# Topological Manin pairs and (n, s) -type series



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# Topological Manin pairs and $(n, s)$-type series

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## Abstract

Lie subalgebras of $L = \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]$, complementary to the diagonal embedding $\Delta$ of $\mathfrak{g}[\![x]\!]$ and Lagrangian with respect to some particular form, are in bijection with formal classical $r$-matrices and topological Lie bialgebra structures on the Lie algebra of formal power series $\mathfrak{g}[\![x]\!]$. In this work we consider arbitrary subspaces of $L$ complementary to $\Delta$ and associate them with so-called series of type $(n, s)$. We prove that Lagrangian subspaces are in bijection with skew-symmetric $(n, s)$-type series and topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$. Using the classificaiton of Manin pairs we classify up to twisting and coordinate transformations all quasi-Lie bialgebra structures. Series of type $(n, s)$, solving the generalized classical Yang-Baxter equation, correspond to subalgebras of $L$. We discuss their possible utility in the theory of integrable systems.


**Keywords** Lie bialgebras · quasi-Lie bialgebras · Manin pairs · Yang-Baxter equations · r-matrices · Lie algebra splittings

**Mathematics Subject Classification** 17B62 · 17B38 (Primary) · 17B80 (Secondary)

Dedicated to the memory of Yuri Manin.

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## 1 Introduction

Let $F$ be an algebraically closed field of characteristic 0 equipped with the discrete topology and $\mathfrak{g}$ be a simple Lie algebra over $F$. We define the Lie algebra $\mathfrak{g}[\![x]\!]$ to be the space $\mathfrak{g} \otimes F[\![x]\!]$ with the bracket

$$[a \otimes f, b \otimes g] = [a, b] \otimes fg$$

and we equip it with the $(x)$-adic topology. The continuous dual of $\mathfrak{g}[\![x]\!]$ is denoted by $\mathfrak{g}[\![x]\!]'$ and it is endowed with the discrete topology.

A topological Manin pair is a pair $(L, \mathfrak{g}[\![x]\!])$ where

1. $L$ is a Lie algebra equipped with an invariant non-degenerate symmetric bilinear form $B$;
2. $\mathfrak{g}[\![x]\!] \subset L$ is a Lagrangian subalgebra with respect to $B$;
3. for any continuous functional $T\colon \mathfrak{g}[\![x]\!] \to F$ there is $f \in L$ such that $T = B(f, -)$.

Topological Manin pairs were classified in [1] using the tools from [8]. More precisely, if $(L, \mathfrak{g}[\![x]\!])$ is a topological Manin pair, then $L$ is isomorphic, as a Lie algebra with form, to either $L(\infty)$ or $L(n, \alpha)$. In this paper we consider only the "non-degenerate" case, namely $L \cong L(n, \alpha)$.

As a Lie algebra

$$L(n, \alpha) = \mathfrak{g}(\!(x)\!) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x].$$

The bilinear form $B$ on $L(n, \alpha)$ is completely determined by the sequence $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$. For example, when $n = 0$ the form is given by

$$B(a \otimes f, b \otimes g) = \kappa(a, b)\mathrm{res}_0 \{\alpha(x) fg\},$$

where $\kappa$ is the Killing form on $\mathfrak{g}$ and $\alpha(x) := 1 + \alpha_{-2}x + \alpha_{-3}x^2 + \cdots \in F(\!(x)\!)$. In case $n > 0$ the form is given by a similar formula; see Sect. 2.

It was established in [1], that the following objects are in one-to-one correspondence

- Lagrangian subalgebras $W \subseteq L(n, 0), 0 \leq n \leq 2$, complementary to the diagonal

$$\Delta := \{(f, [f]) \mid f \in \mathfrak{g}[\![x]\!]\},$$

  i.e. $\Delta \dotplus W = L(n, 0)$;
- non-degenerate topological Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ and
- formal solutions to the classical Yang-Baxter equation (CYBE) in the form

$$\frac{y^n\Omega}{x-y} + g(x, y) = \Omega \sum_{k\geq 0} x^{-k-1}y^{k+n} + g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!], \tag{1}$$

  where $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is the quadratic Casimir element and $g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$.

Furthermore, the proof of the above-mentioned correspondence reveals that series Eq. (1) can be viewed as a generating series for the corresponding subalgebra $W$. The present paper can be thus considered as a continuation of [1], where we extend the preceding correspondence using series of type $(n, s)$.

To define a series of type $(n, s)$ fix a basis $\{b_i\}_{i=1}^d$ of $\mathfrak{g}$, orthonormal with respect to its Killing form $\kappa$, and interpret $y^n\Omega/(x-y)$ as a series

$$\frac{y^n\Omega}{x-y} = \sum_{k=0}^{\infty}\sum_{i=1}^{d} w_{k,i} \otimes b_i y^k \in \big((\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]) \otimes \mathfrak{g}\big)[\![y]\!]. \tag{2}$$

This expression might be understood as a Taylor series expansion. Elements $w_{k,i} \in \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]$ are presented explicitly in Eq. (20). A series of type $(n, s)$ is a series of the form

$$\frac{s(x)y^n\Omega}{x-y} + g(x, y) \in \big((\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]) \otimes \mathfrak{g}\big)[\![y]\!], \tag{3}$$

where $s \in F[\![x]\!]^\times$ and $g \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$; See Definition 3.2. For each series $r$ of type $(n, s)$ we define another series $\overline{r}$ of the same type as follows

$$\overline{r} := \frac{s(y)x^n\Omega}{x-y} - \tau(g(y, x)), \tag{4}$$

where $\tau$ is the $F[\![x, y]\!]$-linear extension of the map $a \otimes b \mapsto b \otimes a$.

