)$. Then $\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle = \sum_{j=1}^p \lambda_j \gamma_j^2$ and by independence of the γ_j 's

$$\mathbb{E}\left\{\frac{\mu}{2}\langle B\boldsymbol{\gamma},\boldsymbol{\gamma}\rangle\right\} = \prod_{j=1}^{p} \mathbb{E}\exp\left(\frac{\mu}{2}\lambda_{j}\varepsilon_{j}^{2}\right) = \prod_{j=1}^{p} \frac{1}{\sqrt{1-\mu\lambda_{j}}} = \det\left(I-\mu B\right)^{-1/2}.$$

Below we use the simple bound:

$$-\log(1-u) - u = \sum_{k=2}^{\infty} \frac{u^k}{k} \le \frac{u^2}{2} \sum_{k=0}^{\infty} u^k = \frac{u^2}{2(1-u)}, \qquad u \in (0,1),$$
$$-\log(1-u) + u = \sum_{k=2}^{\infty} \frac{u^k}{k} \le \frac{u^2}{2}, \qquad u \in (-1,0).$$

Now it holds

$$\log \mathbb{E}\left\{\frac{\mu}{2}(\langle B\gamma,\gamma\rangle - \mathbf{p})\right\} = \log \det(I - \mu B)^{-1/2} - \frac{\mu \mathbf{p}}{2}$$
$$= -\frac{1}{2}\sum_{j=1}^{p}\left\{\log(1 - \mu\lambda_j) + \mu\lambda_j\right\} \le \sum_{j=1}^{p}\frac{\mu^2\lambda_j^2}{4(1 - \mu)} = \frac{\mu^2\mathbf{v}^2}{4(1 - \mu)}.$$

The last statement can be proved similarly.

Now we consider the case of a non-centered quadratic form $\langle B\gamma, \gamma \rangle/2 + \langle A, \gamma \rangle$ for a fixed vector A.

Lemma A.3. Let $\lambda_{\max}(B) < 1$. Then for any A

$$\mathbb{E}\exp\left\{\frac{1}{2}\langle B\boldsymbol{\gamma},\boldsymbol{\gamma}\rangle+\langle \boldsymbol{A},\boldsymbol{\gamma}\rangle\right\}=\exp\left\{\frac{\|(\boldsymbol{I}-\boldsymbol{B})^{-1/2}\boldsymbol{A}\|^{2}}{2}\right\}\det(\boldsymbol{I}-\boldsymbol{B})^{-1/2}.$$

Moreover, for any $\mu \in (0,1)$

$$\log \mathbb{E} \exp\left\{\frac{\mu}{2} \left(\langle B\gamma, \gamma \rangle - p\right) + \langle A, \gamma \rangle\right\}$$
$$= \frac{\|(I - \mu B)^{-1/2} A\|^2}{2} + \log \det(I - \mu B)^{-1/2} - \mu p$$
$$\leq \frac{\|(I - \mu B)^{-1/2} A\|^2}{2} + \frac{\mu^2 v^2}{4(1 - \mu)}.$$
(A.3)

Proof. Denote $\boldsymbol{a} = (\boldsymbol{I} - B)^{-1/2} \boldsymbol{A}$. It holds by change of variables $(\boldsymbol{I} - B)^{1/2} \boldsymbol{x} = \boldsymbol{u}$ for

$$\begin{split} \mathbf{C}_p &= (2\pi)^{-p/2} \\ \mathbb{E} \exp\Big\{\frac{1}{2}\langle B\boldsymbol{\gamma},\boldsymbol{\gamma}\rangle + \langle \boldsymbol{A},\boldsymbol{\gamma}\rangle\Big\} = \mathbf{C}_p \int \exp\Big\{-\frac{1}{2}\langle (\boldsymbol{I}-B)\boldsymbol{x},\boldsymbol{x}\rangle + \langle \boldsymbol{A},\boldsymbol{x}\rangle\Big\} d\boldsymbol{x} \\ &= \mathbf{C}_p \det(\boldsymbol{I}-B)^{-1/2} \int \exp\Big\{-\frac{1}{2}\|\boldsymbol{u}\|^2 + \langle \boldsymbol{a},\boldsymbol{u}\rangle\Big\} d\boldsymbol{u} = \det(\boldsymbol{I}-B)^{-1/2} e^{\|\boldsymbol{a}\|^2/2}. \end{split}$$

The last inequality (A.3) follows by (A.1).

