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Korovkin-type approximation of set-valued functions with convex graphs

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Abstract

Different Korovkin type results have been obtained in cones of set-valued continuous functions. Here we show that if we consider the subcone of set-valued continuous functions having a convex graph, then we can consider a Korovkin system which contains only affine functions. In this way we give a non trivial example where the number of functions to be used in a Korovkin system can be reduced.

1 Introduction and notation

The problem of considering the smallest number of functions in a Korovkin system has been considered for a long time (see e.g. [1]) and it is well-known that in many cases this number cannot be reduced.

For example, the classical Korovkin theorem states the set $K := \{\mathbf{1}, \mathrm{id}, \mathrm{id}^2\}$ is a Korovkin system in $C[0,1]$, that is we have the strong convergence of a sequence of positive linear operators $(L_n)_{n\geq 1}$ to the identity operator on $C([0,1])$ if and only if

$$\lim_{n\to+\infty} L_n(\mathbf{1}) = \mathbf{1}\,, \qquad \lim_{n\to+\infty} L_n(\mathrm{id}) = \mathrm{id}\,, \qquad \lim_{n\to+\infty} L_n(\mathrm{id}^2) = \mathrm{id}^2$$

uniformly on $[0,1]$. Many different examples of Korovkin systems have been obtained and it is well known that the number of their elements cannot be less than three.

Many extensions of the Korovkin system have been obtained in more general settings (see. e.g., [1, 2]) and also for cones of set-valued continuous functions, obtained starting from the pioneristic work by Keimel and Roth [13, 14], and continued until today [3, 12].

In many cases it has been shown that a Korovkin system must have a minimum number of elements which cannot be reduced. However, we can ask if it is enough to consider less elements in a Korovkin system by requiring the convergence property only on a subspace rather than on the whole space.

This question has only trivial answers for example in the space $C([0,1])$; if we consider only the two functions $\mathbf{1}, \mathrm{id}$ and a positive linear operator $L : C([0,1]) \to C([0,1])$ satisfying $L(\mathbf{1}) = \mathbf{1}$ and $L(\mathrm{id}) = \mathrm{id}$, then it is not possible to predict its value at any non linear function.

On the contrary, in the context of cones of set-valued continuous functions, we may have some non trivial results and our aim is to show that even in the real case we can obtain some interesting results.

First, we shall denote by $\mathcal{K}(\mathbb{R})$ the cone of all non-empty compact convex subsets (i.e., non-empty closed bounded intervals) of $\mathbb{R}$ endowed with the natural addition and multiplication by positive scalars; in $\mathcal{K}(\mathbb{R})$ it is defined the Hausdorff distance

$$d_H(A,B) := \max\left\{\sup_{x\in A}\inf_{y\in B}|x-y|, \sup_{y\in B}\inf_{x\in A}|x-y|\right\}.$$

Moreover, if $[a,b]$ is a real interval, we recall that a function $F \in \mathcal{F}([a,b],\mathcal{K}(\mathbb{R}))$ is Hausdorff continuous if it is continuous with respect to the Hausdorff distance on $\mathcal{K}(\mathbb{R})$ at each $x_0 \in [a,b]$, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_H(F(x),F(x_0)) < \varepsilon$ (or equivalently $F(x) \subset F(x_0) + \varepsilon\cdot[-1,1]$ and $F(x_0) \subset F(x) + \varepsilon\cdot[-1,1]$) for every $x \in [a,b]$ satisfying $\|x - x_0\| < \delta$.

We shall denote by $\mathcal{C}([a,b],\mathcal{K}(\mathbb{R}))$ the cone of all set-valued Hausdorff continuous functions defined on $[a,b]$ and with values in $\mathcal{K}(\mathbb{R})$.

The cone $\mathcal{C}([a,b],\mathcal{K}(\mathbb{R}))$ is naturally ordered by inclusion, that is

$$F \leq G \iff \forall x \in [a,b]:\ F(x) \subset G(x)\,,$$

whenever $F, G \in \mathcal{C}([a,b],\mathcal{K}(\mathbb{R}^d))$.

