# Sharp deviation bounds and concentration phenomenon for the squared norm of a sub-gaussian vector 

Vladimir Spokoiny*<br>Weierstrass Institute and HU Berlin, Mohrenstr. 39, 10117 Berlin, Germany spokoiny@wias-berlin.de

May 16, 2023


#### Abstract

Let $\boldsymbol{X}$ be a Gaussian zero mean vector with $\operatorname{Var}(\boldsymbol{X})=B$. Then $\|\boldsymbol{X}\|^{2}$ well concentrates around its expectation $\mathrm{p}=\mathbb{E}\|\boldsymbol{X}\|^{2}=\operatorname{tr} B$ provided that the latter is sufficiently large. Namely, $\mathbb{P}\left(\|\boldsymbol{X}\|^{2}-\operatorname{tr} B>2 \sqrt{\mathrm{x} \operatorname{tr}\left(B^{2}\right)}+2\|B\| \mathrm{x}\right) \leq \mathrm{e}^{-\mathrm{x}}$ and $\mathbb{P}\left(\|\boldsymbol{X}\|^{2}-\operatorname{tr} B<\right.$ $\left.-2 \sqrt{\mathrm{x} \operatorname{tr}\left(B^{2}\right)}\right) \leq \mathrm{e}^{-\mathrm{x}}$; see Laurent and Massart (2000). This note provides an extension of these bounds to the case of a sub-gaussian vector $\boldsymbol{X}$. The results are based on the recent advances in Laplace approximation from Spokoiny (2022).


AMS Subject Classification: Primary 62E15. Secondary 62E10
Keywords: exponential moments, quadratic forms, Laplace approximation

## 1 Introduction

Let $\boldsymbol{X}$ be a zero mean Gaussian vector in $\mathbb{R}^{p}$ for $p$ large. Denote $B=\operatorname{Var}(\boldsymbol{X})$. Then for the squared norm $\|\boldsymbol{X}\|^{2}$, it holds $\mathbb{E}\|\boldsymbol{X}\|^{2}=\operatorname{tr} B, \operatorname{Var}\left(\|\boldsymbol{X}\|^{2}\right)=\operatorname{tr}\left(B^{2}\right)$, and this random variable concentrates around its expectation $\operatorname{tr} B$ in the sense that for any $\mathrm{x}>0$

$$
\begin{array}{r}
\mathbb{P}\left(\|\boldsymbol{X}\|^{2}-\operatorname{tr} B>2 \sqrt{\mathrm{x} \operatorname{tr}\left(B^{2}\right)}+2\|B\| \mathrm{x}\right)
\end{array} \leq \mathrm{e}^{-\mathrm{x}}, ~ 子 \begin{array}{r} 
 \tag{1.1}\\
\mathbb{P}\left(\|\boldsymbol{X}\|^{2}-\operatorname{tr} B<-2 \sqrt{\mathrm{x} \operatorname{tr}\left(B^{2}\right)}\right)
\end{array} \leq \mathrm{e}^{-\mathrm{x}} ;
$$

see e.g. Laurent and Massart (2000). The upper bound here can easily be extended to the sub-gaussian case; see e.g. Hsu et al. (2012) or Section 2.1 later. Rudelson and Vershynin

[^0](2013) described the effect of sun-gaussian concentration and established deviation bounds for the centered quadratic form $\|\boldsymbol{X}\|^{2}-\mathbb{E}\|\boldsymbol{X}\|^{2}$ by extending the Hanson-Wright inequality (see Hanson and Wright (1971)). In the recent years, a number of new results were obtained in this direction. We refer to Klochkov and Zhivotovskiy (2020) for an extensive overview and advanced results on Hanson-Wright type concentration inequalities. This note aims at extending the concentration result from (1.1) to a non-gaussian case under possibly mild conditions. Namely, we establish a version of the upper bound in (1.1) using local smoothness of the moment generating function $\mathbb{E} \mathrm{e}^{\langle\boldsymbol{u}, \boldsymbol{X}\rangle}$ and the recent advances in Laplace approximation from Spokoiny (2022). The lower bound is obtained by similar arguments applied to the characteristic function $\mathbb{E} \mathrm{e}^{\mathrm{i}\{\boldsymbol{u}, \boldsymbol{X}\rangle}$.

The paper is organized as follows. Section 2.1 provides a simple but rough upper bound under sub-gaussian condition on $\boldsymbol{X}$. The main results about concentration of $\|\boldsymbol{X}\|^{2}$ are collected in Section 2.2. Section 2.3 specifies the results to the case when $\boldsymbol{X}$ is a normalized sum of independent random vectors. In Section 2.4 we extend the upper bound to a sub-exponential case. Some useful technical facts about Gaussian quadratic forms are collected in the Appendix A and Appendix B.

## 2 Deviation bounds for non-Gaussian quadratic forms

This section collects some probability bounds for non-Gaussian quadratic forms starting from the sub-gaussian case. Then we extend the result to the case of exponential tails.

### 2.1 Sub-gaussian upper bound

Let $\boldsymbol{\xi}$ be a random vector in $\mathbb{R}^{p}, p \leq \infty$ satisfying $\mathbb{E} \boldsymbol{\xi}=0$. We suppose that there exists an operator $\mathbb{V}$ in $\mathbb{R}^{p}$ such that

$$
\begin{equation*}
\log \mathbb{E} \exp \left(\left\langle\boldsymbol{u}, \mathbb{V}^{-1} \boldsymbol{\xi}\right\rangle\right) \leq \frac{\|\boldsymbol{u}\|^{2}}{2}, \quad \boldsymbol{u} \in \mathbb{R}^{p} \tag{2.1}
\end{equation*}
$$

In the Gaussian case, one obviously takes $\mathbb{V}^{2}=\operatorname{Var}(\boldsymbol{\xi})$. In general, $\mathbb{V}^{2} \geq \operatorname{Var}(\boldsymbol{\xi})$. We consider a quadratic form $\|Q \boldsymbol{\xi}\|^{2}$, where $\boldsymbol{\xi}$ satisfies (2.1) and $Q$ is a given linear operator $\mathbb{R}^{p} \rightarrow \mathbb{R}^{q}$ such that $B=Q \mathbb{V}^{2} Q^{\top}$ is a trace operator. Denote

$$
\mathrm{p}=\operatorname{tr}(B), \quad \mathrm{v}^{2} \stackrel{\text { def }}{=} \operatorname{tr}\left(B^{2}\right)
$$

We show that under (2.1), the quadratic form $\|Q \boldsymbol{\xi}\|^{2}$ follows the same upper deviation bound $\mathbb{P}\left(\|Q \boldsymbol{\xi}\|^{2} \geq z^{2}(B, \mathrm{x})\right) \leq \mathrm{e}^{-\mathrm{x}}$ with $z^{2}(B, \mathrm{x})$ from (B.3) as in the Gaussian case.

Similar results can be found e.g. in Hsu et al. (2012). We present an independent proof for reference convenience.

Theorem 2.1. Suppose (2.1). With $B=Q \mathbb{V}^{2} Q^{\top}$, it holds for any $\mu<1 /\|B\|$

$$
\mathbb{E} \exp \left(\frac{\mu}{2}\|Q \boldsymbol{\xi}\|^{2}\right) \leq \exp \left(\frac{\mu^{2} \operatorname{tr}\left(B^{2}\right)}{4(1-\|B\| \mu)}+\frac{\mu \operatorname{tr}(B)}{2}\right)
$$

and for any $\mathrm{x}>0$

$$
\begin{equation*}
\mathbb{P}\left(\|Q \boldsymbol{\xi}\|^{2}>\mathrm{p}+2 \mathrm{v} \sqrt{\mathrm{x}}+2 \mathrm{x}\right) \leq \mathrm{e}^{-\mathrm{x}} . \tag{2.2}
\end{equation*}
$$

The bounds (B.9) through (B.10) of Theorem B. 5 continue to apply as well.
Proof. Normalization by $\|B\|$ reduces the proof to $\|B\|=1$. For $\mu \in(0,1)$, we use

$$
\begin{equation*}
\mathbb{E} \exp \left(\mu\|Q \boldsymbol{\xi}\|^{2} / 2\right)=\mathbb{E} \mathbb{E}_{\boldsymbol{\gamma}} \exp \left(\mu^{1 / 2}\left\langle\mathbb{V} Q^{\top} \boldsymbol{\gamma}, \mathbb{V}^{-1} \boldsymbol{\xi}\right\rangle\right), \tag{2.3}
\end{equation*}
$$

where $\boldsymbol{\gamma}$ is standard Gaussian under $\mathbb{E}_{\boldsymbol{\gamma}}$ independent on $\boldsymbol{\xi}$. Application of Fubini's theorem, (2.1), and (B.5) yields

$$
\mathbb{E} \exp \left(\frac{\mu}{2}\|Q \boldsymbol{\xi}\|^{2}\right) \leq \mathbb{E}_{\gamma} \exp \left(\frac{\mu}{2}\left\|\mathbb{V} Q^{\top} \boldsymbol{\gamma}\right\|^{2}\right) \leq \exp \left(\frac{\mu^{2} \operatorname{tr}\left(B^{2}\right)}{4(1-\mu)}+\frac{\mu \operatorname{tr}(B)}{2}\right)
$$

Further we proceed as in the Gaussian case.
The bound (2.2) looks identical to the Gaussian case, however, there is an essential difference: $\mathrm{p}=\operatorname{tr}(B)$ can be much larger than $\mathbb{E}\|Q \boldsymbol{\xi}\|^{2}=Q^{\top} \operatorname{Var}(\boldsymbol{\xi}) Q$. For supporting the concentration phenomenon of $\|Q \boldsymbol{\xi}\|^{2}$ around its expectation $\mathbb{E}\|Q \boldsymbol{\xi}\|^{2}=$ $\operatorname{tr}\left\{Q^{\top} \operatorname{Var}(\boldsymbol{\xi}) Q\right\}$, the result from (2.2) is not accurate enough. Rudelson and Vershynin (2013) established deviation bounds for the centered quadratic form $\|Q \boldsymbol{\xi}\|^{2}-\mathbb{E}\|Q \boldsymbol{\xi}\|^{2}$ by applying Hanson-Wright inequality (see Hanson and Wright (1971)) to its absolute value. The next section presents some sufficient conditions for obtaining sharp Gaussianlike deviation bounds.

### 2.2 Sharp deviation bounds for the norm of a sub-gaussian vector

Let $\boldsymbol{\xi}$ be a centered random vector in $\mathbb{R}^{p}$ with sub-gaussian tails. We study concentration effect of the squared norm $\|Q \boldsymbol{X}\|^{2}$ for a linear mapping $Q$ and for $\boldsymbol{X}=\mathbb{V}^{-1} \boldsymbol{\xi}$ being the standardized version of $\boldsymbol{\xi}$, where $\mathbb{V}^{2}=\operatorname{Var}(\boldsymbol{\xi})$. More generally, we allow $\mathbb{V}^{2} \geq \operatorname{Var}(\boldsymbol{\xi})$ yielding $\operatorname{Var}(\boldsymbol{X}) \leq I_{p}$ to incorporate the case when $\operatorname{Var}(\boldsymbol{\xi})$ is ill-posed. Later we assume the following condition.
( $\boldsymbol{X})$ A random vector $\boldsymbol{X} \in \mathbb{R}^{p}$ satisfies $\mathbb{E} \boldsymbol{X}=0, \operatorname{Var}(\boldsymbol{X}) \leq I_{p}$. The function $\phi(\boldsymbol{u}) \stackrel{\text { def }}{=} \log \mathbb{E} \mathrm{e}^{\langle\boldsymbol{u}, \boldsymbol{X}\rangle}$ is finite and fulfills for some $\mathbf{C}_{\phi}$

$$
\begin{equation*}
\phi(\boldsymbol{u}) \stackrel{\text { def }}{=} \log \mathbb{E} \mathrm{e}^{\langle\boldsymbol{u}, \boldsymbol{X}\rangle} \leq \frac{\mathrm{C}_{\phi}\|\boldsymbol{u}\|^{2}}{2}, \quad \boldsymbol{u} \in \mathbb{R}^{p} \tag{2.4}
\end{equation*}
$$

The constant $\mathrm{C}_{\phi}$ can be quite large, it does not show up in the leading term of the obtained bound. Also we will only use this condition for $\|\boldsymbol{u}\| \geq \mathrm{g}$ for some sufficiently large g .

Given a linear mapping $Q: \mathbb{R}^{p} \rightarrow \mathbb{R}^{q}$ with $\|Q\| \leq 1$, we expect the quadratic form $\|Q \boldsymbol{X}\|^{2}$ behaves nearly as $\boldsymbol{X}$ were a Gaussian vector. In that case for $\mu<1$

$$
\mathbb{E} \exp \left(\mu\|Q \boldsymbol{X}\|^{2} / 2\right)=\operatorname{det}\left(I_{p}-\mu B\right)^{-1 / 2}, \quad B \stackrel{\text { def }}{=} Q^{\top} \operatorname{Var}(\boldsymbol{X}) Q
$$

Our results provide a bound on $\left|\mathbb{E} \exp \left(\mu\|Q \boldsymbol{X}\|^{2} / 2\right)-\operatorname{det}\left(I_{p}-\mu B\right)^{-1 / 2}\right|$ for $\boldsymbol{X}$ nonGaussian but ( $\boldsymbol{X}$ ) is fulfilled. Define

$$
\begin{aligned}
\mathrm{p} & \stackrel{\text { def }}{=} \mathbb{E}\|Q \boldsymbol{X}\|^{2}=\operatorname{tr}\left\{Q^{\top} \operatorname{Var}(\boldsymbol{X}) Q\right\}=\operatorname{tr} B \\
\mathrm{p}_{Q} & \stackrel{\text { def }}{=} \mathbb{E}\|Q \boldsymbol{\gamma}\|^{2}=\operatorname{tr}\left(Q^{\top} Q\right)
\end{aligned}
$$

Fix $g$ and define for $\boldsymbol{u} \in \mathbb{R}^{p}$ with $\|\boldsymbol{u}\| \leq g$ a measure $\mathbb{E}_{\boldsymbol{u}}$ by

$$
\begin{equation*}
\mathbb{E}_{\boldsymbol{u}} \eta \stackrel{\text { def }}{=} \frac{\mathbb{E}\left(\eta \mathrm{e}^{\langle\boldsymbol{u}, \boldsymbol{X}\rangle}\right)}{\mathbb{E} \mathrm{e}^{\langle\boldsymbol{u}, \boldsymbol{X}\rangle}} \tag{2.5}
\end{equation*}
$$

Also define

$$
\begin{align*}
& \tau_{3} \stackrel{\text { def }}{=} \sup _{\|\boldsymbol{u}\| \leq \mathrm{g}} \frac{1}{\|\boldsymbol{u}\|^{3}}\left|\mathbb{E}_{\boldsymbol{u}}\left\langle\boldsymbol{u}, \boldsymbol{X}-\mathbb{E}_{\boldsymbol{u}} \boldsymbol{X}\right\rangle^{3}\right| \\
& \tau_{4} \stackrel{\text { def }}{=} \sup _{\|\boldsymbol{u}\| \leq \mathrm{g}} \frac{1}{\|\boldsymbol{u}\|^{4}}\left|\mathbb{E}_{\boldsymbol{u}}\left\langle\boldsymbol{u}, \boldsymbol{X}-\mathbb{E}_{\boldsymbol{u}} \boldsymbol{X}\right\rangle^{4}-3\left\{\mathbb{E}_{\boldsymbol{u}}\left\langle\boldsymbol{u}, \boldsymbol{X}-\mathbb{E}_{\boldsymbol{u}} \boldsymbol{X}\right\rangle^{2}\right\}^{2}\right| \tag{2.6}
\end{align*}
$$

These quantities are typically not only finite but also very small. Indeed, for $\boldsymbol{X}$ Gaussian they just vanish. If $\boldsymbol{X}$ is a normalized sum of $n$ i.i.d. centred random vectors $\boldsymbol{\xi}_{i}$ then $\tau_{m} \asymp n^{-m / 2+1}$; see Section 2.3.