B Deviation bounds for Gaussian quadratic forms

The next result explains the concentration effect of $||Q\boldsymbol{\xi}||^2$ for a centered Gaussian vector $\boldsymbol{\xi} \sim \mathcal{N}(0, \mathbb{V}^2)$ and a linear operator $Q \colon \mathbb{R}^p \to \mathbb{R}^q$, $p, q \leq \infty$. We use a version from Laurent and Massart (2000). For completeness, we present a simple proof of the upper bound.

Theorem B.1. Let $\boldsymbol{\xi} \sim \mathcal{N}(0, \mathbb{V}^2)$ be a Gaussian element in \mathbb{R}^p and let $Q: \mathbb{R}^p \to \mathbb{R}^q$ be such that $B = Q \mathbb{V}^2 Q^\top$ is a trace operator in \mathbb{R}^q . Then with $\mathbf{p} = \operatorname{tr}(B)$, $\mathbf{v}^2 = \operatorname{tr}(B^2)$, and $\lambda = ||B||$, it holds for each $\mathbf{x} \ge 0$

$$\mathbb{P}\left(\|Q\boldsymbol{\xi}\|^2 - \mathbf{p} > 2\mathbf{v}\sqrt{\mathbf{x}} + 2\lambda\mathbf{x}\right) \le e^{-\mathbf{x}},\tag{B.1}$$

$$\mathbb{P}\left(\|Q\boldsymbol{\xi}\|^2 - p \le -2v\sqrt{x}\right) \le e^{-x}.$$
(B.2)

It also implies

$$\mathbb{P}\left(\left|\|Q\boldsymbol{\xi}\|^2 - \mathbf{p}\right| > z_2(B, \mathbf{x})\right) \le 2\mathrm{e}^{-\mathbf{x}},$$

with

$$z_2(B,\mathbf{x}) \stackrel{\text{def}}{=} 2\mathbf{v}\sqrt{\mathbf{x}} + 2\lambda\mathbf{x} . \tag{B.3}$$

Proof. W.l.o.g. assume that $\lambda = ||B|| = 1$. We use the identity $||Q\boldsymbol{\xi}||^2 = \langle B\boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle$ with $\boldsymbol{\gamma} \sim \mathcal{N}(0, I_q)$. We apply the exponential Chebyshev inequality: with $\mu > 0$

$$I\!\!P\Big(\langle Bm{\gamma},m{\gamma}
angle-{\tt p}>z_2(B,{\tt x})\Big)\leq I\!\!E\exp\Bigl(rac{\mu}{2}ig(\langle Bm{\gamma},m{\gamma}
angle-{\tt p}ig)-rac{\mu\,z_2(B,{\tt x})}{2}\Bigr)$$

Given $\mathbf{x} > 0$, fix $\mu < 1$ by the equation

$$\frac{\mu}{1-\mu} = \frac{2\sqrt{\mathbf{x}}}{\mathbf{v}} \quad \text{or} \quad \mu^{-1} = 1 + \frac{\mathbf{v}}{2\sqrt{\mathbf{x}}}.$$
 (B.4)

22 Sharp deviation bounds for the squared norm of a sub-gaussian vector

Let λ_j be the eigenvalues of B, $|\lambda_j| \leq 1$. It holds with $\mathbf{p} = \operatorname{tr} B$ in view of (A.1)

$$\log \mathbb{E}\left\{\frac{\mu}{2}\left(\langle B\boldsymbol{\gamma},\boldsymbol{\gamma}\rangle - \mathbf{p}\right)\right\} \leq \frac{\mu^2 \mathbf{v}^2}{4(1-\mu)}.$$
(B.5)

