Finite-strain poro-visco-elasticity with degenerate mobility

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Abstract

A quasistatic nonlinear model for poro-visco-elastic solids at finite strains is considered in the Lagrangian frame using the concept of second-order nonsimple materials. The elastic stresses satisfy static frame-indifference, while the viscous stresses satisfy dynamic frame-indifference. The mechanical equation is coupled to a diffusion equation for a solvent or fluid content. The latter is pulled-back to the reference configuration. To treat the nonlinear dependence of the mobility tensor on the deformation gradient, the result by Healey & Krömer is used to show that the determinant of the deformation gradient is bounded away from zero. Moreover, the focus is on the physically relevant case of degenerate mobilities. The existence of weak solutions is shown using a staggered time-incremental scheme and suitable energy-dissipation inequalities.

1 Introduction

The coupling of the mechanical deformations of solids to other physical processes such as heat conduction or diffusion of chemical species is relevant in many applications in technology or biology. We refer to [Ger99; Wan+20; Lat04; CSS22; Hon+08; CA10] and the references therein for applications in thermo-mechanics, solid-state batteries, poroelasticity in biological tissue, hydrogen storage, and elastomeric materials.

In many of these applications, the assumption of small strains is no longer justified and nonlinear, finite-strain, theories have to be considered. While the static theory for finite-strain elasticity developed rapidly after the seminal work of Ball [Bal77], the mathematical analysis for time-dependent processes in the case of large strains is just currently receiving increased attention, see e.g. [MRS18] for a result on finite-strain visco-plasticity, [MR20; BFK23] for finite-strain thermo-visco-elasticity, and [Rou21; SP23] for phase-field models in the finite-strain setting.

In this text, we consider a coupled model for the visco-elastic evolution of the deformation $\chi(t, x)$ of a solid in the reference configuration $\Omega \subseteq \mathbb{R}^d$ and the diffusion of a solvent, fluid content, or other chemical species through the (visco-)elastic body. We shall call such materials *poroelastic* (although this name is usually used to refer to the interaction between fluid flow and solids deformation, see [Bio41]).

For a time horizon $T > 0$ and $\Omega \subseteq \mathbb{R}^d$ a bounded, open reference configuration, we establish the existence of a deformation $\chi : [0, T] \times \Omega \to \mathbb{R}^d$ and a concentration $c : [0, T] \times \Omega \to \mathbb{R}^+$ satisfying the quasi-static system

$$-\operatorname{div}\big(\sigma_{\mathsf{el}}(\nabla\chi, c) + \sigma_{\mathsf{vi}}(\nabla\chi, \nabla\dot{\chi}, c) - \operatorname{div}\left(\mathfrak{h}(\mathrm{D}^2\chi)\right)\big) = f(t), \quad \text{in } [0, T] \times \Omega, \tag{1.1a}$$

$$\dot{c} - \operatorname{div}\big(\mathcal{M}(\nabla\chi, c)\nabla\mu\big) = 0, \qquad \text{in } [0, T] \times \Omega, \tag{1.1b}$$

where *quasistatic* refers to the neglect of kinetic energy (and therefore inertial forces $\rho\ddot{\chi}$) such that there will be no mechanical oscillations. The total stress $\Sigma_{\mathsf{tot}} := \sigma_{\mathsf{el}} + \sigma_{\mathsf{vi}} - \operatorname{div}\mathfrak{h}$ consists of the elastic stress $\sigma_{\mathsf{el}}(F, c) = \partial_F\Phi(F, c)$, coming from a free energy density $\Phi(\nabla\chi, c)$, the viscous stress

$\sigma_{\mathsf{vi}}(F, \dot{F}, c) = \partial_{\dot{F}}\zeta(F, \dot{F}, c)$, given in terms of a viscous dissipation potential $\zeta(\nabla\chi, \nabla\dot{\chi}, c)$, and the hyperstress $\mathfrak{h}(G) = \partial_G\mathscr{H}(G)$ with potential $\mathscr{H}(\mathrm{D}^2\chi)$, which gives higher regularity of the deformation. Furthermore, f is a body force, $\mathcal{M}$ is the mobility tensor, and $\mu(F, c) = \partial_c\Phi(F, c)$ is the chemical potential. The system is completed by suitable boundary conditions. Note that the equation for the deformation is of fourth order, due to the hyperstress regularization. Thus, the Neumann boundary condition contains an additional contribution related to the surface divergence of $\mathfrak{h}$, and a further boundary condition has to be satisfied, see (2.3b) and (2.3c). For the diffusion equation, we assume the following Robin-type boundary condition

$$\mathcal{M}(\nabla\chi, c)\nabla\mu \cdot \vec{n} = \kappa(\mu_{\mathrm{ext}}(t) - \mu) \quad \text{on } \partial\Omega,$$

where $\kappa \geq 0$ is a transmission coefficient and μ_{ext} is a fixed external potential. As is the case in previous works [MR20], [RT20], we do not put any convexity assumptions on the energy density Φ, but rather add the higher order convex term $\mathscr{H}(\mathrm{D}^2\chi)$, leading to the hyperstress $\mathfrak{h}$. This addition makes it into a second-grade non-simple material, a notion which was introduced by Toupin [Tou62].

We highlight that the diffusion processes takes place in the actual (deformed) configuration, which leads to a nontrivial description when pulled back to the reference configuration Ω for the mathematical analysis. In the actual configuration, the diffusion equation takes the form

$$\dot{\mathsf{c}} + \mathrm{Div}(\mathsf{c}v - \mathbb{M}(F, \mathsf{c})\nabla\mu) = 0, \tag{1.2}$$

where $\mathsf{c}(t, \chi(t, x)) = c(t, x)/\det\nabla\chi(t, x)$ is the spatial concentration, $v(t, \chi(t, x)) = \dot{\chi}(t, x)$ is the Eulerian velocity, $F(t, \chi(t, x)) = \nabla\chi(t, x)$ is the deformation gradient, $\mu(t, \chi(t, x)) = \mu(t, x)$ is the spatial chemical potential, and $\mathbb{M}$ is the Eulerian mobility tensor. The Lagrangian mobility tensor $\mathcal{M}$ is obtained from the pull-back of $\mathbb{M}$. The two are related by the formula

$$\mathcal{M}(F, c) = \frac{(\mathrm{Cof}F^\top)\mathbb{M}(F, c/\det F)\mathrm{Cof}F}{\det F} \quad \text{for } (F, c) \in \mathrm{GL}^+(d) \times \mathbb{R}^+,$$

where $\mathrm{Cof}F = (\det F)F^{-\top}$ denotes the cofactor matrix associated with F. Equation (1.1b) now follows from (1.2) via the pullback under χ. As a consequence of the transformation, we need to ensure that the determinant of the deformation gradient is bounded away from zero, which follows from a result by Healey and Krömer [HK09]. Indeed, the hyperstress regularization in connection with growth assumptions for the elastic energy contribution in Φ in the form $\Phi(F, c) \geq C_1/(\det F)^q - C_2$, yields a uniform constant $\delta > 0$ such that $\det\nabla\chi \geq \delta$ in Ω for all deformations with finite energy, see also [MR20, Thm. 3.1]. This blowup of the free energy, if the determinant of the deformation gradient approaches 0 from above, gives rise to local non-selfpenetration; however, we do not enforce global non-selfpenetration.

In [DNS22] a model for the growth of biological tissue was considered. Therein, the evolution of the deformation is coupled to the diffusion of a nutrient and the evolution of a growth variable which plays the role of a plastic tensor. However, the diffusion equation for the nutrient density is assumed to take place in the reference configuration and the deformation χ only appears in a source/sink term. In particular, no viscous stresses and no hyperstress regularization were considered. The missing temporal compactness of the deformation (gradient) is compensated by introducing a suitable time convoluted deformation $K * \chi$ instead of χ in the source terms. In [RS22], a similar system as (1.1) was considered. However, there the authors work purely in the Eulerian setting, which restricts them to assuming that the actual domain is not evolving in time, i.e., $\chi(t, \Omega) = \Omega$.

Our new contribution is that we allow for degenerate mobilities, i.e. $\mathcal{M}(F, 0) \sim c^m$ for some power $m > 0$. Until now, to the best of the authors' knowledge, the mobility was always assumed to be

uniformly positive definite, see e.g. [Rou17b; RT20], [MR20] (which uses conductivities instead of mobilities), or [Rou21]. Our assumptions allow, for example, for the linear (actual) mobility $\mathbb{M}(c) = c\mathbb{M}_0$, or more generally, for a polynomial mobility. This type of mobility is physically relevant, as it models a higher species permeability whenever the material opens up (i.e., swells) due to an increase in species concentration, see [CA10]. See Example 2.4 for the case of a Biot-type model.

Finally, an important modeling assumption is the static frame-indifference of the free energy, namely $\Phi(RF, c) = \Phi(F, c)$ for all $R \in \mathrm{SO}(d)$ and $F \in \mathrm{GL}^+(d)$, and the dynamic frame-indifference of the viscous stress potential, i.e., $\zeta(RF, \dot{R}F + R\dot{F}, c) = \zeta(F, \dot{F}, c)$ for all smooth $t \mapsto R(t) \in \mathrm{SO}(d)$, and $F, \dot{F} \in \mathbb{R}^d$. For simplicity, we will assume that the viscous stress is linear in $(\nabla\chi^\top \nabla\chi)^{\boldsymbol{\cdot}}$, which is used to model non-activated dissipative processes with moderate rates, see e.g., [MR20, Sect. 2]. To control $\nabla\dot{\chi}$ via $(\nabla\chi^\top \nabla\chi)^{\boldsymbol{\cdot}}$, we exploit results for generalized Korn's inequalities by Neff [Nef02] and the extension by Pompe [Pom03]. Here, again it is used that we can control the determinant of the deformation gradient via the hyperstress regularization.

The main result of this paper is that under certain conditions our model always admits weak solutions (see Definition 2.5 and Theorem 2.7). To prove the existence of solutions, we use a time-incremental scheme consisting of two steps. First, the elastic equation is solved to obtain the new deformation χ, with fixed concentration c from the previous time level, by reformulating the equation as minimization problem. Then, the diffusion equation is solved using a fixed point argument to obtain the new chemical potential μ, which also implies the existence of a concentration c. Here, it is important to note that due to the degenerate mobility an extra regularization is needed, cf. [Jün15]. This regularization takes the form $\eta(-\Delta)^\theta \mu$, with regularization parameter $\eta > 0$ and exponent $\theta > \frac{d}{2}$ (such that there is a compact embedding $H^\theta(\Omega) \hookrightarrow L^\infty(\Omega)$). Next, we show that these solutions satisfy an energy-dissipation balance, from which suitable a priori estimates are derived. Due to the degenerate mobility, the classical Aubin–Lions lemma no longer suffices to pass to the limit in the concentration terms. Instead we use a more general compactness result due to Dubinskiĭ [Dub65] to pass to the limit $(\tau, \eta) \to 0$, see also Theorem A.3.

The paper is structured as follows. In Section 2 we introduce our model and assumptions in a mathematically rigorous way, and state the main result. In Section 3, we start the proof of the main result by introducing a regularization term and a time-incremental scheme. After showing existence of these approximate solutions, we proceed by showing an energy-dissipation inequality, from which suitable a priori estimates are deduced. Finally, in Section 4, we pass to the simultaneous limit with respect to the time discretization and regularization to obtain the main result.

2 Mathematical setting and main result

Our model is described in the Lagrangian setting in the reference configuration $\Omega \subseteq \mathbb{R}^d$. We assume that Ω is an open, bounded domain with Lipschitz boundary, and that $\partial\Omega = \Gamma_D \cup \Gamma_N$ (disjoint) such that the Dirichlet part has positive surface measures $\int_{\Gamma_D} 1 \, \mathrm{d}S > 0$. We denote by $L^p(\Omega)$, $H^k(\Omega)$, and $W^{k,p}(\Omega)$ the usual Lebesgue and Sobolev spaces with the standard norms. Moreover, the (closed) subspace $W_0^{k,p}(\Omega)$ denotes the functions in $W^{k,p}(\Omega)$ with zero trace on Γ_D. We consider deformations χ on Ω that are fixed on the Dirichlet part Γ_D, namely, we consider the space

$$W_{\mathrm{id}}^{2,p}(\Omega; \mathbb{R}^d) := \{\chi \in W^{2,p}(\Omega; \mathbb{R}^d) \mid \chi|_{\Gamma_D} = \mathrm{id}\}.$$

We denote by "$a \cdot b$", "$A : B$", and "$G \vdots H$" the scalar products between vectors $a, b \in \mathbb{R}^d$, matrices $A, B \in \mathbb{R}^{d \times d}$, and third-order tensors $G, H \in \mathbb{R}^{d \times d \times d}$, respectively.