Theorem 2.2. Fix a linear mapping $Q: \mathbb{R}^{p} \rightarrow \mathbb{R}^{q}$ s.t. $\|Q\|=1$. Let a random vector $\boldsymbol{X} \in \mathbb{R}^{p}$ satisfy $\mathbb{E} \boldsymbol{X}=0, \operatorname{Var}(\boldsymbol{X}) \leq I_{p}$, and $(\boldsymbol{X})$. Let also $\tau_{3}$ and $\tau_{4}$ be given by (2.6) and g be fixed to ensure $\omega \stackrel{\text { def }}{=} \mathrm{g} \tau_{3} / 2 \leq 1 / 3$. Consider $\mu>0$ satisfying

$$
\begin{equation*}
\mathrm{C}_{\phi} \mu \leq 1 / 3, \quad \mu^{-1} \mathrm{~g}^{2} \geq 9 \mathrm{C}_{\phi} \mathrm{p}_{Q} \tag{2.7}
\end{equation*}
$$

with $\mathrm{p}_{Q}=\operatorname{tr}\left(Q^{\top} Q\right)$. Further, define

$$
\begin{align*}
& \mathrm{x}_{\mu} \stackrel{\text { def }}{=} \frac{1}{4}\left(\sqrt{\mathrm{C}_{\phi}^{-1} \mu^{-1} \mathrm{~g}^{2}}-\sqrt{\mathrm{P}_{Q}}\right)^{2} \\
& \epsilon_{\mu} \stackrel{\text { def }}{=} \mathrm{C}_{\phi} \mu+\mathrm{C}_{\phi} \mu \sqrt{\mathrm{p}_{Q} / \mathrm{x}_{\mu}} \tag{2.8}
\end{align*}
$$

It holds with $B=Q^{\top} \operatorname{Var}(\boldsymbol{X}) Q$

$$
\begin{equation*}
\left|\mathbb{E} \exp \left(\mu\|Q \boldsymbol{X}\|^{2} / 2\right)-\operatorname{det}\left(I_{p}-\mu B\right)^{-1 / 2}\right| \leq \Delta_{\mu} \operatorname{det}\left(I_{p}-\mu B\right)^{-1 / 2} \tag{2.9}
\end{equation*}
$$

where, with some small quantities $\diamond_{4}$ and $\rho_{\mu}$ given below,

$$
\begin{equation*}
\Delta_{\mu} \leq \diamond_{4}+\rho_{\mu}+\frac{1}{1-\epsilon_{\mu}} \exp \left\{\mathrm{C}_{\phi} \mu \mathrm{p}_{Q} / 2-\left(1-\epsilon_{\mu}\right) \mathrm{x}_{\mu}\right\} \tag{2.10}
\end{equation*}
$$

Remark 2.1. Conditions (2.7) imply $\mathrm{x}_{\mu} \geq \mathrm{p}_{Q}$ and $\epsilon_{\mu} \leq 2 / 3$. If $\mu^{-1} \mathrm{~g}^{2} \gg \mathrm{p}_{Q}$ then $\mathrm{x}_{\mu} \gg \mathrm{p}_{Q}$, and hence the last term in (2.10) is quite small. The same holds for the value $\rho_{\mu}$; see later in the proof.

Proof. We use representation (2.3) and Fubini theorem: with $\mathbb{E}_{\boldsymbol{\gamma}}=\mathbb{E}_{\boldsymbol{\gamma} \sim \mathcal{N}(0, I)}$

$$
\begin{equation*}
\mathbb{E} \exp \left(\mu\|Q \boldsymbol{X}\|^{2} / 2\right)=\mathbb{E} \mathbb{E}_{\boldsymbol{\gamma}} \exp \left(\mu^{1 / 2}\left\langle Q^{\top} \boldsymbol{\gamma}, \boldsymbol{X}\right\rangle\right)=\mathbb{E}_{\boldsymbol{\gamma}} \exp \phi\left(\mu^{1 / 2} Q^{\top} \boldsymbol{\gamma}\right) \tag{2.11}
\end{equation*}
$$

Further,

$$
\begin{align*}
\mathbb{E}_{\boldsymbol{\gamma}} \exp \phi\left(\mu^{1 / 2} Q^{\top} \boldsymbol{\gamma}\right)= & \mathbb{E}_{\boldsymbol{\gamma}} \exp \phi\left(\mu^{1 / 2} Q^{\top} \boldsymbol{\gamma}\right) \mathbb{I}\left(\left\|\mu^{1 / 2} Q^{\top} \boldsymbol{\gamma}\right\| \leq \mathrm{g}\right) \\
& +\mathbb{E}_{\boldsymbol{\gamma}} \exp \phi\left(\mu^{1 / 2} Q^{\top} \boldsymbol{\gamma}\right) \mathbb{I}\left(\left\|\mu^{1 / 2} Q^{\top} \boldsymbol{\gamma}\right\|>\mathrm{g}\right) \tag{2.12}
\end{align*}
$$

Each summand here will be bounded separately starting from the second one. By (2.4) and (B.11) of Theorem B.5, it holds under the condition $\epsilon_{\mu}<1$ for $\epsilon_{\mu}$ from (2.8)

$$
\begin{align*}
& \mathbb{E}_{\gamma} \exp \phi\left(\mu^{1 / 2} Q^{\top} \gamma\right) \mathbb{I}\left(\left\|\mu^{1 / 2} Q^{\top} \gamma\right\|>\mathrm{g}\right) \\
& \quad \leq \mathbb{E}_{\gamma} \exp \left(\mathrm{C}_{\phi} \mu\left\|Q^{\top} \gamma\right\|^{2} / 2\right) \mathbb{I}\left(\left\|Q^{\top} \gamma\right\|^{2}>\mu^{-1} \mathrm{~g}^{2}\right) \\
& \quad \leq \exp \left(\mathrm{C}_{\phi} \mu \mathrm{p}_{Q} / 2\right) \mathbb{E}_{\gamma} \exp \left(\mathrm{C}_{\phi} \mu\left(\left\|Q^{\top} \gamma\right\|^{2}-\mathrm{p}_{Q}\right) / 2\right) \mathbb{I}\left(\left\|Q^{\top} \gamma\right\|^{2}>\mu^{-1} \mathrm{~g}^{2}\right) \\
& \quad \leq \frac{1}{1-\epsilon_{\mu}} \exp \left\{\mathrm{C}_{\phi} \mu \mathrm{p}_{Q} / 2-\left(1-\epsilon_{\mu}\right) \mathrm{x}_{\mu}\right\} . \tag{2.13}
\end{align*}
$$

Now we check that $\phi(\boldsymbol{u})$ satisfies $\left(\mathcal{T}_{3}\right)$ and $\left(\mathcal{T}_{4}\right)$ : for any $\|\boldsymbol{u}\| \leq \mathrm{g}$

$$
\begin{align*}
& \left|\delta_{3}(\boldsymbol{u})\right| \stackrel{\text { def }}{=}\left|\phi(\boldsymbol{u})-\frac{1}{2}\left\langle\phi^{\prime \prime}(0), \boldsymbol{u}^{\otimes 2}\right\rangle\right| \leq \frac{\tau_{3}}{6}\|\boldsymbol{u}\|^{3}, \\
& \left|\delta_{4}(\boldsymbol{u})\right| \stackrel{\text { def }}{=}\left|\phi(\boldsymbol{u})-\frac{1}{2}\left\langle\phi^{\prime \prime}(0), \boldsymbol{u}^{\otimes 2}\right\rangle-\frac{1}{6}\left\langle\phi^{\prime \prime \prime}(0), \boldsymbol{u}^{\otimes 3}\right\rangle\right| \leq \frac{\tau_{4}}{24}\|\boldsymbol{u}\|^{4} . \tag{2.14}
\end{align*}
$$

Consider first the univariate case. Let a r.v. $X$ satisfy $\mathbb{E} X=0$ and $\mathbb{E} X^{2} \leq \sigma^{2}$. Define for any $t \in[0, \mathrm{~g}]$ a measure $\mathbb{P}_{t}$ s.t. for any r.v. $\eta$

$$
\mathbb{E}_{t} \eta \stackrel{\text { def }}{=} \frac{\mathbb{E}\left(\eta \mathrm{e}^{t X}\right)}{\mathbb{E} \mathrm{e}^{t X}}
$$

Consider $\phi(t) \stackrel{\text { def }}{=} \log \mathbb{E e}^{t X}$ as a function of $t \in[0, \lambda]$. It is well defined and satisfies $\phi(0)=\phi^{\prime}(0)=0, \phi^{\prime \prime}(0)=\mathbb{E} X^{2} \leq \sigma^{2}$,

$$
\begin{aligned}
\phi^{\prime}(t) & =\mathbb{E}_{t} X \\
\phi^{\prime \prime}(t) & =\mathbb{E}_{t}\left(X-\mathbb{E}_{t} X\right)^{2} \\
\phi^{\prime \prime \prime}(t) & =\mathbb{E}_{t}\left(X-\mathbb{E}_{t} X\right)^{3} \\
\phi^{(4)}(t) & =\mathbb{E}_{t}\left(X-\mathbb{E}_{t} X\right)^{4}-3\left\{\mathbb{E}_{t}\left(X-\mathbb{E}_{t} X\right)^{2}\right\}^{2}
\end{aligned}
$$

Therefore, conditions $\left(\mathcal{T}_{3}\right)$ and $\left(\mathcal{T}_{4}\right)$ follow from (2.6). The multivariate case can be reduced to the univariate one by fixing a direction $\boldsymbol{u} \in \mathbb{R}^{p}$ and considering the function $\phi(t u)$ of $t$.

Next consider the first term in the right hand-side of (2.12). Define $\mathcal{U}=\left\{\boldsymbol{u}:\left\|\mu^{1 / 2} Q^{\top} \boldsymbol{u}\right\| \leq\right.$ $\mathrm{g}\}$. Then with $\mathrm{C}_{p}=(2 \pi)^{-p / 2}$

$$
\mathbb{E}_{\boldsymbol{\gamma}} \exp \phi\left(\mu^{1 / 2} Q^{\top} \boldsymbol{\gamma}\right) \mathbb{I}\left(\left\|\mu^{1 / 2} Q^{\top} \gamma\right\| \leq \mathrm{g}\right)=\mathrm{C}_{p} \int_{\mathcal{U}} \mathrm{e}^{f_{\mu}(\boldsymbol{u})} d \boldsymbol{u}
$$

where

$$
f_{\mu}(\boldsymbol{u})=\phi\left(\mu^{1 / 2} Q^{\top} \boldsymbol{u}\right)-\|\boldsymbol{u}\|^{2} / 2
$$

so that $f_{\mu}(0)=0, \nabla f_{\mu}(0)=0$. Also define

$$
D_{\mu}^{2} \stackrel{\text { def }}{=}-\nabla^{2} f_{\mu}(0)=-\mu Q^{\top} \operatorname{Var}(\boldsymbol{X}) Q+I_{p}=I_{p}-\mu B
$$

The function $f_{\mu}(\boldsymbol{u})$ inherits smoothness properties of $\phi\left(\mu^{1 / 2} Q^{\top} \boldsymbol{u}\right)$. In particular,

$$
\left|f_{\mu}(\boldsymbol{u})-\frac{1}{2}\left\|D_{\mu} \boldsymbol{u}\right\|^{2}\right| \leq \frac{\tau_{3}}{6}\left\|\mu^{1 / 2} Q \boldsymbol{u}\right\|^{3} .
$$

We apply Proposition A8 from Spokoiny (2022) to $f_{\mu}(\boldsymbol{u})$ yielding

$$
\begin{gather*}
\left|\frac{\int_{\mathcal{U}} \mathrm{e}^{f_{\mu}(\boldsymbol{u})} d \boldsymbol{u}-\int_{\mathcal{U}} \mathrm{e}^{-\left\|D_{\mu} \boldsymbol{u}\right\|^{2} / 2} d \boldsymbol{u}}{\int \mathrm{e}^{-\left\|D_{\mu} \boldsymbol{u}\right\|^{2} / 2} d \boldsymbol{u}}\right| \leq \diamond_{4}, \\
\diamond_{4}=\frac{1}{16(1-\omega)^{2}}\left\{\tau_{3}^{2}\left(\mathrm{p}_{\mu}+2 \alpha_{\mu}\right)^{3}+2 \tau_{4}\left(\mathrm{p}_{\mu}+\alpha_{\mu}\right)^{2}\right\}, \tag{2.15}
\end{gather*}
$$

with $\omega=\mathrm{g} \tau_{3} / 2 \leq 1 / 3$ and

$$
\begin{aligned}
& \mathrm{p}_{\mu} \stackrel{\text { def }}{=} \operatorname{tr}\left\{D_{\mu}^{-2}\left(\mu Q^{\top} Q\right)\right\} \\
& \alpha_{\mu} \stackrel{\text { def }}{=}\left\|D_{\mu}^{-1}\left(\mu Q^{\top} Q\right) D_{\mu}^{-1}\right\| .
\end{aligned}
$$

Note that $\|B\| \leq 1$ implies with $\mathrm{p}_{Q}=\operatorname{tr}\left(Q^{\top} Q\right)$

$$
\mathrm{p}_{\mu} \leq \frac{\mu}{1-\mu} \mathrm{p}_{Q}, \quad \alpha_{\mu} \leq \frac{\mu}{1-\mu}
$$

and

$$
\begin{equation*}
\diamond_{4} \leq \frac{1}{16(1-\omega)^{2}}\left\{\frac{\tau_{3}^{2} \mu^{3}\left(\mathrm{p}_{Q}+2\right)^{3}}{(1-\mu)^{3}}+\frac{2 \tau_{4} \mu^{2}\left(\mathrm{p}_{Q}+1\right)^{2}}{(1-\mu)^{2}}\right\} . \tag{2.16}
\end{equation*}
$$

Furthermore, it holds

$$
\begin{align*}
\rho_{\mu} & \stackrel{\text { def }}{=} 1-\frac{\int_{\mathcal{U}} \mathrm{e}^{-\left\|D_{\mu} \boldsymbol{u}\right\|^{2} / 2} d \boldsymbol{u}}{\int \mathrm{e}^{-\left\|D_{\mu} u\right\|^{2} / 2} d \boldsymbol{u}}=\mathbb{P}\left(\left\|\mu^{1 / 2} Q^{\top} D_{\mu}^{-1} \gamma\right\|>\mathrm{g}\right) \\
& \leq \mathbb{P}\left(\left\|Q^{\top} \gamma\right\|^{2}>(1-\mu) \mu^{-1} \mathrm{~g}^{2}\right) \tag{2.17}
\end{align*}
$$

and the latter value is small provided $\mu^{-1} \mathrm{~g}^{2} \gg \mathrm{p}_{Q}$. This and (2.15) yield the bound

$$
\begin{equation*}
\left|\frac{\int_{\mathcal{U}} \mathrm{e}^{f_{\mu}(\boldsymbol{u})} d \boldsymbol{u}}{\int \mathrm{e}^{-\left\|D_{\mu} \boldsymbol{u}\right\|^{2} / 2} d \boldsymbol{u}}-1\right| \leq \diamond_{4}+\rho_{\mu} . \tag{2.18}
\end{equation*}
$$

It remains to note that

$$
\mathrm{C}_{p} \int \mathrm{e}^{-\left\|D_{\mu} u\right\|^{2} / 2} d \boldsymbol{u}=\frac{1}{\operatorname{det} D_{\mu}}=\operatorname{det}\left(I_{p}-\mu B\right)^{-1 / 2}
$$

and (2.9) follows from (2.13) and (2.18) in view of $\operatorname{det}\left(I_{p}-\mu B\right) \leq 1$.
Upper deviation bounds for $\|Q \boldsymbol{X}\|^{2}$ can now be derived as in the Gaussian case by applying (2.9) with a proper choice of $\mu$. This leads to a surprisingly sharp bound on the upper deviation probability which almost repeats bound (B.8) for $\boldsymbol{X}$ Gaussian.