For (B.1), it remains to check that the choice μ by (B.4) yields

$$\frac{\mu^2 \mathbf{v}^2}{4(1-\mu)} - \frac{\mu z_2(B,\mathbf{x})}{2} = \frac{\mu^2 \mathbf{v}^2}{4(1-\mu)} - \mu \big(\mathbf{v}\sqrt{\mathbf{x}} + \mathbf{x} \big) = \mu \Big(\frac{\mathbf{v}\sqrt{\mathbf{x}}}{2} - \mathbf{v}\sqrt{\mathbf{x}} - \mathbf{x} \Big) = -\mathbf{x}.$$
 (B.6)

The bound (B.2) is obtained similarly by applying the exponential Chebyshev inequality to $-\langle B\gamma, \gamma \rangle + p$ with $\mu = 2v^{-1}\sqrt{x}$. The use of (A.2) yields

$$\begin{split} \mathbb{P}\Big(\langle B\boldsymbol{\gamma},\boldsymbol{\gamma}\rangle - \mathrm{p} < -2\mathrm{v}\sqrt{\mathrm{x}}\Big) &\leq \mathbb{E}\exp\Big\{\frac{\mu}{2}\big(-\langle B\boldsymbol{\gamma},\boldsymbol{\gamma}\rangle + \mathrm{p}\big) - \mu\,\mathrm{v}\sqrt{\mathrm{x}}\Big\} \\ &\leq \exp\Big(\frac{\mu^2\mathrm{v}^2}{4} - \mu\,\mathrm{v}\sqrt{\mathrm{x}}\Big) = \mathrm{e}^{-\mathrm{x}} \end{split}$$

as required.

Corollary B.2. Assume the conditions of Theorem B.1. Then for z > v

$$\mathbb{P}\left(\left|\|Q\boldsymbol{\xi}\|^{2}-\mathbf{p}\right| \geq z\right) \leq 2\exp\left\{-\frac{z^{2}}{\left(\mathbf{v}+\sqrt{\mathbf{v}^{2}+2\lambda z}\right)^{2}}\right\} \leq 2\exp\left(-\frac{z^{2}}{4\mathbf{v}^{2}+4\lambda z}\right).$$
(B.7)

Proof. Given z, define x by $2v\sqrt{x} + 2\lambda x = z$ or $2\lambda\sqrt{x} = \sqrt{v^2 + 2\lambda z} - v$. Then

$$\mathbb{I}\left(\|Q\boldsymbol{\xi}\|^2 - \mathbf{p} \ge z\right) \le e^{-\mathbf{x}} = \exp\left\{-\frac{\left(\sqrt{\mathbf{v}^2 + 2\lambda z} - \mathbf{v}\right)^2}{4\lambda^2}\right\} = \exp\left\{-\frac{z^2}{\left(\mathbf{v} + \sqrt{\mathbf{v}^2 + 2\lambda z}\right)^2}\right\}.$$

This yields (B.7) by direct calculus.

Of course, bound (B.7) is sensible only if $z \gg v$.

Corollary B.3. Assume the conditions of Theorem B.1. If also $B \ge 0$, then

$$\mathbb{P}\Big(\|Q\boldsymbol{\xi}\|^2 \ge z^2(B,\mathbf{x})\Big) \le e^{-\mathbf{x}}$$
(B.8)

with

$$z^2(B,\mathbf{x}) \stackrel{\text{def}}{=} \mathbf{p} + 2\mathbf{v}\sqrt{\mathbf{x}} + 2\lambda\mathbf{x} \le \left(\sqrt{\mathbf{p}} + \sqrt{2\lambda\mathbf{x}}\right)^2.$$

Also

$$\mathbb{P}\Big(\|Q\boldsymbol{\xi}\|^2 - p < -2v\sqrt{x}\Big) \le e^{-x}.$$

Proof. The definition implies $v^2 \leq p\lambda$. One can use a sub-optimal choice of the value $\mu(\mathbf{x}) = \left\{1 + 2\sqrt{\lambda p/\mathbf{x}}\right\}^{-1}$ yielding the statement of the corollary.