To introduce the system of partial differential equations coupling the evolution of the deformation χ and the concentration c of some species, we consider a free energy density $\Phi = \Phi(\nabla\chi, c)$, a higher-order regularization $\mathscr{H} = \mathscr{H}(\mathrm{D}^2\chi)$, a viscous dissipation potential $\zeta = \zeta(\nabla\chi, \nabla\dot\chi, c)$, and a (Lagrangian) mobility tensor $\mathcal{M} = \mathcal{M}(\nabla\chi, c)$. The free energy density Φ gives rise to the first Piola–Kirchhoff stress σ_{el} and the chemical potential μ, the viscous dissipation potential ζ to the viscous stress σ_{vi} via

$$\sigma_{\mathsf{el}}(F, c) := \partial_F\Phi(F, c), \quad \mu(F, c) := \partial_c\Phi(F, c), \quad \text{and} \quad \sigma_{\mathsf{vi}}(F, \dot F, c) := \partial_{\dot F}\zeta(F, \dot F, c), \quad (2.1)$$

and the potential $\mathscr{H}$ to the hyperstress $\mathfrak{h}(G) := \partial_G\mathscr{H}(G)$, where we have used the placeholders F for $\nabla\chi$, $\dot F$ for $\nabla\dot\chi$, and G for $\mathrm{D}^2\chi$. The model is then given by the evolutionary system in the reference domain Ω

$$-\mathrm{div}\big(\sigma_{\mathsf{el}}(\nabla\chi, c) + \sigma_{\mathsf{vi}}(\nabla\chi, \nabla\dot\chi, c) - \mathrm{div}\,\mathfrak{h}(\mathrm{D}^2\chi)\big) = f(t), \quad (2.2a)$$

$$\dot c - \mathrm{div}\big(\mathcal{M}(\nabla\chi, c)\nabla\mu\big) = 0, \quad (2.2b)$$

completed with the boundary conditions

$$\begin{aligned}
\chi &= \mathrm{id} & \text{on } \Gamma_D, & \quad (2.3a)\\
\big(\sigma_{\mathsf{el}}(\nabla\chi, c) + \sigma_{\mathsf{vi}}(\nabla\chi, \nabla\dot\chi, c)\big)\vec n - \mathrm{div}_{\mathsf{s}}(\mathfrak{h}(\mathrm{D}^2\chi)\vec n) &= g(t) & \text{on } \Gamma_N, & \quad (2.3b)\\
\mathfrak{h}(\mathrm{D}^2\chi) : (\vec n \otimes \vec n) &= 0 & \text{on } \partial\Omega, & \quad (2.3c)\\
\mathcal{M}(\nabla\chi, c)\nabla\mu \cdot \vec n &= \kappa(\mu_{\mathsf{ext}}(t) - \mu) & \text{on } \partial\Omega, & \quad (2.3d)
\end{aligned}$$

where $\vec n$ denotes the unit normal vector on $\partial\Omega$, and $\kappa(x) \geq 0$ and $\mu_{\mathsf{ext}}(t, x)$ are a given permeability and an external potential, respectively. Here, $\mathrm{div}_{\mathsf{s}}$ denotes the surface divergence, defined by $\mathrm{div}_{\mathsf{s}}(\cdot) = \mathrm{tr}(\nabla_{\mathsf{s}}(\cdot))$, i.e., the trace of the surface gradient $\nabla_{\mathsf{s}}v = (I - \vec n \otimes \vec n)\nabla v = \nabla v - \frac{\partial v}{\partial\vec n}\vec n$. Finally, we consider initial conditions

$$\chi(0) = \chi_0, \quad c(0) = c_0 \quad \text{on } \Omega. \quad (2.4)$$

For any $R > 0$, let us denote the set

$$\mathsf{F}_R := \big\{ F \in \mathrm{GL}^+(d) \,\big|\, |F| \leq R,\ |F^{-1}| \leq R,\ \text{and } \det F \geq 1/R \big\}. \quad (2.5)$$

We impose the following assumptions on our model:

(A1) The hyperstress potential is a convex, frame-indifferent C^1 function $\mathscr{H} : \mathbb{R}^{d \times d \times d} \to \mathbb{R}^+$ such that the hyperstress is given by $\mathfrak{h}(G) = \partial_G\mathscr{H}(G) \in \mathbb{R}^{d \times d \times d}$. Moreover, there exist $p \in (d, \infty) \cap [3, \infty)$ and constants $C_{\mathcal{H},1}, C_{\mathcal{H},2}, C_{\mathcal{H},3} > 0$ such that

$$C_{\mathcal{H},1}|G|^p \leq \mathscr{H}(G) \leq C_{\mathcal{H},2}(1 + |G|^p), \quad |\partial_G\mathscr{H}(G)| \leq C_{\mathcal{H},3}|G|^{p-1} \quad \text{for all } G \in \mathbb{R}^{d \times d \times d}.$$

(A2) The mobility tensor $\mathcal{M} : \mathrm{GL}^+(d) \times \mathbb{R}^+ \to \mathbb{R}^{d \times d}_{\mathsf{sym}}$ is a continuous map. There exist an exponent $m > 0$, and for all $R > 0$ there exist constants $C_{0,\mathcal{M},R}, C_{1,\mathcal{M},R} > 0$ such that

$$\xi \cdot \mathcal{M}(F, c)\xi \geq C_{0,\mathcal{M},R}c^m|\xi|^2 \quad \text{and} \quad |\mathcal{M}(F, c)| \leq C_{1,\mathcal{M},R}c^m$$
$$\text{for all } \xi \in \mathbb{R}^d,\ F \in \mathsf{F}_R,\ c \in \mathbb{R}^+. \quad (2.6)$$

The admissible range of the exponent $m > 0$ depends on the growth properties of (the derivatives of) Φ and is fixed in assumption **(A3)**.

(A3) The free energy $\Phi : \mathrm{GL}^+(d) \times \mathbb{R}^+ \to \mathbb{R}$ is bounded from below, continuous, and C^2 on $\mathrm{GL}^+(d) \times (0, \infty)$, i.e., for strictly positive concentrations. It is frame indifferent, and satisfies the following assumptions:

(i) For any $c \in \mathbb{R}^+$ there exists constants $C_{\Phi,0}, C_{\Phi,1} > 0$ such that

$$\Phi(F, c) \geq C_{\Phi,0}|F| + \frac{C_{\Phi,0}}{(\det F)^q} - C_{\Phi,1} \quad \text{for all } F \in \mathrm{GL}^+(d).$$

(ii) There exist an exponent $-1 < r < \infty$ such that $r + m \geq 0$, and for all $R > 0$ constants $C_i := C_{\Phi,i,R} > 0$ ($1 \leq i \leq 2$) and constants $\gamma_i := \gamma_{\Phi,i,R} \geq 0$ ($1 \leq i \leq 2$) such that

$$\frac{C_1}{c} + \gamma_1 c^r \leq \partial_{cc}^2 \Phi(F, c) \leq \frac{C_3}{c} + \gamma_2 c^r \quad \text{for all } c \in \mathbb{R}^+, \ F \in \mathsf{F}_R.$$

Concerning the constants γ_i, we distinguish two cases (see also Remark 2.2(ii)):

Case I: We assume $\gamma_1 = \gamma_2 = 0$, and also require $1 \leq m \leq 2$.

Case IIa: For $\gamma_2 \geq \gamma_1 > 0$, we require $0 < m \leq 3 + r$.

Case IIb: For $\gamma_2 \geq \gamma_1 > 0$, we require $0 < m \leq 2$.

(iii) There exist an exponent $\alpha \in \mathbb{R}$, and for all $R > 0$ a constant $C_{\Phi,5,R} > 0$ such that

$$\left|\partial_{Fc}^2 \Phi(F, c)\right| \leq C_{\Phi,5,R} c^\alpha \quad \text{for all } c \in \mathbb{R}^+, \ F \in \mathsf{F}_R$$

In **Case I** above, α is such that $0 \leq m + \alpha \leq \frac{p-s}{ps}$, where $1 < s = \frac{md+2}{md+1} < 2$ and $0 \leq m + 2\alpha$.

In both **Case IIa** and **Case IIb** $\alpha \geq -1$ is such that $0 \leq m + \alpha \leq (2+r)\frac{p-s}{ps}$, where $1 < s = \min\{\frac{md+2(r+2)}{md+r+2}, \frac{d(m+r+1)+2(r+2)}{d(m+r+1)+r+2}\} < 2$. Furthermore, in case **Case IIa** we require that $0 \leq m + 2\alpha < m + 1 + r$, while in **Case IIb** we require that $0 \leq m + 2\alpha < m + 2 + 2r$

(A4) For all $R > 0$ there exists a concentration $c_R \in \mathbb{R}^+$ such that $\Phi(F, c_R) < \infty$ and $|\partial_c \Phi(F, c_R)| < \infty$ for all $F \in \mathsf{F}_R$.

(A5) The viscous stress potential $\zeta : \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \times \mathbb{R}^+ \to \mathbb{R}^+$ is such that $\zeta(F, \dot{F}, c) = \widehat{\zeta}(F^\top F, (F^\top F)^\cdot, c)$, where $\widehat{\zeta} : \mathbb{R}_{\mathrm{sym}}^{d \times d} \times \mathbb{R}_{\mathrm{sym}}^{d \times d} \times \mathbb{R}^+ \to \mathbb{R}^+$ is quadratic in the second variable, namely

$$\widehat{\zeta}(\mathsf{C}, \dot{\mathsf{C}}, c) = \frac{1}{2}\dot{\mathsf{C}} : \widetilde{\mathbb{D}}(\mathsf{C}, c)\dot{\mathsf{C}}.$$

This quadratic form is such that there exist constants $C_{\zeta,1}, C_{\zeta,2} > 0$ such that

$$C_{\zeta,1}|(F^\top F)^\cdot|^2 \leq \widehat{\zeta}(F^\top F, (F^\top F)^\cdot, c) \leq C_{\zeta,2}|(F^\top F)^\cdot|^2 \quad \text{for all } c \in \mathbb{R}^+, \ F \in \mathbb{R}^{d \times d}.$$

(A6) The external forces satisfy $f \in W^{1,\infty}(0, T; L^2(\Omega; \mathbb{R}^d))$, $g \in W^{1,\infty}(0, T; L^2(\partial\Omega; \mathbb{R}^d))$. We set

$$\langle \ell(t), \chi \rangle := \int_\Omega f(t) \cdot \chi \, \mathrm{d}x + \int_{\Gamma_N} g(t) \cdot \chi \, \mathrm{d}S$$

such that $\ell \in W^{1,\infty}(0, T; H^1(\Omega; \mathbb{R}^d)^*)$.

(A7) The permeability $\kappa \in L^\infty(\partial\Omega)$ is nonnegative and strictly positive on a part of the boundary $\partial\Omega$ with positive surface measure, i.e., $\int_{\partial\Omega} \kappa \, \mathrm{d}S \geq \kappa_* > 0$. We assume that the external chemical potential is such that $\mu_{\mathrm{ext}} \in L^2([0, T] \times \partial\Omega)$.

(A8) The initial conditions satisfy $\chi_0 \in W^{2,p}_{\mathrm{id}}(\Omega; \mathbb{R}^d)$ with $\det \nabla \chi_0 \geq \rho_0 > 0$ and $c_0 \in L^1(\Omega)$ with $c_0 \geq 0$ and are such that $\int_\Omega \Phi(\nabla \chi_0, c_0) \, \mathrm{d}x < \infty$.

As an immediate consequence of these assumptions, we note the following lower bound for the free energy.

Lemma 2.1 (Lower bound for free energy). *If $\nabla \chi \in \mathsf{F}_R$ for some $R > 0$, then there exist $C_1, C_2, C_3 > 0$ such that*

$$\int_\Omega \Phi(\nabla \chi, c) \, \mathrm{d}x \geq C_1 \|c\|_{L \log L(\Omega)} + \gamma_1 C_2 \|c\|^{2+r}_{L^{2+r}} - C_3. \tag{2.7}$$

Proof. Using first the lower bound $\partial^2_{cc} \Phi(F, c) \geq \frac{C_{\Phi,1,R}}{c}$ of Assumption **(A3)**(ii) and Assumption **(A4)**, we obtain by integrating once

$$\partial_c \Phi(F, c) \geq C_{1,\Phi,R} \log(c) + \partial_c \Phi_{\mathrm{mix}}(F, c_R) - C_{1,\Phi,R} \log(c_R).$$

Integrating again, and using Young's inequality with ϵ to absorb the lower order terms in c in the highest order term, we thus obtain

$$\int_\Omega \Phi(\nabla \chi, c) \, \mathrm{d}x \geq C_1 \|c\|_{L \log L(\Omega)} - C_3.$$

The second part of the lower bound now follows from integrating the lower bound $\partial^2_{cc} \Phi \geq \gamma_1 c^r$ twice. $\qquad\square$

Remark 2.2. (i) (Growth of free energy) Note that Assumption **(A3)**(ii) implies that $c \mapsto \Phi(c, F)$ is superlinear for every $F \in \mathrm{GL}^+(d)$. Moreover, we see that $\Phi(F, c) \sim c \log c - c + \gamma_1 c^{r+2}$.

 (ii) (Differences between Case I and II) We remark that the limit $r \downarrow -1$ in the conditions of Case II almost reduces to the conditions of Case I. However, in Case I, we additionally need to assume the lower bound $m \geq 1$. Without this bound, it is not possible to obtain the strong convergence of the concentrations, which is needed to pass to the limit, see Prop. 4.1. Note that as a result, Case I is restricted to negative exponents $\alpha < 0$. Indeed, we have $\alpha \leq \frac{p(1-ms)-s}{ps} < 0$ since $m \geq 1$ and $s > 1$. In fact, for $p \to \infty$, we get $\alpha \leq \frac{1}{s} - m$.

 (iii) (Relation between μ and c) From Assumption **(A3)**(ii) and the resulting strict monotonicity of $c \mapsto \partial_c \Phi(F, c)$ it follows that the equation $\mu = \partial_c \Phi(F, c)$ for $F \in \mathrm{GL}^+(d)$ fixed is uniquely solvable for given $\mu \in \mathbb{R}$. By the implicit function theorem, the map $c = c(F, \mu)$ is continuous on $\mathrm{GL}^+(d) \times \mathbb{R}$. In particular, for any $F \in \mathrm{GL}^+(d)$ and $\mu \in \mathbb{R}$ there exist constants $M, \delta > 0$ depending only on $|\mu|, |F|$, and $\det F$ such that $\delta \leq c(F, \mu) \leq M$.

 (iv) (Constraint $c \geq 0$) To obtain the constraint that the concentration stays positive, we note that $\partial^2_{cc} \Phi(F, c) \geq \frac{C_1}{c}$ (cf. **(A3)**(ii)) implies that $\lim_{c \to 0^+} \partial_c \Phi(F, c) = -\infty$. Thus, we can extend Φ by setting $\Phi(F, c) = +\infty$ for $c \leq 0$. Note that this implies that $\partial_c \Phi(F, c) = \emptyset$ for $c \leq 0$ (and in particular for $c = 0$).

 Another possibility (which we will not use) is discussed in [Rou17a, Rem. 2], where instead the mobility is extended by setting $\mathcal{M}(F, c) = 0$ for $c < 0$. However, this will only give the positivity of the concentration for solutions (2.2), not for the regularized problem. Indeed, since our existence proof uses an extra regularization for the diffusion part, it is not clear if the regularized time-discrete solution c_k obtained from (3.1) is positive. Consequently, several modifications would have to be made to the proof.

(v) (Degenerate & non-degenerate mobility) If we assume that the mobility is uniformly positive definite, continuous and bounded, i.e., if $m = 0$ (see [Rou17b], [MR20] (conductivities instead of mobilities), or [RT20]), then it is possible to prove existence of solutions without using an additional regularization. Indeed, in this case one can test the diffusion equation with μ and use the estimate $\mathcal{M}\nabla\mu \cdot \nabla\mu \geq C|\nabla\mu|^2$ to obtain a bound for $\nabla\mu$ in $L^2(\Omega)$. Here, we restrict the discussion to the degenerate case, where $m > 0$.