Corollary 2.3. Let $B=Q^{\top} \operatorname{Var}(\boldsymbol{X}) Q$. With $\mathrm{x}>0$ fixed, define $\mu=\mu(\mathrm{x})$ by $\mu^{-1}=$ $1+\sqrt{\operatorname{tr}\left(B^{2}\right) /(4 \mathrm{x})}$. Assume the condition of Theorem 2.2 for this choice of $\mu$. Then

$$
\mathbb{P}(\|Q \boldsymbol{X}\|>z(B, \mathrm{x}))=\mathbb{P}\left(\|Q \boldsymbol{X}\|^{2}>\operatorname{tr} B+2 \sqrt{\mathrm{x} \operatorname{tr}\left(B^{2}\right)}+2 \mathrm{x}\right) \leq\left(1+\Delta_{\mu}\right) \mathrm{e}^{-\mathrm{x}}
$$

Remark 2.2. Theorem 2.2 requires $\mu$ to be a small number to ensure (2.7). Alternatively, we need $\operatorname{tr}\left(B^{2}\right) \gg \mathrm{x}$. This is an important message: concentration of the squared
norm $\|Q \boldsymbol{X}\|^{2}$ is only possible in high dimension when $\operatorname{tr}\left(B^{2}\right)$ is sufficiently large. In typical situations it holds $\operatorname{tr} B \approx \operatorname{tr}\left(Q^{\top} Q\right)$ and also $\operatorname{tr}\left(B^{2}\right) \asymp \operatorname{tr} B \approx \mathrm{p}_{Q}$. Then the effective trace of $Q^{\top} Q$ should be large. The choice of $\mu$ by $\mu^{-1}=1+\sqrt{\operatorname{tr}\left(B^{2}\right) /(4 \mathrm{x})}$ leads to $\mu \asymp \sqrt{\mathrm{xp}_{Q}}$. This helps to evaluate the term $\diamond_{4}$ from (2.16) in the bound (2.10). Namely, (2.16) yields

$$
\diamond_{4} \lesssim \tau_{3}^{2} \mathrm{x}^{3 / 2} \mathrm{p}_{Q}^{3 / 2}+\tau_{4} \mathrm{xp}_{Q} .
$$

This value is small provided $\tau_{3}^{2} \ll \mathrm{p}_{Q}^{-3 / 2}$ and $\tau_{4} \ll \mathrm{p}_{Q}^{-1}$.
For getting the bound on the lower deviation probability, we need an analog of (2.9) for $\mu$ negative. Representation (2.11) reads as

$$
\begin{equation*}
\mathbb{E} \mathrm{e}^{-\mu\|Q \boldsymbol{X}\|^{2} / 2}=\mathbb{E} \mathbb{E}_{\boldsymbol{\gamma}} \mathrm{e}^{\mathrm{i} \sqrt{\mu}\left\langle Q^{\top} \boldsymbol{\gamma}, \boldsymbol{X}\right\rangle}=\mathbb{E}_{\boldsymbol{\gamma}} \mathbb{E} \mathrm{e}^{\mathrm{i} \sqrt{\mu}\left\langle Q^{\top} \boldsymbol{\gamma}, \boldsymbol{X}\right\rangle} \tag{2.19}
\end{equation*}
$$

with $i=\sqrt{-1}$. Our technique requires that the characteristic function $\mathbb{E} \exp (i\langle\boldsymbol{u}, \boldsymbol{X}\rangle)$ does not vanish. This allows to define

$$
\mathrm{f}(\boldsymbol{u}) \stackrel{\text { def }}{=} \log \mathbb{E} \mathrm{e}^{\mathrm{i}\langle\boldsymbol{u}, \boldsymbol{X}\rangle}
$$

Later we assume that the function $\mathrm{f}(\boldsymbol{u})$ satisfies the condition similar to ( $\boldsymbol{X}$ ).
(iX) For some fixed g and $\mathrm{C}_{\mathrm{f}}$, the function $\mathrm{f}(\boldsymbol{u})=\log \mathbb{E} \mathrm{e}^{\mathrm{i}\langle\boldsymbol{u}, \boldsymbol{X}\rangle}$ satisfies

$$
|\mathrm{f}(\boldsymbol{u})|=\left|\log \mathbb{E} \mathrm{e}^{\mathrm{i}\langle\boldsymbol{u}, \boldsymbol{X}\rangle}\right| \leq \mathrm{C}_{\mathrm{f}}, \quad\|\boldsymbol{u}\| \leq \mathrm{g}
$$

Note that this condition can easily be ensured by replacing $\boldsymbol{X}$ with $\boldsymbol{X}+\alpha \boldsymbol{\gamma}$ for any positive $\alpha$ and $\gamma \sim \mathcal{N}\left(0, I_{p}\right)$. The constant $\mathrm{C}_{\mathrm{f}}$ is unimportant, it does not show up in our results. It, however, enables us to define similarly to (2.6)

$$
\begin{align*}
& \tau_{3} \stackrel{\text { def }}{=} \sup _{\|\boldsymbol{u}\| \leq \mathrm{g}} \frac{1}{\|\boldsymbol{u}\|^{3}}\left|\mathbb{E}_{\mathrm{i} \boldsymbol{u}}\left\langle\mathrm{i} \boldsymbol{u}, \boldsymbol{X}-\mathbb{E}_{\mathrm{i} \boldsymbol{u}} \boldsymbol{X}\right\rangle^{3}\right| \\
& \tau_{4} \stackrel{\text { def }}{=} \sup _{\|\boldsymbol{u}\| \leq \mathrm{g}} \frac{1}{\|\boldsymbol{u}\|^{4}}\left|\mathbb{E}_{\mathrm{i} \boldsymbol{u}}\left\langle\mathrm{i} \boldsymbol{u}, \boldsymbol{X}-\mathbb{E}_{\mathrm{i} \boldsymbol{u}} \boldsymbol{X}\right\rangle^{4}-3\left\{\mathbb{E}_{\mathrm{i} \boldsymbol{u}}\left\langle\mathrm{i} \boldsymbol{u}, \boldsymbol{X}-\mathbb{E}_{\mathrm{i} \boldsymbol{u}} \boldsymbol{X}\right\rangle^{2}\right\}^{2}\right| \tag{2.20}
\end{align*}
$$

Theorem 2.4. Let $\|Q\|=1, \mathrm{p}_{Q}=\operatorname{tr}\left(Q^{\top} Q\right)$. Let $\boldsymbol{X}$ satisfy $\mathbb{E} \boldsymbol{X}=0, \operatorname{Var}(\boldsymbol{X}) \leq I_{p}$, and $(\mathrm{i} \boldsymbol{X})$ for a fixed g . Let also $\tau_{3}$ and $\tau_{4}$ be given by (2.20) and $\omega \stackrel{\text { def }}{=} \mathrm{g} \tau_{3} / 2 \leq 1 / 3$. For any $\mu>0$ s.t. $\mu^{-1} \mathrm{~g}^{2} \geq 4 \mathrm{p}_{Q}$, it holds with $B=Q^{\top} \operatorname{Var}(\boldsymbol{X}) Q$

$$
\begin{gather*}
\left|\mathbb{E} \mathrm{e}^{-\mu\|Q \boldsymbol{X}\|^{2} / 2}-\operatorname{det}\left(I_{p}+\mu B\right)^{-1 / 2}\right| \leq\left(\diamond_{4}+\rho_{\mu}\right) \operatorname{det}\left(I_{p}+\mu B\right)^{-1 / 2}+\rho_{\mu} \\
\rho_{\mu} \leq \mathbb{P}_{\gamma}\left(\|Q \gamma\|^{2} \geq \mu^{-1} \mathrm{~g}^{2}\right) \leq \frac{1}{4}\left(\sqrt{\mu^{-1} \mathrm{~g}^{2}}-\sqrt{\mathrm{P}_{Q}}\right)^{2} \tag{2.21}
\end{gather*}
$$

Proof. We follow the line of the proof of Theorem 2.2 replacing everywhere $\phi(\boldsymbol{u})$ with $\mathrm{f}(\boldsymbol{u})$. In particular, we start with representation (2.19) and apply

$$
\begin{aligned}
& \mathbb{E} \mathrm{e}^{-\mu\|Q \boldsymbol{X}\|^{2} / 2}=\mathbb{E}_{\boldsymbol{\gamma}} \mathrm{e}^{\mathrm{f}\left(\sqrt{\mu} Q^{\top} \gamma\right)} \\
& \quad=\mathbb{E}_{\boldsymbol{\gamma}} \mathrm{e}^{\mathrm{f}\left(\sqrt{\mu} Q^{\top} \boldsymbol{\gamma}\right)} \mathbb{I}\left(\left\|\sqrt{\mu} Q^{\top} \gamma\right\| \leq \mathrm{g}\right)+\mathbb{E}_{\boldsymbol{\gamma}} \mathrm{e}^{\mathrm{f}\left(\sqrt{\mu} Q^{\top} \boldsymbol{\gamma}\right)} \mathbb{I}\left(\left\|\sqrt{\mu} Q^{\top} \boldsymbol{\gamma}\right\|>\mathrm{g}\right) .
\end{aligned}
$$

It holds

$$
\mathrm{f}(0)=0, \quad \nabla \mathrm{f}(0)=0, \quad-\nabla^{2} \mathrm{f}(0)=\operatorname{Var}(\boldsymbol{X}) \leq I_{p} .
$$

Moreover, smoothness conditions (2.14) are automatically fulfilled for $f(\boldsymbol{u})$ with the same $\tau_{3}$ and $\tau_{4}$. The most important observation for the proof is that the bound (2.18) continues to apply for $\mu<0$ and

$$
f_{\mu}(\boldsymbol{u})=\mathrm{f}\left(\sqrt{\mu} Q^{\top} \boldsymbol{u}\right)-\|\boldsymbol{u}\|^{2} / 2
$$

with $\diamond_{4}$ from (2.15) and

$$
\begin{array}{ll}
D_{\mu}^{2} \stackrel{\text { def }}{=}-\nabla^{2} f_{\mu}(0) & =\mu Q^{\top} \operatorname{Var}(\boldsymbol{X}) Q+I_{p}=I_{p}+\mu B, \\
\mathrm{p}_{\mu} \stackrel{\text { def }}{=} \operatorname{tr}\left\{D_{\mu}^{-2}\left(\mu Q^{\top} Q\right)\right\} & \leq \frac{\mu}{1+\mu} \operatorname{tr}\left(Q^{\top} Q\right) \leq \mu \mathrm{p}_{Q}, \\
\alpha_{\mu} \stackrel{\text { def }}{=}\left\|D_{\mu}^{-1}\left(\mu Q^{\top} Q\right) D_{\mu}^{-1}\right\| \leq \frac{\mu}{1+\mu},
\end{array}
$$

and $\rho_{\mu} \leq \mathbb{P}\left(\|Q \gamma\|^{2} \geq \mu^{-1} \mathrm{~g}^{2}\right)$; cf. (2.17). This yields

$$
\left|\mathbb{E}_{\boldsymbol{\gamma}} \mathrm{e}^{\mathrm{f}\left(\sqrt{\mu} Q^{\top} \gamma\right)} \mathbb{I}\left(\left\|\sqrt{\mu} Q^{\top} \gamma\right\| \leq \mathrm{g}\right)-\frac{1}{\operatorname{det}\left(I_{p}+\mu B\right)^{1 / 2}}\right| \leq \frac{\diamond_{4}+\rho_{\mu}}{\operatorname{det}\left(I_{p}+\mu B\right)^{1 / 2}}
$$

Finally we use $\left|\mathrm{e}^{\mathrm{f}(u)}\right| \leq 1$ and thus,

$$
\left|\mathbb{E}_{\gamma} \mathrm{e}^{\mathrm{f}\left(\sqrt{\mu} Q^{\top} \gamma\right)} \mathbb{I}\left(\left\|\sqrt{\mu} Q^{\top} \gamma\right\|>\mathrm{g}\right)\right| \leq \mathbb{P}\left(\left\|\sqrt{\mu} Q^{\top} \gamma\right\|>\mathrm{g}\right)
$$

and (2.21) follows.

Corollary 2.5. With $\mathrm{x}>0$ fixed, define $\mu=2 \mathrm{v}^{-1} \sqrt{\mathrm{x}}$ for $\mathrm{v}^{2}=\operatorname{tr} B^{2}$. Assume the condition of Theorem 2.4 for this choice of $\mu$. Then with $\rho_{\mu}=\mathbb{P}\left(\|Q \gamma\|^{2} \geq \mu^{-1} \mathrm{~g}^{2}\right)$

$$
\mathbb{P}\left(\|Q \boldsymbol{X}\|^{2}<\operatorname{tr} B-2 \mathrm{v} \sqrt{\mathrm{x}}\right) \leq\left(1+\diamond_{4}+\rho_{\mu}\right) \mathrm{e}^{-\mathrm{x}}+\rho_{\mu} \exp \left(\mathrm{v}^{-1} \operatorname{tr} B \sqrt{\mathrm{x}}-2 \mathrm{x}\right)
$$

Proof. By the exponential Chebyshev inequality and (2.21)

$$
\begin{aligned}
& \mathbb{P}\left(\operatorname{tr} B-\|Q \boldsymbol{X}\|^{2}>2 \mathrm{v} \sqrt{\mathrm{x}}\right) \leq \exp (-\mu \mathrm{v} \sqrt{\mathrm{x}}) \mathbb{E} \exp \left\{\mu \operatorname{tr} B / 2-\mu\|Q \boldsymbol{X}\|^{2} / 2\right\} \\
& \quad \leq \exp (\mu \operatorname{tr} B / 2-\mu \mathrm{v} \sqrt{\mathrm{x}})\left\{\left(\diamond_{4}+\rho_{\mu}\right) \operatorname{det}\left(I_{p}+\mu B\right)^{-1 / 2}+\rho_{\mu}\right\} .
\end{aligned}
$$

It remains to note that by $x-\log (1+x) \leq x^{2} / 2$ and $\mu=2 \mathrm{v}^{-1} \sqrt{\mathrm{x}}$, it holds

$$
-\mu \mathrm{v} \sqrt{\mathrm{x}}+\mu \operatorname{tr} B / 2+\log \operatorname{det}\left(I_{p}+\mu B\right)^{-1 / 2} \leq-\mu \mathrm{v} \sqrt{\mathrm{x}}+\mu^{2} \mathrm{v}^{2} / 4=-\mathrm{x}
$$

and also $\mu \operatorname{tr} B / 2-\mu \mathrm{v} \sqrt{\mathrm{x}}=\mathrm{v}^{-1} \operatorname{tr} B \sqrt{\mathrm{x}}-2 \mathrm{x}$.
Remark 2.3. The statement of Corollary 2.5 is meaningful and informative if $\mu^{-1} \mathrm{~g}^{2} \gg$ $\mathrm{p}_{Q}$. If $\mathrm{v}^{2}=\operatorname{tr} B^{2} \asymp \operatorname{tr} B \asymp \mathrm{p}_{Q}$, it suffices to ensure $\mathrm{g}^{2} \gg \mathrm{p}_{Q}^{1 / 2}$.