As a special case, we present a bound for the chi-squared distribution corresponding to $Q = \mathbb{V}^2 = I_p$, $p < \infty$. Then $B = I_p$, $\operatorname{tr}(B) = p$, $\operatorname{tr}(B^2) = p$ and $\lambda(B) = 1$.

Corollary B.4. Let γ be a standard normal vector in \mathbb{R}^p . Then for any x > 0

$$\begin{split} \mathbb{P}\left(\|\boldsymbol{\gamma}\|^2 \geq p + 2\sqrt{p\,\mathbf{x}} + 2\mathbf{x}\right) &\leq \mathrm{e}^{-\mathbf{x}}, \\ \mathbb{P}\left(\|\boldsymbol{\gamma}\| \geq \sqrt{p} + \sqrt{2\mathbf{x}}\right) &\leq \mathrm{e}^{-\mathbf{x}}, \\ \mathbb{P}\left(\|\boldsymbol{\gamma}\|^2 \leq p - 2\sqrt{p\,\mathbf{x}}\right) &\leq \mathrm{e}^{-\mathbf{x}}. \end{split}$$

The bound of Theorem B.1 can be represented as a usual deviation bound.

Theorem B.5. Assume the conditions of Theorem B.1. For y > 0, define

$$\mathbf{x}(\mathbf{y}) \stackrel{\text{def}}{=} \frac{(\sqrt{\mathbf{y} + \mathbf{p}} - \sqrt{\mathbf{p}})^2}{4\lambda}$$

Then

$$\mathbb{P}(\|Q\boldsymbol{\xi}\|^2 \ge p + y) \le e^{-x(y)},\tag{B.9}$$

$$\mathbb{E}\left\{ \left(\|Q\boldsymbol{\xi}\|^2 - \mathbf{p} \right) \mathbb{I}\left(\|Q\boldsymbol{\xi}\|^2 \ge \mathbf{p} + \mathbf{y} \right) \right\} \le 2 \left(\frac{\mathbf{y} + \mathbf{p}}{\lambda \, \mathbf{x}(\mathbf{y})} \right)^{1/2} \, \mathrm{e}^{-\mathbf{x}(\mathbf{y})} \,. \tag{B.10}$$

Moreover, let $\mu > 0$ fulfill $\epsilon = \mu \lambda + \mu \sqrt{\lambda p / x(y)} < 1$. Then

$$\mathbb{E}\left\{\mathrm{e}^{\mu(\|Q\boldsymbol{\xi}\|^2-\mathbf{p})/2}\,\mathbb{I}(\|Q\boldsymbol{\xi}\|^2\geq\mathbf{p}+\mathbf{y})\right\}\leq\frac{1}{1-\epsilon}\,\exp\{-(1-\epsilon)\mathbf{x}(\mathbf{y})\}\,.\tag{B.11}$$

Proof. Normalizing by λ reduces the statements to the case with $\lambda = 1$. Define $\eta = ||Q\boldsymbol{\xi}||^2 - p$ and

$$z(\mathbf{x}) = 2\sqrt{\mathbf{p}\,\mathbf{x}} + 2\mathbf{x}.\tag{B.12}$$

Then by (B.1) $I\!\!P(\eta \ge z(\mathbf{x})) \le e^{-\mathbf{x}}$. Inverting the relation (B.12) yields

$$\mathbf{x}(z) = \frac{1}{4} \left(\sqrt{z + \mathbf{p}} - \sqrt{\mathbf{p}} \right)^2$$

and (B.9) follows by applying z = y. Further,

$$\mathbb{E}\left\{\eta\,\mathbb{I}(\eta\geq y)\right\} = \int_{y}^{\infty}\mathbb{P}(\eta\geq z)\,dz \leq \int_{y}^{\infty} e^{-\mathbf{x}(z)}\,dz = \int_{\mathbf{x}(y)}^{\infty} e^{-\mathbf{x}}\,z'(\mathbf{x})\,d\mathbf{x}\,.$$

