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RESEARCH ARTICLE

On the symmetries of Siklos spacetimes

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Abstract
We consider the large class of Siklos spacetimes and investigate their relevant symmetries (homothetic and affine vector fields, Ricci, curvature, Weyl and matter collineations). We prove some general results and obtain complete classifications for homogeneous Siklos spacetimes.

Keywords Siklos spacetimes · Killing vector fields · Affine vector fields · Ricci collineations · Curvature and Weyl collineations · Matter collineations

Mathematics Subject Classification 53C50 · 53B30

Contents

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1 Introduction

Let (M, g) denote a Lorentzian manifold. If T is a tensor on (M, g), which codifies some either mathematical or physical quantity, a *symmetry* of T is a one-parameter group of diffeomorphisms of (M, g), which leaves T invariant.

Thus, a symmetry corresponds to a vector field X for which $\mathcal{L}_X T = 0$, where $\mathcal{L}$ denotes the Lie derivative. For example, for $T = g$, symmetries are isometries and the corresponding vector fields X are Killing. Homotheties and conformal motions on (M, g) are further examples of symmetries. In recent years, *curvature collineations* (T=R being the curvature tensor), *Weyl collineations* (T=W is the Weyl conformal curvature tensor) and *Ricci collineations* (for which T=ϱ is the Ricci tensor) have been studied. We may refer to [15] for a thorough survey and further information on symmetries. Examples of investigations of Ricci collineations in some classes of Lorentzian manifolds may be found in [9, 10, 13, 16–20, 31] and references therein.

Matter collineations are the symmetries of the energy-momentum tensor $T = \varrho - \frac{1}{2}\tau g$ (with τ denoting the scalar curvature) of a Lorentzian manifold (M, g). Both matter collineations and Ricci collineations are interesting objects to investigate, for their physical and geometric meaning respectively (see for example [11, 17]).

The purpose of this paper is to investigate symmetries within the large class of *Siklos spacetimes* [30]. They are solutions of Einstein field equations with an Einstein–Maxwell source. Siklos metrics appear in Petrov classification as spacetimes of type N with cosmological constant $\Lambda < 0$ and all of them admit a null non-twisting Killing field. The general form of Siklos metrics in global coordinates $(x_1, x_2, x_3, x_4) = (v, u, x, y)$ is given by

$$g = -\frac{3}{\Lambda x_3^2}\left(2dx_1dx_2 + H dx_2^2 + dx_3^2 + dx_4^2\right), \tag{1.1}$$

for an arbitrary defining smooth function $H = H(x_2, x_3, x_4)$ ([24, 30]). These metrics naturally appear in several context in Mathematical Physics and in Geometry. For example:

- the study of the behaviour of free particles in these spacetimes led to prove that they can been interpreted as exact gravitational waves propagating in the anti-de Sitter universe [24];
- plane-fronted waves in spacetimes were classified in [23], depending on the sign of the cosmological constant Λ and that (determined by the sign of some constant k) of some second-order invariant depending on the congruence of null rays. Siklos spacetimes occur in this classification as one of the two subcases with $\Lambda < 0$ and $k = 0$, and coincide with the subclass $(IV)_0$ of Kundt spacetimes;
- all nontwisting type N solutions of Einstein's vacuum equations, including Siklos spacetimes, were investigated in [1] and [2], obtaining a physical interpretation by analysing the equation of geodesic deviation;
- impulsive gravitational waves propagating either in a de Sitter or an anti-de Sitter background were studied in [26]. In the latter case, they are Siklos metrics;

- the equations of vacuum polarization for photons propagating in a general Siklos spacetime were solved in [21], in order to study the effect of one-loop vacuum polarization in the geometric optics limit.

Moreover, several particular subclasses and representatives of the class of Siklos spacetimes, are well known and their physical and geometrical properties have been investigated in literature. Such examples include, among others: Defrise spacetime [12], Kaigodorov spacetime [19], generalized Defrise spacetimes [25] (see also Remark 3.2). Ricci solitons ([4–7]) and conformal geometry [8] were studied within the class of Siklos metrics.

We may observe that this paper can also be considered as a natural prolongation of the pioneering work [30], where Killing vector fields of metrics (1.1) were completely classified.

The paper is organized in the following way. In Sect 2 we shall provide some basic information on symmetries and on Siklos metrics. In Sect. 3 we shall deduce some general conclusions concerning the symmetries of Siklos metrics. On the one hand, we prove that Siklos metrics do not admit any proper curvature collineations (in particular, neither proper homothetic nor proper affine Killing vector fields). On the other hand, we clarify the conditions ensuring the existence of Ricci and matter collineations and we establish the existence of proper Weyl collineations for all Siklos metrics. Finally, in Sect. 4 we shall focus on the subclasses of homogeneous Siklos metrics and obtain complete classifications of their symmetries. Calculations have been checked using the software *Maple 16*©.

2 Preliminaries

2.1 Symmetries

Let (M, g) denote a Lorentzian manifold. A vector field X on M preserving its metric tensor g, the Levi–Civita connection ∇ of g, its curvature tensor R or its Ricci tensor ϱ, is respectively known as a *Killing vector field*, an *affine vector field*, a *curvature collineation* or a *Ricci collineation*. A *Weyl collineation* is a vector field preserving the Weyl conformal curvature tensor W.

As the Levi–Civita connection and the curvature are completely determined by the metric tensor g, if a vector field X preserves g, then it also preserves ∇, R, ϱ, while the converse does not hold in general. Similarly, if X preserves ∇ (respectively, R), then it also preserves R (respectively, ϱ), but not conversely. Even homothetic vector fields (i.e., vector fields X satisfying $\mathcal{L}_X g = \lambda g$ for some real constant λ) are necessarily curvature collineations (hence, in particular, Ricci collineations).

Although conditions defining curvature, Ricci and matter collineations are formally similar to the ones defining Killing or affine vector fields, the behaviours of the corresponding classes of vector fields may show some relevant differences. In particular (see for example [15, 17]), we emphasize that:

(a) while Killing and affine vector fields are smooth, for any positive integer k there exist spacetimes admitting Ricci (and curvature, matter) collineations, which are C^k but not C^{k+1}.
(b) differently from the ones determined by Killing, homothetic and conformal vector fields, the vector space of Ricci (and curvature, matter) collineations may be infinite-dimensional. Moreover, because of the above point (a), this vector space need not be a Lie algebra: when X, Y are Ricci (curvature, matter) collineations, $[X, Y]$ may be not differentiable.
(c) two Ricci (respectively, curvature, matter) collineations that agree on an non-empty subset of M may not agree on M. In fact, differently from Killing, homothetic and conformal vector fields, they are not uniquely determined by the value of X and its covariant derivatives of any order at a point.

Remark 2.1 Restricting to smooth vector fields, the sets of Ricci, curvature, Weyl and matter collineations of a given manifold form a Lie algebra. Moreover, if ϱ (respectively, $T = \varrho - \frac{1}{2}\tau g$) is nondegenerate, then the Lie algebra of Ricci (respectively, matter) collineations is finite-dimensional. In fact, in such a case, $\tilde{g} = \varrho$ (respectively, $\tilde{g} = \varrho - \frac{1}{2}\tau g$) itself is a nondegenerate metric tensor, whose Killing vector fields are exactly the Ricci (respectively, matter) collineations of g [16]. This general observation will be particularly useful in the case of Siklos metrics.

With regard to Weyl collineations, it is worthwhile to emphasize that in spite of the Weyl tensor being completely determined by the curvature, in general the classes of curvature collineations and Weyl collineations are not reducible to one another [18]. Siklos spacetimes will provide further examples of this behaviour.

2.2 Geometry of Siklos metrics

We may refer to [24] and [4] for the description of the Levi–Civita connection and curvature of Siklos metrics. We shall report here the information we shall use for the investigation of the symmetries of Siklos metrics.

Consider an arbitrary Siklos metric g, as described in (1.1) with respect to a system of global coordinates (x_1, x_2, x_3, x_4). Throughout the paper, we shall use notations $\partial_i = \frac{\partial}{\partial x_i}, f_{,i} = \frac{\partial f}{\partial x_i}, f_{,ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$,... for all indices i, j, ... The Levi–Civita connection ∇ of g is completely determined by the following possibly non-vanishing components:

$$\begin{aligned}
&\nabla_{\partial_1}\partial_2 = \tfrac{1}{x_3}\partial_3, && \nabla_{\partial_1}\partial_3 = -\tfrac{1}{x_3}\partial_1,\\
&\nabla_{\partial_2}\partial_2 = \tfrac{1}{2}(H_{,2})\partial_1 + \tfrac{1}{2x_3}(2H - x_3(H_{,3}))\partial_3 - \tfrac{1}{2}(H_{,4})\partial_4, && \nabla_{\partial_2}\partial_3 = \tfrac{1}{2}(H_{,3})\partial_1 - \tfrac{1}{x_3}\partial_2,\\
&\nabla_{\partial_2}\partial_4 = \tfrac{1}{2}(H_{,4})\partial_1, && \nabla_{\partial_3}\partial_3 = -\tfrac{1}{x_3}\partial_3,\\
&\nabla_{\partial_3}\partial_4 = -\tfrac{1}{x_3}\partial_4, && \nabla_{\partial_4}\partial_4 = \tfrac{1}{x_3}\partial_3.
\end{aligned} \tag{2.1}$$