### 2.3 Sum of i.i.d. random vectors

Here we specify the obtained results to the case when $\boldsymbol{X}=n^{-1 / 2} \sum_{i=1}^{n} \boldsymbol{\xi}_{i}$ and $\boldsymbol{\xi}_{i}$ are i.i.d. in $\mathbb{R}^{p}$ with $\mathbb{E} \boldsymbol{\xi}_{i}=0$ and $\operatorname{Var}\left(\boldsymbol{\xi}_{i}\right) \leq I_{p}$. In fact, the i.i.d. structure of the $\boldsymbol{\xi}_{i}$ 's is not used, it suffices to check that all the moment conditions later on are satisfied uniformly over $i \leq n$. However, the formulation slightly simplifies in the i.i.d case. Let some $Q: \mathbb{R}^{p} \rightarrow \mathbb{R}^{q}$ be fixed with $\|Q\|=1$. It holds

$$
\mathrm{p}=\mathbb{E}\|Q \boldsymbol{X}\|^{2}=\operatorname{tr} B, \quad B=Q^{\top} \Sigma Q .
$$

Also define $\mathrm{p}_{Q}=Q^{\top} Q$. We study the concentration phenomenon for $\|Q \boldsymbol{X}\|^{2}$ under two basic. Later we assume that $\mathrm{p} \approx \mathrm{p}_{Q}$ is a large number and $\mathrm{v}^{2}=\operatorname{tr}\left(B^{2}\right) \approx \mathrm{p} \approx \mathrm{p}_{Q}$. The goal is to apply Corollary 2.3 and Corollary 2.5 claiming that $\|Q \boldsymbol{X}\|^{2}-\mathrm{p}$ can be sandwiched between $-2 \mathrm{v} \sqrt{\mathrm{x}}$ and $2 \mathrm{v} \sqrt{\mathrm{x}}+2 \mathrm{x}$ with probability at least $1-2 \mathrm{e}^{-\mathrm{x}}$. The major required condition is sub-gaussian behavior of $\boldsymbol{\xi}_{1}$. The whole list is given here.
$\left(\xi_{1}\right)$ A random vector $\boldsymbol{\xi}_{1} \in \mathbb{R}^{p}$ satisfies $\mathbb{E} \boldsymbol{\xi}_{1}=0, \operatorname{Var}\left(\boldsymbol{\xi}_{1}\right) \leq I_{p}$. Also

1. The function $\phi_{1}(\boldsymbol{u}) \stackrel{\text { def }}{=} \log \mathbb{E} \mathrm{e}^{\left\langle\boldsymbol{u}, \boldsymbol{\xi}_{1}\right\rangle}$ is finite and fulfills for some $\mathrm{C}_{\phi}$

$$
\phi_{1}(\boldsymbol{u}) \stackrel{\text { def }}{=} \log \mathbb{E} \mathrm{e}^{\left\langle\boldsymbol{u}, \boldsymbol{\xi}_{1}\right\rangle} \leq \frac{\mathrm{C}_{\phi}\|\boldsymbol{u}\|^{2}}{2}, \quad \boldsymbol{u} \in \mathbb{R}^{p}
$$

2. For some $\varrho>0$ and some constants $\mathbf{c}_{3}, \mathbf{c}_{\mathbf{4}}$, it holds with $\mathbb{E}_{\boldsymbol{u}}$ from (2.5)

$$
\begin{aligned}
& \sup _{\|\boldsymbol{u}\| \leq \varrho} \frac{1}{\|\boldsymbol{u}\|^{3}}\left|\mathbb{E}_{\boldsymbol{u}}\left\langle\boldsymbol{u}, \boldsymbol{\xi}_{1}-\mathbb{E}_{\boldsymbol{u}} \boldsymbol{\xi}_{1}\right\rangle^{3}\right| \leq \mathrm{c}_{3}, \\
& \sup _{\|\boldsymbol{u}\| \leq \varrho} \frac{1}{\|\boldsymbol{u}\|^{4}}\left|\mathbb{E}_{\boldsymbol{u}}\left\langle\boldsymbol{u}, \boldsymbol{\xi}_{1}-\mathbb{E}_{\boldsymbol{u}} \boldsymbol{\xi}_{1}\right\rangle^{4}-3\left\{\mathbb{E}_{\boldsymbol{u}}\left\langle\boldsymbol{u}, \boldsymbol{\xi}_{1}-\mathbb{E}_{\boldsymbol{u}} \boldsymbol{\xi}_{1}\right\rangle^{2}\right\}^{2}\right| \leq \mathrm{c}_{4} .
\end{aligned}
$$

3. The function $\log \mathbb{E} \mathrm{e}^{\mathrm{i}\left\{\boldsymbol{u}, \boldsymbol{\xi}_{1}\right\rangle}$ is well defined and

$$
\begin{aligned}
& \sup _{\|\boldsymbol{u}\| \leq \varrho} \frac{1}{\|\boldsymbol{u}\|^{3}}\left|\mathbb{E}_{\mathrm{i} \boldsymbol{u}}\left\langle\mathrm{i} \boldsymbol{u}, \boldsymbol{\xi}_{1}-\mathbb{E}_{\mathrm{i} \boldsymbol{u}} \boldsymbol{\xi}_{1}\right\rangle^{3}\right| \leq \mathrm{c}_{3}, \\
& \sup _{\|\boldsymbol{u}\| \leq \varrho} \frac{1}{\|\boldsymbol{u}\|^{4}}\left|\mathbb{E}_{\mathrm{i} \boldsymbol{u}}\left\langle\mathrm{i} \boldsymbol{u}, \boldsymbol{\xi}_{1}-\mathbb{E}_{\mathrm{i} \boldsymbol{u}} \boldsymbol{\xi}_{1}\right\rangle^{4}-3\left\{\mathbb{E}_{\mathrm{i} \boldsymbol{u}}\left\langle\mathrm{i} \boldsymbol{u}, \boldsymbol{\xi}_{1}-\mathbb{E}_{\mathrm{i} \boldsymbol{u}} \boldsymbol{\xi}_{1}\right\rangle^{2}\right\}^{2}\right| \leq \mathrm{c}_{4} .
\end{aligned}
$$

We are now well prepared to state the result for the i.i.d. case. Apart $\left(\xi_{1}\right)$, we need $\mathrm{p}_{Q}$ to be sufficiently large to ensure the condition $\mathrm{C}_{\phi} \mu \leq 1 / 3$; see (2.7). Also we require $n$ to be large enough for the relation $\mathrm{p}_{Q}^{3 / 2} \ll n$, where $a \ll b$ means that $a / b \leq \mathrm{c}$ for some small absolute constant $c$. Similarly $a \lesssim b$ means $a / b \leq C$ for an absolute constant C.

Theorem 2.6. Let $\boldsymbol{X}=n^{-1 / 2} \sum_{i=1}^{n} \boldsymbol{\xi}_{i}$, $\boldsymbol{\xi}_{i}$ are i.i.d. in $\mathbb{R}^{p}$ with $\mathbb{E} \boldsymbol{\xi}_{1}=0$ and $\operatorname{Var}\left(\boldsymbol{\xi}_{1}\right) \leq I_{p}$. For a fixed x , assume $\left(\xi_{1}\right)$ with $n \varrho^{2} \gg \mathrm{x}_{Q}$. Let also $\mathrm{p}_{Q} \gg \mathrm{C}_{\phi}^{2} \mathrm{x}$ and $n \gg \mathrm{p}_{Q}^{3 / 2}$. Then

$$
\begin{aligned}
\mathbb{P}\left(\|Q \boldsymbol{X}\|^{2}>\operatorname{tr} B+2 \sqrt{\mathrm{x} \operatorname{tr}\left(B^{2}\right)}+2 \mathrm{x}\right) & \leq\left(1+\Delta_{\mu}\right) \mathrm{e}^{-\mathrm{x}} \\
\mathbb{P}\left(\|Q \boldsymbol{X}\|^{2}<\operatorname{tr} B-2 \mathrm{v} \sqrt{\mathrm{x}}\right) & \leq\left(1+\Delta_{\mu}\right) \mathrm{e}^{-\mathrm{x}}
\end{aligned}
$$

with

$$
\Delta_{\mu} \lesssim \frac{\mathrm{x}^{3 / 2} \mathrm{p}_{Q}^{3 / 2}}{n}
$$

Proof. The definition and i.i.d structure of the $\boldsymbol{\xi}_{i}$ 's yield

$$
\phi(\boldsymbol{u})=\log \mathbb{E} \mathrm{e}^{\langle\boldsymbol{X}, \boldsymbol{u}\rangle}=n \phi_{1}\left(n^{-1 / 2} \boldsymbol{u}\right) .
$$

Moreover, for any $\boldsymbol{u}$

$$
\begin{aligned}
& \mathbb{E}_{\boldsymbol{u}}\left\langle\boldsymbol{u}, \boldsymbol{X}-\mathbb{E}_{\boldsymbol{u}} \boldsymbol{X}\right\rangle^{2}=\mathbb{E}_{\boldsymbol{u}}\left\langle\boldsymbol{u}, \boldsymbol{\xi}_{1}-\mathbb{E}_{\boldsymbol{u}} \boldsymbol{\xi}_{1}\right\rangle^{2} \\
& \mathbb{E}_{\boldsymbol{u}}\left\langle\boldsymbol{u}, \boldsymbol{X}-\mathbb{E}_{\boldsymbol{u}} \boldsymbol{X}\right\rangle^{3}=n^{-1 / 2} \mathbb{E}_{\boldsymbol{u}}\left\langle\boldsymbol{u}, \boldsymbol{\xi}_{1}-\mathbb{E}_{\boldsymbol{u}} \boldsymbol{\xi}_{1}\right\rangle^{3},
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathbb{E}_{\boldsymbol{u}}\left\langle\boldsymbol{u}, \boldsymbol{X}-\mathbb{E}_{\boldsymbol{u}} \boldsymbol{X}\right\rangle^{4}-3\left\{\mathbb{E}_{\boldsymbol{u}}\left\langle\boldsymbol{u}, \boldsymbol{X}-\mathbb{E}_{\boldsymbol{u}} \boldsymbol{X}\right\rangle^{2}\right\}^{2} \\
& \quad=n^{-1} \mathbb{E}_{\boldsymbol{u}}\left\langle\boldsymbol{u}, \boldsymbol{\xi}_{1}-\mathbb{E}_{\boldsymbol{u}} \boldsymbol{\xi}_{1}\right\rangle^{4}-3 n^{-1}\left\{\mathbb{E}_{\boldsymbol{u}}\left\langle\boldsymbol{u}, \boldsymbol{\xi}_{1}-\mathbb{E}_{\boldsymbol{u}} \boldsymbol{\xi}_{1}\right\rangle^{2}\right\}^{2}
\end{aligned}
$$

This implies (2.6) for any g with $\mathrm{g} / \sqrt{n} \leq \varrho$ and

$$
\tau_{3} \leq n^{-1 / 2} \mathrm{c}_{3}, \quad \tau_{4} \leq n^{-1} \mathrm{c}_{4} .
$$

Moreover, the quantity $\diamond_{4}$ from (2.16) satisfies

$$
\diamond_{4} \lesssim \frac{\mathrm{x}^{3 / 2} \mathrm{p}_{Q}^{3 / 2}}{n}
$$

while the other terms like $\rho_{\mu}$ in the definition (2.10) of $\Delta_{\mu}$ are exponentially small. Now the upper bound follows from Corollary 2.3. Similar arguments can be used for checking the lower bound by Corollary 2.5 .

### 2.4 Light exponential tails

Now we turn to the main case of light exponential tails of $\boldsymbol{\xi}$. Namely, we suppose that $\mathbb{E} \boldsymbol{\xi}=0$ and for some fixed $\mathrm{g}>0$

$$
\begin{equation*}
\phi(\boldsymbol{u}) \stackrel{\text { def }}{=} \log \mathbb{E} \exp \left(\left\langle\boldsymbol{u}, \mathbb{V}^{-1} \boldsymbol{\xi}\right\rangle\right) \leq \frac{\|\boldsymbol{u}\|^{2}}{2}, \quad \boldsymbol{u} \in \mathbb{R}^{p},\|\boldsymbol{u}\| \leq \mathrm{g} \tag{2.22}
\end{equation*}
$$

for some self-adjoint operator $\mathbb{V}$ in $\mathbb{R}^{p}, \mathbb{V} \geq I_{p}$. In fact, it suffices to assume that

$$
\begin{equation*}
\sup _{\|\boldsymbol{u}\| \leq g} \mathbb{E} \exp \left(\left\langle\boldsymbol{u}, \mathbb{V}^{-1} \boldsymbol{\xi}\right\rangle\right) \leq \mathrm{C} \tag{2.23}
\end{equation*}
$$

The quantity C can be very large but it is not important and does not enter in the established bounds. In fact, condition (2.23) implies an analog of (2.22) for a $\mathrm{g}<\mathrm{g}$ : by (2.14)

$$
\phi(\boldsymbol{u}) \leq \frac{\|\boldsymbol{u}\|^{2}}{2}+\frac{\tau_{3}\|\boldsymbol{u}\|^{3}}{6} \leq \frac{\|\boldsymbol{u}\|^{2}}{2}\left(1+\frac{\tau_{3} \mathrm{~g}}{3}\right), \quad\|\boldsymbol{u}\| \leq \mathrm{g}
$$

for a small value $\tau_{3}$. Moreover, reducing $g$ allows to take $\mathbb{V}^{2}$ equal or close to $\operatorname{Var}(\boldsymbol{\xi})$.
Now we continue with a vector $\boldsymbol{\xi}$ satisfying (2.22). As previously, the goal is to establish possibly sharp deviation bounds on $\|Q \boldsymbol{\xi}\|^{2}$ for a given linear mapping $Q: \mathbb{R}^{p} \rightarrow$ $\mathbb{R}^{q}$. Remind the notation $B=Q \mathbb{V}^{2} Q^{\top}$. By normalization, one can easily reduce the study to the case $\|B\|=1$. Let $\mathrm{p}=\operatorname{tr}(B), \mathrm{v}^{2}=\operatorname{tr}\left(B^{2}\right)$, and $\mu(\mathrm{x})$ be defined by $\mu(\mathrm{x})=\left(1+\frac{\mathrm{v}}{2 \sqrt{\mathrm{x}}}\right)^{-1}$; see (B.4). Obviously $\mu(\mathrm{x})$ grows with x . Define the value $\mathrm{x}_{c}$ as the root of the equation

$$
\begin{equation*}
\frac{\mathrm{g}-\sqrt{\mathrm{p} \mu(\mathrm{x})}}{\mu(\mathrm{x})}=z(B, \mathrm{x})+1 \tag{2.24}
\end{equation*}
$$

The left hand-side here decreases with x , while the right hand-side is increasing in x to infinity. Therefore, the solution exists and is unique. Also denote $\mu_{c}=\mu\left(\mathrm{x}_{c}\right)$ and

$$
\begin{equation*}
\mathrm{g}_{c}=\mathrm{g}-\sqrt{\mathrm{p} \mu_{c}} \tag{2.25}
\end{equation*}
$$

so that

$$
\mathrm{g}_{c} / \mu_{c}=z\left(B, \mathbf{x}_{c}\right)+1 .
$$

Theorem 2.7. Let (2.22) hold and let $Q$ be such that $B=Q \mathbb{V}^{2} Q^{\top}$ satisfies $\|B\|=1$ and $\mathrm{p}=\operatorname{tr}(B)<\infty$. Define $\mathrm{x}_{c}$ by (2.24) and $\mathrm{g}_{c}$ by (2.25), and suppose $\mathrm{g}_{c} \geq 1$. Then for any $\mathrm{x}>0$

$$
\begin{equation*}
\mathbb{P}\left(\|Q \boldsymbol{\xi}\|^{2} \geq z_{c}^{2}(B, \mathrm{x})\right) \leq 2 \mathrm{e}^{-\mathrm{x}}+\mathrm{e}^{-\mathrm{x}_{c}} \mathbb{I}\left(\mathrm{x}<\mathrm{x}_{c}\right) \leq 3 \mathrm{e}^{-\mathrm{x}} \tag{2.26}
\end{equation*}
$$

where $z_{c}(B, \mathrm{x})$ is defined by

$$
\begin{aligned}
& z_{c}(B, \mathrm{x}) \stackrel{\text { def }}{=} \begin{cases}\sqrt{\mathrm{p}+2 \mathrm{v} \mathrm{x}^{1 / 2}+2 \mathrm{x}}, & \mathrm{x} \leq \mathrm{x}_{c}, \\
\mathrm{~g}_{c} / \mu_{c}+2\left(\mathrm{x}-\mathrm{x}_{c}\right) / \mathrm{g}_{c}, & \mathrm{x}>\mathrm{x}_{c},\end{cases} \\
& \quad \leq \begin{cases}\sqrt{\mathrm{p}}+\sqrt{2 \mathrm{x}}, & \mathrm{x} \leq \mathrm{x}_{c}, \\
\mathrm{~g}_{c} / \mu_{c}+2\left(\mathrm{x}-\mathrm{x}_{c}\right) / \mathrm{g}_{c}, & \mathrm{x}>\mathrm{x}_{c} .\end{cases}
\end{aligned}
$$

Moreover, if, given x , it holds

$$
\begin{equation*}
\mathrm{g} \geq \mathrm{x}^{1 / 2} / 2+(\mathrm{px} / 4)^{1 / 4} \tag{2.27}
\end{equation*}
$$

then

$$
\begin{equation*}
\mathbb{P}(\|Q \boldsymbol{\xi}\| \geq \sqrt{\mathrm{p}}+\sqrt{2 \mathrm{x}}) \leq 3 \mathrm{e}^{-\mathrm{x}} \tag{2.28}
\end{equation*}
$$

Remark 2.4. Depending on the value x , we have two types of tail behavior of the quadratic form $\|Q \boldsymbol{\xi}\|^{2}$. For $\mathrm{x} \leq \mathbf{x}_{c}$, we have essentially the same deviation bounds as in the Gaussian case with the extra-factor two in the deviation probability. For $\mathrm{x}>\mathrm{x}_{c}$, we switch to the special regime driven by the exponential moment condition (2.22). Usually $\mathrm{g}^{2}$ is a large number (of order $n$ in the i.i.d. setup) yielding $\mathrm{x}_{c}$ also large, and the second term in (2.26) can be simply ignored. The function $z_{c}(B, \mathrm{x})$ is discontinuous at the point $\mathrm{x}_{c}$. Indeed, $z_{c}(B, \mathrm{x})=z(B, \mathrm{x})$ for $\mathrm{x}<\mathrm{x}_{c}$, while by $(2.24)$, it holds $\mathrm{g}_{c} / \mu_{c}=z\left(B, \mathrm{x}_{c}\right)+1$. However, the jump at $\mathrm{x}_{c}$ is at most one.