As $z'(\mathbf{x}) = 2 + \sqrt{\mathbf{p}/\mathbf{x}}$ monotonously decreases with \mathbf{x} , we derive

$$\mathbb{E}\left\{\eta\,\mathbb{I}(\eta\geq \mathsf{y})\right\} \leq z'(\mathsf{x}(\mathsf{y}))\mathrm{e}^{-\mathsf{x}(\mathsf{y})} = \frac{1}{\mathsf{x}'(\mathsf{y})}\,\mathrm{e}^{-\mathsf{x}(\mathsf{y})} = \frac{4\sqrt{\mathsf{y}+\mathsf{p}}}{\sqrt{\mathsf{y}+\mathsf{p}}-\sqrt{\mathsf{p}}}\,\mathrm{e}^{-\mathsf{x}(\mathsf{y})}$$

and (B.10) follows.

In a similar way, define z(x) from the relation $\mu^{-1} \log z(x) = \sqrt{p x} + x$ yielding

$$\mathsf{z}(\mathsf{x}) = \exp\bigl(\mu\sqrt{\mathsf{p}\,\mathsf{x}} + \mu\,\mathsf{x}\bigr).$$

The inverse relation reads

$$x_{e}(z) = (\sqrt{\mu^{-1}\log z + p/4} - \sqrt{p/4})^{2}.$$

Then with $x(y) = x_e(e^{\mu y/2}) = (\sqrt{y+p} - \sqrt{p})^2/4$

$$\begin{split} I\!\!E \big\{ \mathrm{e}^{\mu\eta/2} \, \mathrm{I\!I}(\eta \ge \mathrm{y}) \big\} &= \int_{\mathrm{e}^{\mu\mathrm{y}/2}}^{\infty} I\!\!P(\mathrm{e}^{\mu\eta/2} \ge \mathrm{z}) \, d\mathrm{z} = \int_{\mathrm{e}^{\mu\mathrm{y}/2}}^{\infty} I\!\!P(\eta \ge 2\mu^{-1}\log\mathrm{z}) \, d\mathrm{z} \\ &\leq \int_{\mathrm{e}^{\mu\mathrm{y}/2}}^{\infty} \mathrm{e}^{-\mathrm{x}_{\mathrm{e}}(\mathrm{z})} \, d\mathrm{z} = \int_{\mathrm{x}(\mathrm{y})}^{\infty} \mathrm{e}^{-\mathrm{x}} \, \mathrm{z}'(\mathrm{x}) \, d\mathrm{x}. \end{split}$$

Further, in view of $\mu + 0.5 \,\mu \sqrt{p/x} < \mu + \mu \sqrt{p/x(y)} = \epsilon < 1$ for $x \ge x(y)$, it holds

$$\mathbf{z}'(\mathbf{x}) = \left(\mu + 0.5\,\mu\sqrt{\mathbf{p}/\mathbf{x}}\right)\exp\left(\mu\sqrt{\mathbf{p}\,\mathbf{x}} + \mu\,\mathbf{x}\right) \le \exp\left(\mu\,\mathbf{x}\sqrt{\mathbf{p}/\mathbf{x}(\mathbf{y})} + \mu\,\mathbf{x}\right) = \exp(\epsilon\,\mathbf{x})$$

and

$$\mathbb{E}\left\{\mathrm{e}^{\mu\eta/2}\,\mathbb{I}(\eta\geq \mathtt{y})\right\} \leq \int_{\mathtt{x}(\mathtt{y})}^{\infty} \mathrm{e}^{-(1-\epsilon)\mathtt{x}}\,d\mathtt{x} = \frac{1}{1-\epsilon}\,\mathrm{e}^{-(1-\epsilon)\mathtt{x}(\mathtt{y})}$$

and (B.11) follows.