(vi) (Gradient structure) The viscous-elastic evolution and the diffusion process can be formally written in terms of a gradient-flow structure. Indeed, define the free energy functional $\mathcal{E}_0(\chi, c) = \int_\Omega \Phi(\nabla\chi, c) + \mathscr{H}(\mathrm{D}^2\chi)\,\mathrm{d}x$ as well as the convex dissipation potential

$$\mathcal{R}_{\text{tot}}(\chi, c, \dot{\chi}, \dot{c}) = \int_\Omega \zeta(\nabla\chi, \nabla\dot{\chi}, c)\,\mathrm{d}x$$
$$+ \int_\Omega \mathcal{M}(\nabla\chi, c)\nabla(-\Delta^{-1}_{\mathcal{M}(\nabla\chi,c),\kappa}\dot{c}) \cdot \nabla(-\Delta^{-1}_{\mathcal{M}(\nabla\chi,c),\kappa}\dot{c})\,\mathrm{d}x + \int_\Omega \kappa(-\Delta^{-1}_{\mathcal{M}(\nabla\chi,c),\kappa}\dot{c})^2\,\mathrm{d}S$$

with $-\Delta^{-1}_{\mathcal{M}(\nabla\chi,c),\kappa} : v \mapsto \tilde{\mu}$ denoting the linear operator defined formally by the weak solution $\tilde{\mu}$ of the elliptic equation $-\mathrm{div}\,(\mathcal{M}(\nabla\chi, c)\nabla\tilde{\mu}) = v$ with boundary conditions $\mathcal{M}(\nabla\chi, c)\nabla\mu \cdot \vec{n} + \kappa\tilde{\mu} = 0$. The evolutionary system in (1.1) can be formally rewritten in the form

$$\partial_{(\dot{\chi},\dot{c})}\mathcal{R}_{\text{tot}}(\chi, c, \dot{\chi}, \dot{c}) + \mathrm{D}_{(\chi,c)}\mathcal{E}_0(\chi, c) = \xi_{\text{ext}},$$

where ξ_{ext} contains the mechanical loading and the external potential μ_{ext}. It is possible to exploit the gradient structure of the equations to obtain time-discrete solutions via a staggered incremental scheme. However, since this scheme would require additional assumptions to derive suitable a priori estimates, we instead use a fixed point theorem to obtain the existence of time-discrete concentrations.

Example 2.3 (Strong coupling in free energy). Consider the multiplicative decomposition of the deformation gradient as $F = F_{\text{el}}F_c$, see e.g. [Lub04]. In this case, the free energy is formulated in terms of $F_{\text{el}} = FF_c^{-1}$, e.g., $\Phi(F, c) = \Phi_1(FF_c^{-1}) + \Phi_2(c)$, where $\Phi_1 : \mathrm{GL}^+(d) \to [0, \infty)$ and $\Phi_2 : (0, \infty) \to [0, \infty)$ describe the elastic and the chemical contribution to the free energy, respectively. A standard choice for F_c, which models isotropic swelling, is e.g., $F_c = a(c)I$ for some C^1 function $a : [0, \infty) \to (0, \infty)$ with $0 < a_* \leq a(c) \leq a^*$.

We assume that $\Phi_1(F_{\text{el}}, c)$ satisfies the coercivity estimate $\Phi_1(F_{\text{el}}, c) \geq C_0|F_{\text{el}}| + \frac{C_0}{(\det F_{\text{el}})^q} - C_1$ with exponent $1 < q < \infty$ as in Assumption *(A3)*(i). Then, it follows that

$$\Phi_1(\nabla\chi F_c^{-1}, c) \geq \frac{C_0}{a(c)}|\nabla\chi| + \frac{C_0 a(c)^q}{(\det F_{\text{el}})^q} - C_1 \geq \frac{C_0}{a^*}|F_{\text{el}}| + \frac{C_0 a_*}{(\det F_{\text{el}})^q} - C_1,$$

i.e., $\Phi(F, c)$ also satisfies the coercivity estimate. However, conditions on Φ_1 and Φ_2 for convexity with respect to c and the growth conditions in Assumption *(A3)*(ii) and (iii) are hard to formulate at this abstract level and have to be confirmed for more concrete examples.

Example 2.4 (Biot model and Fick/Darcy's law). The Biot model [Bio41] (cf. [RT20, Sect. 4]) with Boltzmann entropy is given as

$$\Phi(F, c) = \Phi_{\text{el}}(F) + \frac{1}{2}M_B(c - c_{\text{eq}} - \beta(\det F - 1))^2 + kc\big(\log\big(\frac{c}{c_{\text{eq}}}\big) - 1\big),$$

for some suitable elastic energy Φ_{el} and constants $M_B, \beta, k, c_{\mathsf{eq}} > 0$. In this case, the assumptions for Case II are satisfied with $\alpha = 0$ and $r = 0$. Defining the (Eulerian) flux as $\boldsymbol{j} = -\mathbb{M}(F, c)\nabla\mu$, and assuming the mobility is linear in c, namely $\mathbb{M}(F, c) = c\mathbb{M}_0$ (i.e., $m = 1$), we then obtain

$$\boldsymbol{j} = -k\mathbb{M}_0\nabla c - c\mathbb{M}_0\nabla p,$$

where $p = M_B(c - c_{\mathsf{eq}} - \beta(\det F - 1))$ is the pressure. The first term corresponds to Fick's law, while the second is related to Darcy's law.

Definition 2.5 (Weak solution). Let $1 < s < 2$ be as in **(A3)**(iii). We call a pair (χ, c) a weak solution of the initial-boundary-value problem (2.2)–(2.4) if $\chi \in L^\infty(0, T; W_{\mathrm{id}}^{2,p}(\Omega; \mathbb{R}^d))$, $\dot\chi \in L^2(0, T; H^1(\Omega; \mathbb{R}^d))$ and $c \in L^\infty(0, T; L\log L(\Omega))$, $\dot c \in L^s(0, T; W^{1,s}(\Omega)^*)$ with $\nabla c^{\frac{m}{2}} \in L^2(0, T; L^2(\Omega))$ (Case I). In Case II, we additionally require that $c \in L^\infty(0, T; L^{2+r}(\Omega))$ and $\nabla c^{\frac{m+1+r}{2}}$, $\nabla c^{\frac{m}{2}+1+r} \in L^2(0, T; L^2(\Omega))$. The pair satisfies the integral equations

$$\int_0^T \int_\Omega \left(\sigma_{\mathsf{el}}(\nabla\chi, c) + \sigma_{\mathsf{vi}}(\nabla\chi, \nabla\dot\chi, c)\right) : \nabla\phi + \mathfrak{h}(\mathrm{D}^2\chi) \vdots \mathrm{D}^2\phi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \langle \ell(t), \phi \rangle \, \mathrm{d}t \quad (2.8a)$$

for all $\phi \in L^2(0, T; W_0^{2,p}(\Omega; \mathbb{R}^d))$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{2,p}(\Omega; \mathbb{R}^d)^*$ and $W^{2,p}(\Omega; \mathbb{R}^d)$, and

$$\int_0^T \langle \dot c, \psi \rangle \, \mathrm{d}t + \int_0^T \int_\Omega \mathcal{M}(\nabla\chi, c)\nabla\mu \cdot \nabla\psi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\partial\Omega} \kappa(\mu - \mu_{\mathsf{ext}})\psi \, \mathrm{d}S \, \mathrm{d}t = 0 \quad (2.9)$$

for all $\psi \in L^{s'}(0, T; W^{1,s'}(\Omega))$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{1,s}(\Omega)^*$ and $W^{1,s'}(\Omega)$.

Furthermore, we require that $\mu \in \partial_c\Phi(\nabla\chi, c)$ almost everywhere in Ω, and that $\mu \in L^2([0, T] \times \partial\Omega)$.

We note that sufficiently smooth weak solutions indeed satisfy the classical formulation (2.2) with boundary conditions (2.3). For the derivation of (2.2a) and (2.3a)-(2.3c), we refer to [MR20, Eqn. (2.28)-(2.29)]. The derivation of (2.2b) and (2.3d) follows directly from integration by parts.

Remark 2.6. Note that we not state any regularity conditions for the chemical potential μ, but only for the concentration c. Using the relation $\nabla\mu = \partial_{Fc}^2\Phi\mathrm{D}^2\chi + \partial_{cc}^2\Phi\nabla c$ and the bounds in **(A2)** and **(A3)**, we see that this gives a well-defined concept of weak solution.

Theorem 2.7 (Existence of weak solutions). *Suppose that the assumptions* **(A1)**–**(A8)** *hold. Then, the system in* (2.2)–(2.4) *possesses at least one weak solution in the sense of Definition 2.5.*

3 Time-discretization of a regularized problem

For the time discretization, we consider an equidistant partition $\{0 = t_0 < t_1 < ... < t_N = T\}$ of $[0, T]$, where $N \in \mathbb{N}$, $t_k = k\tau$ with $k = 0, \dots, N$ and $\tau = T/N > 0$. We construct approximate solutions (χ_k, μ_k) such that $\chi_k \approx \chi(t_k)$ and $\mu_k \approx \mu(t_k)$ using a staggered scheme. In the following, we use for the time discretization the difference notation

$$\delta_\tau h_k = \frac{h_k - h_{k-1}}{\tau} \quad \text{for} \quad k = 1, \dots, N.$$

We consider the following time-discrete system for $k = 1, \ldots, N$:

$$\int_\Omega \big(\sigma_{\mathsf{el}}(\nabla\chi_k, c_{k-1}) + \sigma_{\mathsf{vi}}(\nabla\chi_{k-1}, \delta_\tau \nabla\chi_k, c_{k-1})\big) : \nabla\phi + \mathfrak{h}(\mathrm{D}^2\chi_k) \vdots \mathrm{D}^2\phi \,\mathrm{d}x = \langle \ell_k, \phi \rangle, \quad (3.1\mathrm{a})$$

$$\langle \delta_\tau c_k, \psi \rangle + \int_\Omega \mathcal{M}(\nabla\chi_k, c_k)\nabla\mu_k \cdot \nabla\psi \,\mathrm{d}x + \int_{\partial\Omega} \kappa(\mu_k - \mu_{k,\mathrm{ext}})\psi \,\mathrm{d}S = 0, \quad (3.1\mathrm{b})$$

$$\text{where } \mu_k \in \partial_c \Phi(\chi_k, c_k) \text{ a.e. in } \Omega. \quad (3.1\mathrm{c})$$

Moreover, we set $\ell_k := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} \ell(t)\,\mathrm{d}t$ and $\mu_{k,\mathrm{ext}} := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} \mu_{\mathrm{ext}}(t)\,\mathrm{d}t$ for $k = 1, \ldots N$ and set $\ell_0 := \ell_1$ such that $\delta_\tau \ell_1 = 0$.

We note that this scheme is constructed in a specific way: Starting from the initial conditions $\chi_{k=0} = \chi_0$ and $c_{k=0} = c_0$ (cf. **(A8)**), we first solve the mechanical equation using the concentration c_{k-1} from the previous time step, and then we solve the diffusion equation using the just obtained deformation χ_k. The existence of time-discrete solutions (χ_k, c_k) follows from the formulation of the mechanical equation as the Euler–Lagrange equation of a minimization problem and for the diffusion equation via a fixed-point argument.

Due to the degeneracy of the mobility, we cannot expect to obtain the existence and suitable estimates for $\nabla\mu_k$. Thus, following [Jün15], we add an elliptic regularization $\eta(-\Delta)^\theta\mu$ to the diffusion equation, for some small parameter $0 < \eta \ll 1$ and $\theta \in \mathbb{N}$. This regularization ensures that $\mu_k \in H^\theta(\Omega)$. The exponent θ is chosen sufficiently large such that $H^\theta(\Omega) \hookrightarrow L^\infty(\Omega)$ compactly, i.e., $\theta > \frac{d}{2}$. We will use the notation $|\beta| := \sum_{i=1}^d \beta_i$ for multi-indices $\beta \in \mathbb{N}_0$. With this regularization, we replace the equation in (3.1b) with

$$\langle \delta_\tau c_k, \psi \rangle + \int_\Omega \mathcal{M}(\nabla\chi_k, c_k)\nabla\mu_k \cdot \nabla\psi \,\mathrm{d}x$$

$$+ \eta \int_\Omega \sum_{|\beta|=\theta} \partial^\beta\mu_k \cdot \partial^\beta\psi \,\mathrm{d}x + \int_{\partial\Omega} \kappa(\mu_k - \mu_{k,\mathrm{ext}})\psi \,\mathrm{d}S = 0 \quad (3.1\mathrm{b}^\star)$$

for all $\psi \in H^\theta(\Omega)$ and with $\mu_k \in \partial_c \Phi(\chi_k, c_k)$ a.e. in Ω. Note that this weak formulation is still well-defined since $c_k \in L^\infty(\Omega)$ (see Remark 2.2(iii)) and thus $\mathcal{M}(\nabla\chi_k, c_k) \in L^\infty(\Omega)$ and $\nabla\mu_k, \nabla\psi \in L^2(\Omega)$.

Remark 3.1. We highlight that we do not require an additional regularization term $\eta\delta_\tau\chi_k$ in the mechanical equation to deal with the frame-indifferent viscous dissipation potential ζ as e.g. in [MR20].

Lemma 3.2 (Existence of time-discrete solutions)**.** *Let the assumptions in **(A1)**–**(A8)** hold. We fix $\mu_0^\eta \in H^\theta(\Omega)$ and $\chi_0^\eta \in W_{\mathrm{id}}^{2,p}(\Omega; \mathbb{R}^d)$ such that $\det \nabla\chi_0^\eta \geq \rho_0 > 0$, and we define c_0^η as the unique density with $\mu_0^\eta = \partial_c \Phi(\nabla\chi_0^\eta, c_0^\eta)$. Then, starting for $k = 0$ from χ_0^η and c_0^η, we can iteratively find weak solutions $(\chi_k^\eta, c_k^\eta, \mu_k^\eta) \in W_{\mathrm{id}}^{2,p}(\Omega; \mathbb{R}^d) \times L^\infty(\Omega) \times H^\theta(\Omega)$, for $k = 1, \ldots, N$, for the mechanical equation (3.1a) and the regularized diffusion equation (3.1b*), respectively, with $c_k^\eta > 0$ and $\mu_k^\eta = \partial_c \Phi(\nabla\chi_k^\eta, c_k^\eta)$ a.e. in Ω.*

Proof. We drop the index η throughout the proof.