As a corollary, we state the result for the norm of $\boldsymbol{\xi} \in \mathbb{R}^{p}$ corresponding to the case $\mathbb{V}^{-2}=Q=I_{p}$ and $p<\infty$. Then

$$
\mathrm{p}=\mathrm{v}^{2}=p
$$

Corollary 2.8. Let (2.22) hold with $\mathbb{V}=I_{p}$. Then for each $\mathrm{x}>0$

$$
\mathbb{P}\left(\|\boldsymbol{\xi}\| \geq z_{c}(p, \mathrm{x})\right) \leq 2 \mathrm{e}^{-\mathrm{x}}+\mathrm{e}^{-\mathrm{x}_{c}} \mathbb{I}\left(\mathrm{x}<\mathrm{x}_{c}\right)
$$

where $z_{c}(p, \mathrm{x})$ is defined by

$$
z_{c}(p, \mathrm{x}) \stackrel{\text { def }}{=} \begin{cases}(p+2 \sqrt{p \mathrm{x}}+2 \mathrm{x})^{1 / 2}, & \mathrm{x} \leq \mathrm{x}_{c} \\ \mathrm{~g}_{c} / \mu_{c}+2 \mathrm{~g}_{c}^{-1}\left(\mathrm{x}-\mathrm{x}_{c}\right), & \mathrm{x}>\mathrm{x}_{c}\end{cases}
$$

If $\mathrm{g} \geq \mathrm{x}^{1 / 2} / 2+(p \mathrm{x} / 4)^{1 / 4}$, then

$$
\mathbb{P}(\|\boldsymbol{\xi}\| \geq z(p, \mathrm{x})) \leq 3 \mathrm{e}^{-\mathrm{x}}
$$

Proof of Theorem 2.7. First we consider the most interesting case $\mathrm{x} \leq \mathrm{x}_{c}$. We expect to get Gaussian type deviation bounds for such x . The main tool of the proof is the following lemma.

Lemma 2.9. Let $\mu \in(0,1)$ and $\mathfrak{z}(\mu)=\mathrm{g} / \mu-\sqrt{\mathrm{p} / \mu}>0$. Then (2.22) implies

$$
\begin{equation*}
\mathbb{E} \exp \left(\mu\|Q \boldsymbol{\xi}\|^{2} / 2\right) \mathbb{I}\left(\left\|\mathbb{V} Q^{\top} Q \boldsymbol{\xi}\right\| \leq \mathfrak{z}(\mu)\right) \leq 2 \exp \left(\frac{\mu^{2} \mathrm{v}^{2}}{4(1-\mu)}+\frac{\mu \mathrm{p}}{2}\right) \tag{2.29}
\end{equation*}
$$

Proof. Let us fix for a moment some $\boldsymbol{\xi} \in \mathbb{R}^{p}$ and $\mu<1$ and define

$$
\boldsymbol{a}=\mathbb{V}^{-1} \boldsymbol{\xi}, \quad \Sigma=\mu \mathbb{V} Q^{\top} Q \mathbb{V} .
$$

Consider the Gaussian measure $\mathbb{P}_{\boldsymbol{a}, \Sigma}=\mathcal{N}\left(\boldsymbol{a}, \Sigma^{-1}\right)$, and let $\boldsymbol{U} \sim \mathcal{N}\left(0, \Sigma^{-1}\right)$. By the Girsanov formula

$$
\log \frac{d \mathbb{P}_{\boldsymbol{a}, \Sigma}}{d \mathbb{P}_{0, \Sigma}}(\boldsymbol{u})=\langle\Sigma \boldsymbol{a}, \boldsymbol{u}\rangle-\frac{1}{2}\langle\Sigma \boldsymbol{a}, \boldsymbol{a}\rangle
$$

and for any set $A \in \mathbb{R}^{p}$

$$
\mathbb{P}_{\boldsymbol{a}, \Sigma}(A)=\mathbb{P}_{0, \Sigma}(A-\boldsymbol{a})=\mathbb{E}_{0, \Sigma}\left[\exp \left\{\langle\Sigma \boldsymbol{U}, \boldsymbol{a}\rangle-\frac{1}{2}\langle\Sigma \boldsymbol{a}, \boldsymbol{a}\rangle\right\} \mathbb{I}(A)\right] .
$$

Now we select $A=\{\boldsymbol{u}:\|\Sigma \boldsymbol{u}\| \leq \mathrm{g}\}$. Under $\mathbb{P}_{0, \Sigma}$, one can represent $\Sigma \boldsymbol{U}=\Sigma^{1 / 2} \boldsymbol{\gamma}$ with a standard Gaussian $\gamma$. Therefore,

$$
\begin{aligned}
\mathbb{P}_{0, \Sigma}(A-\boldsymbol{a}) & =\mathbb{P}_{\gamma \sim \mathcal{N}(0, I)}\left(\left\|\Sigma^{1 / 2}\left(\boldsymbol{\gamma}-\Sigma^{1 / 2} \boldsymbol{a}\right)\right\| \leq \mathrm{g}\right) \\
& \geq \mathbb{P}_{\gamma \sim \mathcal{N}(0, I)}\left(\left\|\Sigma^{1 / 2} \gamma\right\| \leq \mathrm{g}-\|\Sigma \boldsymbol{a}\|\right)
\end{aligned}
$$

We now use that $\mathbb{P}_{\gamma \sim \mathcal{N}(0, I)}\left(\left\|\Sigma^{1 / 2} \gamma\right\|^{2} \leq \operatorname{tr}(\Sigma)\right) \geq 1 / 2$ with $\operatorname{tr}(\Sigma)=\mu \operatorname{tr}(B)=\mu \mathrm{p}$. Therefore, the condition $\|\Sigma \boldsymbol{a}\|+\sqrt{\mu \mathrm{p}} \leq \mathrm{g}$ implies in view of $\langle\Sigma \boldsymbol{a}, \boldsymbol{a}\rangle=\mu\|Q \boldsymbol{\xi}\|^{2}$

$$
1 / 2 \leq \mathbb{P}_{\boldsymbol{a}, \Sigma}(A)=\mathbb{E}_{0, \Sigma}\left[\exp \left\{\left\langle\Sigma \boldsymbol{U}, \mathbb{V}^{-1} \boldsymbol{\xi}\right\rangle-\mu\|Q \boldsymbol{\xi}\|^{2} / 2\right\} \mathbb{I}(\|\Sigma \boldsymbol{U}\| \leq \mathrm{g})\right]
$$

or

$$
\begin{aligned}
& \exp \left(\mu\|Q \boldsymbol{\xi}\|^{2} / 2\right) \mathbb{I}\left(\left\|\Sigma \mathbb{V}^{-1} \boldsymbol{\xi}\right\| \leq \mathrm{g}-\sqrt{\mu \mathrm{p}}\right) \\
& \quad \leq 2 \mathbb{E}_{0, \Sigma}\left[\exp \left\{\left\langle\Sigma \boldsymbol{U}, \mathbb{V}^{-1} \boldsymbol{\xi}\right\rangle \mathbb{I}(\|\Sigma \boldsymbol{U}\| \leq \mathrm{g})\right]\right.
\end{aligned}
$$

We now take the expectation of the each side of this equation w.r.t. $\boldsymbol{\xi}$, change the integration order, and use (2.22) yielding

$$
\begin{aligned}
& \mathbb{E} \exp \left(\mu\|Q \boldsymbol{\xi}\|^{2} / 2\right) \mathbb{I}\left(\left\|\Sigma \mathbb{V}^{-1} \boldsymbol{\xi}\right\| \leq \mathrm{g}-\sqrt{\mu \mathrm{p}}\right) \leq 2 \mathbb{E}_{0, \Sigma} \exp \left(\|\Sigma \boldsymbol{U}\|^{2} / 2\right) \\
& \quad=2 \mathbb{E}_{\gamma \sim \mathcal{N}(0, I)} \exp \left(\left\|\Sigma^{1 / 2} \boldsymbol{\gamma}\right\|^{2} / 2\right)=2 \operatorname{det}(I-\Sigma)^{-1 / 2}=2 \operatorname{det}(I-\mu B)^{-1 / 2}
\end{aligned}
$$

We also use that for any $\mu>0$

$$
\log \operatorname{det}(I-\mu B)^{-1 / 2}-\frac{\mu \operatorname{tr}(B)}{2} \leq \frac{\mu^{2} \operatorname{tr}\left(B^{2}\right)}{4(1-\mu)}
$$

see (B.5), and the first statement follows in view of $\Sigma \mathbb{V}^{-1} \boldsymbol{\xi}=\mu \mathbb{V} Q^{\top} Q \boldsymbol{\xi}$.
The use of $\mu$ from (B.4) in (2.29) yields similarly to the proof of Theorem B. 1

$$
\begin{equation*}
\mathbb{P}\left(\|Q \boldsymbol{\xi}\|^{2}>z^{2}(B, \mathrm{x}),\left\|\mathbb{V} Q^{\top} Q \boldsymbol{\xi}\right\| \leq \mathfrak{z}(\mu)\right) \leq 2 \mathrm{e}^{-\mathrm{x}} \tag{2.30}
\end{equation*}
$$

It remains to consider the probability of large deviation $\mathbb{P}\left(\left\|\mathbb{V} Q^{\top} Q \boldsymbol{\xi}\right\|>\mathfrak{z}(\mu)\right)$.
Lemma 2.10. For any $\mathrm{x}_{c}>0$ such that $z\left(B, \mathrm{x}_{c}\right)+1 \leq \mathrm{g}_{c} / \mu_{c}$, it holds with $\mu_{c}=$ $\left\{1+\mathrm{v} /\left(2 \sqrt{\mathrm{x}_{c}}\right)\right\}^{-1}$ and $z_{c}=\mathfrak{z}\left(\mu_{c}\right)=\mathrm{g} / \mu_{c}-\sqrt{\mathrm{p} / \mu_{c}}$

$$
\mathbb{P}\left(\left\|\mathbb{V} Q^{\top} Q \boldsymbol{\xi}\right\|>z_{c}\right) \leq \mathbb{P}\left(\|Q \boldsymbol{\xi}\|^{2}>z_{c}^{2}\right) \leq \mathrm{e}^{-\mathrm{x}_{c}}
$$

Proof. Define

$$
\Phi(\mu) \stackrel{\text { def }}{=} \frac{\mu^{2} \mathrm{v}^{2}}{4(1-\mu)}+\frac{\mu \mathrm{p}}{2} .
$$

It follows due to (B.4) and (B.6) for any $\mu \leq \mu_{c}$

$$
\Phi(\mu) \leq \Phi\left(\mu_{c}\right) \leq \frac{\mu_{c} z^{2}\left(B, \mathrm{x}_{c}\right)}{2}-\mathrm{x}_{c},
$$

where the right hand-side does not depend on $\mu$. Denote $\eta=\|Q \boldsymbol{\xi}\|$ and use that $\left\|\mathbb{V} Q^{\top} Q \boldsymbol{\xi}\right\| \leq\left\|Q \mathbb{V}^{2} Q^{\top}\right\|^{1 / 2}\|Q \boldsymbol{\xi}\| \leq \eta$. Then by (2.29)

$$
\begin{equation*}
\mathbb{E} \exp \left(\mu \eta^{2} / 2\right) \mathbb{I}(\eta \leq \mathfrak{z}(\mu)) \leq 2 \exp \Phi(\mu) \leq 2 \exp \Phi\left(\mu_{c}\right) \tag{2.31}
\end{equation*}
$$

Define the inverse function $\mu(\mathfrak{z})=\mathfrak{z}^{-1}(\mu)$. For any $\mathfrak{z} \geq z_{c}$, it follows from (2.31) with $\mu=\mu(\mathfrak{z})$

$$
\mathbb{E} \exp \left\{\mu(\mathfrak{z})(\mathfrak{z}-1)^{2} / 2\right\} \mathbb{I}(\mathfrak{z}-1 \leq \eta \leq \mathfrak{z}) \leq 2 \exp \Phi\left(\mu_{c}\right)
$$

yielding

$$
\mathbb{P}(\mathfrak{z}-1 \leq \eta \leq \mathfrak{z}) \leq 2 \exp \left(-\mu(\mathfrak{z})(\mathfrak{z}-1)^{2} / 2+\Phi\left(\mu_{c}\right)\right)
$$

and hence,

$$
\mathbb{P}(\eta>\mathfrak{z}) \leq 2 \int_{\mathfrak{z}}^{\infty} \exp \left\{-\mu(z)(z-1)^{2} / 2+\Phi\left(\mu_{c}\right)\right\} d z
$$

Further, $\mu_{\mathfrak{z}}(\mu)=\mathrm{g}-\sqrt{\mathrm{p} \mu}$ and

$$
\mathrm{g}_{c}=\mu_{c} z_{c} \leq \mu_{\mathfrak{z}}(\mu) \leq \mathrm{g}, \quad \mu \leq \mu_{c}
$$

This implies the same bound for the inverse function:

$$
\mathrm{g}_{c} \leq \mathfrak{z} \mu(\mathfrak{z}) \leq \mathrm{g}, \quad \mathfrak{z} \geq z_{c}
$$

and for $\mathfrak{z} \geq 2$

$$
\begin{align*}
\mathbb{P}(\eta>\mathfrak{z}) & \leq 2 \int_{\mathfrak{z}}^{\infty} \exp \left\{-\mu(z)\left(z^{2} / 2-z\right)+\Phi\left(\mu_{c}\right)\right\} d z \\
& \leq 2 \int_{\mathfrak{z}}^{\infty} \exp \left\{-\mathrm{g}_{c}(z / 2-1)+\Phi\left(\mu_{c}\right)\right\} d z \\
& \leq \frac{4}{\mathrm{~g}_{c}} \exp \left\{-\mathrm{g}_{c}(\mathfrak{z} / 2-1)+\Phi\left(\mu_{c}\right)\right\} \tag{2.32}
\end{align*}
$$

Conditions $\mathrm{g}_{c} z_{c}=\mu_{c}^{-1} \mathbf{g}_{c}^{2} \geq \mu_{c}\left\{z\left(B, \mathbf{x}_{c}\right)+1\right\}^{2}$ and $\mathrm{g}_{c} \geq 1$ ensure that $\mathbb{P}\left(\eta>z_{c}\right) \leq$ $\mathrm{e}^{-\mathrm{x}_{c}}$.