C Local smoothness conditions

This section discusses different local smoothness characteristics of a multivariate function $f(\boldsymbol{v}) = \mathbb{E}L(\boldsymbol{v}), \ \boldsymbol{v} \in \mathbb{R}^p$.

C.1 Smoothness and self-concordance in Gateaux sense

Below we assume the function $f(\boldsymbol{v})$ to be strongly concave with the negative Hessian $\mathbb{F}(\boldsymbol{v}) \stackrel{\text{def}}{=} -\nabla^2 f(\boldsymbol{v}) \in \mathfrak{M}_p$ positive definite. Also assume $f(\boldsymbol{v})$ three or sometimes even four times Gateaux differentiable in $\boldsymbol{v} \in \Upsilon$. For any particular direction $\boldsymbol{u} \in \mathbb{R}^p$, we consider the univariate function $f(\boldsymbol{v} + t\boldsymbol{u})$ and measure its smoothness in t. Local smoothness of f will be described by the relative error of the Taylor expansion of the third or four order. Namely, define

$$\delta_3(\boldsymbol{v},\boldsymbol{u}) = f(\boldsymbol{v}+\boldsymbol{u}) - f(\boldsymbol{v}) - \langle \nabla f(\boldsymbol{v}), \boldsymbol{u} \rangle - \frac{1}{2} \langle \nabla^2 f(\boldsymbol{v}), \boldsymbol{u}^{\otimes 2} \rangle,$$

$$\delta'_3(\boldsymbol{v},\boldsymbol{u}) = \langle \nabla f(\boldsymbol{v}+\boldsymbol{u}), \boldsymbol{u} \rangle - \langle \nabla f(\boldsymbol{v}), \boldsymbol{u} \rangle - \langle \nabla^2 f(\boldsymbol{v}), \boldsymbol{u}^{\otimes 2} \rangle,$$

and

$$\delta_4(\boldsymbol{\upsilon},\boldsymbol{u}) \stackrel{\text{def}}{=} f(\boldsymbol{\upsilon}+\boldsymbol{u}) - f(\boldsymbol{\upsilon}) - \langle \nabla f(\boldsymbol{\upsilon}),\boldsymbol{u} \rangle - \frac{1}{2} \langle \nabla^2 f(\boldsymbol{\upsilon}),\boldsymbol{u}^{\otimes 2} \rangle - \frac{1}{6} \langle \nabla^3 f(\boldsymbol{\upsilon}),\boldsymbol{u}^{\otimes 3} \rangle.$$

Now, for each \boldsymbol{v} , suppose to be given a positive symmetric operator $\mathsf{D}(\boldsymbol{v}) \in \mathfrak{M}_p$ with $\mathsf{D}^2(\boldsymbol{v}) \leq \mathbb{F}(\boldsymbol{v}) = -\nabla^2 f(\boldsymbol{v})$ defining a local metric and a local vicinity around \boldsymbol{v} :

$$\mathcal{U}(oldsymbol{v}) = \left\{oldsymbol{u} \in I\!\!R^p \colon \|\mathsf{D}(oldsymbol{v})oldsymbol{u}\| \leq \mathtt{r}
ight\}$$

for some radius r.

Local smoothness properties of f are given via the quantities

$$\omega(\boldsymbol{v}) \stackrel{\text{def}}{=} \sup_{\boldsymbol{u}: \|\mathsf{D}(\boldsymbol{v})\boldsymbol{u}\| \le \mathbf{r}} \frac{2|\delta_3(\boldsymbol{v}, \boldsymbol{u})|}{\|\mathsf{D}(\boldsymbol{v})\boldsymbol{u}\|^2}, \qquad \omega'(\boldsymbol{v}) \stackrel{\text{def}}{=} \sup_{\boldsymbol{u}: \|\mathsf{D}(\boldsymbol{v})\boldsymbol{u}\| \le \mathbf{r}} \frac{2|\delta'_3(\boldsymbol{v}, \boldsymbol{u})|}{\|\mathsf{D}(\boldsymbol{v})\boldsymbol{u}\|^2}.$$
(C.1)