Mechanical step. First, we consider the equation (3.1a), where $\chi_{k-1} \in W_{\mathrm{id}}^{2,p}(\Omega; \mathbb{R}^d)$ and $c_{k-1} \in L^\infty(\Omega)$ with $c_{k-1} > 0$ a.e. in Ω are fixed. Let us define the energy functional $\mathcal{E}_0$ and the dissipation potential $\mathcal{R}$ via

$$\mathcal{E}_0(\chi, c) := \int_\Omega \Phi(\nabla\chi, c) + \mathscr{H}(\mathrm{D}^2\chi)\,\mathrm{d}x, \quad \text{and} \quad \mathcal{R}(\chi, \dot\chi, c) := \int_\Omega \zeta(\nabla\chi, \nabla\dot\chi, c)\,\mathrm{d}x.$$

We assume that χ_{k-1} and c_{k-1} are such that $\mathcal{E}_0(\chi_{k-1}, c_{k-1}) < \infty$. The deformation $\chi_k \in W^{2,p}_{\mathrm{id}}(\Omega; \mathbb{R}^d)$ is obtained as the solution of the minimization problem

$$\chi_k \in \operatorname{Arg Min}\Big\{ \mathcal{E}_0(\widetilde{\chi}, c_{k-1}) + \tau \mathcal{R}\Big(\chi_{k-1}, \frac{\widetilde{\chi} - \chi_{k-1}}{\tau}, c_{k-1} \Big) - \langle \ell_k, \widetilde{\chi} \rangle \,\Big|\, \widetilde{\chi} \in W^{2,p}_{\mathrm{id}}(\Omega; \mathbb{R}^d) \Big\}. \quad (3.2)$$

Note that the corresponding weak Euler–Lagrange equation is exactly (3.1a). Indeed, the Gâteaux differentiability of $\chi \mapsto \mathcal{E}_0(\chi, c_{k-1})$ follows as in [MR20, Proposition 3.2].

To show that there exists a minimizer $\chi_k \in W^{2,p}_{\mathrm{id}}(\Omega; \mathbb{R}^d)$, we first note that the functional in (3.2) is bounded from below due to the assumptions *(A1)*, and *(A3)–(A6)*. This boundedness follows from the embedding $W^{2,p}_{\mathrm{id}}(\Omega) \hookrightarrow W^{1,\infty}(\Omega; \mathbb{R}^d)$, which gives a bound for $\nabla \chi$ in $L^\infty(\Omega; \mathbb{R}^{d\times d})$. In fact, using the assumptions on the hyperstress, we also obtain that the functional in (3.2) is coercive on $W^{2,p}_{\mathrm{id}}(\Omega; \mathbb{R}^d)$. To show that it is weak lower semicontinuous, we note that $\mathscr{H}$ is convex by assumption. As the rest of the functional consists of a (non-)convex perturbation of lower order, Ioffe's theorem [FL07, Theorem 7.5] now gives the weak lower semicontinuity. Note that to apply Ioffe's theorem, we use the compact embedding $W^{2,p}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, which follows from $p > d$ (comp. *(A1)*). Finally, since the minimization is feasible as by assumption $\mathcal{E}_0(\chi_{k-1}, c_{k-1}) < \infty$, we obtain the existence of a minimizer χ_k. In particular, we also have that $\mathcal{E}_0(\chi_k, c_{k-1}) < \infty$.

Diffusion step. Given $\chi_k \in W^{2,p}(\Omega; \mathbb{R}^d)$ from the mechanical step and $c_{k-1} \in L^\infty(\Omega)$ from the previous time step, we now solve the diffusion equation (3.1b*) via a fixed-point argument, analogously to e.g. [Jün15; Fis+22]. Let us fix $\widetilde{\mu} \in L^\infty(\Omega)$ and define $\widetilde{c}$ such that $\widetilde{\mu} \in \partial_c \Phi(\nabla \chi_k, \widetilde{c})$ a.e. in Ω (see Remark 2.2(iii)). We solve the linear problem $a_{\widetilde{c}}(\mu, \psi) = \langle \xi_{\widetilde{c}}, \psi \rangle$ for all $\psi \in H^\theta(\Omega)$, where the bilinear and linear forms $a_{\widetilde{c}} : H^\theta(\Omega) \times H^\theta(\Omega) \to [0, \infty)$ and $\xi_{\widetilde{c}} \in H^\theta(\Omega)^*$ are given via

$$a_{\widetilde{c}}(\mu, \psi) = \int_\Omega \Big\{ \mathcal{M}(\nabla \chi_k, \widetilde{c}) \nabla \mu \cdot \nabla \psi + \eta \sum_{|\beta|=\theta} \partial^\beta \mu \cdot \partial^\beta \psi \Big\} \, \mathrm{d}x + \int_{\partial\Omega} \kappa \mu \psi \, \mathrm{d}S,$$

$$\langle \xi_{\widetilde{c}}, \psi \rangle = - \int_\Omega \frac{\widetilde{c} - c_{k-1}}{\tau} \psi \, \mathrm{d}x + \int_{\partial\Omega} \kappa \mu_{k,\mathrm{ext}} \psi \, \mathrm{d}S.$$

The existence of a unique solution $\mu \in H^\theta(\Omega)$ follows from the generalized Poincaré inequality for the space $H^\theta(\Omega)$, see e.g. [Tem97, Ch. 2.1.4]. Since $H^\theta(\Omega)$ embeds compactly into $L^\infty(\Omega)$ (recall that $\theta > d/2$), we can define the map $\mathcal{S} : L^\infty(\Omega) \to L^\infty(\Omega)$, that maps $\widetilde{\mu}$ to μ. We show that $\mathcal{S}$ is continuous, compact, and that for some $\lambda \in (0, 1]$ the set $S_\lambda := \{ \mu \in L^\infty(\Omega) \,|\, \mu = \lambda \mathcal{S}(\mu) \}$ is bounded. Then, $\mathcal{S}$ has a fixed-point $\mu_k \in H^\theta(\Omega) \subset L^\infty(\Omega)$ by Schauder's fixed point theorem, see [GT01, Thm. 11.3].

To show the continuity of $\mathcal{S}$, consider a sequence of chemical potentials $\widetilde{\mu}^{(n)} \to \widetilde{\mu}$ in $L^\infty(\Omega)$ as $n \to \infty$ and the associated densities $\widetilde{c}^{(n)}$. In particular, we have that $\|\widetilde{\mu}^{(n)}\|_{L^\infty(\Omega)} \leq K$ for some $K > 0$ independent of n and thus also $\|\widetilde{c}^{(n)}\|_{L^\infty(\Omega)} \leq C(K)$. Let now $\mu^{(n)} := \mathcal{S}(\widetilde{\mu}^{(n)})$. We have to show that this sequence has a limit $\mu \in L^\infty(\Omega)$ such that $\mathcal{S}(\widetilde{\mu}) = \mu$. Choosing $\psi = \mu^{(n)}$ in the equation for $\mu^{(n)}$ gives the uniform estimate $\|\mu^{(n)}\|_{H^\theta(\Omega)} \leq C$ via standard arguments. Thus, we can find a (non-relabeled) subsequence and a limit $\mu \in H^\theta(\Omega)$ such that $\mu^{(n)} \overset{w}{\rightharpoonup} \mu$ in $H^\theta(\Omega)$. Moreover, by possibly passing to another subsequence we can assume that $\widetilde{c}^{(n)} \to \widetilde{c}$ a.e. in Ω and by dominated convergence also in $L^{\tilde{p}}(\Omega)$ for every $\tilde{p} \in [1, \infty)$. Thus, we can pass to the limit in $a_{\widetilde{c}^{(n)}}(\mu^{(n)}, \psi) = \langle \xi_{\widetilde{c}^{(n)}}, \psi \rangle$ to find that μ is the unique solution of $a_{\widetilde{c}}(\mu, \psi) = \langle \xi_{\widetilde{c}}, \psi \rangle$. In particular, since it is unique, we get that $\widetilde{\mu} = \mu$ and all converging subsequences converge to the same limit μ. Thus, $\mathcal{S}$ is continuous. The compactness of $\mathcal{S}$ follows similarly, as the images of bounded sets in $L^\infty(\Omega)$ under $\mathcal{S}$ are bounded in $H^\theta(\Omega)$. Due to the compact embedding of $H^\theta(\Omega)$ into $L^\infty(\Omega)$ the claim follows.

Finally, we have to show that the set S_λ for some $\lambda \in (0,1]$ is bounded. Let $\mu_\lambda \in S_\lambda$, i.e., μ_λ satisfies $a_{c_\lambda}(\mu_\lambda, \psi) = \lambda \langle \xi_{c_\lambda}, \psi \rangle$ for all $\psi \in H^\theta(\Omega)$, where $\mu_\lambda = \partial_c \Phi(\nabla \chi_k, c_\lambda)$. Choosing $\psi = \mu_\lambda$ in the equation yields

$$a_{c_\lambda}(\mu_\lambda, \mu_\lambda) = \int_\Omega \left\{ \mathcal{M}(\nabla \chi_k, c_\lambda) \nabla \mu_\lambda \cdot \nabla \mu_\lambda + \eta \sum_{|\beta| = \theta} |\partial^\beta \mu_\lambda|^2 \right\} \mathrm{d}x + \int_{\partial \Omega} \kappa \mu_\lambda^2 \, \mathrm{d}S$$

$$= \langle \xi_{c_\lambda}, \mu_\lambda \rangle = \lambda \int_{\partial \Omega} \kappa \mu_\lambda \mu_{k,\mathrm{ext}} \, \mathrm{d}S - \lambda \int_\Omega \frac{c_\lambda - c_{k-1}}{\tau} \mu_\lambda \, \mathrm{d}x$$

$$\leq \lambda \int_{\partial \Omega} \kappa \mu_\lambda \mu_{k,\mathrm{ext}} \, \mathrm{d}S + \frac{\lambda}{\tau} \int_\Omega \Phi(\nabla \chi_k, c_{k-1}) \, \mathrm{d}x - \frac{\lambda}{\tau} \int_\Omega \Phi(\nabla \chi_k, c_\lambda) \, \mathrm{d}x.$$

Applying standard Hölder's and Young's inequalities for the boundary integral on the right-hand side, we conclude that

$$\lambda \int_\Omega \Phi(\nabla \chi_k, c_k) + \tau \int_\Omega \left\{ \mathcal{M}(\nabla \chi_k, c_\lambda) \nabla \mu_\lambda \cdot \nabla \mu_\lambda + \eta \sum_{|\beta| = \theta} |\partial^\beta \mu_\lambda|^2 \right\} \mathrm{d}x + \tau \kappa \left(1 - \frac{\lambda}{2} \right) \int_{\partial \Omega} \mu_\lambda^2 \, \mathrm{d}S$$

$$\leq \int_{\partial \Omega} \frac{\kappa \lambda}{2} \mu_{k,\mathrm{ext}}^2 \, \mathrm{d}S + \lambda \int_\Omega \Phi(\nabla \chi_k, c_{k-1}) \, \mathrm{d}x,$$

and thus that $\|\mu_\lambda\|_{L^\infty(\Omega)} \leq C_{\infty,\theta} \|\mu_\lambda\|_{H^\theta(\Omega)} \leq C$ since Φ is bounded from below and the right-hand side is bounded.

Thus, Schauder's fixed-point theorem yields the existence of a fixed point $\mu_k \in H^\theta(\Omega)$. With c_k defined as above, this fixed-point is a solution to the regularized diffusion equation (3.1b*). Moreover, we also obtain from the estimate above for $\lambda = 1$ that $\mathcal{E}_0(\chi_k, c_k) < \infty$. Note that by Remark 2.2(iii), we have that $c_k \in L^\infty(\Omega)$ and $c_k > 0$ a.e. in Ω.

$\square$

To pass to the limit $(\tau, \eta) \to 0$, we now introduce the piecewise constant and piecewise affine interpolants with respect to the time-discrete solutions obtained in Lemma 3.2 by

$$\overline{\chi}_{\eta,\tau}(0) = \chi_0^\eta, \quad \overline{\chi}_{\eta,\tau}(t) = \chi_k^\eta \qquad \qquad \text{for } t \in (t_{k-1}, t_k], \qquad (3.3)$$

$$\underline{\chi}_{\eta,\tau}(0) = \chi_0^\eta, \quad \underline{\chi}_{\eta,\tau}(t) = \chi_{k-1}^\eta \qquad \qquad \text{for } t \in [t_{k-1}, t_k), \qquad (3.4)$$

$$\widehat{\chi}_{\eta,\tau}(0) = \chi_0^\eta, \quad \widehat{\chi}_{\eta,\tau}(t) = \frac{t - t_{k-1}}{\tau} \chi_k^\eta + \frac{t_k - t}{\tau} \chi_{k-1}^\eta \quad \text{for } t \in (t_{k-1}, t_k], \qquad (3.5)$$

where, as before, $t_k = k\tau$ for $k = 0, \ldots, N$. Similarly, we define the interpolants with respect to c_k, μ_k, ℓ_k, and $\mu_{k,\mathrm{ext}}$.

Remark 3.3. Note that the strong convergence of the piecewise affine interpolants together with a suitable estimate for the time derivative of the latter also implies the strong convergence of the piecewise constant interpolants, as for any Banach space X the following estimates hold:

$$\|\widehat{\chi}_\tau - \underline{\chi}_\tau\|_{L^2(0,T;X)} \leq \frac{\tau}{\sqrt{3}} \|\dot{\widehat{\chi}}_\tau\|_{L^2(0,T;X)}, \qquad (3.6)$$

$$\|\widehat{\chi}_\tau - \underline{\chi}_\tau\|_{L^\infty(0,T;X)} \leq \tau^{1/2} \|\dot{\widehat{\chi}}_\tau\|_{L^2(0,T;X)}. \qquad (3.7)$$

Of course, similar estimates also hold for $\overline{\chi}_\tau$.