The possibly non-vanishing components (up to symmetries) of the Riemann–Christoffel curvature tensor R of g are then given by

$$\begin{aligned}
&R_{1212} = -\frac{3}{\Lambda x_3^4}, && R_{1323} = \frac{3}{\Lambda x_3^4}, \quad R_{1424} = \frac{3}{\Lambda x_3^4},\\
&R_{2323} = \frac{3\big(2H - x_3(H_{,3}) + x_3^2(H_{,33})\big)}{2\Lambda x_3^4}, && R_{2324} = \frac{3(H_{,34})}{2\Lambda x_3^2}, \quad R_{2424} = \frac{3\big(2H - x_3(H_{,3}) + x_3^2(H_{,44})\big)}{2\Lambda x_3^4},\\
&R_{3434} = \frac{3}{\Lambda x_3^4}
\end{aligned} \tag{2.2}$$

and the *Ricci tensor* of g, defined by $\varrho(X, Y) = \mathrm{tr}(Z \mapsto R(Z, X)Y)$, is completely described by the matrix

$$\varrho = (\varrho_{ij}) = \begin{pmatrix} 0 & -3x_3^{-2} & 0 & 0 \\ -3x_3^{-2} & -\frac{6H - 2x_3(H_{,3}) + x_3^2(H_{,33} + H_{,44})}{2x_3^2} & 0 & 0 \\ 0 & 0 & -3x_3^{-2} & 0 \\ 0 & 0 & 0 & -3x_3^{-2} \end{pmatrix}, \tag{2.3}$$

where $\varrho_{ij} = \varrho(\partial_i, \partial_j)$. The *scalar curvature* of a Siklos metric is given by $\tau = \mathrm{tr}\varrho = 4\Lambda$. Next, the *Weyl conformal curvature tensor field* W of a pseudo-Riemannian manifold (M^n, g) is defined by

$$\begin{aligned} W(X, Y)Z = R(X, Y)Z &- \frac{1}{n-2}(QX \wedge Y + X \wedge QY)Z \\ &+ \frac{\tau}{(n-1)(n-2)}(X \wedge Y)Z, \end{aligned} \tag{2.4}$$

where $(X \wedge Y)(Z) = g(Y, Z)X - g(X, Z)Y$ and Q denotes the Ricci operator. It is well known that the vanishing of W completely characterizes (locally) conformally flat metrics in dimension greater than three. The Weyl tensor of type (0, 4) of a Siklos metric is completely determined, up to symmetries, by the following components W_{ijkh} with respect to $\{\partial_i\}$:

$$W_{2323} = -W_{2424} = \frac{3}{4\Lambda x_3^2}\left(H_{,33} - H_{,44}\right), \quad W_{2324} = \frac{3}{2\Lambda x_3^2}(H_{,34}). \tag{2.5}$$

We now report the classification of Einstein and conformally flat Siklos metrics.

Proposition 2.2 ([4, 24, 30]) *For an arbitrary Siklos metric g, as described in* (1.1)*, the following conditions are equivalent:*

(i) g is Einstein. More precisely, $\varrho = \Lambda g$;
(ii) g is Ricci-parallel (that is, $\nabla\varrho = 0$);
(iii) the defining function $H = H(x_2, x_3, x_4)$ satisfies the PDE

$$\frac{2}{x_3}(H_{,3}) - H_{,33} - H_{,44} = 0. \tag{2.6}$$

Whenever H does not satisfy (2.6)*, a Siklos spacetime (being not Ricci-parallel) is not locally symmetric, and its Ricci operator is of Segre type* $[(11, 2)]$*, having an eigenvalue of multiplicity four associated to a three-dimensional eigenspace.*

Proposition 2.3 ([4, 5, 27]) *A Siklos metric g, as described in* (1.1)*, is locally conformally flat if and only if the defining function $H = H(x_2, x_3, x_4)$ satisfies the system of PDEs*

$$H_{,33} - H_{,44} = H_{,34} = 0, \tag{2.7}$$

that is, when H is explicitly given by

$$H(x_2, x_3, x_4) = \frac{1}{2}\, T(x_2) \left(x_3^2 + x_4^2\right) + L(x_2)\, x_3 + M(x_2)\, x_4 + N(x_2), \tag{2.8}$$

where T, L, M, N are arbitrary smooth functions.

Remark 2.4 Explicit form (2.8) for conformally flat Siklos metric was first obtained in [27], where it was also shown how to obtain these examples from the general description of all conformally flat Kundt spacetimes with a cosmological constant, which are either vacuum or contain pure radiation, as described in [23]. We may refer to [14, Section 18.3.3] and references therein for a comprehensive description of conformally flat Kundt solutions.

If H satisfies both conditions (2.6) and (2.7), the general form of the solution is

$$H = P(x_2)(x_3^2 + x_4^2) + Q(x_2)x_4 + R(x_2)$$

and the corresponding Siklos metric is isometric to the anti-de Sitter space [30].

3 General results on the symmetries of Siklos metrics

As we recalled in the previous Section, some kinds of symmetries need not be of class C^∞, and this also implies that they do not necessarily form a Lie algebra. For this reason, when investigating the symmetries of a given metric, it is customary to restrict to smooth symmetries, which always form a Lie algebra.

The strongest symmetries of Siklos metrics are well known. In fact, as proved in [4, Proposition 2.1], Siklos spacetimes never admit any parallel vector field, while Killing vector fields of Siklos metrics were classified by Siklos in the original paper [30]. We now prove the following general non-existence result for some more general symmetries.

Theorem 3.1 *A Siklos metric g, as described by* (1.1) *for some defining function H, does not admit neither proper homothetic vector fields, proper affine vector fields, nor proper curvature collineations.*

Proof Consider the global coordinates (x_1, x_2, x_3, x_4) used in (1.1) to describe Siklos metrics. Let $X = X^i\partial_i$ denote an arbitrary vector field, where $X^i = X^i(x_1, x_2, x_3, x_4)$, $i = 1, \ldots, 4$, are smooth functions. We use (2.1) to calculate the components $(\mathcal{L}_X g)_{ij} = (\mathcal{L}_X g)(\partial_i, \partial_j), i \leq j = 1, \ldots, 4$, of the Lie derivative $\mathcal{L}_X g$. Explicitly, we get

$$
\begin{aligned}
(\mathcal{L}_X g)_{11} &= -\frac{6}{\Lambda x_3^2}\,(X^2{}_{,1}),\\
(\mathcal{L}_X g)_{12} &= -\frac{3}{\Lambda x_3^3}\left\{x_3(X^1{}_{,1}) + x_3 H(X^2{}_{,1}) + x_3(X^2{}_{,2}) - 2X^3\right\},\\
(\mathcal{L}_X g)_{13} &= -\frac{3}{\Lambda x_3^2}\left\{X^2{}_{,3} + X^3{}_{,1}\right\},\\
(\mathcal{L}_X g)_{14} &= -\frac{3}{\Lambda x_3^2}\left\{X^2{}_{,4} + X^4{}_{,1}\right\},\\
(\mathcal{L}_X g)_{22} &= -\frac{3}{\Lambda x_3^3}\Big\{2x_3(X^1{}_{,2}) + x_3(H_{,2})\,X^2 + 2x_3 H(X^2{}_{,2})\\
&\qquad -2H\,X^3 + x_3(H_{,3})\,X^3 + x_3(H_{,4})\,X^4\Big\},\\
(\mathcal{L}_X g)_{23} &= -\frac{3}{\Lambda x_3^2}\left\{X^1{}_{,3} + H(X^2{}_{,3}) + X^3{}_{,2}\right\},\\
(\mathcal{L}_X g)_{24} &= -\frac{3}{\Lambda x_3^2}\left\{X^1{}_{,4} + H(X^2{}_{,4}) + X^4{}_{,2}\right\},\\
(\mathcal{L}_X g)_{33} &= -\frac{6}{\Lambda x_3^3}\left\{x_3(X^3{}_{,3}) - X^3\right\},\\
(\mathcal{L}_X g)_{34} &= -\frac{3}{\Lambda x_3^2}\left\{X^3{}_{,4} + X^4{}_{,3}\right\},\\
(\mathcal{L}_X g)_{44} &= -\frac{6}{\Lambda x_3^3}\left\{x_3(X^4{}_{,4}) - X^3\right\}.
\end{aligned}
\tag{3.1}
$$

Suppose now that X is a homothetic vector field, that is, $\mathcal{L}_X g = \lambda g$ for some real constant λ. We shall prove that necessarily $\lambda = 0$, so that no proper homothetic vector fields occur, as a homothetic vector field is indeed Killing.