The Taylor expansion yields for any \boldsymbol{u} with $\|\mathsf{D}(\boldsymbol{v})\boldsymbol{u}\| \leq r$

$$\left|\delta_{3}(\boldsymbol{v},\boldsymbol{u})\rangle\right| \leq \frac{\omega(\boldsymbol{v})}{2} \|\mathsf{D}(\boldsymbol{v})\boldsymbol{u}\|^{2}, \qquad \left|\delta_{3}'(\boldsymbol{v},\boldsymbol{u})\right| \leq \frac{\omega'(\boldsymbol{v})}{2} \|\mathsf{D}(\boldsymbol{v})\boldsymbol{u}\|^{2}.$$
 (C.2)

The introduced quantities $\omega(v)$, $\omega'(v)$ strongly depend on the radius \mathbf{r} of the local vicinity $\mathcal{U}(v)$. The results about Laplace approximation can be improved provided a homogeneous upper bound on the error of Taylor expansion. Assume a subset Υ° of Υ to be fixed.

 (\mathcal{T}_3) There exists au_3 such that for all $v \in \Upsilon^\circ$

$$\left|\delta_3(\boldsymbol{v}, \boldsymbol{u})\right| \leq rac{ au_3}{6} \|\mathsf{D}(\boldsymbol{v})\,\boldsymbol{u}\|^3, \quad \left|\delta_3'(\boldsymbol{v}, \boldsymbol{u})\right| \leq rac{ au_3}{2} \|\mathsf{D}(\boldsymbol{v})\,\boldsymbol{u}\|^3, \quad \boldsymbol{u} \in \mathcal{U}(\boldsymbol{v}).$$

 (\mathcal{T}_4) There exists au_4 such that for all $v \in \Upsilon^\circ$

$$\left|\delta_4(\boldsymbol{v}, \boldsymbol{u})\right| \leq rac{ au_4}{24} \|\mathsf{D}(\boldsymbol{v})\, \boldsymbol{u}\|^4\,, \qquad \boldsymbol{u}\in \mathcal{U}(\boldsymbol{v}).$$

Lemma C.1. Under (\mathcal{T}_3) , the values $\omega(v)$ and $\omega'(v)$ from (C.1) satisfy

$$\omega(\boldsymbol{v}) \leq \frac{\tau_3 \, \mathbf{r}}{3}, \qquad \omega'(\boldsymbol{v}) \leq \tau_3 \, \mathbf{r}, \qquad \boldsymbol{v} \in \Upsilon^{\circ}.$$

Proof. For any $u \in \mathcal{U}(v)$ with $||\mathsf{D}(v)u|| \leq r$

$$\left|\delta_3(\boldsymbol{v}, \boldsymbol{u})\right| \leq rac{ au_3}{6} \|\mathsf{D}(\boldsymbol{v})\boldsymbol{u}\|^3 \leq rac{ au_3\,\mathbf{r}}{6} \,\|\mathsf{D}(\boldsymbol{v})\boldsymbol{u}\|^2,$$

 \Box

and the bound for $\omega(v)$ follows. The proof for $\omega'(v)$ is similar.

The values τ_3 and τ_4 are usually very small. Some quantitative bounds are given later in this section under the assumption that the function $f(\boldsymbol{v}) = \mathbb{E}L_G(\boldsymbol{v})$ can be written in the form $-f(\boldsymbol{v}) = nh(\boldsymbol{v})$ for a fixed smooth function $h(\boldsymbol{v})$ with the Hessian $\nabla^2 h(\boldsymbol{v})$. The factor n has meaning of the sample size.