Remind that $\mathbf{x}_{c}$ is the largest $\mathbf{x}$-value ensuring the condition $\mathbf{g}_{c} \geq z\left(B, \mathbf{x}_{c}\right)+1$. We also use that for $\mathrm{x} \leq \mathrm{x}_{c}$, it holds $\mathfrak{z}(\mu) \geq \mathfrak{z}\left(\mu_{c}\right)=z_{c}$. Therefore, by (2.30) and

Lemma 2.10

$$
\begin{aligned}
\mathbb{P}\left(\|Q \boldsymbol{\xi}\|^{2} \geq z^{2}(B, \mathbf{x})\right) & \leq \mathbb{P}\left(\|Q \boldsymbol{\xi}\|^{2} \geq z^{2}(B, \mathbf{x}),\left\|\mathbb{V} Q^{\top} Q \boldsymbol{\xi}\right\| \leq \mathfrak{z}(\mu)\right)+\mathbb{P}\left(\|Q \boldsymbol{\xi}\|^{2} \geq z_{c}^{2}\right) \\
& \leq 2 \mathrm{e}^{-\mathrm{x}}+\mathrm{e}^{-\mathbf{x}_{c}}
\end{aligned}
$$

Finally we consider $\mathrm{x}>\mathbf{x}_{c}$. Applying (2.32) yields by $\mathfrak{z} \geq z_{c}$

$$
\begin{aligned}
\mathbb{P}(\eta>\mathfrak{z}) & \leq \frac{2}{\mu_{c} z_{c}} \exp \left\{-\mu_{c} z_{c}^{2} / 2+\mathrm{g}+\mu_{c} z^{2}\left(B, \mathrm{x}_{c}\right) / 2-\mathrm{x}_{c}\right\} \exp \left\{-\mu_{c} z_{c}\left(\mathfrak{z}-z_{c}\right) / 2\right\} \\
& \leq \mathrm{e}^{-\mathrm{x}_{c}} \exp \left\{-\mathrm{g}_{c}\left(\mathfrak{z}-z_{c}\right) / 2\right\}
\end{aligned}
$$

The choice $\mathfrak{z}$ by

$$
\mathrm{g}_{c}\left(\mathfrak{z}-z_{c}\right) / 2=\mathrm{x}-\mathrm{x}_{c}
$$

ensures the desired bound.
Now, for a prescribed x , we evaluate the minimal value g ensuring the bound (2.26) with $\mathrm{x}_{c} \geq \mathrm{x}$. For simplicity we apply the sub-optimal choice $\mu(\mathrm{x})=(1+2 \sqrt{\mathrm{p} / \mathrm{x}})^{-1}$; see Remark B.3. Then for any $\mathrm{x} \geq 1$

$$
\begin{aligned}
\mu(\mathrm{x})\{z(B, \mathrm{x})+1\} & \leq \frac{\sqrt{\mathrm{x}}}{\sqrt{\mathrm{x}}+2 \sqrt{\mathrm{p}}}\left(\sqrt{\mathrm{p}+2(\mathrm{xp})^{1 / 2}+2 \mathrm{x}}+1\right) \\
\mathrm{p} \mu(\mathrm{x}) & =\frac{\sqrt{\mathrm{x}} \mathrm{p}}{\sqrt{\mathrm{x}}+2 \sqrt{\mathrm{p}}}
\end{aligned}
$$

It is now straightforward to check that

$$
\mu(\mathrm{x})\{z(B, \mathrm{x})+1\}+\sqrt{\mathrm{p} \mu(\mathrm{x})} \leq \sqrt{\mathrm{x}} / 2+(\mathrm{x} \mathrm{p} / 4)^{1 / 4}
$$

Therefore, if (2.27) holds for the given x , then (2.24) is fulfilled with $\mathrm{x}_{c} \geq \mathrm{x}$ yielding (2.28).

## A Moments of a Gaussian quadratic form

Let $\gamma$ be standard normal in $\mathbb{R}^{p}$ for $p \leq \infty$. Given a self-adjoint trace operator $B$, consider a quadratic form $\langle B \gamma, \gamma\rangle$.

Lemma A.1. It holds

$$
\begin{aligned}
\mathbb{E}\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle & =\operatorname{tr} B \\
\operatorname{Var}\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle & =2 \operatorname{tr} B^{2} .
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
& \mathbb{E}(\langle B \gamma, \gamma\rangle-\operatorname{tr} B)^{2}=2 \operatorname{tr} B^{2} \\
& \mathbb{E}(\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle-\operatorname{tr} B)^{3}=8 \operatorname{tr} B^{3} \\
& \mathbb{E}(\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle-\operatorname{tr} B)^{4}=48 \operatorname{tr} B^{4}+12\left(\operatorname{tr} B^{2}\right)^{2}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\langle B \gamma, \gamma\rangle^{2} & =(\operatorname{tr} B)^{2}+2 \operatorname{tr} B^{2}, \\
\mathbb{E}\langle B \gamma, \gamma\rangle^{3} & =(\operatorname{tr} B)^{3}+6 \operatorname{tr} B \operatorname{tr} B^{2}+8 \operatorname{tr} B^{3}, \\
\mathbb{E}\langle B \gamma, \gamma\rangle^{4} & =(\operatorname{tr} B)^{4}+12(\operatorname{tr} B)^{2} \operatorname{tr} B^{2}+32(\operatorname{tr} B) \operatorname{tr} B^{3}+48 \operatorname{tr} B^{4}+12\left(\operatorname{tr} B^{2}\right)^{2}, \\
\operatorname{Var}\langle B \gamma, \gamma\rangle^{2} & =8(\operatorname{tr} B)^{2} \operatorname{tr} B^{2}+32(\operatorname{tr} B) \operatorname{tr} B^{3}+48 \operatorname{tr} B^{4}+8\left(\operatorname{tr} B^{2}\right)^{2}
\end{aligned}
$$

Moreover, if $B \leq I_{p}$ and $\mathrm{p}=\operatorname{tr} B$, then $\operatorname{tr} B^{m} \leq \mathrm{p}\|B\|^{m-1}$ for $m \geq 1$ and

$$
\begin{array}{rlcl}
\mathbb{E}\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle^{2} & \leq & \mathrm{p}^{2}+2 \mathrm{p}\|B\| & \leq(\mathrm{p}+\|B\|)^{2}, \\
\mathbb{E}\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle^{3} & \leq & \mathrm{p}^{3}+6 \mathrm{p}^{2}\|B\|+8 \mathrm{p}\|B\|^{2} & \leq(\mathrm{p}+2\|B\|)^{3}, \\
\mathbb{E}\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle^{4} \leq \mathrm{p}^{4}+12 \mathrm{p}^{3}\|B\|+44 \mathrm{p}^{2}\|B\|^{2}+48 \mathrm{p}\|B\|^{3} & \leq(\mathrm{p}+3\|B\|)^{4}, \\
\operatorname{Var}\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle^{2} & \leq & 8 \mathrm{p}^{3}+40 \mathrm{p}^{2}\|B\|+48 \mathrm{p}\|B\|^{2} . &
\end{array}
$$

Proof. Let $\chi=\gamma^{2}-1$ for $\gamma$ standard normal. Then $\mathbb{E} \chi=0, \mathbb{E} \chi^{2}=2, \mathbb{E} \chi^{3}=8$, $\mathbb{E} \chi^{4}=60$. Without loss of generality assume $B$ diagonal: $B=\operatorname{diag}\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{p}\right)$. Then

$$
\xi \stackrel{\text { def }}{=}\langle B \gamma, \gamma\rangle-\operatorname{tr} B=\sum_{j=1}^{p} \lambda_{j}\left(\gamma_{j}^{2}-1\right),
$$

where $\gamma_{j}$ are i.i.d. standard normal. This easily yields

$$
\begin{aligned}
\mathbb{E} \xi^{2} & =\sum_{j=1}^{p} \lambda_{j}^{2} \mathbb{E}\left(\gamma_{j}^{2}-1\right)^{2}=\mathbb{E} \chi^{2} \operatorname{tr} B^{2}=2 \operatorname{tr} B^{2} \\
\mathbb{E} \xi^{3} & =\sum_{j=1}^{p} \lambda_{j}^{3} \mathbb{E}\left(\gamma_{j}^{2}-1\right)^{3}=\mathbb{E} \chi^{3} \operatorname{tr} B^{3}=8 \operatorname{tr} B^{3} \\
\mathbb{E} \xi^{4} & =\sum_{j=1}^{p} \lambda_{j}^{4}\left(\gamma_{j}^{2}-1\right)^{4}+\sum_{i \neq j} \lambda_{i}^{2} \lambda_{j}^{2} \mathbb{E}\left(\gamma_{i}^{2}-1\right)^{2} \mathbb{E}\left(\gamma_{j}^{2}-1\right)^{2} \\
& =\left(\mathbb{E} \chi^{4}-3\left(\mathbb{E} \chi^{2}\right)^{2}\right) \operatorname{tr} B^{4}+3\left(\mathbb{E} \chi^{2} \operatorname{tr} B^{2}\right)^{2}=48 \operatorname{tr} B^{4}+12\left(\operatorname{tr} B^{2}\right)^{2}
\end{aligned}
$$

ensuring

$$
\begin{aligned}
\mathbb{E}\langle B \gamma, \gamma\rangle^{2} & =(\mathbb{E}\langle B \gamma, \gamma\rangle)^{2}+\mathbb{E} \xi^{2}=(\operatorname{tr} B)^{2}+2 \operatorname{tr} B^{2}, \\
\mathbb{E}\langle B \gamma, \gamma\rangle^{3} & =\mathbb{E}(\xi+\operatorname{tr} B)^{3}=(\operatorname{tr} B)^{3}+\mathbb{E} \xi^{3}+3 \operatorname{tr} B \mathbb{E}^{2} \\
& =(\operatorname{tr} B)^{3}+6 \operatorname{tr} B \operatorname{tr} B^{2}+8 \operatorname{tr} B^{3},
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{Var}\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle^{2} & =\mathbb{E}(\xi+\operatorname{tr} B)^{4}-(\mathbb{E}\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle)^{2} \\
& =(\operatorname{tr} B)^{4}+6(\operatorname{tr} B)^{2} \mathbb{E} \xi^{2}+4 \operatorname{tr} B \mathbb{E} \xi^{3}+\mathbb{E} \xi^{4}-\left((\operatorname{tr} B)^{2}+2 \operatorname{tr} B^{2}\right)^{2} \\
& =8(\operatorname{tr} B)^{2} \operatorname{tr} B^{2}+32(\operatorname{tr} B) \operatorname{tr} B^{3}+48 \operatorname{tr} B^{4}+8\left(\operatorname{tr} B^{2}\right)^{2} .
\end{aligned}
$$

This implies the results of the lemma.
Now we compute the exponential moments of centered and non-centered quadratic forms.

Lemma A.2. Let $\|B\|_{\mathrm{op}}=\lambda$ and $\gamma \sim \mathcal{N}\left(0, I_{p}\right)$. Then for any $\mu \in\left(0, \lambda^{-1}\right)$,

$$
\mathbb{E} \exp \left\{\frac{\mu}{2}(\langle B \gamma, \gamma\rangle-\mathrm{p})\right\}=\operatorname{det}(I-\mu B)^{-1 / 2}
$$

Moreover, with $\mathrm{p}=\operatorname{tr} B$ and $\mathrm{v}^{2}=\operatorname{tr} B^{2}$

$$
\begin{equation*}
\log \mathbb{E} \exp \left\{\frac{\mu}{2}(\langle B \gamma, \boldsymbol{\gamma}\rangle-\mathrm{p})\right\} \leq \frac{\mu^{2} \mathrm{v}^{2}}{4(1-\lambda \mu)} \tag{A.1}
\end{equation*}
$$

If $B$ is positive semidefinite, $\lambda_{j} \geq 0$, then

$$
\begin{equation*}
\log \mathbb{E} \exp \left\{-\frac{\mu}{2}(\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle-\mathrm{p})\right\} \leq \frac{\mu^{2} \mathrm{v}^{2}}{4} \tag{A.2}
\end{equation*}
$$

Proof. W.l.o.g. assume $\lambda=1$. Let $\lambda_{j}$ be the eigenvalues of $B,\left|\lambda_{j}\right| \leq 1$. By an orthogonal transform, one can reduce the statement to the case of a diagonal matrix $B=\operatorname{diag}\left(\lambda_{j}\right)$. Then $\langle B \gamma, \gamma\rangle=\sum_{j=1}^{p} \lambda_{j} \gamma_{j}^{2}$ and by independence of the $\gamma_{j}{ }^{\prime}$ s

$$
\mathbb{E}\left\{\frac{\mu}{2}\langle B \boldsymbol{\gamma}, \gamma\rangle\right\}=\prod_{j=1}^{p} \mathbb{E} \exp \left(\frac{\mu}{2} \lambda_{j} \varepsilon_{j}^{2}\right)=\prod_{j=1}^{p} \frac{1}{\sqrt{1-\mu \lambda_{j}}}=\operatorname{det}(I-\mu B)^{-1 / 2}
$$

Below we use the simple bound:

$$
\begin{aligned}
-\log (1-u)-u & =\sum_{k=2}^{\infty} \frac{u^{k}}{k} \leq \frac{u^{2}}{2} \sum_{k=0}^{\infty} u^{k}=\frac{u^{2}}{2(1-u)}, \quad u \in(0,1) \\
-\log (1-u)+u=\sum_{k=2}^{\infty} \frac{u^{k}}{k} \leq \frac{u^{2}}{2}, & u \in(-1,0)
\end{aligned}
$$

Now it holds

$$
\begin{aligned}
& \log \mathbb{E}\left\{\frac{\mu}{2}(\langle B \gamma, \gamma\rangle-\mathrm{p})\right\}=\log \operatorname{det}(I-\mu B)^{-1 / 2}-\frac{\mu \mathrm{p}}{2} \\
& \quad=-\frac{1}{2} \sum_{j=1}^{p}\left\{\log \left(1-\mu \lambda_{j}\right)+\mu \lambda_{j}\right\} \leq \sum_{j=1}^{p} \frac{\mu^{2} \lambda_{j}^{2}}{4(1-\mu)}=\frac{\mu^{2} \mathrm{v}^{2}}{4(1-\mu)}
\end{aligned}
$$

The last statement can be proved similarly.

Now we consider the case of a non-centered quadratic form $\langle B \gamma, \gamma\rangle / 2+\langle\boldsymbol{A}, \gamma\rangle$ for a fixed vector $\boldsymbol{A}$.