Let $\mathcal{C}$ be a subcone of $\mathcal{C}([a,b],\mathcal{K}(\mathbb{R}^d))$. An operator $L : \mathcal{C} \to \mathcal{C}([a,b],\mathcal{K}(\mathbb{R}^d))$ is linear if preserves addition and multiplication by positive scalars and is monotone (or positive) if preserves the order relation on $\mathcal{C}([a,b],\mathcal{K}(\mathbb{R}^d))$.

Now, we can recall the notion of Korovkin system.

If $\mathcal{C}$ is a subcone of $\mathcal{C}([a,b],\mathcal{K}(\mathbb{R}^d))$, a Korovkin system for the identity operator in $\mathcal{C}$ with respect to linear positive operators is a subset $\mathcal{M}$ of $\mathcal{C}$ which satisfies the following condition:

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- *If $(L_n)_{n\geq 1}$ is a sequence of positive linear operators from $\mathcal{C}$ into $C([a,b],\mathcal{K}(\mathbb{R}))$ satisfying*

$$\lim_{n\to+\infty} L_n(H) = H \qquad \textit{i.e.,} \quad \lim_{n\to+\infty} d_H(L_n(H),H) = 0 \quad \textit{for every } H \in \mathcal{M}\,,$$

then we also have

$$\lim_{n\to+\infty} L_n(F) = F \qquad \textit{for every } F \in \mathcal{C}\,.$$

First, we recall the following Korovkin type result which is contained in [4, Theorem 2.4 and Corollary 2.5] where it has been established even in the more general case where the limit is not the identity operator.

Theorem 1.1. ([4, Theorem 2.4]) *Let $\mathcal{H}$ be a subset of $C([a,b],\mathcal{K}(\mathbb{R}))$ such that, for every $F \in C([a,b],\mathcal{K}(\mathbb{R}))$, $x_0 \in [a,b]$ and $\varepsilon > 0$, there exists $H \in \mathcal{H}$ satisfying the following conditions*

$$F \leq H, \qquad H(x_0) \subset F(x_0) + \varepsilon\cdot[-1,1]\,. \tag{1}$$

Then H is a Korovkin system for the identity operator in $C([a,b],\mathcal{K}(\mathbb{R}))$ with respect to linear positive operators.

As observed in [4, Remark 2.6], if $\mathcal{H}$ contains the constant set-valued functions, then condition (1) can be weakened with the following condition

$$F \leq H + \varepsilon\cdot[-1,1], \qquad H(x_0) \subset F(x_0) + \varepsilon\cdot[-1,1]\,. \tag{2}$$

As a consequence of the above result, many examples of Korovkin systems have been obtained in [9]-[11].

We limit ourselves to state only the following consequence (see [11, Theorems 2.5] and [11, Corollary 2.1]) since it will be useful in the next section in order to state our main results.

Corollary 1.2. *Let $\mathcal{H}$ be a subset of $C([a,b],\mathcal{K}(\mathbb{R}))$ which contains the functions*

$$x \mapsto K + \lambda(x-x_0)^2\cdot[-1,1]\,,$$

whenever $K \in \mathcal{K}(\mathbb{R})$, $x_0 \in [a,b]$ and $\lambda \geq 0$ (in particular $\mathcal{H}$ contains the constant functions).

Then $\mathcal{H}$ is a Korovkin system in $C([a,b],\mathcal{K}(\mathbb{R}))$ for the identity operator with respect to linear positive operators .

If $X = [0,1]$, we obtain that the subcone $\mathcal{H}$ of $C([0,1],\mathcal{K}(\mathbb{R}))$ containing the functions

$$x \mapsto (\lambda + \mu(x-x_0)^2)\cdot[-1,1], \qquad x_0 \in [0,1]\,,$$

for every $x_0 \in [0,1]$, is a Korovkin system in $C([0,1],\mathcal{K}(\mathbb{R}))$ for the identity operator with respect to linear positive operators.

In the next section, we shall consider Korovkin systems for suitable subcones of $C([a,b],\mathcal{K}(\mathbb{R}))$.