 $(\boldsymbol{\mathcal{S}_3}) \quad -f(\boldsymbol{\upsilon}) = nh(\boldsymbol{\upsilon}) \ \text{for } h(\boldsymbol{\upsilon}) \ \text{convex with } \nabla^2 h(\boldsymbol{\upsilon}) \geq \mathsf{m}^2(\boldsymbol{\upsilon}) = \mathsf{D}^2(\boldsymbol{\upsilon})/n \ \text{and}$

$$\sup_{\boldsymbol{u}: \, \|\mathsf{m}(\boldsymbol{v})\boldsymbol{u}\| \leq \mathsf{r}/\sqrt{n}} \frac{\left| \langle \nabla^3 h(\boldsymbol{v}+\boldsymbol{u}), \boldsymbol{u}^{\otimes 3} \rangle \right|}{\|\mathsf{m}(\boldsymbol{v})\boldsymbol{u}\|^3} \leq \mathsf{c}_3 \, .$$

 (\mathcal{S}_4) the function $h(\cdot)$ satisfies (\mathcal{S}_3) and

$$\sup_{\boldsymbol{u}: \, \|\boldsymbol{\mathsf{m}}(\boldsymbol{v})\boldsymbol{u}\| \leq \mathbf{r}/\sqrt{n}} \frac{\left| \langle \nabla^4 h(\boldsymbol{v}+\boldsymbol{u}), \boldsymbol{u}^{\otimes 4} \rangle \right|}{\|\boldsymbol{\mathsf{m}}(\boldsymbol{v})\boldsymbol{u}\|^4} \leq \mathsf{c}_4$$

 (\mathcal{S}_3) and (\mathcal{S}_4) are local versions of the so called self-concordance condition; see Nesterov (1988). In fact, they require that each univariate function $h(\boldsymbol{v} + t\boldsymbol{u})$ of $t \in \mathbb{R}$ is selfconcordant with some universal constants c_3 and c_4 . Under (\mathcal{S}_3) and (\mathcal{S}_4) , we can use $\mathsf{D}^2(\boldsymbol{v}) = n \,\mathsf{m}^2(\boldsymbol{v})$ and easily bound the values $\delta_3(\boldsymbol{v}, \boldsymbol{u})$, $\delta_4(\boldsymbol{v}, \boldsymbol{u})$, and $\omega(\boldsymbol{v})$, $\omega'(\boldsymbol{v})$.

Lemma C.2. Suppose (S_3) . Then (T_3) follows with $\tau_3 = c_3 n^{-1/2}$. Moreover, for $\omega(v)$ and $\omega'(v)$ from (C.1), it holds

$$\omega(\boldsymbol{v}) \leq \frac{\mathbf{c}_3 \,\mathbf{r}}{3n^{1/2}}, \qquad \omega'(\boldsymbol{v}) \leq \frac{\mathbf{c}_3 \,\mathbf{r}}{n^{1/2}}. \tag{C.3}$$

Also (\mathcal{T}_4) follows from (\mathcal{S}_4) with $\tau_4 = c_4 n^{-1}$.

Proof. For any $u \in \mathcal{U}(v)$ and $t \in [0,1]$, by the Taylor expansion of the third order

$$\begin{split} \delta(\boldsymbol{v},\boldsymbol{u}) &| \leq \frac{1}{6} \left| \langle \nabla^3 f(\boldsymbol{v} + t\boldsymbol{u}), \boldsymbol{u}^{\otimes 3} \rangle \right| = \frac{n}{6} \left| \langle \nabla^3 h(\boldsymbol{v} + t\boldsymbol{u}), \boldsymbol{u}^{\otimes 3} \rangle \right| \leq \frac{n \, \mathsf{c}_3}{6} \, \|\mathsf{m}(\boldsymbol{v})\boldsymbol{u}\|^3 \\ &= \frac{n^{-1/2} \, \mathsf{c}_3}{6} \, \|\mathsf{D}(\boldsymbol{v})\boldsymbol{u}\|^3 \leq \frac{n^{-1/2} \, \mathsf{c}_3 \, \mathsf{r}}{6} \, \|\mathsf{D}(\boldsymbol{v})\boldsymbol{u}\|^2 \, . \end{split}$$

This implies (\mathcal{T}_3) as well as (C.3); see (C.2). The statement about (\mathcal{T}_4) is similar. \Box

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