Lemma A.3. Let $\lambda_{\max }(B)<1$. Then for any $\boldsymbol{A}$

$$
\mathbb{E} \exp \left\{\frac{1}{2}\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle+\langle\boldsymbol{A}, \boldsymbol{\gamma}\rangle\right\}=\exp \left\{\frac{\left\|(I-B)^{-1 / 2} \boldsymbol{A}\right\|^{2}}{2}\right\} \operatorname{det}(I-B)^{-1 / 2}
$$

Moreover, for any $\mu \in(0,1)$

$$
\begin{align*}
\log & \mathbb{E} \exp \left\{\frac{\mu}{2}(\langle B \gamma, \gamma\rangle-\mathrm{p})+\langle\boldsymbol{A}, \gamma\rangle\right\} \\
& =\frac{\left\|(I-\mu B)^{-1 / 2} \boldsymbol{A}\right\|^{2}}{2}+\log \operatorname{det}(I-\mu B)^{-1 / 2}-\mu \mathrm{p} \\
& \leq \frac{\left\|(I-\mu B)^{-1 / 2} \boldsymbol{A}\right\|^{2}}{2}+\frac{\mu^{2} \mathrm{v}^{2}}{4(1-\mu)} \tag{A.3}
\end{align*}
$$

Proof. Denote $\boldsymbol{a}=(I-B)^{-1 / 2} \boldsymbol{A}$. It holds by change of variables $(I-B)^{1 / 2} \boldsymbol{x}=\boldsymbol{u}$ for

$$
\begin{aligned}
& \mathrm{C}_{p}=(2 \pi)^{-p / 2} \\
& \qquad \begin{array}{l}
\mathbb{E} \exp \left\{\frac{1}{2}\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle+\langle\boldsymbol{A}, \boldsymbol{\gamma}\rangle\right\}=\mathrm{C}_{p} \int \exp \left\{-\frac{1}{2}\langle(I-B) \boldsymbol{x}, \boldsymbol{x}\rangle+\langle\boldsymbol{A}, \boldsymbol{x}\rangle\right\} d \boldsymbol{x} \\
\quad=\mathrm{C}_{p} \operatorname{det}(I-B)^{-1 / 2} \int \exp \left\{-\frac{1}{2}\|\boldsymbol{u}\|^{2}+\langle\boldsymbol{a}, \boldsymbol{u}\rangle\right\} d \boldsymbol{u}=\operatorname{det}(I-B)^{-1 / 2} \mathrm{e}^{\|\boldsymbol{a}\|^{2} / 2} .
\end{array}
\end{aligned}
$$

The last inequality (A.3) follows by (A.1).

## B Deviation bounds for Gaussian quadratic forms

The next result explains the concentration effect of $\|Q \boldsymbol{\xi}\|^{2}$ for a centered Gaussian vector $\boldsymbol{\xi} \sim \mathcal{N}\left(0, \mathbb{V}^{2}\right)$ and a linear operator $Q: \mathbb{R}^{p} \rightarrow \mathbb{R}^{q}, p, q \leq \infty$. We use a version from Laurent and Massart (2000). For completeness, we present a simple proof of the upper bound.

Theorem B.1. Let $\boldsymbol{\xi} \sim \mathcal{N}\left(0, \mathbb{V}^{2}\right)$ be a Gaussian element in $\mathbb{R}^{p}$ and let $Q: \mathbb{R}^{p} \rightarrow \mathbb{R}^{q}$ be such that $B=Q \mathbb{V}^{2} Q^{\top}$ is a trace operator in $\mathbb{R}^{q}$. Then with $\mathrm{p}=\operatorname{tr}(B), \mathrm{v}^{2}=$ $\operatorname{tr}\left(B^{2}\right)$, and $\lambda=\|B\|$, it holds for each $\mathrm{x} \geq 0$

$$
\begin{align*}
\mathbb{P}\left(\|Q \boldsymbol{\xi}\|^{2}-\mathrm{p}>2 \mathrm{v} \sqrt{\mathrm{x}}+2 \lambda \mathrm{x}\right) & \leq \mathrm{e}^{-\mathrm{x}}  \tag{B.1}\\
\mathbb{P}\left(\|Q \boldsymbol{\xi}\|^{2}-\mathrm{p} \leq-2 \mathrm{v} \sqrt{\mathrm{x}}\right) & \leq \mathrm{e}^{-\mathrm{x}} \tag{B.2}
\end{align*}
$$

It also implies

$$
\mathbb{P}\left(\left|\|Q \boldsymbol{\xi}\|^{2}-\mathrm{p}\right|>z_{2}(B, \mathrm{x})\right) \leq 2 \mathrm{e}^{-\mathrm{x}}
$$

with

$$
\begin{equation*}
z_{2}(B, \mathrm{x}) \stackrel{\text { def }}{=} 2 \mathrm{v} \sqrt{\mathrm{x}}+2 \lambda \mathrm{x} . \tag{B.3}
\end{equation*}
$$

Proof. W.l.o.g. assume that $\lambda=\|B\|=1$. We use the identity $\|Q \boldsymbol{\xi}\|^{2}=\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle$ with $\gamma \sim \mathcal{N}\left(0, I_{q}\right)$. We apply the exponential Chebyshev inequality: with $\mu>0$

$$
\mathbb{P}\left(\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle-\mathrm{p}>z_{2}(B, \mathrm{x})\right) \leq \mathbb{E} \exp \left(\frac{\mu}{2}(\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle-\mathrm{p})-\frac{\mu z_{2}(B, \mathrm{x})}{2}\right)
$$

Given $\mathbf{x}>0$, fix $\mu<1$ by the equation

$$
\begin{equation*}
\frac{\mu}{1-\mu}=\frac{2 \sqrt{x}}{v} \quad \text { or } \quad \mu^{-1}=1+\frac{v}{2 \sqrt{x}} . \tag{B.4}
\end{equation*}
$$

Let $\lambda_{j}$ be the eigenvalues of $B,\left|\lambda_{j}\right| \leq 1$. It holds with $\mathrm{p}=\operatorname{tr} B$ in view of (A.1)

$$
\begin{equation*}
\log \mathbb{E}\left\{\frac{\mu}{2}(\langle B \gamma, \gamma\rangle-\mathrm{p})\right\} \leq \frac{\mu^{2} \mathrm{v}^{2}}{4(1-\mu)} \tag{B.5}
\end{equation*}
$$

For (B.1), it remains to check that the choice $\mu$ by (B.4) yields

$$
\begin{equation*}
\frac{\mu^{2} \mathrm{v}^{2}}{4(1-\mu)}-\frac{\mu z_{2}(B, \mathrm{x})}{2}=\frac{\mu^{2} \mathrm{v}^{2}}{4(1-\mu)}-\mu(\mathrm{v} \sqrt{\mathrm{x}}+\mathrm{x})=\mu\left(\frac{\mathrm{v} \sqrt{\mathrm{x}}}{2}-\mathrm{v} \sqrt{\mathrm{x}}-\mathrm{x}\right)=-\mathrm{x} \tag{B.6}
\end{equation*}
$$

The bound (B.2) is obtained similarly by applying the exponential Chebyshev inequality to $-\langle B \gamma, \gamma\rangle+\mathrm{p}$ with $\mu=2 \mathrm{v}^{-1} \sqrt{\mathrm{x}}$. The use of (A.2) yields

$$
\begin{aligned}
& \mathbb{P}(\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle-\mathrm{p}<-2 \mathrm{v} \sqrt{\mathrm{x}}) \leq \mathbb{E} \exp \left\{\frac{\mu}{2}(-\langle B \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle+\mathrm{p})-\mu \mathrm{v} \sqrt{\mathrm{x}}\right\} \\
& \quad \leq \exp \left(\frac{\mu^{2} \mathrm{v}^{2}}{4}-\mu \mathrm{v} \sqrt{\mathrm{x}}\right)=\mathrm{e}^{-\mathrm{x}}
\end{aligned}
$$

as required.
Corollary B.2. Assume the conditions of Theorem B.1. Then for $z>\mathrm{v}$

$$
\begin{equation*}
\mathbb{P}\left(\left|\|Q \boldsymbol{\xi}\|^{2}-\mathrm{p}\right| \geq z\right) \leq 2 \exp \left\{-\frac{z^{2}}{\left(\mathrm{v}+\sqrt{\mathrm{v}^{2}+2 \lambda z}\right)^{2}}\right\} \leq 2 \exp \left(-\frac{z^{2}}{4 \mathrm{v}^{2}+4 \lambda z}\right) \tag{B.7}
\end{equation*}
$$

Proof. Given $z$, define x by $2 \mathrm{v} \sqrt{\mathrm{x}}+2 \lambda \mathrm{x}=z$ or $2 \lambda \sqrt{\mathrm{x}}=\sqrt{\mathrm{v}^{2}+2 \lambda z}-\mathrm{v}$. Then

$$
\mathbb{P}\left(\|Q \boldsymbol{\xi}\|^{2}-\mathrm{p} \geq z\right) \leq \mathrm{e}^{-\mathrm{x}}=\exp \left\{-\frac{\left(\sqrt{\mathrm{v}^{2}+2 \lambda z}-\mathrm{v}\right)^{2}}{4 \lambda^{2}}\right\}=\exp \left\{-\frac{z^{2}}{\left(\mathrm{v}+\sqrt{\mathrm{v}^{2}+2 \lambda z}\right)^{2}}\right\}
$$

This yields (B.7) by direct calculus.
Of course, bound (B.7) is sensible only if $z \gg \mathrm{v}$.
Corollary B.3. Assume the conditions of Theorem B.1. If also $B \geq 0$, then

$$
\begin{equation*}
\mathbb{P}\left(\|Q \boldsymbol{\xi}\|^{2} \geq z^{2}(B, \mathrm{x})\right) \leq \mathrm{e}^{-\mathrm{x}} \tag{B.8}
\end{equation*}
$$

with

$$
z^{2}(B, \mathrm{x}) \stackrel{\text { def }}{=} \mathrm{p}+2 \mathrm{v} \sqrt{\mathrm{x}}+2 \lambda \mathrm{x} \leq(\sqrt{\mathrm{p}}+\sqrt{2 \lambda \mathrm{x}})^{2}
$$

Also

$$
\mathbb{P}\left(\|Q \boldsymbol{\xi}\|^{2}-\mathrm{p}<-2 \mathrm{v} \sqrt{\mathrm{x}}\right) \leq \mathrm{e}^{-\mathrm{x}}
$$

Proof. The definition implies $\mathrm{v}^{2} \leq \mathrm{p} \lambda$. One can use a sub-optimal choice of the value $\mu(\mathrm{x})=\{1+2 \sqrt{\lambda \mathrm{p} / \mathrm{x}}\}^{-1}$ yielding the statement of the corollary.

As a special case, we present a bound for the chi-squared distribution corresponding to $Q=\mathbb{V}^{2}=I_{p}, p<\infty$. Then $B=I_{p}, \operatorname{tr}(B)=p, \operatorname{tr}\left(B^{2}\right)=p$ and $\lambda(B)=1$.

Corollary B.4. Let $\gamma$ be a standard normal vector in $\mathbb{R}^{p}$. Then for any $\mathrm{x}>0$

$$
\begin{aligned}
\mathbb{P}\left(\|\boldsymbol{\gamma}\|^{2} \geq p+2 \sqrt{p \mathrm{x}}+2 \mathrm{x}\right) & \leq \mathrm{e}^{-\mathrm{x}} \\
\mathbb{P}(\|\boldsymbol{\gamma}\| \geq \sqrt{p}+\sqrt{2 \mathrm{x}}) & \leq \mathrm{e}^{-\mathrm{x}} \\
\mathbb{P}\left(\|\boldsymbol{\gamma}\|^{2} \leq p-2 \sqrt{p \mathrm{x}}\right) & \leq \mathrm{e}^{-\mathrm{x}}
\end{aligned}
$$

The bound of Theorem B. 1 can be represented as a usual deviation bound.
Theorem B.5. Assume the conditions of Theorem B.1. For y $>0$, define

$$
\mathrm{x}(\mathrm{y}) \stackrel{\text { def }}{=} \frac{(\sqrt{\mathrm{y}+\mathrm{p}}-\sqrt{\mathrm{p}})^{2}}{4 \lambda}
$$

Then

$$
\begin{align*}
\mathbb{P}\left(\|Q \boldsymbol{\xi}\|^{2} \geq \mathrm{p}+\mathrm{y}\right) & \leq \mathrm{e}^{-\mathrm{x}(\mathrm{y})}  \tag{B.9}\\
\mathbb{E}\left\{\left(\|Q \boldsymbol{\xi}\|^{2}-\mathrm{p}\right) \mathbb{I}\left(\|Q \boldsymbol{\xi}\|^{2} \geq \mathrm{p}+\mathrm{y}\right)\right\} & \leq 2\left(\frac{\mathrm{y}+\mathrm{p}}{\lambda \mathrm{x}(\mathrm{y})}\right)^{1 / 2} \mathrm{e}^{-\mathrm{x}(\mathrm{y})} . \tag{B.10}
\end{align*}
$$

Moreover, let $\mu>0$ fulfill $\epsilon=\mu \lambda+\mu \sqrt{\lambda \mathrm{p} / \mathrm{x}(\mathrm{y})}<1$. Then

$$
\begin{equation*}
\mathbb{E}\left\{\mathrm{e}^{\mu\left(\|Q \boldsymbol{\xi}\|^{2}-\mathrm{p}\right) / 2} \mathbb{I}\left(\|Q \boldsymbol{\xi}\|^{2} \geq \mathrm{p}+\mathrm{y}\right)\right\} \leq \frac{1}{1-\epsilon} \exp \{-(1-\epsilon) \mathrm{x}(\mathrm{y})\} \tag{B.11}
\end{equation*}
$$

Proof. Normalizing by $\lambda$ reduces the statements to the case with $\lambda=1$. Define $\eta=$ $\|Q \xi\|^{2}-\mathrm{p}$ and

$$
\begin{equation*}
z(\mathrm{x})=2 \sqrt{\mathrm{px}}+2 \mathrm{x} . \tag{B.12}
\end{equation*}
$$

Then by (B.1) $\mathbb{P}(\eta \geq z(\mathrm{x})) \leq \mathrm{e}^{-\mathrm{x}}$. Inverting the relation (B.12) yields

$$
\mathrm{x}(z)=\frac{1}{4}(\sqrt{z+\mathrm{p}}-\sqrt{\mathrm{p}})^{2}
$$

and (B.9) follows by applying $z=\mathrm{y}$. Further,

$$
\mathbb{E}\{\eta \mathbb{I}(\eta \geq \mathrm{y})\}=\int_{\mathrm{y}}^{\infty} \mathbb{P}(\eta \geq z) d z \leq \int_{\mathrm{y}}^{\infty} \mathrm{e}^{-\mathrm{x}(z)} d z=\int_{\mathrm{x}(\mathrm{y})}^{\infty} \mathrm{e}^{-\mathrm{x}} z^{\prime}(\mathrm{x}) d \mathrm{x}
$$

As $z^{\prime}(\mathrm{x})=2+\sqrt{\mathrm{p} / \mathrm{x}}$ monotonously decreases with x , we derive

$$
\mathbb{E}\{\eta \mathbb{I}(\eta \geq \mathrm{y})\} \leq z^{\prime}(\mathrm{x}(\mathrm{y})) \mathrm{e}^{-\mathrm{x}(\mathrm{y})}=\frac{1}{\mathrm{x}^{\prime}(\mathrm{y})} \mathrm{e}^{-\mathrm{x}(\mathrm{y})}=\frac{4 \sqrt{\mathrm{y}+\mathrm{p}}}{\sqrt{\mathrm{y}+\mathrm{p}}-\sqrt{\mathrm{p}}} \mathrm{e}^{-\mathrm{x}(\mathrm{y})}
$$

and (B.10) follows.
In a similar way, define $z(x)$ from the relation $\mu^{-1} \log z(x)=\sqrt{p x}+x$ yielding

$$
\mathrm{z}(\mathrm{x})=\exp (\mu \sqrt{\mathrm{px}}+\mu \mathrm{x})
$$

The inverse relation reads

$$
\mathrm{x}_{\mathrm{e}}(\mathrm{z})=\left(\sqrt{\mu^{-1} \log \mathrm{z}+\mathrm{p} / 4}-\sqrt{\mathrm{p} / 4}\right)^{2} .
$$

Then with $\mathrm{x}(\mathrm{y})=\mathrm{x}_{\mathrm{e}}\left(\mathrm{e}^{\mu \mathrm{y} / 2}\right)=(\sqrt{\mathrm{y}+\mathrm{p}}-\sqrt{\mathrm{p}})^{2} / 4$

$$
\begin{aligned}
\mathbb{E}\left\{\mathrm{e}^{\mu \eta / 2} \mathbb{I}(\eta \geq \mathrm{y})\right\} & =\int_{\mathrm{e}^{\mu y / 2}}^{\infty} \mathbb{P}\left(\mathrm{e}^{\mu \eta / 2} \geq \mathrm{z}\right) d \mathrm{z}=\int_{\mathrm{e}^{\mu \mathrm{y}} / 2}^{\infty} \mathbb{P}\left(\eta \geq 2 \mu^{-1} \log \mathrm{z}\right) d \mathrm{z} \\
& \leq \int_{\mathrm{e}^{\mu \mathrm{y} / 2}}^{\infty} \mathrm{e}^{-\mathrm{x}_{\mathrm{e}}(\mathrm{z})} d \mathrm{z}=\int_{\mathrm{x}(\mathrm{y})}^{\infty} \mathrm{e}^{-\mathrm{x}} \mathrm{z}^{\prime}(\mathrm{x}) d \mathrm{x}
\end{aligned}
$$

Further, in view of $\mu+0.5 \mu \sqrt{\mathrm{p} / \mathrm{x}}<\mu+\mu \sqrt{\mathrm{p} / \mathrm{x}(\mathrm{y})}=\epsilon<1$ for $\mathrm{x} \geq \mathrm{x}(\mathrm{y})$, it holds

$$
\mathrm{z}^{\prime}(\mathrm{x})=(\mu+0.5 \mu \sqrt{\mathrm{p} / \mathrm{x}}) \exp (\mu \sqrt{\mathrm{px}}+\mu \mathrm{x}) \leq \exp (\mu \mathrm{x} \sqrt{\mathrm{p} / \mathrm{x}(\mathrm{y})}+\mu \mathrm{x})=\exp (\epsilon \mathrm{x})
$$

and

$$
\mathbb{E}\left\{\mathrm{e}^{\mu \eta / 2} \mathbb{I}(\eta \geq \mathrm{y})\right\} \leq \int_{\mathrm{x}(\mathrm{y})}^{\infty} \mathrm{e}^{-(1-\epsilon) \mathrm{x}} d \mathrm{x}=\frac{1}{1-\epsilon} \mathrm{e}^{-(1-\epsilon) \mathbf{x}(\mathrm{y})}
$$

and (B.11) follows.