2 The main result

We shall consider the subcone $\mathcal{G}$ of $C([a,b],\mathcal{K}(\mathbb{R}))$ consisting of all functions in $F \in C([a,b],\mathcal{K}(\mathbb{R}))$ having a convex graph in $\mathbb{R}^2$, i.e.

$$\begin{aligned}\mathcal{G} := \{F \in C([a,b],\mathcal{K}(\mathbb{R})) \mid x,y \in [a,b], u \in F(x), v \in F(y), t \in [0,1] \\ \Rightarrow\ tu + (1-t)v \in F(tx + (1-t)y)\}\,.\end{aligned}$$

In the case of a single-valued real continuous function $\varphi : [a,b] \to \mathbb{R}$, the graph is convex if and only if φ is an affine function, while for set-valued functions we have a more general situation and even non affine functions may have a convex graph.

For example, the set-valued continuous function $F : [0,1] \to \mathcal{K}(\mathbb{R})$ defined by setting

$$F(x) = x(1-x)\cdot[-1,1]$$

has a convex graph but it is not affine.

If $F \in C([a,b],\mathcal{K}(\mathbb{R}))$ has a convex graph, then we can consider the functions $\varphi_F, \psi_F : [a,b] \to \mathbb{R}$ defined by setting, for every $x \in [a,b]$,

$$\varphi_F(x) := \min F(x)\,, \qquad \psi_F(x) = \max F(x)\,.$$

Since F is Hausdorff continuous, we have $\varphi_F, \psi_F \in C([a,b])$ and further φ_F is a convex function and ψ_F is a concave function. It is well-known that φ_F and ψ_F have finite left and right derivatives ${\varphi_F}'_-(x_0) \leq {\varphi_F}'_+(x_0)$ and ${\psi_F}'_-(x_0) \geq {\psi_F}'_+(x_0)$ at every $x_0 \in]a,b[$, and admit a derivative at the endpoints which may be not finite.

At this point we can state our main result. In order to obtain a Korovkin system in $\mathcal{G}$, we don't need to consider all the set-valued functions in Corollary 1.2, but only the set-valued functions $H_{h,k} : [a,b] \to \mathcal{K}(\mathbb{R})$ defined by setting

$$H_{h,k}(x) := [h(x), k(x)]\,, \qquad x \in [a,b]\,, \tag{3}$$

where $h(x) := p_1 x + q_1$, $k(x) := p_2 x + q_2$ and $h(x) \leq k(x)$ for every $x \in [a,b]$.

These functions play the role of affine functions for single-valued real functions.

Theorem 2.1. *Let $\mathcal{M}$ be a subset of $\mathcal{G}$ which contains the functions $H_{h,k}$ defined in (3), whenever $h(x) := p_1 x + q_1$, $k(x) := p_2 x + q_2$ and $h(x) \leq k(x)$ for every $x \in [a,b]$.*

Then $\mathcal{M}$ is a Korovkin system in $\mathcal{G}$ for the identity operator with respect to linear positive operators.

Proof. Let $F:[a,b]\to\mathcal{K}(\mathbb{R})$ be a Hausdorff continuous function with a convex graph. Then, we can consider the continuous functions

$$\varphi_F(x):=\min F(x)\,,\qquad \psi_F(x)=\max F(x)\,,\qquad x\in[a,b]\,.$$

Moreover φ_F is convex, ψ_F is concave and both have finite left and right derivatives at each $x_0\in]a,b[$ and admit a derivative at the endpoints which may be not finite.

Let $\varepsilon>0$ and $x_0\in[a,b]$. If $x_0\in]a,b[$, we can consider $p_1\in[{\varphi_F}'_-(x_0),{\varphi_F}'_+(x_0)]$ and $p_2\in[{\psi_F}'_+(x_0),{\psi_F}'_-(x_0)]$ and the tangents $h(x):=p_1(x-x_0)+\varphi(x_0)$ to the graph of φ at x_0 and $k(x):=p_2(x-x_0)+\psi(x_0)$ to the graph of ψ at x_0.

Now, consider the function $H_{h,k}\in\mathcal{M}$ (see (3)). Since φ is convex and ψ is concave, we necessarily have $F\le H_{h,k}$. Moreover we obviously have $H_{h,k}(x_0)=[\varphi(x_0),\psi(x_0)]=F(x_0)\subset F(x_0)+\varepsilon\cdot[-1,1]$ and therefore in this case condition (1) of Theorem 1.1 is satisfied.