Using the piecewise affine and constant interpolants defined above, we can rewrite equation (3.1a) for the deformation in the form

$$\int_0^T \int_\Omega \big(\sigma_{\mathsf{el}}(\nabla \overline{\chi}_{\eta,\tau}, \underline{c}_{\eta,\tau}) + \sigma_{\mathsf{vi}}(\nabla \underline{\chi}_{\eta,\tau}, \nabla \dot{\widehat{\chi}}_{\eta,\tau}, \underline{c}_{\eta,\tau}) \big) : \nabla \phi \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^T \int_\Omega \mathfrak{h}(\mathrm{D}^2 \overline{\chi}_{\eta,\tau}) \vdots \mathrm{D}^2 \phi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \langle \overline{\ell}_\tau, \phi \rangle \, \mathrm{d}t, \quad \text{(3.8a)}$$

for $\phi \in L^2(0, T; W_0^{2,p}(\Omega; \mathbb{R}^d))$ and (3.1b*) for the concentration as

$$\int_0^T \langle \dot{\widehat{c}}_{\eta,\tau}, \psi \rangle \, \mathrm{d}t + \int_0^T \int_\Omega \mathcal{M}(\nabla \overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau}) \nabla \overline{\mu}_{\eta,\tau} \cdot \nabla \psi + \eta \sum_{|\beta|=\theta} \partial^\beta \overline{\mu}_{\eta,\tau} \cdot \partial^\beta \psi \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^T \int_{\partial\Omega} \kappa(\overline{\mu}_{\eta,\tau} - \overline{\mu}_{\tau,\mathrm{ext}}) \psi \, \mathrm{d}S \, \mathrm{d}t = 0, \quad \text{(3.8b)}$$

for $\psi \in L^{s'}(0, T; H^\theta(\Omega))$, where $\overline{\mu}_{\eta,\tau} \in \partial_c \Phi(\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})$ a.e. in Ω.

We start by showing an energy-dissipation inequality.

Lemma 3.4 (Energy-dissipation inequality). *Let $\overline{\chi}_{\eta,\tau}$, $\overline{c}_{\eta,\tau}$, $\underline{\chi}_{\eta,\tau}$, $\underline{c}_{\eta,\tau}$, $\overline{\mu}_{\eta,\tau}$, and $\widehat{\chi}_{\eta,\tau}$ denote the piecewise constant and affine interpolants with respect to the time-discrete solutions obtained in Lemma 3.2. For $\mathcal{E}_\tau(t, \chi, c) := \int_\Omega \Phi(\nabla \chi, c) + \mathscr{H}(\mathrm{D}^2 \chi) \, \mathrm{d}x - \langle \overline{\ell}_\tau(t), \chi \rangle$, the following time-discrete energy-dissipation inequality holds for $t_k = k\tau$, $k = 1, \ldots, N$:*

$$\mathcal{E}_\tau(t_k, \overline{\chi}_{\eta,\tau}(t_k), \overline{c}_{\eta,\tau}(t_k)) + \int_0^{t_k} \int_\Omega \Big\{ \mathcal{M}(\nabla \overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau}) \nabla \overline{\mu}_{\eta,\tau} \cdot \nabla \overline{\mu}_{\eta,\tau} + \eta \sum_{|\beta|=\theta} |\partial^\beta \overline{\mu}_{\eta,\tau}|^2 \Big\} \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^{t_k} \int_{\partial\Omega} \kappa \overline{\mu}_{\eta,\tau}^2 \, \mathrm{d}S \, \mathrm{d}t + \int_0^{t_k} \mathcal{R}\big(\underline{\chi}_{\eta,\tau}, \dot{\widehat{\chi}}_{\eta,\tau}, \underline{c}_{\eta,\tau} \big) \, \mathrm{d}t$$

$$\leq \mathcal{E}_\tau(0, \chi_0^\eta, c_0^\eta) + \int_0^{t_k} \int_{\partial\Omega} \kappa \overline{\mu}_{\eta,\tau} \overline{\mu}_{\tau,\mathrm{ext}} \, \mathrm{d}S \, \mathrm{d}t - \int_0^{t_k} \langle \dot{\widehat{\ell}}_\tau, \underline{\chi}_{\eta,\tau} \rangle \, \mathrm{d}t. \quad \text{(3.9)}$$

Proof. We drop the index η throughout the proof.

Using the definition of the subdifferential, we get with $\mu_k = \partial_c \Phi(\nabla \chi_k, c_k)$ that a.e. in Ω

$$\Phi(\nabla \chi_k, c_{k-1}) \geq \Phi(\nabla \chi_k, c_k) + \mu_k(c_{k-1} - c_k).$$

Thus, it follows for all $k = 1, \ldots, N$, after integration over Ω (and adding the hyperstress potential $\mathscr{H}$ to both sides) that

$$\int_\Omega \Phi(\nabla \chi_k, c_k) + \mathscr{H}(\mathrm{D}^2 \chi_k) \, \mathrm{d}x - \int_\Omega \Phi(\nabla \chi_k, c_{k-1}) + \mathscr{H}(\mathrm{D}^2 \chi_k) \, \mathrm{d}x - \int_\Omega \mu_k(c_k - c_{k-1}) \, \mathrm{d}x \leq 0.$$

Using the time-discrete diffusion equation in (3.1b*) with test function $\psi = \mu_k \in H^\theta(\Omega)$ for the last term on the left-hand side, we now obtain the discrete energy inequality

$$\mathcal{E}_0(\chi_k, c_k) - \mathcal{E}_0(\chi_k, c_{k-1}) + \tau \int_\Omega \mathcal{M}(\nabla \chi_k, c_k) \nabla \mu_k \cdot \nabla \mu_k \, \mathrm{d}x$$

$$+ \tau \int_{\partial\Omega} \kappa(\mu_k^2 - \mu_k \mu_{k,\mathrm{ext}}) \, \mathrm{d}S + \tau \eta \int_\Omega \sum_{|\beta|=\theta} |\partial^\beta \mu_k|^2 \, \mathrm{d}x \leq 0, \quad \text{(3.10)}$$

where, as before, $\mathcal{E}_0(\chi, c) = \int_\Omega \Phi(\nabla\chi, c) + \mathscr{H}(\mathrm{D}^2\chi)\,\mathrm{d}x$.

Next, we establish the analogous estimate for the mechanical step. This estimate, however, follows directly from χ_k being the solution of the minimization problem (3.2). Indeed, choosing χ_{k-1} as competitor in the minimization gives

$$\mathcal{E}_0(\chi_k, c_{k-1}) + \tau\mathcal{R}(\chi_{k-1}, \delta_\tau\chi_k, c_{k-1}) - \langle \ell_k, \chi_k \rangle \leq \mathcal{E}_0(\chi_{k-1}, c_{k-1}) - \langle \ell_k, \chi_{k-1} \rangle. \qquad (3.11)$$

To obtain the estimate in (3.9), we use the estimate (3.11) for $\mathcal{E}_0(\chi_k, c_{k-1})$ in the estimate (3.10) to get

$$\mathcal{E}_0(\chi_i, c_i) + \tau\int_\Omega \left\{ \mathcal{M}(\nabla\chi_i, c_i)\nabla\mu_i \cdot \nabla\mu_i + \eta\sum_{|\beta|=\theta}|\partial^\beta\mu_i|^2 \right\}\mathrm{d}x$$

$$+ \tau\int_{\partial\Omega} \kappa(\mu_i^2 - \mu_i\mu_{i,\mathrm{ext}})\,\mathrm{d}S + \tau\mathcal{R}(\chi_{i-1}, \delta_\tau\chi_i, c_{i-1}) \leq \mathcal{E}_0(\chi_{i-1}, c_{i-1}) + \langle \ell_i, \chi_i - \chi_{i-1}\rangle.$$

The inequality in (3.9) then follows from summing over $i = 1, \dots, k$, summation by parts for the loading part, and the definition of the piecewise constant and affine interpolants. In particular, the summation-by-parts formula reads

$$\sum_{i=1}^{k} \tau\langle \ell_i, \delta_\tau\chi_i \rangle = \langle \ell_k, \chi_k \rangle - \langle \ell_0, \chi_0 \rangle - \sum_{i=1}^{k}\tau\langle \delta_\tau\ell_i, \chi_{i-1} \rangle.$$

$\square$

The time-discrete energy-dissipation inequality gives rise to uniform a priori bounds (with respect to τ and η) for the interpolants of the time-discrete solutions (χ_k, μ_k, c_k) obtained in Lemma 3.2. We assume in the following that the initial conditions satisfy

$$\chi_0^\eta \to \chi_0 \text{ in } W_{\mathrm{id}}^{2,p}(\Omega; \mathbb{R}^d), \quad c_0^\eta \to c_0 \text{ in } L^1(\Omega), \quad \text{and} \quad \mathcal{E}_0(\chi_0^\eta, c_0^\eta) \to \mathcal{E}_0(\chi_0, c_0) \qquad (3.12)$$

as $\eta \to 0$.

Lemma 3.5 (A priori estimates). *Assuming that* **(A1)–(A8)**, *as well as* (3.12) *are satisfied, there exists $C > 0$ (independent of τ and η) such that for $\tau, \eta > 0$ small enough, the following uniform estimates are satisfied*

$$\|\overline{\chi}_{\eta,\tau}\|_{L^\infty(0,T;W_{\mathrm{id}}^{2,p}(\Omega))} + \|\dot{\hat{\chi}}_{\eta,\tau}\|_{L^2(0,T;H_0^1(\Omega))} + \|(\det\nabla\overline{\chi}_{\eta,\tau})^{-1}\|_{L^\infty(0,T;L^q(\Omega))} \leq C, \qquad (3.13a)$$

$$\sqrt{\eta}\|\overline{\mu}_{\eta,\tau}\|_{L^2(0,T;H^\theta(\Omega))} + \|\sqrt{\kappa}\overline{\mu}_{\eta,\tau}\|_{L^2([0,T]\times\partial\Omega)} \leq C, \qquad (3.13b)$$

$$\|\overline{c}_{\eta,\tau}\|_{L^\infty(0,T;L\log L(\Omega))} + \|\nabla\overline{c}_{\eta,\tau}^{\frac{m}{2}}\|_{L^2(0,T;L^2(\Omega))} + \|\dot{\hat{c}}_{\eta,\tau}\|_{L^s(0,T;H^\theta(\Omega)^*)} \leq C, \qquad (3.13c)$$

$$\|\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})\nabla\overline{\mu}_{\eta,\tau}\|_{L^s([0,T]\times\Omega)} \leq C_\eta. \qquad (3.13d)$$

where $s = \frac{md+2}{md+1} > 1$.

If additionally $\gamma_1 > 0$ in Assumption **(A3)***(ii) then we also have that*

$$\|\overline{c}_{\eta,\tau}\|_{L^\infty(0,T;L^{2+r}(\Omega))} + \sum_{\omega\in\{0,\frac{1+r}{2},1+r\}}\|\nabla\overline{c}_{\eta,\tau}^{\frac{m}{2}+\omega}\|_{L^2(0,T;L^2(\Omega))}^2 + \|\dot{\hat{c}}_{\eta,\tau}\|_{L^s(0,T;H^\theta(\Omega)^*)} \leq C,$$

$$(3.13e)$$

$$\|\mathcal{M}(\nabla\chi_\eta, c_\eta)\nabla\mu_\eta\|_{L^s([0,T]\times\Omega)} \leq C, \qquad (3.13f)$$

where $s = \min\{\frac{md+2(r+2)}{md+r+2}, \frac{d(m+r+1)+2(r+2)}{d(m+r+1)+r+2}\} > 1$.

Proof. Step 1. To obtain the a priori estimate for $\overline{\chi}_{\eta,\tau}$, we exploit the energy-dissipation estimate in (3.9). We absorb the boundary integral on the right-hand side via Young's inequality in the corresponding boundary terms on the left-hand side. Moreover, neglecting any nonnegative terms on the left-hand side gives

$$\mathcal{E}_\tau(t_k, \overline{\chi}_{\eta,\tau}(t_k), \overline{c}_{\eta,\tau}(t_k)) \leq \mathcal{E}_\tau(0, \chi_0, c_0) + \frac{1}{2}\int_0^{t_k} \|\sqrt{\kappa}\overline{\mu}_{\tau,\mathrm{ext}}\|_{L^2(\partial\Omega)}^2 \, \mathrm{d}t$$

$$+ \int_0^{t_k} \|\dot{\overline{\ell}}_\tau\|_{W^{2,p}(\Omega;\mathbb{R}^d)^*} \|\underline{\chi}_{\eta,\tau}\|_{W_{\mathrm{id}}^{2,p}(\Omega;\mathbb{R}^d)} \, \mathrm{d}t \leq C_0 + \Lambda_1 \int_0^{t_k} \|\underline{\chi}_{\eta,\tau}(t)\|_{W_{\mathrm{id}}^{2,p}(\Omega;\mathbb{R}^d)} \, \mathrm{d}t, \quad (3.14)$$

where we used Assumptions *(A7)* and *(A6)* for the data μ_{ext} and ℓ, respectively, and set $\Lambda_1 := \|\dot{\ell}\|_{L^\infty(0,T;W^{2,p}(\Omega;\mathbb{R}^d)^*)}$. However, due to Assumptions *(A1)*, *(A3)*(i), and *(A6)*, we find a constant $C_\mathcal{E} > 0$ such that the energy $\mathcal{E}_\tau$ satisfies the lower estimate

$$\mathcal{E}_\tau(t, \chi, c) \geq C_\mathcal{E}\|\chi\|_{W_{\mathrm{id}}^{2,p}(\Omega;\mathbb{R}^d)} + C_{\Phi,0}\|(\det\nabla\chi)^{-1}\|_{L^q(\Omega)}^q - 1/C_\mathcal{E}. \tag{3.15}$$

Thus, setting $e_k := \mathcal{E}_\tau(t_k, \overline{\chi}_{\eta,\tau}(t_k), \overline{c}_{\eta,\tau}(t_k))$, we have the time-discrete estimate

$$e_k \leq C_0 + \frac{\Lambda_1 T}{C_\mathcal{E}^2} + \frac{\Lambda_1}{C_\mathcal{E}}\sum_{i=0}^{k-1}\tau e_i, \quad k = 1, \dots, N.$$

An application of the discrete Gronwall lemma (see e.g. [QV94, Lemma 1.4.2]) now gives the bound

$$e_k \leq \left(C_0 + \frac{\Lambda_1 T}{C_\mathcal{E}^2}\right)\exp\left(\frac{\Lambda_1}{C_\mathcal{E}}k\tau\right). \tag{3.16}$$

Hence, estimate (3.15) gives the bound for $\overline{\chi}_{\eta,\tau}$ in $L^\infty(0, T; W_{\mathrm{id}}^{2,p}(\Omega;\mathbb{R}^d))$ (cf. (3.13a)).