By (3.1) and (1.1) we get that in terms of components, $\mathcal{L}_X g = \lambda g$ is equivalent to the following system of ten PDEs:

$$
\begin{aligned}
&X^2{}_{,1} = 0,\\
&x_3(X^1{}_{,1}) + x_3 H(X^2{}_{,1}) + x_3(X^2{}_{,2}) - 2X^3 - \lambda x_3 = 0,\\
&X^2{}_{,3} + X^3{}_{,1} = 0,\\
&X^2{}_{,4} + X^4{}_{,1} = 0,\\
&2x_3(X^1{}_{,2}) + x_3(H_{,2})\,X^2 + 2x_3 H(X^2{}_{,2}) - 2H\,X^3 + x_3(H_{,3})\,X^3\\
&\quad + x_3(H_{,4})\,X^4 - \lambda x_3 H = 0,
\end{aligned}
$$

$$
\begin{aligned}
&X^1{}_{,3} + H(X^2{}_{,3}) + X^3{}_{,2} = 0,\\
&X^1{}_{,4} + H(X^2{}_{,4}) + X^4{}_{,2} = 0,\\
&2x_3(X^3{}_{,3}) - 2X^3 - \lambda x_3 = 0,\\
&X^3{}_{,4} + X^4{}_{,3} = 0,\\
&2x_3(X^4{}_{,4}) - 2X^3 - \lambda x_3 = 0.
\end{aligned}
\tag{3.2}
$$

By integration, the first equation in (3.2) yields

$$X^2 = F^2\,(x_2, x_3, x_4)\,,$$

for some smooth function F^2. Substituting into the third and fourth equations of (3.2) and integrating, we obtain

$$
\begin{aligned}
X^3 &= -x_1(F^2{}_{,3}) + F^3\,(x_2, x_3, x_4)\,,\\
X^4 &= -x_1(F^2{}_{,4}) + F^4\,(x_2, x_3, x_4)\,.
\end{aligned}
$$

The ninth equation of (3.2) then becomes

$$2x_1(F^2{}_{,34}) - \left(F^3{}_{,4} + F^4{}_{,3}\right) = 0,$$

for all values of x_1. In particular, $F^2{}_{,34} = 0$ and so, by integration we have

$$F^2 = G^2\,(x_2, x_3) + H^2\,(x_2, x_4)\,.$$

From the eighth equation of (3.2) we then get

$$\big(2x_3(G^2{}_{,33}) - 2(G^2{}_{,3})\big)x_1 + \big(2F^3 - 2x_3(F^3{}_{,3}) + \lambda x_3\big) = 0,$$

for all values of x_1. In particular, $2F^3 - 2x_3(F^3{}_{,3}) + \lambda x_3 = 0$ whence, integrating,

$$F^3 = \left(\frac{1}{2}\lambda \ln(x_3) + G^3(x_2, x_4)\right) x_3,$$

for a smooth function G^3. The ninth equation in (3.2) then reduces to

$$x_3(G^3{}_{,4}) + F^4{}_{,3} = 0,$$

whose integral is given by

$$F^4 = -\frac{1}{2}x_3^2(G^3{}_{,4}) + G^4(x_2, x_4).$$

We now substitute all the above expressions into the last equation of (3.2), which becomes

$$2\big(x_3(H^2{}_{,44}) - (G^2{}_{,3})\big)x_1 + \big(2G^3 + x_3^2(G^3{}_{,44}) - 2(G^3{}_{,4}) + \lambda\ln(x_3) + \lambda\big)x_3 = 0.$$

Since the above equation must hold for all values of x_1, in particular it yields

$$\lambda\ln(x_3) + (G^3{}_{,44})\,x_3^2 + \big(2G^3 - 2(G^3{}_{,4}) + \lambda\big) = 0,$$

for all values of x_3, so that necessarily $\lambda = 0$. Therefore, X is a Killing vector field.

A similar argument applies to the case of curvature collineations of Siklos metrics. Starting from (2.2), we find that if X is a curvature collineation, then its components X^i satisfy the following system of PDEs:

$$\begin{aligned}
&X^3{}_{,1} + X^2{}_{,3} = 0, \quad X^4{}_{,1} + X^2{}_{,4} = 0, \quad X^4{}_{,3} + X^3{}_{,4} = 0,\\
&x_3(X^3{}_{,3}) - X^3 = 0, \quad x_3(X^4{}_{,4}) - X^3 = 0,\\
&X^3{}_{,2} + X^1{}_{,3} + H(X^2{}_{,3}) = 0, \quad X^4{}_{,2} + X^1{}_{,4} + H(X^2{}_{,4}) = 0,\\
&\Big(X^2{}_{,2} + X^1{}_{,1} + H(X^2{}_{,1})\Big)\,x_3 - 2X^3 = 0,\\
&\Big(2H(X^2{}_{,2}) + 2X^1{}_{,2} + X^2(H_{,2}) + X^3(H_{,3}) + X^4(H_{,4})\Big)\,x_3 - 2X^3H = 0,\\
&\Big((X^2{}_{,3})(H_{,34}) - (X^2{}_{,4})(H_{,33})\Big)\,x_3 + (X^2{}_{,4})(H_{,3}) = 0,\\
&\Big((X^4{}_{,1})(H_{,44}) + (X^3{}_{,1})(H_{,34})\Big)\,x_3 - (X^4{}_{,1})(H_{,3}) = 0,\\
&\Big((X^2{}_{,3})(H_{,34}) + (X^4{}_{,1})(H_{,33})\Big)\,x_3 - (X^4{}_{,1})(H_{,3}) = 0,\\
&\Big((X^2{}_{,4})(H_{,34}) - (X^2{}_{,3})(H_{,44})\Big)\,x_3 + (X^2{}_{,3})(H_{,3}) = 0,\\
&\Big((X^2{}_{,4})(H_{,34}) + (X^3{}_{,1})(H_{,44})\Big)\,x_3 - (X^3{}_{,1})(H_{,3}) = 0,\\
&\Big((X^4{}_{,1})(H_{,34}) + (X^3{}_{,1})(H_{,33})\Big)\,x_3 - (X^3{}_{,1})(H_{,3}) = 0,\\
&(X^2{}_{,1})(H_{,34})x_3^2 - 2X^3{}_{,4} - 2X^4{}_{,3} = 0,\\
&(X^2{}_{,1})(H_{,34})x_3^2 + 2X^4{}_{,3} + 2X^3{}_{,4} = 0,\\
&(X^2{}_{,1})(H_{,33})x_3^3 - (X^2{}_{,1})(H_{,3})x_3^2 + 4(X^3{}_{,3})x_3 - 4X^3 = 0,\\
&(X^2{}_{,1})(H_{,33})x_3^3 - (X^2{}_{,1})(H_{,3})x_3^2 - 4(X^3{}_{,3})x_3 + 4X^3 = 0,\\
&(X^2{}_{,1})(H_{,44})x_3^3 - (X^2{}_{,1})(H_{,3})x_3^2 + 4(X^4{}_{,4})x_3 - 4X^3 = 0,\\
&(X^2{}_{,1})(H_{,44})x_3^3 - (X^2{}_{,1})(H_{,3})x_3^2 - 4(X^4{}_{,4})x_3 + 4X^3 = 0,\\
&(X^2{}_{,1})(H_{,33})x_3^3 - (X^2{}_{,1})(H_{,3})x_3^2 + \Big(2X^2{}_{,2} + 2(X^2{}_{,1})H + 2X^1{}_{,1}\Big)\,x_3 - 4X^3 = 0,\\
&(X^2{}_{,1})(H_{,44})x_3^3 - (X^2{}_{,1})(H_{,3})x_3^2 + \Big(2X^2{}_{,2} + 2(X^2{}_{,1})H + 2X^1{}_{,1}\Big)\,x_3 - 4X^3 = 0,\\
&\Big((X^4{}_{,2})(H_{,44}) + (X^3{}_{,2})(H_{,34}) + (X^1{}_{,4})(H_{,44}) + (X^1{}_{,3})(H_{,34})\Big)\,x_3
\end{aligned}$$