## C Local smoothness conditions

This section discusses different local smoothness characteristics of a multivariate function $f(\boldsymbol{v})=\mathbb{E} L(\boldsymbol{v}), \boldsymbol{v} \in \mathbb{R}^{p}$.

## C. 1 Smoothness and self-concordance in Gateaux sense

Below we assume the function $f(\boldsymbol{v})$ to be strongly concave with the negative Hessian $\mathbb{F}(\boldsymbol{v}) \stackrel{\text { def }}{=}-\nabla^{2} f(\boldsymbol{v}) \in \mathfrak{M}_{p}$ positive definite. Also assume $f(\boldsymbol{v})$ three or sometimes even four times Gateaux differentiable in $\boldsymbol{v} \in \Upsilon$. For any particular direction $\boldsymbol{u} \in \mathbb{R}^{p}$, we consider the univariate function $f(\boldsymbol{v}+t \boldsymbol{u})$ and measure its smoothness in $t$. Local
smoothness of $f$ will be described by the relative error of the Taylor expansion of the third or four order. Namely, define

$$
\begin{aligned}
& \delta_{3}(\boldsymbol{v}, \boldsymbol{u})=f(\boldsymbol{v}+\boldsymbol{u})-f(\boldsymbol{v})-\langle\nabla f(\boldsymbol{v}), \boldsymbol{u}\rangle-\frac{1}{2}\left\langle\nabla^{2} f(\boldsymbol{v}), \boldsymbol{u}^{\otimes 2}\right\rangle, \\
& \delta_{3}^{\prime}(\boldsymbol{v}, \boldsymbol{u})=\langle\nabla f(\boldsymbol{v}+\boldsymbol{u}), \boldsymbol{u}\rangle-\langle\nabla f(\boldsymbol{v}), \boldsymbol{u}\rangle-\left\langle\nabla^{2} f(\boldsymbol{v}), \boldsymbol{u}^{\otimes 2}\right\rangle,
\end{aligned}
$$

and

$$
\delta_{4}(\boldsymbol{v}, \boldsymbol{u}) \stackrel{\text { def }}{=} f(\boldsymbol{v}+\boldsymbol{u})-f(\boldsymbol{v})-\langle\nabla f(\boldsymbol{v}), \boldsymbol{u}\rangle-\frac{1}{2}\left\langle\nabla^{2} f(\boldsymbol{v}), \boldsymbol{u}^{\otimes 2}\right\rangle-\frac{1}{6}\left\langle\nabla^{3} f(\boldsymbol{v}), \boldsymbol{u}^{\otimes 3}\right\rangle .
$$

Now, for each $\boldsymbol{v}$, suppose to be given a positive symmetric operator $\mathrm{D}(\boldsymbol{v}) \in \mathfrak{M}_{p}$ with $\mathrm{D}^{2}(\boldsymbol{v}) \leq \mathbb{F}(\boldsymbol{v})=-\nabla^{2} f(\boldsymbol{v})$ defining a local metric and a local vicinity around $\boldsymbol{v}$ :

$$
\mathcal{U}(\boldsymbol{v})=\left\{\boldsymbol{u} \in \mathbb{R}^{p}:\|\mathrm{D}(\boldsymbol{v}) \boldsymbol{u}\| \leq \mathrm{r}\right\}
$$

for some radius $r$.
Local smoothness properties of $f$ are given via the quantities

$$
\begin{equation*}
\omega(\boldsymbol{v}) \stackrel{\text { def }}{=} \sup _{u:\|\mathrm{D}(\boldsymbol{v}) \boldsymbol{u}\| \leq \mathrm{r}} \frac{2\left|\delta_{3}(\boldsymbol{v}, \boldsymbol{u})\right|}{\|\mathrm{D}(\boldsymbol{v}) \boldsymbol{u}\|^{2}}, \quad \omega^{\prime}(\boldsymbol{v}) \stackrel{\text { def }}{=} \sup _{u:\|\mathrm{D}(\boldsymbol{v}) \boldsymbol{u}\| \leq \mathrm{r}} \frac{2\left|\delta_{3}^{\prime}(\boldsymbol{v}, \boldsymbol{u})\right|}{\|\mathrm{D}(\boldsymbol{v}) \boldsymbol{u}\|^{2}} . \tag{C.1}
\end{equation*}
$$

The Taylor expansion yields for any $\boldsymbol{u}$ with $\|\mathrm{D}(\boldsymbol{v}) \boldsymbol{u}\| \leq \mathrm{r}$

$$
\begin{equation*}
\left.\left|\delta_{3}(\boldsymbol{v}, \boldsymbol{u})\right\rangle\left|\leq \frac{\omega(\boldsymbol{v})}{2}\|\mathrm{D}(\boldsymbol{v}) \boldsymbol{u}\|^{2}, \quad\right| \delta_{3}^{\prime}(\boldsymbol{v}, \boldsymbol{u}) \right\rvert\, \leq \frac{\omega^{\prime}(\boldsymbol{v})}{2}\|\mathrm{D}(\boldsymbol{v}) \boldsymbol{u}\|^{2} . \tag{C.2}
\end{equation*}
$$

The introduced quantities $\omega(\boldsymbol{v}), \omega^{\prime}(\boldsymbol{v})$ strongly depend on the radius r of the local vicinity $\mathcal{U}(\boldsymbol{v})$. The results about Laplace approximation can be improved provided a homogeneous upper bound on the error of Taylor expansion. Assume a subset $\Upsilon^{\circ}$ of $\Upsilon$ to be fixed.
$\left(\boldsymbol{T}_{3}\right)$ There exists $\tau_{3}$ such that for all $\boldsymbol{v} \in \Upsilon^{\circ}$

$$
\left|\delta_{3}(\boldsymbol{v}, \boldsymbol{u})\right| \leq \frac{\tau_{3}}{6}\|\mathrm{D}(\boldsymbol{v}) \boldsymbol{u}\|^{3}, \quad\left|\delta_{3}^{\prime}(\boldsymbol{v}, \boldsymbol{u})\right| \leq \frac{\tau_{3}}{2}\|\mathrm{D}(\boldsymbol{v}) \boldsymbol{u}\|^{3}, \quad \boldsymbol{u} \in \mathcal{U}(\boldsymbol{v}) .
$$

$\left(\boldsymbol{T}_{4}\right) \quad$ There exists $\tau_{4}$ such that for all $\boldsymbol{v} \in \Upsilon^{\circ}$

$$
\left|\delta_{4}(\boldsymbol{v}, \boldsymbol{u})\right| \leq \frac{\tau_{4}}{24}\|\mathrm{D}(\boldsymbol{v}) \boldsymbol{u}\|^{4}, \quad \boldsymbol{u} \in \mathcal{U}(\boldsymbol{v})
$$

Lemma C.1. Under $\left(\mathcal{T}_{3}\right)$, the values $\omega(\boldsymbol{v})$ and $\omega^{\prime}(\boldsymbol{v})$ from (C.1) satisfy

$$
\omega(\boldsymbol{v}) \leq \frac{\tau_{3} r}{3}, \quad \omega^{\prime}(\boldsymbol{v}) \leq \tau_{3} r, \quad \boldsymbol{v} \in \Upsilon^{\circ}
$$

Proof. For any $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{v})$ with $\|\mathrm{D}(\boldsymbol{v}) \boldsymbol{u}\| \leq \mathrm{r}$

$$
\left|\delta_{3}(\boldsymbol{v}, \boldsymbol{u})\right| \leq \frac{\tau_{3}}{6}\|\mathrm{D}(\boldsymbol{v}) \boldsymbol{u}\|^{3} \leq \frac{\tau_{3} r}{6}\|\mathrm{D}(\boldsymbol{v}) \boldsymbol{u}\|^{2},
$$

and the bound for $\omega(\boldsymbol{v})$ follows. The proof for $\omega^{\prime}(\boldsymbol{v})$ is similar.
The values $\tau_{3}$ and $\tau_{4}$ are usually very small. Some quantitative bounds are given later in this section under the assumption that the function $f(\boldsymbol{v})=\mathbb{E} L_{G}(\boldsymbol{v})$ can be written in the form $-f(\boldsymbol{v})=n h(\boldsymbol{v})$ for a fixed smooth function $h(\boldsymbol{v})$ with the Hessian $\nabla^{2} h(\boldsymbol{v})$. The factor $n$ has meaning of the sample size.
$\left(\mathcal{S}_{3}\right) \quad-f(\boldsymbol{v})=n h(\boldsymbol{v})$ for $h(\boldsymbol{v})$ convex with $\nabla^{2} h(\boldsymbol{v}) \geq \mathrm{m}^{2}(\boldsymbol{v})=\mathrm{D}^{2}(\boldsymbol{v}) / n$ and

$$
\sup _{\boldsymbol{u}:\|\mathrm{m}(\boldsymbol{v}) \boldsymbol{u}\| \leq \mathrm{r} / \sqrt{n}} \frac{\left|\left\langle\nabla^{3} h(\boldsymbol{v}+\boldsymbol{u}), \boldsymbol{u}^{\otimes 3}\right\rangle\right|}{\|\mathrm{m}(\boldsymbol{v}) \boldsymbol{u}\|^{3}} \leq \mathrm{c}_{3} .
$$

$\left(\mathcal{S}_{4}\right)$ the function $h(\cdot)$ satisfies $\left(\mathcal{S}_{3}\right)$ and

$$
\sup _{\boldsymbol{u}:\|\mathrm{m}(\boldsymbol{v}) \boldsymbol{u}\| \leq \mathrm{r} / \sqrt{n}} \frac{\left|\left\langle\nabla^{4} h(\boldsymbol{v}+\boldsymbol{u}), \boldsymbol{u}^{\otimes 4}\right\rangle\right|}{\|\mathrm{m}(\boldsymbol{v}) \boldsymbol{u}\|^{4}} \leq \mathrm{c}_{4} .
$$

$\left(\mathcal{S}_{3}\right)$ and $\left(\mathcal{S}_{4}\right)$ are local versions of the so called self-concordance condition; see Nesterov (1988). In fact, they require that each univariate function $h(\boldsymbol{v}+t \boldsymbol{u})$ of $t \in \mathbb{R}$ is selfconcordant with some universal constants $c_{3}$ and $c_{4}$. Under $\left(\mathcal{S}_{3}\right)$ and $\left(\mathcal{S}_{4}\right)$, we can use $\mathrm{D}^{2}(\boldsymbol{v})=n \mathrm{~m}^{2}(\boldsymbol{v})$ and easily bound the values $\delta_{3}(\boldsymbol{v}, \boldsymbol{u}), \delta_{4}(\boldsymbol{v}, \boldsymbol{u})$, and $\omega(\boldsymbol{v}), \omega^{\prime}(\boldsymbol{v})$.

Lemma C.2. Suppose $\left(\mathcal{S}_{3}\right)$. Then $\left(\mathcal{T}_{3}\right)$ follows with $\tau_{3}=\mathrm{c}_{3} n^{-1 / 2}$. Moreover, for $\omega(\boldsymbol{v})$ and $\omega^{\prime}(\boldsymbol{v})$ from (C.1), it holds

$$
\begin{equation*}
\omega(\boldsymbol{v}) \leq \frac{\mathrm{c}_{3} r}{3 n^{1 / 2}}, \quad \omega^{\prime}(\boldsymbol{v}) \leq \frac{\mathrm{c}_{3} r}{n^{1 / 2}} . \tag{C.3}
\end{equation*}
$$

Also $\left(\mathcal{T}_{4}\right)$ follows from $\left(\mathcal{S}_{4}\right)$ with $\tau_{4}=\mathrm{c}_{4} n^{-1}$.
Proof. For any $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{v})$ and $t \in[0,1]$, by the Taylor expansion of the third order

$$
\begin{aligned}
|\delta(\boldsymbol{v}, \boldsymbol{u})| & \leq \frac{1}{6}\left|\left\langle\nabla^{3} f(\boldsymbol{v}+t \boldsymbol{u}), \boldsymbol{u}^{\otimes 3}\right\rangle\right|=\frac{n}{6}\left|\left\langle\nabla^{3} h(\boldsymbol{v}+t \boldsymbol{u}), \boldsymbol{u}^{\otimes 3}\right\rangle\right| \leq \frac{n \mathrm{c}_{3}}{6}\|\mathrm{~m}(\boldsymbol{v}) \boldsymbol{u}\|^{3} \\
& =\frac{n^{-1 / 2} \mathrm{c}_{3}}{6}\|\mathrm{D}(\boldsymbol{v}) \boldsymbol{u}\|^{3} \leq \frac{n^{-1 / 2} \mathrm{c}_{3} r}{6}\|\mathrm{D}(\boldsymbol{v}) \boldsymbol{u}\|^{2} .
\end{aligned}
$$

This implies $\left(\mathcal{T}_{3}\right)$ as well as (C.3); see (C.2). The statement about $\left(\mathcal{T}_{4}\right)$ is similar.

## References

Hanson, D. L. and Wright, F. T. (1971). A Bound on Tail Probabilities for Quadratic Forms in Independent Random Variables. The Annals of Mathematical Statistics, 42(3):1079-1083.

Hsu, D., Kakade, S., and Zhang, T. (2012). A tail inequality for quadratic forms of subgaussian random vectors. Electronic Communications in Probability, 17(none):1-6.

Klochkov, Y. and Zhivotovskiy, N. (2020). Uniform Hanson-Wright type concentration inequalities for unbounded entries via the entropy method. Electronic Journal of Probability, 25(none):1-30. https://arxiv.org/abs/1812.03548.

Laurent, B. and Massart, P. (2000). Adaptive estimation of a quadratic functional by model selection. Ann. Statist., 28(5):1302-1338.

Nesterov, Y. E. (1988). Polynomial methods in the linear and quadratic programming. Sov. J. Comput. Syst. Sci., 26(5):98-101.

Rudelson, M. and Vershynin, R. (2013). Hanson-Wright inequality and sub-gaussian concentration. Electronic Communications in Probability, 18(none):1-9.

Spokoiny, V. (2022). Dimension free non-asymptotic bounds on the accuracy of high dimensional Laplace approximation. https://arxiv.org/abs/2204.11038.


[^0]:    *Financial support by the German Research Foundation (DFG) through the Collaborative Research Center 1294 "Data assimilation" is gratefully acknowledged.