Consider now $x_0=a$. If the derivatives ${\varphi_F}'(a)$ and ${\psi_F}'(a)$ are both finite we can consider the tangents at a of φ_F and ψ_F and we can proceed as in the case $x_0\in]a,b[$.

If ${\varphi_F}'(a)=-\infty$ we can consider $\delta>0$ such that $|\varphi_F(x)-\varphi_F(a)|\le\varepsilon/2$ for every $x\in[a,a+\delta]$. Define the affine function $h(x):=p_1(x-a)+\varphi_F(a)-\varepsilon/2$ with $p_1\in[{\varphi_F}'_-(a+\delta),{\varphi_F}'_+(a+\delta)]$. Then $h(x)\le\varphi_F(x)$ for every $x\in[a,b]$. Indeed this is obvious in the interval $[a,a+\delta]$ and is ensured by the fact that the function $t(x):=p_1(x-a-\delta)+\varphi_F(a+\delta)$ satisfies $t\le\varphi_F$ (for the same argument used before in the case $x_0\in]a,b[$) and $h(x)\le t(x)-\varepsilon/2$.

Now, consider ${\psi_F}'(a)$; if ${\psi_F}'(a)<+\infty$ we set $p_2:={\psi_F}'(a)$ and define $k(x):=p_2(x-a)+\psi(a)$. If ${\psi_F}'(a)=+\infty$ we can consider $\sigma>0$ such that $|\psi_F(x)-\psi_F(a)|\le\varepsilon/2$ for every $x\in[a,a+\sigma]$. In this case we define $k(x):=p_2(x-a)+\psi_F(a)+\varepsilon/2$ with $p_2\in[{\psi_F}'_+(a+\sigma),{\psi_F}'_-(a+\sigma)]$. Then $k(x)\ge\psi_F(x)$ for every $x\in[a,b]$. This is obvious in the interval $[a,a+\sigma]$ and is ensured by the fact that the function $s(x):=p_2(x-a-\sigma)+\psi_F(a+\sigma)$ satisfies $s\ge\psi_F$ and $k(x)\ge t(x)+\varepsilon/2$.

Now, consider the function $H_{h,k}\in\mathcal{M}$; as above, we necessarily have $F\le H_{h,k}$ and $H_{h,k}(a)=[\varphi(a),\psi(a)]=F(a)\subset F(a)+\varepsilon\cdot[-1,1]$ and therefore also in this case condition (1) of Theorem 1.1 is satisfied.

Obviously the same reasoning can be applied to the point b and this completes the proof. □ □

The above result is a general Korovkin-type result which allows us to ensure the convergence of an arbitrary sequence of positive linear operators to the identity operator on the set-valued functions having a convex graph.

Now, we consider the same question for a class of set-valued operators which are more general than linear positive operators and we obtain a deeper result which holds also in a multi-dimensional setting.

Namely, a continuous monotone linear operator

$$L:C([a,b],\mathcal{K}(\mathbb{R}^d))\to C([a,b],\mathcal{K}(\mathbb{R}^d))$$

is said to be *convexity monotone* if it satisfies the condition

$$\varphi\in\mathrm{co}(\varphi_1,\varphi_2)\ \Rightarrow\ L(\{\varphi\})\subset\mathrm{co}(L(\{\varphi_1\}),L(\{\varphi_2\}))\tag{4}$$

for every $\varphi_1,\varphi_2\in C([a,b],\mathbb{R}^d)$, where $\mathrm{co}(\varphi_1,\varphi_2):[a,b]\to\mathcal{K}(\mathbb{R}^d)$ is the set-valued (continuous) function defined by setting, for every $x\in[a,b]$,

$$\mathrm{co}(\varphi_1,\varphi_2)(x):=\mathrm{co}(\varphi_1(x),\varphi_2(x))\,,$$

and $\mathrm{co}(\varphi_1(x),\varphi_2(x))$ denotes the convex hull of $\{\varphi_1(x),\varphi_2(x)\}$. Moreover $\{\varphi\}$ is the set-valued function defined by setting $\{\varphi\}(x):=\{\varphi(x)\}$ for every $x\in[a,b]$.

Convexity monotone operators have been studied in details in [9]-[11] and we refer also to [5, 6, 7] for other interesting properties.