Note that as an immediate consequence of (3.15) and (3.16), we also obtain that $1/(\det\nabla\overline{\chi}_{\eta,\tau})$ is uniformly bounded in $L^\infty(0, T; L^q(\Omega))$ for $q \geq pd/(p-d)$ (comp. Assumption *(A3)*(i)). Consequently, the Healey–Krömer lemma [MR20, Thm. 3.1] can be applied, which gives a uniform constant $C_{\mathsf{HK}} > 0$ and the lower bound

$$\det\nabla\overline{\chi}_{\eta,\tau}(t, x) \geq C_{\mathsf{HK}} > 0 \quad \text{for all } (t, x) \in [0, T] \times \Omega.$$

(Note that by assumption, also $\det\nabla\chi_0(x) \geq \rho_0 > 0$.) In particular, we see that for some $R > 0$ large enough, $\nabla\overline{\chi}_{\eta,\tau}(t, x) \in \mathsf{F}_R$ for all $(t, x) \in [0, T] \times \overline{\Omega}$.

Step 2. To obtain the a priori estimate for $\dot{\widehat{\chi}}_{\eta,\tau}$ in $L^2(0, T; H^1(\Omega))$, we use assumption *(A5)* to estimate

$$\int_0^T \mathcal{R}(\underline{\chi}_{\eta,\tau}, \dot{\widehat{\chi}}_{\eta,\tau}, \underline{c}_{\eta,\tau}) \, \mathrm{d}t = \int_0^T \int_\Omega \zeta(\nabla\underline{\chi}_{\eta,\tau}, \nabla\dot{\widehat{\chi}}_{\eta,\tau}, \underline{c}_{\eta,\tau}) \, \mathrm{d}x \, \mathrm{d}t$$

$$= \int_0^T \int_\Omega \widehat{\zeta}\left((\nabla\underline{\chi}_{\eta,\tau})^\top\nabla\underline{\chi}_{\eta,\tau}, (\nabla\underline{\chi}_{\eta,\tau})^\top\nabla\dot{\widehat{\chi}}_{\eta,\tau} + (\nabla\dot{\widehat{\chi}}_{\eta,\tau})^\top\nabla\underline{\chi}_{\eta,\tau}, \underline{c}_{\eta,\tau}\right) \mathrm{d}x \, \mathrm{d}t$$

$$\geq C_{\zeta,1}\int_0^T \int_\Omega \left|(\nabla\underline{\chi}_{\eta,\tau})^\top\nabla\dot{\widehat{\chi}}_{\eta,\tau} + (\nabla\dot{\widehat{\chi}}_{\eta,\tau})^\top\nabla\underline{\chi}_{\eta,\tau}\right|^2 \mathrm{d}x \, \mathrm{d}t$$

$$\geq C_{\zeta,1}\int_0^T \|\dot{\widehat{\chi}}_{\eta,\tau}\|_{H^1(\Omega)}^2 \, \mathrm{d}t = C_{\zeta,1}\|\dot{\widehat{\chi}}_{\eta,\tau}\|_{L^2(0,T;H^1(\Omega))}^2,$$

where the last inequality follows from the generalized Korn's inequality as in [MR20, Cor. 3.4]. The uniform boundedness of the left-hand side now follows directly from the energy-dissipation inequality (3.9) and hence the proof of (3.13a) is complete.

Step 3. To obtain the a priori estimates for $\overline{\mu}_{\eta,\tau}$, we again exploit the energy-dissipation inequality (3.9) as well as the generalized Poincaré inequality (see [Tem97, Section 2.1.4]) to obtain the estimate

$$\sqrt{\eta}\|\overline{\mu}_{\eta,\tau}\|_{L^2(0,T;H^\theta(\Omega))} + \|\sqrt{\kappa}\overline{\mu}_{\eta,\tau}\|_{L^2([0,T]\times\partial\Omega)} \leq C$$

for some constant $C > 0$ independent of τ and η. Thus, we have shown (3.13b).

Step 4. The first estimate of (3.13c), i.e., the estimate for $\overline{c}_{\eta,\tau}$ in $L^\infty(0,T;L\log L(\Omega))$ follows directly from the $L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^{d\times d}))$-bound for $\nabla\overline{\chi}_{\eta,\tau}$, Lemma 2.1, and the energy-dissipation inequality (3.9). Similarly, when $\gamma_1 > 0$, we also obtain the bound for $\overline{c}_{\eta,\tau}$ in $L^\infty(0,T;L^{2+r}(\Omega))$.

To obtain the bounds for the gradient terms, we now use that

$$\nabla\overline{\mu}_{\eta,\tau} = \partial^2_{Fc}\Phi(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\mathrm{D}^2\overline{\chi}_{\eta,\tau} + \partial^2_{cc}\Phi(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\nabla\overline{c}_{\eta,\tau},$$

which by the binomial formula implies that

$$\int_0^T \int_\Omega \mathcal{M}(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\nabla\overline{\mu}_{\eta,\tau} \cdot \nabla\overline{\mu}_{\eta,\tau} \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_\Omega \mathcal{M}(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\Big\{\big|\partial^2_{Fc}\Phi(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\mathrm{D}^2\overline{\chi}_{\eta,\tau}\big|^2 + \big|\partial^2_{cc}\Phi(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\nabla\overline{c}_{\eta,\tau}\big|^2$$
$$+ 2(\partial^2_{Fc}\Phi(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\mathrm{D}^2\overline{\chi}_{\eta,\tau}) \cdot \partial^2_{cc}\Phi(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\nabla\overline{c}_{\eta,\tau}\Big\} \,\mathrm{d}x\,\mathrm{d}t.$$

Thus, using Young's inequality with $\epsilon > 0$ for the last term in the brackets, we see that

$$\int_0^T \int_\Omega \mathcal{M}(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\nabla\overline{\mu}_{\eta,\tau} \cdot \nabla\overline{\mu}_{\eta,\tau} \,\mathrm{d}x\,\mathrm{d}t$$
$$\geq C\int_0^T \int_\Omega \partial^2_{cc}\Phi(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})^2\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\nabla\overline{c}_{\eta,\tau} \cdot \nabla\overline{c}_{\eta,\tau} \,\mathrm{d}x\,\mathrm{d}t$$
$$- C\int_0^T \int_\Omega |\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})|\big|\partial^2_{Fc}\Phi(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\mathrm{D}^2\overline{\chi}_{\eta,\tau}\big|^2 \,\mathrm{d}x\,\mathrm{d}t. \quad (3.17)$$

We now discuss the two cases from Assumption *(A3)*(ii), separately, i.e., $\gamma_1 = \gamma_2 = 0$ or $0 < \gamma_1 \leq \gamma_2$.

Case I: $\gamma_1 = \gamma_2 = 0$. To estimate the first integral on the right-hand side of (3.17), we use that $\nabla\overline{\chi}_{\eta,\tau} \in \overline{\mathsf{F}_R}$ and Assumptions *(A2)*, *(A3)*(ii) to obtain

$$\int_0^T \int_\Omega \partial^2_{cc}\Phi(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})^2\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\nabla\overline{c}_{\eta,\tau} \cdot \nabla\overline{c}_{\eta,\tau} \,\mathrm{d}x\,\mathrm{d}t$$
$$\geq C\int_0^T \int_\Omega \overline{c}_{\eta,\tau}^{m-2}|\nabla\overline{c}_{\eta,\tau}|^2 \,\mathrm{d}x\,\mathrm{d}t \geq C\big\|\nabla\overline{c}_{\eta,\tau}^{\frac{m}{2}}\big\|_{L^2(0,T;L^2(\Omega))}. \quad (3.18)$$

For the second integral on the right-hand side of (3.17), we note that by **(A3)**(iii), Hölder's inequality, and the uniform bound for $\mathrm{D}^2\overline{\chi}_{\eta,\tau}$ in $L^\infty(0,T;L^p(\Omega;\mathbb{R}^{d\times d\times d}))$

$$\int_0^T\int_\Omega |\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})|\big|\partial^2_{Fc}\Phi(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\mathrm{D}^2\overline{\chi}_{\eta,\tau}\big|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\le \int_0^T\int_\Omega \overline{c}_{\eta,\tau}^{m+2\alpha}|\mathrm{D}^2\overline{\chi}_{\eta,\tau}|^2\,\mathrm{d}x\,\mathrm{d}t \le C\big\|\overline{c}_{\eta,\tau}^{m+2\alpha}\big\|_{L^1(0,T;L^{\frac{p}{p-2}}(\Omega))}.$$

We now distinguish two cases: First, if $0\le \frac{p}{p-2}(m+2\alpha)\le 1$, a uniform upper estimate for the right-hand side follows immediately from the $L^\infty(0,T;L^1(\Omega))$-bound for $\overline{c}_{\eta,\tau}$ (cf. (3.13c)). For the second case, when $\frac{p}{p-2}(m+2\alpha)>1$, we note that $\frac{p}{p-2}\le \frac{m}{2}\frac{2d}{d-2}\frac{1}{m+2\alpha}=:\omega_2\in(1,\infty)$, which follows from Assumption **(A3)**(iii) and $p>d$. Since $\alpha<0$ in this case, we also have that $1<\frac{m}{2}\frac{2}{m+2\alpha}=:\omega_1$. Thus, we now find that

$$\big\|\overline{c}_{\eta,\tau}^{m+2\alpha}\big\|_{L^1(0,T;L^{\frac{p}{p-2}}(\Omega))}\le C\big\|\overline{c}_{\eta,\tau}^{m+2\alpha}\big\|_{L^{\omega_1}(0,T;L^{\omega_2}(\Omega))}=C\big\|\overline{c}_{\eta,\tau}^{\frac{m}{2}}\big\|^{2/\omega_1}_{L^2(0,T;L^{\frac{2d}{d-2}}(\Omega))}.$$

Next, we note that $\frac{2}{\omega_1}<2$, and thus we can use Young's inequality with $\epsilon>0$ to obtain

$$\big\|\overline{c}_{\eta,\tau}^{m+2\alpha}\big\|_{L^1(0,T;L^{\frac{p}{p-2}}(\Omega))}\le C(\epsilon)+\epsilon C\big\|\overline{c}_{\eta,\tau}^{\frac{m}{2}}\big\|^2_{L^2(0,T;L^{\frac{2d}{d-2}}(\Omega))}.$$

Using the Sobolev embedding $H^1(\Omega)\hookrightarrow L^{\frac{2d}{d-2}}(\Omega)$, we obtain

$$\int_0^T\int_\Omega |\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})|\big|\partial^2_{Fc}\Phi(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\mathrm{D}^2\overline{\chi}_{\eta,\tau}\big|^2\,\mathrm{d}x\,\mathrm{d}t\le C(\epsilon)+\epsilon C\big\|\overline{c}_{\eta,\tau}^{\frac{m}{2}}\big\|^2_{L^2(0,T;H^1(\Omega))}.$$

Finally, we note that $\overline{c}_{\eta,\tau}^{\frac{m}{2}}$ is uniformly bounded in $L^2(0,T;L^1(\Omega))$ due to the boundedness of the energy and $m\le 2$. Thus we can use a Poincaré-type inequality to find

$$\int_0^T\int_\Omega |\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})|\big|\partial^2_{Fc}\Phi(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\mathrm{D}^2\overline{\chi}_{\eta,\tau}\big|^2\,\mathrm{d}x\,\mathrm{d}t\le C(\epsilon)+\epsilon C\big\|\nabla\overline{c}_{\eta,\tau}^{\frac{m}{2}}\big\|^2_{L^2(0,T;L^2(\Omega))}.$$
$$(3.19)$$

Choosing $\epsilon>0$ sufficiently small, we can combine (3.17), (3.18), and (3.19) to obtain

$$\int_0^T\int_\Omega \mathcal{M}(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\nabla\overline{\mu}_{\eta,\tau}\cdot\nabla\overline{\mu}_{\eta,\tau}\,\mathrm{d}x\,\mathrm{d}t\ge C\big(\big\|\nabla\overline{c}_{\eta,\tau}^{\frac{m}{2}}\big\|^2_{L^2(0,T;L^2(\Omega))}-1\big).$$

Thus, the energy-dissipation inequality (3.9) now gives the uniform bound for $\nabla\overline{c}_{\eta,\tau}^{\frac{m}{2}}$ in $L^2(0,T;L^2(\Omega))$.

Case II: $\gamma_2\ge\gamma_1>0$. To estimate the first integral, we again use that $\nabla\overline{\chi}_{\eta,\tau}\in\mathsf{F}_R$ and Assumptions **(A2)**, **(A3)**(ii) to obtain

$$\int_0^T\int_\Omega \partial^2_{cc}\Phi(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})^2\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau},\overline{c}_{\eta,\tau})\nabla\overline{c}_{\eta,\tau}\cdot\nabla\overline{c}_{\eta,\tau}\,\mathrm{d}x\,\mathrm{d}t$$

$$\ge C\int_0^T\int_\Omega \overline{c}_{\eta,\tau}^m(\overline{c}_{\eta,\tau}^{-1}+\overline{c}_{\eta,\tau}^r)^2|\nabla\overline{c}_{\eta,\tau}|^2\,\mathrm{d}x\,\mathrm{d}t\ge C\sum_{i\in\{0,\frac{1+r}{2},1+r\}}\big\|\nabla c_{\eta,\tau}^{\frac{m}{2}+i}\big\|^2_{L^2(0,T;L^2(\Omega))}.\quad(3.20)$$

For the second integral, we proceed as in Case I to obtain

$$\int_0^T \int_\Omega |\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})| |\partial_{Fc}^2 \Phi(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau}) D^2\overline{\chi}_{\eta,\tau}|^2 \, dx \, dt \leq C \|\overline{c}_{\eta,\tau}^{m+2\alpha}\|_{L^1(0,T;L^{\frac{p}{p-2}}(\Omega))}.$$

Again, we consider two cases: First, if $0 \leq \frac{p}{p-2}(m+2\alpha) \leq 2+r$ it follows from the $L^\infty(0,T;L^{2+r}(\Omega))$-bound for $\overline{c}_{\eta,\tau}$ that the left-hand side is uniformly bounded. If $\frac{p}{p-2}(m+2\alpha) > 2+r$, we define ω_0 by $\omega_0 = \frac{m+1+r}{2}$ in Case IIa, and $\omega_0 = \frac{m}{2}+1+r$ in Case IIb. Next, we note that $\frac{p}{p-2} \leq \frac{2d}{d-2}\frac{\omega_0}{m+2\alpha} =: \omega_2$, which follows from Assumption *(A3)*(iii) and $p > d$. Also, $1 < \frac{2\omega_0}{m+2\alpha} =: \omega_1$ so that

$$\|\overline{c}_{\eta,\tau}^{m+2\alpha}\|_{L^1(0,T;L^{\frac{p}{p-2}}(\Omega))} \leq C\|\overline{c}_{\eta,\tau}^{m+2\alpha}\|_{L^{\omega_1}(0,T;L^{\omega_2}(\Omega))} \leq C\|\overline{c}_{\eta,\tau}^{\omega_0}\|_{L^2(0,T;L^{\frac{2d}{d-2}}(\Omega))}^{2/\omega_1}$$