$$
\begin{aligned}
&-(X^4{}_{,2})(H_{,3}) - (X^1{}_{,4})(H_{,3}) = 0,\\
&\Big((X^4{}_{,2})(H_{,34}) + (X^3{}_{,2})(H_{,33}) + (X^1{}_{,4})(H_{,34}) + (X^1{}_{,3})(H_{,33})\Big)\, x_3\\
&\quad -(X^3{}_{,2})(H_{,3}) - (X^1{}_{,3})(H_{,3}) = 0,\\
&\Big((X^4{}_{,1})(H_{,34}) + (X^3{}_{,1})(H_{,33})\Big)\, x_3^2 - (X^3{}_{,1})(H_{,3})x_3 - 2(X^2{}_{,3})H\\
&\quad +2X^1{}_{,3} - 2X^3{}_{,2} = 0,\\
&\Big((X^4{}_{,1})(H_{,34}) + (X^2{}_{,3})(H_{,44})\Big)\, x_3^2 - (X^2{}_{,3})(H_{,3})x_3 + 2(X^2{}_{,3})H\\
&\quad +2X^3{}_{,2} + 2X^1{}_{,3} = 0,\\
&\Big((X^2{}_{,3})(H_{,44}) - (X^2{}_{,4})(H_{,34})\Big)\, x_3^2 - (X^2{}_{,3})(H_{,3})x_3 + 2(X^2{}_{,3})H\\
&\quad +2X^3{}_{,2} + 2X^1{}_{,3} = 0,\\
&\Big((X^2{}_{,3})(H_{,33}) + (X^2{}_{,4})(H_{,34})\Big)\, x_3^2 - (X^2{}_{,3})(H_{,3})x_3 + 2(X^2{}_{,3})H\\
&\quad +2X^3{}_{,2} + 2X^1{}_{,3} = 0,\\
&\Big((X^2{}_{,4})(H_{,33}) - (X^2{}_{,3})(H_{,34})\Big)\, x_3^2 - (X^2{}_{,4})(H_{,3})x_3 + 2(X^2{}_{,4})H\\
&\quad +2X^4{}_{,2} + 2X^1{}_{,4} = 0,\\
&\Big((X^2{}_{,4})(H_{,33}) + (X^3{}_{,1})(H_{,34})\Big)\, x_3^2 - (X^2{}_{,4})(H_{,3})x_3 + 2(X^2{}_{,4})H\\
&\quad +2X^4{}_{,2} + 2X^1{}_{,4} = 0,\\
&\Big((X^4{}_{,1})(H_{,44}) + (X^3{}_{,1})(H_{,34})\Big)\, x_3^2\\
&\quad -(X^4{}_{,1})(H_{,3})x_3 - 2(X^2{}_{,4})H - 2X^4{}_{,2} - 2X^1{}_{,4} = 0,\\
&\Big((X^2{}_{,4})(H_{,44}) + (X^2{}_{,3})(H_{,34})\Big)\, x_3^2\\
&\quad -(X^2{}_{,4})(H_{,3})x_3 + 2(X^2{}_{,4})H + 2X^4{}_{,2} + 2X^1{}_{,4} = 0,\\
&\Big((X^2{}_{,3})(H_{,33}) + (X^3{}_{,1})(H_{,33})\Big)\, x_3^2\\
&\quad -\Big((X^2{}_{,3})(H_{,3}) + (X^3{}_{,1})(H_{,3})\Big)\, x_3 + 2(X^2{}_{,3})H + 2X^1{}_{,3} + 2X^3{}_{,2} = 0,\\
&\Big((X^2{}_{,4})(H_{,44}) + (X^4{}_{,1})(H_{,44})\Big)\, x_3^2\\
&\quad -\Big((X^2{}_{,4})(H_{,3}) + (X^4{}_{,1})(H_{,3})\Big)\, x_3 + 2(X^2{}_{,4})H + 2X^4{}_{,2} + 2X^1{}_{,4} = 0,\\
&\big((X^4{}_{,4})(H_{,34}) + (X^4{}_{,3})(\partial_{44}H) + (X^3{}_{,4})(H_{,33})\\
&\quad +(X^3{}_{,3})(H_{,34}) + (X^2{}_{,2})(H_{,34}) - (X^1{}_{,1})(H_{,34})\\
&\quad +X^2(H_{,234}) + X^3(H_{,334}) + X^4(H_{,344})\big)x_3 - (X^3{}_{,4})(H_{,3}) - (X^4{}_{,3})(H_{,3}) = 0,\\
&\big(2(X^4{}_{,3})(H_{,34}) + 2(X^3{}_{,3})(H_{,33}) + (X^2{}_{,2})(H_{,33})\\
&\quad -(X^1{}_{,1})(H_{,33}) + X^2(H_{,233}) + X^3(H_{,333}) + X^4(H_{,334})\big)x_3^2\\
&\quad +\big((X^1{}_{,1})(H_{,3}) - 2(X^3{}_{,3})(H_{,3}) - (X^2{}_{,2})(H_{,3})
\end{aligned}
$$

$$
\begin{aligned}
&\quad -X^2(H_{,23}) - X^3(H_{,33}) - X^4(H_{,34})\big)x_3 + X^3(H_{,3}) = 0,\\
&\big(2(X^3{}_{,4})(H_{,34}) - (X^1{}_{,1})(H_{,44}) + (X^2{}_{,2})(H_{,44}) + 2(X^4{}_{,4})(H_{,44})\\
&\quad +X^3(H_{,344}) + X^2(H_{,244}) + X^4(H_{,444})\big)x_3^2\\
&\quad +\big((X^1{}_{,1})(H_{,3}) - 2(X^4{}_{,4})(H_{,3}) - X^4(H_{,34}) - X^3(H_{,33}) - (X^2{}_{,2})(H_{,3})\\
&\quad -X^2(H_{,23})\big)x_3 + X^3(H_{,3}) = 0,\\
&\big((X^3{}_{,4})(H_{,34}) - (X^4{}_{,3})(H_{,34}) + X^2(H_{,244}) + X^3(H_{,344})\\
&\quad +X^4(H_{,444}) + 2(X^2{}_{,2})(H_{,44})\big)x_3^3\\
&\quad -\big(2(X^2{}_{,2})(H_{,3}) + X^3(H_{,33}) + X^4(H_{,34}) + X^2(H_{,23})\big)x_3^2\\
&\quad +\big(4(X^2{}_{,2})H + 4X^1{}_{,2} + 2X^2(H_{,2}) + 3X^3(H_{,3}) + 2X^4(H_{,4})\big)x_3 - 4X^3H = 0,\\
&\big((X^4{}_{,3})(H_{,34}) + 2(X^2{}_{,2})(H_{,33}) - (X^3{}_{,4})(H_{,34}) + X^2(H_{,233})\\
&\quad +X^3(H_{,333}) + X^4(H_{,334})\big)x_3^3\\
&\quad -\big(2(X^2{}_{,2})(H_{,3}) + X^2(H_{,23}) + X^3(H_{,33}) + X^4(H_{,34})\big)x_3^2\\
&\quad +\big(4(X^2{}_{,2})H + 4X^1{}_{,2} + 2X^2(H_{,2}) + 3X^3(H_{,3}) + 2X^4(H_{,4})\big)x_3 - 4X^3H = 0,\\
&X^2{}_{,1} = 0,\\
&H_{,34}(X^3{}_{,1} + X^2{}_{,3}) = 0, \quad H_{,34}(X^4{}_{,1} + X^2{}_{,4}) = 0,\\
&(X^4{}_{,3})(H_{,44}) + (X^3{}_{,3})(H_{,34}) + 2(X^2{}_{,2})(H_{,34})\\
&\quad -(X^4{}_{,4})(H_{,34}) - (X^4{}_{,3})(H_{,33}) + X^2(H_{,234}) + X^3(H_{,334}) + X^4(H_{,344}) = 0,\\
&(X^4{}_{,4})(H_{,34}) + (X^3{}_{,4})(H_{,33}) + 2(X^2{}_{,2})(H_{,34})\\
&\quad -(X^3{}_{,4})(H_{,44}) - (X^3{}_{,3})(H_{,34}) + X^2(H_{,234}) + X^3(H_{,334}) + X^4(H_{,344}) = 0.
\end{aligned}
$$

In order to simplify the above system, we used its simpler equations into the other ones. In this way, we reduced the above system to the following equivalent system, which contains just ten PDEs:

$$
\begin{aligned}
&X^2{}_{,1} = 0,\\
&X^3{}_{,1} + X^2{}_{,3} = 0,\\
&X^4{}_{,1} + X^2{}_{,4} = 0,\\
&X^4{}_{,3} + X^3{}_{,4} = 0,\\
&(X^3{}_{,3})x_3 - X^3 = 0,\\
&(X^4{}_{,4})x_3 - X^3 = 0,\\
&X^1{}_{,4} + X^4{}_{,2} - (X^4{}_{,1})H = 0,\\
&X^3{}_{,2} + X^1{}_{,3} - (X^3{}_{,1})H = 0,\\
&(X^2{}_{,2} + X^1{}_{,1})x_3 - 2X^3 = 0,\\
&\Big(X^2(H_{,2}) + X^4(H_{,4}) + X^3(H_{,3}) + 2X^1{}_{,2} - 2H(X^1{}_{,1})\Big)x_3\\
&\quad +2X^3H = 0.
\end{aligned}
\tag{3.3}
$$

It is now easily seen that system (3.3) is equivalent to (3.2) with $\lambda = 0$, that is, a curvature collineation is necessarily a Killing vector field. So, no proper curvature collineations occur. In particular, there are not any proper affine vector fields. □

Because of Theorem 3.1, we need to focus on Ricci, matter and Weyl collineations of Siklos metrics. We start reporting the classification of Siklos metrics admitting additional Killing vector fields, as they will play an essential role in the study of Ricci and matter collineations.

All Siklos metrics admit at least one Killing vector field. With respect to coordinates (x_1, x_2, x_3, x_4) we used in (1.1), this Killing vector field is given by

$$K_1 = \partial_1.$$

Some subclasses of Siklos metrics do admit some additional Killing vector fields, as reported in the following Table 1, where we also specified the Einstein and conformally flat cases in these subclasses. In Table I, $A(x_i)$ is an arbitrary function of the specified variables, α is an integer, β is a real parameter, $A_\alpha(x_i)$ represents a homogeneous function of degree α and $K_2, .., K_8$ are the following vector fields:

$$K_2 = -x_1\partial_1 + x_2\partial_2, \quad K_3 = \partial_2, \quad K_4 = \partial_4, \quad K_5 = x_4\partial_1 - x_2\partial_4,$$
$$K_{6,\alpha} = (2+\alpha)x_1\partial_1 + (2-\alpha)x_2\partial_2 + 2x_3\partial_3 + 2x_4\partial_4, \quad K_7 = x_1\partial_1 - x_2\partial_2 + 2\partial_4,$$
$$K_8 = -\tfrac{1}{2}(x_3^2 + x_4^2)\partial_1 + x_2^2\partial_2 + x_2x_3\partial_3 + x_2x_4\partial_4.$$

Remark 3.2 Some of the cases listed in Table I are well known solutions of Einstein fields equations and have been previously studied in literature. In particular:

- Case **11**) with $k = 3/2$ is *Kaigorodov spacetime* [19], which is the only homogeneous type-N solution of the Einstein vacuum field equations with $\Lambda \neq 0$;
- Case **11**) with $k = 2$ is Ozsváth's homogeneous solution to Einstein-Maxwell equations [22].
- Case **12**) is *Defrise spacetime* [12], a pure radiation solution of Petrov type N with a G_6 isometry group;
- *Generalized Defrise spacetimes* [25] are a generalization of case **12**), where the defining function is given by $H = d(x_2)/x_3^2$, for an arbitrary bounded function d. As such, they form a special subclass of case **4**) ;

Siklos metrics corresponding to cases **7**)-**12**) in Table I admit at least four Killing vector fields and so, these cases identify the subclasses of homogeneous Siklos spacetimes. As already observed in [30], one excludes in the above Table I the case of the Anti-de Sitter space, whose Killing vector fields (and, more in general, symmetries) are well known.