If $\mathcal{C}$ is a subcone of $C([a,b],\mathcal{K}(\mathbb{R}^d))$, a subset $\mathcal{M}$ of $\mathcal{C}$ is called a *Korovkin system for the identity operator in $\mathcal{C}$ with respect to convexity monotone operators* if for every sequence $(L_n)_{n\ge1}$ of convexity monotone linear operators from $\mathcal{C}$ into itself satisfying $\lim_{n\to+\infty}L_n(H)=H$ for every $H\in\mathcal{M}$, we also have $\lim_{n\to+\infty}L_n(F)=F$ for every $F\in\mathcal{C}$.

Now, consider the subcone of all continuous set-valued functions having a compact convex graph in $\mathbb{R}^{d+1}$:

$$\begin{aligned}\mathcal{G}:=\{F\in C([a,b],\mathcal{K}(\mathbb{R}^d))\mid x,y\in[a,b],u\in F(x),v\in F(y),t\in[0,1]\\ \Rightarrow\ tu+(1-t)v\in F(tx+(1-t)y)\}\,.\end{aligned}$$

For the class of convexity monotone operators, we have the following result. We denote by $A([a,b],\mathbb{R}^d)$ the subspace of $C([a,b],\mathbb{R}^d)$ consisting of all affine functions on $[a,b]$.

Theorem 2.2. *Let $\mathcal{M}$ be a subset of $\mathcal{G}$ the constant function $1\cdot\mathbf{B}$, where $\mathbf{B}$ denotes the closed unit ball in $\mathbb{R}^d$, and the functions $\{h\}$ for every $h\in A([a,b],\mathbb{R}^d)$.*

Then $\mathcal{M}$ is a Korovkin system in $\mathcal{G}$ for the identity operator with respect to convexity monotone operators.

Proof. Let $(L_n)_{n\ge1}$ be a sequence of convexity monotone linear operators converging to H for every $H\in\mathcal{M}$.

First, we show that if $\varphi_1,\dots,\varphi_m\in A([a,b],\mathbb{R}^d)$, then

$$\lim_{n\to+\infty}L_n(\mathrm{co}(\varphi_1,\dots,\varphi_m))=\mathrm{co}(\varphi_1,\dots,\varphi_m)\,.$$

Let $\varepsilon>0$; since $(L_n(\mathbf{B}))_{n\ge1}$ converges to $\mathbf{B}$ we can find $M\ge1$ such that

$$L_n(1\cdot\mathbf{B})\subset M\cdot\mathbf{B}\,.$$

For every $x \in [a,b]$, the set $\mathrm{co}(\varphi_1(x),\ldots,\varphi_m(x))$ is compact and therefore there exist $\lambda(s) := (\lambda_1(s),\ldots,\lambda_m(s))$, $s = 1,\ldots,p$, such that $\lambda_j(s) \geq 0$ for every $j = 1,\ldots,m$ and $\sum_{j=1}^{p} \lambda_j(s) = 1$ for every $s = 1,\ldots,p$ and further

$$\mathrm{co}(\varphi_1(x),\ldots,\varphi_m(x)) \subset \bigcup_{s=1}^{p} \left(\sum_{j=1}^{m} \lambda_j(s)\{\varphi_j(x)\} + \frac{\varepsilon}{2M} \cdot \mathbf{B} \right).$$

Since $[a,b]$ is compact the preceding formula can be extended to every $x \in [a,b]$.

Since $(L_n(\{\varphi\}))_{n\geq 1}$ converges to $\{\varphi\}$ for every $\varphi \in A([a,b],\mathbb{R}^d)$, the sequences $\left(\sum_{j=1}^{m} \lambda_j(s) L_n(\{\varphi_j\})\right)_{n\geq 1}$ converge to $\sum_{j=1}^{m} \lambda_j(s) L(\{\varphi_j\})$ for every $s = 1,\ldots,p$. Hence we can find $\nu \geq 1$ such that, for every $n \geq \nu$ and $s = 1,\ldots,p$,

$$\sum_{j=1}^{m} \lambda_j(s)\{\varphi_j\} \subset \sum_{j=1}^{m} \lambda_j(s) L_n(\{\varphi_j\}) + \frac{\varepsilon}{2} \cdot \mathbf{B},$$

$$\sum_{j=1}^{m} \lambda_j(s) L_n(\{\varphi_j\}) \subset \sum_{j=1}^{m} \lambda_j(s)\{\varphi_j\} + \frac{\varepsilon}{2} \cdot \mathbf{B}.$$