Since $\frac{2}{\omega_1} < 2$ we can proceed as in Case I and use Young's inequality with ε, the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{\frac{2d}{d-2}}(\Omega)$ and Poincaré's inequality (which is applicable since $\overline{c}_{\eta,\tau}^{\omega_0} \in L^2(0,T;L^1(\Omega))$ as $m \leq 3+r$ (Case IIa) or $m \leq 2$ (Case IIb)) to obtain

$$\int_0^T \int_\Omega |\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})| |\partial_{Fc}^2 \Phi(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau}) D^2\overline{\chi}_{\eta,\tau}|^2 \, dx \, dt \leq C(\epsilon) + \epsilon C \|\nabla \overline{c}_{\eta,\tau}^{\omega_0}\|_{L^2(0,T;L^2(\Omega))}^2.$$
$$(3.21)$$

Thus, choosing ε sufficiently small and combining (3.17), (3.20) and (3.21), we obtain

$$\int_0^T \int_\Omega \mathcal{M}(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau}) \nabla\overline{\mu}_{\eta,\tau} \cdot \nabla\overline{\mu}_{\eta,\tau} \, dx \, dt \geq C \Big(\sum_{i \in \{0, \frac{1+r}{2}, 1+r\}} \|\nabla c_{\eta,\tau}^{\frac{m}{2}+i}\|_{L^2(0,T;L^2(\Omega))}^2 - 1 \Big).$$

The energy-dissipation inequality (3.9) now gives the uniform bound for $\nabla\overline{c}_{\eta,\tau}^{\frac{m}{2}}$, $\nabla\overline{c}_{\eta,\tau}^{\frac{m+1+r}{2}}$ and $\nabla\overline{c}_{\eta,\tau}^{\frac{m}{2}+1+r}$ in $L^2(0,T;L^2(\Omega))$.

Step 5. We show that the flux $\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})\nabla\overline{\mu}_{\eta,\tau}$ is uniformly bounded in $L^s([0,T]\times\Omega)$ (cf. (3.13d)). We distinguish again between the Cases I and II in Assumption *(A3)*(iii).

Case I: $\gamma_1 = \gamma_2 = 0$. We estimate for $1 < s = \frac{md+2}{md+1} < 2$

$$\|\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})\nabla\overline{\mu}_{\eta,\tau}\|_{L^s([0,T]\times\Omega)}^s \leq C \int_0^T \int_\Omega (\overline{c}_{\eta,\tau}^m |\partial_{Fc}^2\Phi D^2\overline{\chi}_{\eta,\tau} + \partial_{cc}^2\Phi\nabla\overline{c}_{\eta,\tau}|)^s \, dx \, dt$$

$$\leq C \int_0^T \int_\Omega (\overline{c}_{\eta,\tau}^{s(m+\alpha)} |D^2\overline{\chi}_{\eta,\tau}|^s + \overline{c}_{\eta,\tau}^{s(m-1)} |\nabla\overline{c}_{\eta,\tau}|^s) \, dx \, dt$$

$$\leq C \int_0^T \big(\|\overline{c}_{\eta,\tau}^{(m+\alpha)}\|_{L^{\frac{ps}{p-s}}(\Omega)}^s \|D^2\overline{\chi}_{\eta,\tau}\|_{L^p(\Omega;\mathbb{R}^{d\times d\times d})}^s$$
$$+ \|\overline{c}_{\eta,\tau}^{\frac{m}{2}}\|_{L^{\frac{2s}{2-s}}(\Omega)}^s \|\nabla\overline{c}_{\eta,\tau}^{\frac{m}{2}}\|_{L^2(\Omega)}^s \big) \, dt,$$

where in the first and second inequality we used Assumptions *(A2)* and *(A3)*, respectively, and in the third inequality, we used Hölder's inequality. We now consider the two terms on the right-hand side separately. For the first term, the boundedness follows from the condition $0 \leq m + \alpha \leq \frac{p-s}{ps}$, i.e., $(m+\alpha)\frac{ps}{p-s} \leq 1$ (see Assumption *(A3)*(iii)), and the $L^\infty(0,T;L^1(\Omega))$-bound for $\overline{c}_{\eta,\tau}$ and $L^\infty(0,T;L^p(\Omega;\mathbb{R}^{d\times d\times d}))$-bound for $D^2\overline{\chi}_{\eta,\tau}$.

For the second term, we use the Gagliardo-Nirenberg-Sobolev inequality to bound

$$\|\overline{c}_{\eta,\tau}^{\frac{m}{2}}\|_{L^{\frac{2s}{2-s}}(\Omega)} \leq C\big(\|\nabla\overline{c}_{\eta,\tau}^{\frac{m}{2}}\|_{L^2(\Omega)}^\lambda \|\overline{c}_{\eta,\tau}^{\frac{m}{2}}\|_{L^{\frac{2}{m}}(\Omega)}^{1-\lambda} + \|\overline{c}_{\eta,\tau}^{\frac{m}{2}}\|_{L^1(\Omega)} \big)$$

$$\leq C\big(1 + \|\nabla\overline{c}_{\eta,\tau}^{\frac{m}{2}}\|_{L^2(\Omega)}^\lambda \big)$$

for almost every $t \in [0,T]$. Here, $\lambda \in (0,1)$ is defined via $\frac{2-s}{2s} = \lambda\frac{d-2}{2d} + (1-\lambda)\frac{m}{2}$, or using that $s = \frac{md+2}{md+1}$, via $s(1+\lambda) = 2$, and we have used that $\bar{c}_{\eta,\tau}$ is bounded in $L^\infty(0,T;L^1(\Omega))$. Thus, using that $1 < s < 2$, we obtain

$$\int_0^T \big\|\bar{c}_{\eta,\tau}^{\frac{m}{2}}\big\|^s_{L^{\frac{2}{2-s}}(\Omega)} \big\|\nabla\bar{c}_{\eta,\tau}^{\frac{m}{2}}\big\|^s_{L^2(\Omega)} \,\mathrm{d}t \le C \int_0^T \big\|\nabla\bar{c}_{\eta,\tau}^{\frac{m}{2}}\big\|^s_{L^2(\Omega)} + \big\|\nabla\bar{c}_{\eta,\tau}^{\frac{m}{2}}\big\|^{s(1+\lambda)}_{L^2(\Omega)} \,\mathrm{d}t$$
$$\le C\big(1 + \big\|\nabla\bar{c}_{\eta,\tau}^{\frac{m}{2}}\big\|^2_{L^2(0,T;L^2(\Omega))}\big).$$

Using the bounds in (3.13c), the bound in (3.13d) follows.

Case II: $\gamma_2 \ge \gamma_1 > 0$. To prove the bound for the flux in (3.13f), we estimate for the exponent $1 < s = \min\{\frac{md+2(r+2)}{md+r+2}, \frac{d(m+r+1)+2(r+2)}{d(m+r+1)+r+2}\} < 2$

$$\|\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau}, \bar{c}_{\eta,\tau})\nabla\overline{\mu}_{\eta,\tau}\|^s_{L^s([0,T]\times\Omega)} \le C\int_0^T \int_\Omega \big(\bar{c}_{\eta,\tau}^m |\partial^2_{Fc}\Phi \mathrm{D}^2\overline{\chi}_{\eta,\tau} + \partial^2_{cc}\Phi\nabla\bar{c}_{\eta,\tau}|\big)^s \,\mathrm{d}x\,\mathrm{d}t$$

$$\le C\int_0^T \int_\Omega \big(\bar{c}_{\eta,\tau}^{s(m+\alpha)} |\mathrm{D}^2\overline{\chi}_{\eta,\tau}|^s + (\bar{c}_{\eta,\tau}^{s(m-1)} + \bar{c}_{\eta,\tau}^{s(m+r)})|\nabla\bar{c}_{\eta,\tau}|^s\big) \,\mathrm{d}x\,\mathrm{d}t$$

$$\le C\int_0^T \big(\|\bar{c}_{\eta,\tau}^{(m+\alpha)}\|^s_{L^{\frac{ps}{p-s}}(\Omega)} \|\mathrm{D}^2\overline{\chi}_{\eta,\tau}\|^s_{L^p(\Omega;\mathbb{R}^{d\times d\times d})}$$

$$+ \big\|\bar{c}_{\eta,\tau}^{\frac{m}{2}}\big\|^s_{L^{\frac{2s}{2-s}}(\Omega)} \big\|\nabla\bar{c}_{\eta,\tau}^{\frac{m}{2}}\big\|^s_{L^2(\Omega)} + \big\|\bar{c}_{\eta,\tau}^{\frac{m+r+1}{2}}\big\|^s_{L^{\frac{2s}{2-s}}(\Omega)} \big\|\nabla\bar{c}^{\frac{m+r+1}{2}}\big\|^s_{L^2(\Omega)}\big) \,\mathrm{d}t,$$

where in the first and second inequality we used Assumptions *(A2)* and *(A3)*, and in the second and third inequality we used Hölder's inequality. Again, we look at the integrals separately. For the first integral, the boundedness follows from the condition $(m+\alpha)\frac{ps}{p-s} \le 2+r$ (cf. Assumption *(A3)*(iii). For the second integral, we argue as before, now using the bound $m \le 4+2r$ (which directly follows from $m \le 3+r$ and $r \ge -1$). Taking $s = \frac{md+2(r+2)}{md+r+2} > 1$ the boundedness follows. The boundedness of the third integral follows from $m \le 3+r$ and $s = \frac{d(m+r+1)+2(r+2)}{d(m+r+1)+r+2} > 1$.

Step 6. To prove the final bound of (3.13c) for the time derivative $\dot{\bar{c}}_{\eta,\tau}$, we now test (3.1b*) with $\psi \in L^{s'}(0,T;H^\theta(\Omega))$ and use Assumption *(A7)* to find

$$\int_0^T \langle\dot{\bar{c}}_{\eta,\tau}, \psi\rangle \,\mathrm{d}t \le C\|\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau}, \bar{c}_{\eta,\tau})\nabla\overline{\mu}_{\eta,\tau}\|_{L^s([0,T]\times\Omega)} \|\psi\|_{L^{s'}(0,T;W^{1,s'}(\Omega))}$$

$$+ C\eta\|\overline{\mu}_{\eta,\tau}\|_{L^2(0,T;H^\theta(\Omega))} \|\psi\|_{L^2(0,T;H^\theta(\Omega))}$$

$$+ C\big(\|\overline{\mu}_{\eta,\tau} - \overline{\mu}_{\tau,\mathrm{ext}}\|_{L^2([0,T]\times\partial\Omega)}\big) \|\psi\|_{L^2([0,T]\times\partial\Omega)}.$$

Thus, using the previously obtained uniform bounds, we conclude that

$$\int_0^T \langle\dot{\bar{c}}_{\eta,\tau}, \psi\rangle \,\mathrm{d}t \le C(\|\psi\|_{L^{s'}(0,T;W^{1,s'}(\Omega))} + \sqrt{\eta}\|\psi\|_{L^2(0,T;H^\theta(\Omega))}), \tag{3.22}$$

for all $\psi \in L^{s'}(0,T;H^\theta(\Omega))$ so that $\|\dot{\bar{c}}_{\eta,\tau}\|_{L^s(0,T;H^\theta(\Omega)^*)} \le C$, finishing the proof of (3.13c). $\quad\square$

4 Limit passage $(\tau,\eta) \to 0$

We now pass to the limit $(\tau,\eta) \to 0$. Note that to obtain strong convergence of the concentration c_η, we cannot directly use the Aubin–Lions lemma since we do not have a bound for $\nabla\bar{c}_{\eta,\tau}$, but instead for

$\nabla \bar{c}_{\tau,\eta}^{\frac{m}{2}}$, and, if $\gamma_1 > 0$, also for $\nabla \bar{c}_{\eta,\tau}^{\frac{m}{2}+i}$ ($i \in \{\frac{1+r}{2}, 1+r\}$). Thus, we use a generalization, Dubinskiĭ's theorem, cf. Theorem A.3.

Proposition 4.1 (Limit passage $(\tau, \eta) \to 0$). *There exist subsequences (not relabeled) such that the interpolants $(\widehat{\chi}_\tau, \widehat{c}_\tau, \overline{\mu}_\tau)$ converge to a weak solution (χ, c, μ) of (2.2)–(2.4) in the sense of Definition 2.5).*

Proof. We show the limit passage for Case I, i.e., $\gamma_1 = \gamma_2 = 0$. The limit passage for Case II, i.e., $\gamma_2 \geq \gamma_1 > 0$ follows in a similar way.

Step 1. Using the a priori estimates from Lemma 3.5, we can extract converging subsequences (not relabeled) and some (χ, c, μ, Θ) such that

$$
\begin{aligned}
\widehat{\chi}_{\eta,\tau} &\xrightarrow{w^*} \chi & &\text{in } L^\infty(0,T; W_{\mathrm{id}}^{2,p}(\Omega; \mathbb{R}^d)) \cap H^1(0,T; H^1(\Omega; \mathbb{R}^d)), \\
\widehat{c}_{\eta,\tau} &\xrightarrow{w^*} c & &\text{in } L^\infty(0,T; L\log L(\Omega)) \cap W^{1,s}(0,T; H^\theta(\Omega)^*), \\
\overline{\mu}_{\eta,\tau} &\xrightarrow{w} \mu_* & &\text{in } L^2([0,T] \times \partial\Omega), \\
\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})\nabla\overline{\mu}_{\eta,\tau} &\xrightarrow{w} \Theta & &\text{in } L^s([0,T] \times \Omega).
\end{aligned}
$$

Using the Aubin–Lions lemma, we can extract a strongly converging subsequence (not relabeled) such that

$$
\widehat{\chi}_{\eta,\tau} \xrightarrow{s} \chi \text{ in } C(0,T; C^{1,\lambda}(\overline{\Omega}; \mathbb{R}^d)), \quad \text{with } \lambda = 1 - \frac{d}{p},
$$

Next, we note that since $\bar{c}_{\eta,\tau}^{\frac{m}{2}} \in L^2(0,T; L^1(\Omega))$ (as $1 \leq m \leq 2$), it follows from Poincaré's inequality that $\bar{c}_{\eta,\tau}^{\frac{m}{2}} \in L^2(0,T; H^1(\Omega))$. Thus, we can apply Corollary A.4 of Dubinskiĭ's theorem to obtain a strongly converging subsequence (not relabeled)

$$
\widehat{c}_{\eta,\tau} \xrightarrow{s} c \quad \text{in } L^m(0,T; L^m(\Omega)).
$$

Note that by Remark 3.3 these convergences also hold for the piecewise constant interpolants $\overline{\chi}_{\eta,\tau}$, $\underline{\chi}_{\eta,\tau}$, $\overline{c}_{\eta,\tau}$ and $\underline{c}_{\eta,\tau}$.