We now turn our attention to Ricci collineations. We already observed (Remark 2.1) that when the Ricci tensor of a pseudo-Riemannian metric g is nondegenerate, the Ricci tensor itself is a metric tensor, whose Killing vector fields are nothing but the Ricci

Table 1 Killing vector fields of Siklos metrics

Defining function	Einstein	Conf. flat	Basis of Killing v.f.
0) $H(x_2, x_3, x_4)$	$2H_{,3} - x_3(H_{,44} + H_{,33}) = 0$	$A = \frac{1}{2}\,T(x_2)\left(x_3^2 + x_4^2\right)$ $+L(x_2)\,x_3 + M(x_2)\,x_4$ $+N(x_2)$	K_1
1) $x_2^{-2}A(x_3, x_4)$	$2A_{,3} - x_3(A_{,44} + A_{,33}) = 0$	$A = \frac{c_1}{2}(x_3^2 + x_4^2)$ $+c_2x_3 + c_3x_4 + c_4$	K_1, K_2
2) $A(x_3, x_4)$	$2A_{,3} - x_3(A_{,44} + A_{,33}) = 0$	$A = \frac{c_1}{2}(x_3^2 + x_4^2)$ $+c_2x_3 + c_3x_4 + c_4$	K_1, K_3
3) $A_2(x_2, x_3, x_4)$	$A = f_1(x_2)x_3x_4$ $+(x_3^2 + x_4^2)\,f_1(x_2)\tan^{-1}(\frac{x_4}{x_3})$ $+(x_3^2 + x_4^2)\,f_2(x_2)$	$A = f_1(x_2)(x_3^2 + x_4^2)$	$K_1, K_{6,2}$
4) $A(x_2, x_3)$	$A = f_1(x_2) + f_2(x_2)x_3^3$	$A = f_1(x_2)x_3 + f_2(x_2)$	K_1, K_4, K_5
5) $A_\alpha(x_3, x_4)$ $\alpha \neq -2$	$2A_{,3} - x_3(A_{,44} + A_{,33}) = 0$	$A = 0$	$K_1, K_3, K_{6,\alpha}$
6) $A(x_3)e^{x_4}$	$-2A' + (A + A'')x_3 = 0$	$A = 0$	K_1, K_3, K_7
7) $A(x_3)$	$A = c_1 + c_2x_3^3$	$A = c_1 + c_2x_3$	K_1, K_3, K_4, K_5
8) $A(x_2)x_3^2$	$A = 0$	$A = 0$	$K_1, K_4, K_5, K_{6,2}$
9) $x_2^{-2\beta-2}\,A(x_2^\beta x_3)$	$A = c_1 + c_2x_2^{3\beta}x_3^3$	$A = c_1 + c_2x_2^\beta x_3$	$K_1, K_4, K_5, K_{6,\alpha}$, $\alpha = 2(1 + \frac{1}{\beta})$
10) $A_{-2}(x_3, x_4)$	$A = \frac{c_1(x_4^4 - 3x_3^4 + 6x_3^2x_4^2) + c_2x_3^3x_4}{(x_3^2 + x_4^2)^3}$	$A = 0$	$K_1, K_3, K_{6,-2}, K_8$
11) $\pm x_3^\alpha, \alpha \neq -2$	$\alpha = 0, 3$	$\alpha = 0, 1$	$K_1, K_3, K_4,$ $K_5.K_{6,\alpha}$
12) $\pm x_3^{-2}$	Never	Never	$K_1, K_3, K_4,$ $K_5, K_{6,-2}, K_8$

collineations of g. Therefore, in such a case, the (smooth) Ricci collineations of g necessarily form a finite-dimensional Lie algebra.

In the case of Siklos metrics, comparing (1.1) with (2.3), it is easily seen that the Ricci tensor ϱ of a Siklos metric g is homothetic to a Siklos metric, that is, it is yet another Siklos metric up to scaling. In fact, $\bar{g} = \frac{1}{\Lambda}\varrho$ is the Siklos metric described as in (1.1) by the defining function

$$\bar{H} = H - \frac{1}{3}x_3(H_{,3}) + \frac{1}{6}x_3^2\left(H_{,33} + H_{,44}\right). \tag{3.4}$$

As two metrics homothetic to one another have the same Killing vector fields, this fact has some relevant consequences:

- the cases where proper Ricci collineations exist are completely identified, for arbitrary Siklos metrics and in each of subclasses **1**)-**12**), by some suitable PDE, which further restricts the subclass for the metric determined by the Ricci tensor. For example:

  (1) for H arbitrary (and so, with the Lie algebra of Killing vector fields spanned by K_1), the Lie algebra of Ricci collineations is spanned by $\{K_1, K_3, K_4, K_5, K_{6,-2}, K_7\}$ if and only if $\bar{H}$ belongs to subclass **12**), that is, by (3.4), when H is a solution of the PDE

  $$H - \frac{1}{3}x_3(H_{,3}) + \frac{1}{6}x_3^2\left(H_{,33} + H_{,44}\right) = \pm\frac{1}{x_3^2}.$$

  (2) If g belongs to subclass **7**), then the Lie algebra of its Killing vector fields is spanned by $\{K_1, K_3, K_4, K_5\}$. In this case, the Lie algebra of Ricci collineations is spanned by $\{K_1, K_3, K_4, K_5, K_{6,-2}, K_7\}$ if and only if $H = A(x_3)$ is a solution of

  $$A - \frac{1}{3}x_3A' + \frac{1}{6}x_3^2A'' = \varepsilon\frac{1}{x_3^2}, \qquad \varepsilon = \pm 1,$$

  whose explicit solutions are

  $$H = A(x_3) = c_1x_3^{3/2}\cos\left(\frac{\sqrt{15}}{2}\ln x_3\right) + c_2x_3^{3/2}\sin\left(\frac{\sqrt{15}}{2}\ln x_3\right) + \frac{3\varepsilon}{8x_3^2}.$$

- when H is "generic", that is, unless H satisfies some PDE forcing $\bar{H}$ to belong to a subclass where more Killing vector fields occur, the Lie algebra of Ricci collineations of g coincides with the one of its Killing vector fields. Observe that in such a general case, by the inclusions we recalled in Sect. 1, all Lie algebras of Killing, homothetic and affine vector fields, curvature and Ricci collineations coincide.

A very similar argument applies to matter collineations. In fact, denoted by $T = \varrho - \frac{1}{2}\tau g$ the energy-momentum tensor of a Siklos spacetime (M, g) and recalling that

$\tau = 4\Lambda$, one has that $\hat{g} = -\frac{1}{\Lambda}T = -\frac{1}{\Lambda}\varrho + 2g$ is the Siklos metric described by (1.1) with the defining function

$$\hat{H} = H + \frac{1}{3}x_3(H_{,3}) - \frac{1}{6}x_3^2\left(H_{,33} + H_{,44}\right). \tag{3.5}$$

The above arguments lead to the following.

Theorem 3.3 *Let g denote a Siklos metric, as described by* (1.1) *for some defining function H. Then, its Ricci tensor ϱ is homothetic to the Siklos metric $\bar{g}$ described by* (1.1) *with defining function $\bar{H}$ given by* (3.4).

Consequently, g does not admit proper Ricci collineations (and so, neither proper homothetic and affine vector fields and curvature collineations), unless H satisfies some PDE, ensuring that $\bar{H}$ belongs to a subclass of Siklos metrics for which more Killing vector fields occur (see Table I*). In any case, the Lie algebra of (smooth) Ricci collineations of any Siklos metric is finite-dimensional.*

In the same way, the energy-momentum tensor T of a Siklos metric g, described by (1.1) *for some defining function H, is homothetic to another Siklos metric $\hat{g}$, described by* (1.1) *with defining function $\hat{H}$ given by* (3.5)*. Therefore, g does not admit proper matter collineations, unless H satisfies a suitable PDE, so that $\hat{H}$ belongs to a subclass of Siklos metrics, for which more Killing vector fields occur. In any case, the Lie algebra of (smooth) matter collineations of any Siklos metric is finite-dimensional.*

Remark 3.4 **(Ricci iterations)** Starting from a given metric $g = g_1$ on a manifold M, a *Ricci iteration* is a sequence of metrics $\{g_i\}_{i\in\mathbb{N}}$, such that $\varrho_{i+1} = g_i$ for all indices i. Ricci iteration is a discrete analogue of the Ricci flow and may be interpreted as a dynamical system on the space of metrics on M.

Ancient Ricci iterations are sequences of metrics $g_1, g_0, g_{-1}, \ldots$, satisfying

$$g_{i-1} = \varrho_i, \quad i \in \{1, 0, -1, \ldots\}.$$

These notions were first introduced with reference to Kähler metrics [29] and then considered in Riemannian settings, with particular regard to homogeneous spaces ([3, 28]). It is essential that the iteration preserves the signature of the metric, which makes Ricci iterations a quite rare phenomenon. In [28], existence of Ricci iterations of Riemannian homogeneous spaces was proved starting from dimension six. Our results on Siklos metrics show the existence of Ricci iterations in dimension four in Lorentzian settings, as this is the case for all Siklos metrics. In fact, we have the following.