It follows, for every $n \geq \nu$,

$$\begin{aligned} \mathrm{co}(\varphi_1,\ldots,\varphi_m) &\subset \bigcup_{s=1}^{p} \left(\sum_{j=1}^{m} \lambda_j(s)\{\varphi_j\} + \frac{\varepsilon}{2M} \cdot \mathbf{B} \right) \\ &\subset \bigcup_{s=1}^{p} \left(\sum_{j=1}^{m} \lambda_j(s)\{\varphi_j\} + \frac{\varepsilon}{2} \cdot \mathbf{B} \right) \\ &\subset \bigcup_{s=1}^{p} \left(\sum_{j=1}^{m} \lambda_j(s) L_n(\{\varphi_j\}) + \varepsilon \cdot \mathbf{B} \right) \\ &\subset L_n(\mathrm{co}(\varphi_1,\ldots,\varphi_m)) + \varepsilon \cdot \mathbf{B}, \end{aligned}$$

and conversely

$$\begin{aligned} L_n(\mathrm{co}(\varphi_1,\ldots,\varphi_m)) &\subset \bigcup_{s=1}^{p} \left(\sum_{j=1}^{m} \lambda_j(s) L_n(\{\varphi_j\}) + L_n\left(\frac{\varepsilon}{2M} \cdot \mathbf{B}\right) \right) \\ &\subset \bigcup_{s=1}^{p} \left(\sum_{j=1}^{m} \lambda_j(s) L_n(\{\varphi_j\}) + \frac{\varepsilon}{2} \cdot \mathbf{B} \right) \\ &\subset \bigcup_{s=1}^{p} \left(\sum_{j=1}^{m} \lambda_j(s)\{\varphi_j\} + \varepsilon \cdot \mathbf{B} \right) \\ &\subset \mathrm{co}(\varphi_1,\ldots,\varphi_m) + \varepsilon \cdot \mathbf{B}, \end{aligned}$$

Hence the net $(L_n(\mathrm{co}(\varphi_1,\ldots,\varphi_m)))_{n\geq 1}$ converges to $\mathrm{co}(\varphi_1,\ldots,\varphi_m))$.

Now, let $F \in \mathcal{G}$, $x_0 \in [a,b]$ and $\varepsilon > 0$. For every $y \in \partial F(x_0)$ ($\partial F(x_0)$ denotes the boundary of $F(x_0)$) it is possible to find $\varphi_y \in A([a,b],\mathcal{K}(\mathbb{R}^d))$ such that $\varphi(x) \cap F(x) = \emptyset$ for every $x \in [a,b]$ and further $\|\varphi_y(x_0) - y\| < \varepsilon$. In this way we obtain a family $(\varphi_y)_{y\in\partial F(x_0)}$ of elements of $A([a,b],\mathcal{K}(\mathbb{R}^d))$ such that $F \leq co(\varphi_y \mid y \in \partial F(x_0))$ and $co(\varphi_y(x_0) \mid y \in \partial F(x_0)) \subset F(x_0) + \varepsilon \cdot \mathbf{B}$. The compactness of $\partial F(x_0)$ yields a finite number $\varphi_1,\ldots,\varphi_m \in A([a,b],\mathcal{K}(\mathbb{R}^d))$ such that $F \leq co(\varphi_1,\ldots,\varphi_m)$ and $co(\varphi_1,\ldots\varphi_m)(x_0) \subset F(x_0) + \varepsilon \cdot \mathbf{B}$.

At this point we observe that the set

$$\{\mathrm{co}(\varphi_1,\ldots,\varphi_m) \mid \varphi_1,\ldots,\varphi_m \in A([a,b],\mathbb{R}^n)\}$$

satisfies the assumptions of [11, Theorem 2.3] (see also [4, Theorem 2.4]). Hence we can proceed just as in the proof of [4, Theorem 2.4] in order to show the convergence of $(L_n(F))_{n\geq 1}$ to F and this completes the proof. □ □

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