Consequently, we also have that

$$
\nabla\bar{c}_{\eta,\tau}^{\frac{m}{2}} \xrightarrow{w} \nabla c^{\frac{m}{2}} \quad \text{in } L^2(0,T; L^2(\Omega)).
$$

Step 2. We now pass to the limit $\tau \to 0$ in the time-discrete mechanical equation in (3.8a). Using the above convergences and the continuity of $\sigma_{\mathsf{el}} : \mathsf{F}_R \times \mathbb{R}^+ \to \mathbb{R}^{d \times d}$ (with $R > 0$ as in Step 3 of the proof of the a priori estimates in Lemma 3.5), the limit in the first term on the left-hand side follows from the dominated convergence theorem. Indeed, using the $L^\infty(0,T; L^\infty(\Omega; \mathbb{R}^{d \times d}))$-bound for $\nabla\overline{\chi}_{\eta,\tau}$ and Assumption **(A3)**(iii), we obtain $|\sigma_{\mathsf{el}}(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})| \leq C(1 + \overline{c}_{\eta,\tau}^{1+\alpha})$, which is integrable since $0 \leq 1 + \alpha \leq 1$ (Case I), or since $0 \leq 1 + \alpha \leq 2 + r$ (Case II). Using the linearity of $\sigma_{\mathsf{vi}}(F, \dot{F}, c)$ with respect to $\dot{F}$, we can also directly pass to the limit in the second term. Recall that by Assumption **(A5)** the tensor $\widetilde{\mathbb{D}}$ is uniformly bounded, hence, the dominated convergence theorem can be applied again. The limit in the loading term on the right-hand side follows directly from the fact that $\overline{\ell}_\tau \to \ell$ in $L^2(0,T; H^1(\Omega)^*)$.

Finally, the limit passage in the hyperstress term follows from the strong-weak closedness of the convex subdifferential of $\mathcal{A} : X \to [0, \infty]$ defined by

$$
\mathcal{A}(\chi) = \begin{cases} \int_0^T \int_\Omega \mathscr{H}(\mathrm{D}^2\chi) \, \mathrm{d}x \, \mathrm{d}t & \text{if } \chi \in L^\infty(0,T; W^{2,p}(\Omega; \mathbb{R}^d)), \\ +\infty & \text{otherwise,} \end{cases}
$$

see Lemma A.1.

Step 3. To pass to the limit $\tau \to 0$ in the diffusion equation (3.8b), we note that the first term on the left-hand side converges by the weak convergence of $\dot{\widehat{c}}_{\eta,\tau}$ in $L^2(0,T;H^\theta(\Omega)^*)$. The third term vanishes in the limit $\eta \to 0$ using the first estimate of (3.13b). The fourth term converges using the weak convergences of $\overline{\mu}_{\eta,\tau}$ in $L^2([0,T] \times \partial\Omega)$. Finally, it remains to pass to the limit in the second integral, i.e., to show that

$$\int_0^T \int_\Omega \mathcal{M}(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})\nabla\overline{\mu}_{\eta,\tau} \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \mathcal{M}(\nabla\chi, c)\nabla\mu \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}t$$

for all $\psi \in L^{s'}(0,T;H^\theta(\Omega))$. Note that we have (up-to a subsequence) $\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})\nabla\overline{\mu}_\tau \overset{w}{\rightharpoonup} \Theta$ in $L^s([0,T] \times \Omega))$ by Step 1.

Using the strong convergences of $\nabla\overline{\chi}_{\eta,\tau}$ and $\overline{c}_{\eta,\tau}$ we can assume that $\nabla\overline{\chi}_{\eta,\tau}(t,x) \to \nabla\chi(t,x)$ in $\mathrm{GL}^+(d)$ and $\overline{c}_{\eta,\tau}(t,x) \to c(t,x)$ for almost every $(t,x) \in [0,T] \times \Omega$.

To pass to the limit, we now introduce the rescaled mobility tensor $\widetilde{\mathcal{M}}$ by defining $\widetilde{\mathcal{M}}(F,c) = c^m\mathcal{M}(F,c)$ for all $F \in \mathsf{F}_R$ (some $R > 0$) and $c > 0$, and the rescaled derivatives of free energy A and B defined by $c^\alpha A(F,c) = \partial^2_{Fc}\Phi(F,c)$ and $B(F,c) = c\partial^2_{cc}\Phi(F,c)$ for all $F \in \mathrm{GL}^+(d)$ and $c > 0$. In particular, note that by Assumption *(A2)* we have $|\widetilde{\mathcal{M}}| \leq C$ and by Assumption *(A3)* that $|A|, |B| \leq C$ for all F and $c > 0$. Then, it follows that

$$\int_0^T \int_\Omega \mathcal{M}(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})\nabla\overline{\mu}_{\eta,\tau} \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_\Omega \widetilde{\mathcal{M}}(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})\big(A(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})c_{\eta,\tau}^{m+\alpha}\mathrm{D}^2\overline{\chi}_{\eta,\tau} + \overline{c}_{\eta,\tau}^{\frac{m}{2}+1}B(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})\nabla\overline{c}_{\eta,\tau}^{\frac{m}{2}}\big) \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}t.$$

Since continuous functions preserve almost everywhere convergence, the limit passage follows by applying Lemma A.5 (therein, stated for the scalar case with the extension to the vectorial case being straightforward) with

$$\mathfrak{a}_{\eta,\tau} = \widetilde{\mathcal{M}}(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})A(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})c_{\eta,\tau}^{m+\alpha} \quad \text{and} \quad \mathcal{V}_{\eta,\tau} = \mathrm{D}^2\overline{\chi}_{\eta,\tau}$$

and

$$\mathfrak{a}_{\eta,\tau} = \widetilde{\mathcal{M}}(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})\overline{c}_{\eta,\tau}^{\frac{m}{2}+1}B(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})\nabla\overline{c}_{\eta,\tau}^{\frac{m}{2}} \quad \text{and} \quad \mathcal{V}_{\eta,\tau} = \nabla\overline{c}_{\eta,\tau}^{\frac{m}{2}}$$

for the first and second integral, respectively. Note, in particular, that this uses the $L^s([0,T] \times \Omega)$-bound for the flux $\mathcal{M}(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau})\nabla\overline{\mu}_{\eta,\tau}$.

The limit passage in the terms arising when $\gamma_2 \geq \gamma_1 > 0$ follows in a similar way, and is therefore omitted.

Step 4. It remains to show that $\mu_* \in \partial_c\Phi(\nabla\chi, c)$. Note that this is not immediately clear since we do not have an $L^2(0,T;H^1(\Omega))$ estimate for μ_η (as is the case for nondegenerate mobilities), and thus cannot use the trace operator $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ to conclude that this μ_* satisfies $\mu_* \in \partial_c\Phi(\nabla\chi, c)$.

Instead we use that $\nabla\widehat{\chi}_{\eta,\tau} \overset{s}{\to} \chi$ in $C(0,T;C(\overline{\Omega};\mathbb{R}^{d\times d}))$ to extract a subsequence (not relabeled) such that $\widehat{\chi}_{\eta,\tau}(t,x) \to \chi(t,x)$ in $\mathrm{GL}^+(d)$ for almost every $(t,x) \in [0,T] \times \partial\Omega$. Using that $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ (see e.g. [Bie09, Theorem 1.1]), we thus obtain using Dubinskiĭ's theorem a strongly converging subsequence (not relabeled) such that $\overline{c}_{\eta,\tau} \overset{s}{\to} c$ in $L^m([0,T] \times \partial\Omega)$. In particular, we can extract another subsequence sucht that $\overline{c}_{\eta,\tau}(t,x) \to c(t,x)$ for almost every

$(t, x) \in [0, T] \times \partial\Omega$. Using the Banach–Saks theorem, we now extract a subsequence (η_n, τ_n) such that $\frac{1}{N}\sum_{n=1}^{N} \mu_{\eta_n, \tau_n} \xrightarrow{s} \mu_*$ in $L^2([0, T] \times \partial\Omega)$. Again extracting a converging (non-relabeled) subsequence, we thus find that $\frac{1}{N}\sum_{n=1}^{N} \mu_{\eta_n, \tau_n}(t, x) \to \mu_*(t, x)$ for all $(t, x) \in [0, T] \times \partial\Omega$. Since continuity of $\partial_c\Phi$ and the almost everywhere convergence of $\nabla\overline{\chi}_{\eta,\tau}$ and $\overline{c}_{\eta,\tau}$ also imply that $\mu_{\eta,\tau} = \partial_c\Phi(\nabla\overline{\chi}_{\eta,\tau}, \overline{c}_{\eta,\tau}) \to \partial_c\Phi(\nabla\chi, c)$ for almost every $(t, x) \in [0, T] \times \partial\Omega$, we thus conclude that $\mu_* \in \partial_c\Phi(\nabla\chi, c)$.

$\square$

A   Tools

Lemma A.1 (Strong-weak closedness of the convex subdifferential)**.** *Let X be a reflexive Banach space, and let $\mathcal{A} : X \to \mathbb{R} \cup \infty$ be convex and lower semicontinuous. If $u_n \xrightarrow{s} u$ in X and $\xi_n \in \partial\mathcal{A}(u_n) \subset X^*$ such that $\xi_n \xrightarrow{w} \xi$ in X^*, then $\xi \in \partial\mathcal{A}(u)$.*

Proof. Using convexity of $\mathcal{A}$, we obtain $\mathcal{A}(v) \geq \mathcal{A}(u_n) + \langle \xi_n, v - u_n \rangle$ for all $v \in X^*$. The result now follows using by taking the $\liminf$ of this inequality, and using the lower semicontinuity and given convergences. $\square$

Here we state Dubinskiĭ's theorem [Dub65] in the form of [BS12].

Definition A.2. Let A be a Banach space. Then, $\mathcal{M}_+$ is a **seminormed nonnegative cone** in A if $\mathcal{M}_+ \subseteq A$ satisfies:

(i)   For all $u \in \mathcal{M}_+, \gamma \geq 0$, we have $\gamma u \in \mathcal{M}_+$.

(ii)   There exists a function $[\cdot] : \mathcal{M}_+ \to [0, \infty)$ (the semi-norm) such that $[u] = 0$ if and only if $u = 0$, and for $\gamma \geq 0$ we have $[\gamma u] = \gamma[u]$.

We say that $\mathcal{M}_+ \hookrightarrow A$ continuously if there exists $C > 0$ such that $\|u\|_A \leq C[u]$ for all $u \in \mathcal{M}_+$, and we say that $\mathcal{M}_+ \hookrightarrow A$ compactly if every bounded sequence in $\mathcal{M}_+$ has a converging subsequence in A.

Theorem A.3 (Dubinskiĭ)**.** *Let $\mathcal{M}_+$ be a seminormed nonnegative cone, and A_0, A_1 Banach spaces such that $\mathcal{M}_+ \hookrightarrow A_0$ compactly, and $A_0 \hookrightarrow A_1$ continuously. Let*

$$\mathcal{Y}_+ = \{\varphi : [0, T] \to \mathcal{M}_+ \mid [\varphi]_{\mathcal{Y}_+} := [\varphi]_{L^p(0,T;\mathcal{M}_+)} + \|\dot{\varphi}\|_{L^{p_1}(0,T;A_1)} \leq C\}$$

for $1 \leq p, p_1 \leq \infty$. Then, $\mathcal{Y}_+$ is a seminormed nonnegative cone in $L^p(0, T; A_0) \cap W^{1,p_1}(0, T; A_1)$, and $\mathcal{Y}_+ \hookrightarrow L^p(0, T; A_0)$ compactly if either $p_1 > 1$ or $p < \infty$.

Corollary A.4. *Let $\omega \geq \frac{1}{2}, \theta > \frac{d}{2}$, and let (φ_k) be a sequence of nonnegative functions on $[0, T] \times \Omega$ such that there exists a constant $C > 0$ for which*

$$\|(\varphi_k)^\omega\|_{L^2(0,T;H^1(\Omega))} + \|\dot{\varphi}_k\|_{L^2(0,T;H^\theta(\Omega)^*)} \leq C.$$

Then, (φ_k) is relatively compact in $L^{2\omega}(0, T; L^{2\omega}(\Omega))$.

Proof. The proof is given in the time-discrete case in [CJL14]. In our case, we just apply Theorem A.3 with $\mathcal{M}_+ = \{\varphi \mid \varphi^\omega \in H^1(\Omega)\}$, $A_0 = L^{2\omega}(\Omega)$, $A_1 = H^\theta(\Omega)^*$, and the seminorm

$$[\varphi]_{\mathcal{Y}_+} = \|\varphi^\omega\|_{L^2(0,T;H^1(\Omega))}^{\frac{1}{\omega}} + \|\dot{\varphi}\|_{L^2(0,T;H^\theta(\Omega)^*)}.$$

$\square$

Lemma A.5 ([Fis+22, Lemma A.3]). *Assume that* $\mathfrak{a}_n, \mathcal{V}_n : Q \to \mathbb{R}$, $n \in \mathbb{N}$ *are measurable functions in* $Q \subset \mathbb{R}^m$ *(open, bounded) such that* $\mathcal{V}_n \overset{w}{\rightharpoonup} \mathcal{V}$ *in* $L^1(Q)$, $\mathfrak{a}_n(x) \to \mathfrak{a}(x)$ *for almost every* $x \in Q$, *and* $\sup_{n \in \mathbb{N}} \|\mathfrak{a}_n \mathcal{V}_n\|_{L^{1+\sigma}(Q)} < \infty$ *for some* $\sigma > 0$. *Then* $\mathfrak{a}_n \mathcal{V}_n \overset{w}{\rightharpoonup} \mathfrak{a}\mathcal{V}$ *in* $L^{1+\sigma}(Q)$.

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