Proposition 3.5 *Ricci iterations and ancient Ricci iterations exist for all Siklos metrics. Admitting the possibility of scaling the metrics (which does not affect their Ricci tensors), the class of Siklos metrics is closed with respect to Ricci iterations.*

It will be interesting to investigate convergence properties of Ricci and ancient Ricci iterations of Siklos metrics.

We conclude this section with a general result concerning Weyl collineations of Siklos metrics. For the general form of Siklos metrics as described in (1.1), by (2.5) we deduce that a smooth vector field $X = X^i \partial_i$ is a Weyl collineation if and only if its components X^i satisfy the following system of PDEs:

$$
\begin{aligned}
&(X^2{}_{,1})(H_{,34}) = 0,\\
&X^2{}_{,1}(H_{,44} - H_{,33}) = 0,\\
&X^3{}_{,1}(H_{,33} - H_{,44}) + 2(X^4{}_{,1})(H_{,34}) = 0,\\
&X^4{}_{,1}(H_{,44} - H_{,33}) + 2(X^3{}_{,1})(H_{,34}) = 0,\\
&X^2{}_{,3}(H_{,33} - H_{,44}) + 2(X^2{}_{,4})(H_{,34}) = 0,\\
&X^2{}_{,4}(H_{,44} - H_{,33}) + 2(X^2{}_{,3})(H_{,34}) = 0,\\
&2X^2(H_{,234})x_3 + 2X^3(H_{,334})x_3 + 2X^4(H_{,344})x_3\\
&\quad -x_3(X^3{}_{,4} - X^4{}_{,3})(H_{,44} - H_{,33})\\
&\quad +(2(X^4{}_{,4})x_3 + 2(X^3{}_{,3})x_3 + 4(X^2{}_{,2})x_3 - 4X^3)(H_{,34}) = 0,\\
&(2(X^3{}_{,3})x_3 + 2(X^2{}_{,2})x_3 - 2X^3)(H_{,44} - H_{,33}) - 4(X^4{}_{,3})(H_{,34})x_3\\
&\quad +X^2x_3(H_{,244} - H_{,233}) + X^3x_3(H_{,344} - H_{,333}) + X^4x_3(H_{,444} - H_{,334}) = 0,\\
&(2(X^4{}_{,4})x_3 + 2(X^2{}_{,2})x_3 - 2X^3)(H_{,44} - H_{,33}) + 4(X^3{}_{,4})(H_{,34})x_3 + X^2x_3\\
&\quad \times(H_{,244} - H_{,233}) + X^3x_3(H_{,344} - H_{,333}) + X^4x_3(H_{,444} - H_{,334}) = 0.
\end{aligned}
\tag{3.6}
$$

In particular, as the equations of the above system do not involve the component X^1, it follows at once that vector fields $X = X^1\partial_1$ are Weyl collineations for any arbitrary smooth function $X^1 = X^1(x_1, x_2, x_3, x_4)$. (It is easily seen that $X = X^1\partial_1$ is a Ricci collineation if and only if X^1 is a constant, that is, exactly when $X = c_1\partial_1$ is a Killing vector field). Therefore, we have the following.

Theorem 3.6 *For any defining function $H = H(x_2, x_3, x_4)$, the corresponding Siklos metric g, as described by (1.1), admits as Weyl collineations all vector fields of the form $X = X^1\partial_1$, for any smooth function $X^1 = X^1(x_1, x_2, x_3, x_4)$.*

Consequently, the Lie algebra of smooth Weyl collineations of any Siklos metric is infinite-dimensional.

4 Symmetries of homogeneous Siklos metrics

We shall now focus on the different classes of homogeneous Siklos spacetimes, corresponding to Cases **7)-12)** in Table I. As we already wrote, we shall always exclude the trivial case corresponding to Siklos metrics isometric to the anti-de Sitter space. We shall apply Theorem 3.3 to identify, for each Case, Siklos metrics admitting some proper Ricci and matter collineations.

Case 7): $H = A(x_3)$. In this case, Eq. (3.4) yields that the Ricci tensor of Siklos metric g is (up to scaling) the Siklos metric $\bar{g} = \frac{1}{\Lambda}\varrho$, of the form (1.1) for the defining function

$$\bar{H} = A - \frac{1}{3}x_3A' + \frac{1}{6}x_3^2A''.$$

Taking into account Table I, we see that g admits some proper Ricci collineations when $\bar{H}$ is of type either **11)** or **12)**, that is, when $A(x_3)$ satisfies the following PDE:

$$A - \frac{1}{3}x_3 A' + \frac{1}{6}x_3^2 A'' = \pm x_3^\alpha,$$

which by integration yields

$$H = A(x_3) = c_1 x_3^{\frac{3}{2}} \cos\left(\frac{\sqrt{15}}{2}\ln x_3\right) + c_2 x_3^{\frac{3}{2}} \sin\left(\frac{\sqrt{15}}{2}\ln x_3\right) \pm \frac{6x_3^\alpha}{\alpha^2 - 3\alpha + 6}. \tag{4.1}$$

For the defining function $A(x_3)$ of the form (4.1), the Lie algebra of Ricci collineations of g corresponds to the Lie algebra of Killing vector fields of type **11)** when $\alpha \neq -2$ and of type **12)** for $\alpha = -2$.

With regard to matter collineations, if $H = A(x_3)$, from (3.5) we get that the energy-momentum tensor of Siklos metric g is (up to scaling) the Siklos metric $\hat{g} = -\frac{1}{\Lambda}T$, of the form (1.1) for the defining function

$$\hat{H} = A + \frac{1}{3}x_3 A' - \frac{1}{6}x_3^2 A''.$$

Then, g admits some proper matter collineations when $\hat{H}$ is of type either **11)** or **12)**, that is, when A satisfies

$$A + \frac{1}{3}x_3 A' - \frac{1}{6}x_3^2 A'' = \pm x_3^\alpha,$$

whence we get

$$H = A(x_3) = c_1 x_3^{\frac{3+\sqrt{33}}{2}} + c_2 x_3^{\frac{3-\sqrt{33}}{2}} \pm \frac{6x_3^\alpha}{\alpha^2 - 3\alpha - 6}. \tag{4.2}$$

Thus, we proved the following.

Theorem 4.1 *Let g denote a Siklos metric described by (1.1) for a defining function $H = A(x_3)$ as in type* **7)**. *Then:*

- *g admits proper Ricci collineations if and only if A is of the form (4.1). In this case, the Lie algebra of Ricci collineations of g coincides with the Lie algebra of Killing vector fields of type* **11)** *when $\alpha \neq -2$ and of type* **12)** *for $\alpha = -2$.*
- *g admits proper matter collineations if and only if A is of the form (4.2). In this case, matter collineations of g correspond to Killing vector fields of type either* **11)** *or* **12)***, depending on the value of α.*

Finally, for $H = A(x_3)$, $X = X^i\partial_i$ is a Weyl collineation if and only if the following system of PDEs is satisfied:

$$
\begin{aligned}
&2x_3A''({X^3}_{,3}+{X^2}_{,2})+X^3(x_3A'''-2A'')=0,\\
&2x_3A''({X^4}_{,4}+{X^2}_{,2})+X^3(x_3A'''-2A'')=0,\\
&A''({X^4}_{,3}-{X^3}_{,4})=0,\\
&({X^2}_{,1})A''=0,\\
&({X^2}_{,3})A''=0,\\
&({X^2}_{,4})A''=0,\\
&({X^3}_{,1})A''=0,\\
&({X^4}_{,1})A''=0.
\end{aligned}
$$

When $A''=0$, by Proposition 2.3 we have that g is conformally flat, so indeed any vector field is, trivially, a Weyl collineation. For A generic, the general solution of the system above is given by $X=X^1\partial_1+k\partial_2+f(x_2)\partial_4$, where k is a real constant and $f(x_2)$ is an arbitrary smooth function. So, we proved the following.

Proposition 4.2 *Let g denote a Siklos metric described by* (1.1) *for a defining function $H=A(x_3)$ as in type* **7)**. *For A generic, $X=X^i\partial_i$ is a Weyl collineation if and only if $X=X^1\partial_1+c_1\partial_2+f(x_2)\partial_4$, where X^1 is arbitrary, c_1 is a real constant and $f(x_2)$ is an arbitrary smooth function.*

Case 8): $H=A(x_2)x_3^2$. Equation (3.4) now implies that the Ricci tensor of Siklos metric g is (up to scaling) the Siklos metric $\bar{g}=\frac{1}{\Lambda}\varrho$, of the form (1.1) for the defining function

$$\bar{H}=\frac{2}{3}x_3^2A(x_2).$$

Clearly, being $\bar{H}$ of the same form of H, in general no proper Ricci collineations occur. By Table I we see that g admits some proper Ricci collineations only in the very special case where $\bar{H}$ is of type either **11)** or **12)**, that is, when

$$\frac{2}{3}x_3^2A(x_2)=\pm x_3^{\alpha},$$

whence, necessarily $\alpha=2$ and $A(x_2)=\pm\frac{3}{2}$, that is,

$$H=\pm\frac{3}{2}x_3^2. \tag{4.3}$$

But then, g is homothetic to type 11) with $\alpha=2$.

From (3.5) we find that the energy-momentum tensor of Siklos metric g defined by $H=A(x_2)x_3^2$ is (up to scaling) the Siklos metric $\hat{g}=-\frac{1}{\Lambda}T$, of the form (1.1) for the defining function

$$\hat{H}=\frac{4}{3}x_3^2A(x_2).$$

Then, g admits some proper matter collineations if and only if $-\frac{4}{3}x_3^2A(x_2) = \pm x_3^{\alpha}$, whence, $\alpha = 2$ and $A(x_2) = \pm\frac{3}{4}$, that is,

$$H = \pm\frac{3}{4}x_3^2. \tag{4.4}$$

In this way, we proved the following.

Theorem 4.3 *Siklos metrics g described by* (1.1) *for a defining function* $H = x_3^2A(x_2)$ *as in type* **8)***:*

- *do not admit proper Ricci collineations;*
- *do not admit proper matter collineations.*

Finally, for $H = x_3^2A(x_2)$, $X = X^i\partial_i$ is a Weyl collineation if and only if

$$\begin{aligned}
&2x_3A(X^3{}_{,3} + X^2{}_{,2}) + x_3X^2A' - 2X^3A = 0,\\
&2x_3A(X^4{}_{,4} + X^2{}_{,2}) + x_3X^2A' - 2X^3A = 0,\\
&A(X^4{}_{,3} - X^3{}_{,4}) = 0,\\
&A(X^2{}_{,1}) = 0,\\
&A(X^2{}_{,3}) = 0,\\
&A(X^2{}_{,4}) = 0,\\
&A(X^3{}_{,1}) = 0,\\
&A(X^4{}_{,1}) = 0.
\end{aligned}$$

Integrating the above system, we prove the following.

Proposition 4.4 *Let g denote a Siklos metric described by* (1.1) *for a defining function* $H = A(x_3)$ *as in type* **8)**. *For A generic,* $X = X^i\partial_i$ *is a Weyl collineation if and only if*

$$\begin{aligned}
X = X^1\partial_1 + \frac{c_1}{\sqrt{A(x_2)}}\partial_2 + (2f_1(x_2)x_4 + f_2(x_2))\,x_3\partial_3\\
+\Big((x_3^2 + x_4^2)f_1(x_2) + f_2(x_2)x_4 + f_3(x_2)\Big)\,\partial_4,
\end{aligned}$$

where X^1 *is arbitrary,* c_1 *is a real constant and* $f_i(x_2), i = 1, 2, 3,$ *are arbitrary smooth functions.*

Case 9) $H = x_2^{-2\beta-2}\,A(x_2^{\beta}x_3)$. By (3.4) we now get that the Ricci tensor of Siklos metric g is (up to scaling) the Siklos metric $\bar{g} = \frac{1}{\Lambda}\varrho$, of the form (1.1) for the defining function

$$\bar{H} = \frac{1}{6x_2^2}\left(6x_2^{-2\beta}A(x_2^\beta x_3) - 2x_2^{-\beta}x_3A'(x_2^\beta x_3) + x_3^2A''(x_2^\beta x_3)\right).$$

Observe that $\bar{H}$ is still of the same type of H. In fact,

$$\begin{aligned}\bar{H} &= x_2^{-2\beta-2}\bar{A}(x_2^\beta x_3)\\ &= x_2^{-2\beta-2}\left(A(x_2^\beta x_3) - \frac{1}{3}(x_2^\beta x_3)A'(x_2^\beta x_3) + \frac{1}{6}(x_2^\beta x_3)^2A''(x_2^\beta x_3)\right).\end{aligned}$$

Thus, in general no proper Ricci collineations occur. Taking into account Table I, g admits some proper Ricci collineations only in the very special case where $\bar{H}$ is of type either **11)** or **12)**, that is, for

$$x_2^{-2\beta-2}\bar{A}(x_2^\beta x_3) = \pm x_3^\alpha,$$

whence,

$$\bar{A}(x_2^\beta x_3) = \pm x_3^\alpha x_2^{2\beta+2} = \pm(x_2^{\frac{2\beta+2}{\alpha}}x_3)^\alpha,$$

which necessarily implies that $\frac{2\beta+2}{\alpha} = \beta$, that is, $\alpha = \frac{2\beta+2}{\beta}$ and, setting $t = x_2^\beta x_3$, $A(t) = A(x_2^\beta x_3)$ satisfies

$$A(t) - \frac{1}{3}tA'(t) + \frac{1}{6}t^2A''(t) = \pm t^{\frac{2\beta+2}{\beta}}.$$

Integrating the above equation, we get

$$\begin{aligned}A(x_2^\beta x_3) &= c_1(x_2^\beta x_3)^{\frac{3}{2}}\cos\left(\frac{\sqrt{15}}{2}\ln(x_2^\beta x_3)\right) + c_2(x_2^\beta x_3)^{\frac{3}{2}}\sin\left(\frac{\sqrt{15}}{2}\ln(x_2^\beta x_3)\right)\\ &\quad\pm\frac{3\beta^2(x_2^\beta x_3)^{\frac{2\beta+2}{\beta}}}{2\beta^2+\beta+2}.\end{aligned} \tag{4.5}$$

The Lie algebra of Ricci collineations of g coincides then with the Lie algebra of Killing vector fields of type either **11)** or **12)**, with $\alpha = \frac{2\beta+2}{\beta}$.

Next, the energy-momentum tensor of Siklos metric g defined by $H = x_2^{-2\beta-2}A(x_2^\beta x_3)$ is (up to scaling) the Siklos metric $\hat{g} = -\frac{1}{\Lambda}T$, of the form (1.1) for the defining function

$$\hat{H} = \frac{1}{6x_2^2}\left(6x_2^{-2\beta}A(x_2^\beta x_3) + 2x_2^{-\beta}x_3A'(x_2^\beta x_3) - x_3^2A''(x_2^\beta x_3)\right).$$

Note that

$$
\begin{aligned}
\hat{H} &= x_2^{-2\beta-2}\hat{A}(x_2^\beta x_3)\\
&= x_2^{-2\beta-2}\left(A(x_2^\beta x_3) + \frac{1}{3}(x_2^\beta x_3)A'(x_2^\beta x_3) - \frac{1}{6}(x_2^\beta x_3)^2 A''(x_2^\beta x_3)\right).
\end{aligned}
$$

Thus, in general no proper matter collineations occur. Indeed, g admits some proper matter collineations only in the very special case where $\hat{H}$ is of type either **11)** or **12)**, that is, when

$$
x_2^{-2\beta-2}\hat{A}(x_2^\beta x_3) = \pm x_3^\alpha
$$

and so,

$$
\hat{A}(x_2^\beta x_3) = \pm x_3^\alpha x_2^{2\beta+2} = \pm(x_2^{\frac{2\beta+2}{\alpha}} x_3)^\alpha,
$$

whence, $\frac{2\beta+2}{\alpha} = \beta$, that is, $\alpha = \frac{2\beta+2}{\beta}$ and, setting $t = x_2^\beta x_3$, $A(t) = A(x_2^\beta x_3)$ satisfies

$$
A(t) + \frac{1}{3}tA'(t) - \frac{1}{6}t^2A''(t) = \pm t^{\frac{2\beta+2}{\beta}}.
$$

By integration we conclude that

$$
A(x_3x_2^\beta) = c_1(x_3x_2^\beta)^{\frac{3+\sqrt{33}}{2}} + c_2(x_3x_2^\beta)^{\frac{3-\sqrt{33}}{2}} \pm \frac{3\beta^2(x_3x_2^\beta)^{\frac{2\beta+2}{\beta}}}{4\beta^2-\beta-2}. \tag{4.6}
$$

Matter collineations of g then coincide with Killing vector fields of type either **11)** or **12)** with $\alpha = \frac{2\beta+2}{\beta}$.

In this way, we proved the following.

Theorem 4.5 *Let g denote a Siklos metric described by (1.1) for a defining function $H = x_2^{-2\beta-2}A(x_2^\beta x_3)$ as in type* **9)**. *Then:*

- *g admits proper Ricci collineations if and only if A is of the form (4.5). In this case, the Lie algebra of Ricci collineations of g coincides with the Lie algebra of Killing vector fields of type either* **11)** *or* **12)** *with $\alpha = \frac{2\beta+2}{\beta}$.*
- *g admits proper matter collineations if and only if A is of the form (4.6). In this case, matter collineations of g correspond to Killing vector fields of type either* **11)** *or* **12)** *with $\alpha = \frac{2\beta+2}{\beta}$.*

Finally, for $H = x_2^{-2\beta-2}A(x_2^\beta x_3)$, $X = X^i\partial_i$ is a Weyl collineation if and only if

$$
\begin{aligned}
&2A''(x_2^\beta x_3)(x_2x_3(X^3{}_{,3} + X^2{}_{,2}) - x_3X^2 - x_2X^3)\\
&\quad + A'''(x_2^\beta x_3)(\beta x_3^2 x_2^\beta X^2 + x_3x_2^{\beta+1}X^3) = 0,\\
&2A''(x_2^\beta x_3)(x_2x_3(X^4{}_{,4} + X^2{}_{,2}) - x_3X^2 - x_2X^3)\\
&\quad + A'''(x_2^\beta x_3)(\beta x_3^2 x_2^\beta X^2 + x_3x_2^{\beta+1}X^3) = 0,
\end{aligned}
$$

$$
\begin{aligned}
&A''(x_2^\beta x_3)(X^4{}_{,3} - X^3{}_{,4}) = 0,\\
&A''(x_2^\beta x_3)(X^2{}_{,1}) = 0,\\
&A''(x_2^\beta x_3)(X^3{}_{,1}) = 0,\\
&A''(x_2^\beta x_3)(X^4{}_{,1}) = 0,\\
&A''(x_2^\beta x_3)(X^2{}_{,3}) = 0,\\
&A''(x_2^\beta x_3)(X^2{}_{,4}) = 0.
\end{aligned}
$$

In the general case, that is, assuming $A'' \neq 0$, by integration we obtain the following.

Proposition 4.6 *Let g denote a Siklos metric described by (1.1) for a defining function $H = x_2^{-2\beta-2}A(x_2^\beta x_3)$ as in type* **9)**. *For A generic, $X = X^i\partial_i$ is a Weyl collineation if and only if*

$$X = X^1\partial_1 + c_1x_2\partial_2 - c_1\beta x_3\partial_3 + (f(x_2) - c_1\beta x_4)\partial_4,$$

where X^1 is arbitrary, c_1 is a real constant and $f(x_2)$ is an arbitrary smooth function.

Case 10:) $H = A_{-2}(x_3, x_4)$. This case is more difficult to handle, as being $H = A_{-2}(x_3, x_4)$ homogeneous of degree -2 does not provide an explicit general description of H. To treat this case, we start with the more general case of a smooth function $H = A(x_3, x_4)$, before using the fact that by the well known properties of homogeneous functions, $A_{-2}(x_3, x_4)$ must satisfy the equation

$$2A + x_3(A_{,3}) + x_4(A_{,4}) = 0.$$

When $H = A(x_3, x_4)$, Eq. (3.4) yields that the Ricci tensor of Siklos metric g is (up to scaling) the Siklos metric $\bar{g} = \frac{1}{\Lambda}\varrho$, of the form (1.1) for the defining function

$$\bar{H} = A - \frac{1}{3}x_3(A_{,3}) + \frac{1}{6}x_3^2\left(A_{,33} + A_{,44}\right).$$

Observe that if A is homogeneous of order -2, so is $\bar{H}$. Thus, this does not yield further restrictions on the defining function.

Taking into account Table I, we see that g admits some proper Ricci collineations when $\bar{H}$ is of type either **11)** or **12)**. Indeed, since A will have to be homogeneous of order -2, the only possibility is that $\bar{H}$ is of type **12)**.

The PDE

$$A - \frac{1}{3}x_3(A_{,3}) + \frac{1}{6}x_3^2\left(A_{,33} + A_{,44}\right) = \pm\frac{1}{x_3^2}$$

yields the general solution of the form

$$A(x_3, x_4) = f_1(x_3)f_2(x_4) + f_3(x_3),$$

where f_3 is a function which we omit to write for sake of brevity, and f_1, f_2 are determined as solutions of the ODEs

$$f_1(x_3)'' = c_1 f_1(x_3) + \frac{2}{x_3^2}\left(x_3 f_1'(x_3) - 3 f_1(x_3)\right), \qquad f_2''(x_4) = -c_1 f_2(x_4),$$

for some real constant c_1. This general solution is compatible with $A(x_3, x_4)$ being homogeneous of order -2 only when $A(x_3, x_4) = \pm\frac{3}{8x_3^2}$. But then, g is homothetic to type **12)**.

A similar argument applies to matter collineations. If $H = A(x_3, x_4)$, (3.5) yields that the energy-momentum tensor of Siklos metric g is (up to scaling) the Siklos metric $\hat{g} = -\frac{1}{\Lambda}T$, of the form (1.1) for the defining function

$$\hat{H} = A + \frac{1}{3}x_3(A_{,3}) - \frac{1}{6}x_3^2\left(A_{,33} + A_{,44}\right).$$

Then, g admits some proper Ricci collineations when $\hat{H}$ is of type 12), taking into account that A is homogeneous of order -2. Integrating the PDE

$$A + \frac{1}{3}x_3(A_{,3}) - \frac{1}{6}x_3^2\left(A_{,33} + A_{,44}\right) = \pm\frac{1}{x_3^2}$$

yields an explicit solution, which is compatible with $A(x_3, x_4)$ being homogeneous of order -2 only when $A(x_3, x_4) = \pm\frac{3}{2x_3^2}$, which is again homothetic to case **12)**. Thus, we proved the following.

Theorem 4.7 *Siklos metrics g described by* (1.1) *for a defining homogeneous function $H = A_{-2}(x_3, x_4)$ as in type* **10)***:*

- *do not admit proper Ricci collineations;*
- *do not admit proper matter collineations.*

Due to the complexity of this case, We were not able to classify in full generality Weyl collineations for Siklos metrics of this type. Restricting to the case where $H = A_{-2}(x_3, x_4)$ is a *polynomial* homogeneous function of degree -2, that is, $H = A_{-2}(x_3, x_4) = \frac{k_1}{x_3^2} + \frac{k_2}{x_3 x_4} + \frac{k_3}{x_4^2}$, we find by a direct calculation, starting from Eq. (3.6), that $X = X^i\partial_i$ is a Weyl collineation if and only if

$$X = X^1\partial_1 + f(x_2)\partial_2 + \frac{1}{2}f'(x_2)(x_3\partial_3 + x_4\partial_4),$$

where X^1 is arbitrary and $f(x_2)$ is an arbitrary smooth function.

Cases 11-12) $H = \pm x_3^\alpha$. To exclude trivial cases, we shall assume $\alpha \neq 0$ (and also $\alpha \neq 1$ for the Weyl collineations). In this case, it follows from Eq. (3.4) that the Ricci tensor of Siklos metric g is (up to scaling) the Siklos metric $\bar{g} = \frac{1}{\Lambda}\varrho$, of the form (1.1) for the defining function

$$\bar{H} = \pm \frac{\alpha^2 - 3\alpha + 6}{6} x_3^{\alpha}.$$

Thus, no proper Ricci collineations occur. Indeed, in order to have proper Ricci collineations, we should have

$$\pm \frac{\alpha^2 - 3\alpha + 6}{6} x_3^{\alpha} = \pm x_3^{-2}.$$

But then, $\alpha = -2$, so that we are already in case **12)**, and Killing vector fields and Ricci collineations coincide.

A similar argument holds for matter collineations. If $H = \pm x_3^{\alpha}$, then (3.5) yields that the energy-momentum tensor of Siklos metric g is (up to scaling) the Siklos metric $\hat{g} = -\frac{1}{\Lambda} T$, of the form (1.1) for the defining function

$$\hat{H} = \pm \frac{\alpha^2 - 3\alpha - 6}{6} x_3^{\alpha}$$

and no proper Ricci collineations occur, because in that case one should have

$$\pm \frac{\alpha^2 - 3\alpha - 6}{6} x_3^{\alpha} = \pm x_3^{-2},$$

so that $\alpha = -2$ and we are already in case **12)**.

Thus, we proved the following.

Theorem 4.8 *A Siklos metric g, described by* (1.1) *for a defining function $H = \pm x_3^{\alpha}$ as in types* **11),12)***,*

- *does not admit proper Ricci collineations.*
- *does not admit proper matter collineations.*

Finally, for $H = \pm x_3^{\alpha}$, $X = X^i \partial_i$ is a Weyl collineation if and only if

$$\begin{aligned}
&\alpha(\alpha-1)\left(2x_3^{\alpha-4}(X^2{}_{,2} + X^3{}_{,3}) + X^3 x_3^{\alpha-5}(\alpha-4)\right) = 0,\\
&\alpha(\alpha-1)\left(2x_3^{\alpha-4}(X^2{}_{,2} + X^4{}_{,4}) + X^3 x_3^{\alpha-5}(\alpha-4)\right) = 0,\\
&\alpha(\alpha-1)(X^4{}_{,3} - X^3{}_{,4}) = 0,\\
&\alpha(\alpha-1)X^2{}_{,1} = 0,\\
&\alpha(\alpha-1)X^2{}_{,3} = 0,\\
&\alpha(\alpha-1)X^2{}_{,4} = 0,\\
&\alpha(\alpha-1)X^3{}_{,1} = 0,\\
&\alpha(\alpha-1)X^4{}_{,1} = 0.
\end{aligned}$$

Excluding the trivial cases $\alpha = 0, 1$, by integration of the above system a straightforward calculation leads to the following.

Proposition 4.9 *Let g denote a Siklos metric described by* (1.1) *for a defining function* $H = \pm x_3^\alpha$, *as in types* **11),12)**. $X = X^i \partial_i$ *is a Weyl collineation if and only if*

$$X = X^1\partial_1 + f_1(x_2)\partial_2 + \frac{2}{2-\alpha} f_1'(x_2)x_3\partial_3 + \left(\frac{2}{2-\alpha} f_1'(x_2)x_4 + f_2(x_2)\right)\partial_4, \quad \alpha \neq 2,$$

$$X = X^1\partial_1 + c_1\partial_2 + (2f_1(x_2)x_4 + f_2(x_2))x_3\partial_3 + \left((x_3^2 + x_4^2) f_1(x_2) + f_2(x_2)x_4 + f_3(x_2)\right)\partial_4, \quad \alpha = 2,$$

where c_1 *is an arbitrary constant,* X^1 *and* $f_i(x_2), i = 1, 2, 3$ *are arbitrary smooth functions.*

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