Each series of type $(n, s)$ produces a subspace of $\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]$ complementary to the diagonal embedding $\Delta$ of $\mathfrak{g}[\![x]\!]$. The following results generalize the above-mentioned correspondence from [1].

**Theorem A** *Let $n \in \mathbb{Z}_{\geq 0}$ and $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$ be an arbitrary sequence with the corresponding series $\alpha(x) := x^{-n} + \alpha_{n-2}x^{-n+1} + \cdots + \alpha_0 x^{-1} + \cdots \in F((x))$. For any $(n, s)$-type series*

$$r = \sum_{k=0}^{\infty}\sum_{i=1}^{d} f_{k,i} \otimes b_i y^k \in \big((\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]) \otimes \mathfrak{g}\big)[\![y]\!] \tag{5}$$

*define the space*

$$W(r) := \operatorname{span}_F\{f_{k,i} \mid k \geq 0,\ 1 \leq i \leq d\} \subseteq \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]. \tag{6}$$

*The following results are true:*

1. *$W$ defines a bijection between series of type $\left(n, \frac{1}{x^n\alpha(x)}\right)$ and subspaces $V \subset L(n, \alpha)$ complementary to the diagonal $\Delta$, i.e. $L(n, \alpha) = \Delta \dotplus V$;*
2. *For any series $r$ of type $\left(n, \frac{1}{x^n\alpha(x)}\right)$ we have $W(r)^\perp = W(\overline{r})$ inside $L(n, \alpha)$;*

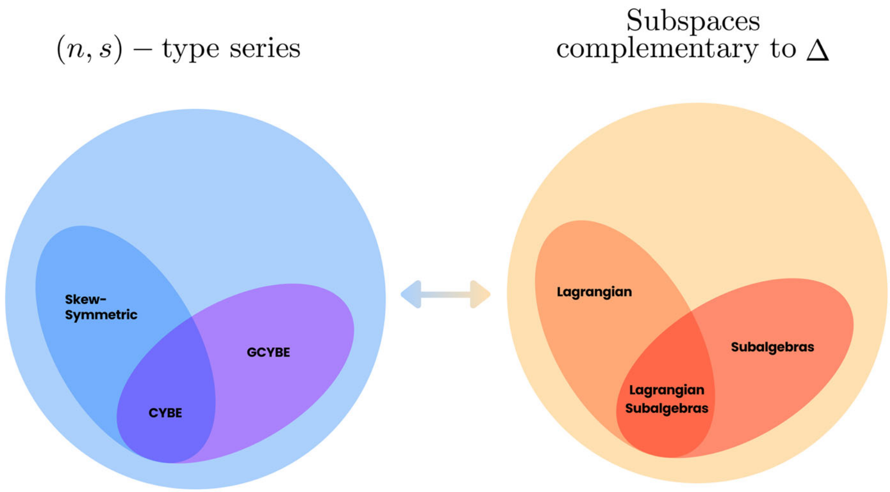


**Fig. 1** Series-subspaces correspondence

3. *Any series* $r$ *of type* $\left(n, \frac{1}{x^n\alpha(x)}\right)$ *satisfies* $\mathrm{GCYB}(r) = \psi$ *(see Definition 3.5 for the meaning of* $\mathrm{GCYB}(r)$*), where* $\psi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[\![x_1, x_2, x_3]\!]$ *is defined by*

$$B(v_1 \otimes v_2 \otimes v_3, \psi) = B(v_1, [v_2, v_3])$$

*for all* $v_1 \in W(\overline{r})$, $v_2, v_3 \in W(r)$.

In particular, considering the case when $r$ is skew-symmetric, meaning $r = \overline{r}$, or when $\psi = 0$ we get the following correspondences.

**Corollary B** *Let* $n \in \mathbb{Z}_{\geq 0}$, $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$ *and* $W$ *be the map from Theorem A. Then*

1. $W$ *defines a bijection between skew-symmtric* $\left(n, \frac{1}{x^n\alpha(x)}\right)$*-type series and Lagrangian subspaces* $V \subseteq L(n, \alpha)$*, complementary to the diagonal* $\Delta$*;*
2. $W$ *defines a bijection between* $\left(n, \frac{1}{x^n\alpha(x)}\right)$*-type series solving the GCYBE and subalgebras* $V \subseteq L(n, \alpha)$ *complementary to the diagonal* $\Delta$.

Observe that an $(n, s)$-type series produces a subspace of $L(n, \alpha)$ for any sequence $\alpha$. However, to obtain the compatibility with the form, given by $\alpha$, we need the equality $s(x) = 1/(x^n\alpha(x))$. In this case, the components $f_{k,i}$ and $b_i y^k$ of the series become dual bases for $W(r)$ and $\Delta$ respectively.

The requirement on a series $r$ of type $(n, s)$ to solve the CYBE is equivalent to being skew-symmetric and to solve the GCYBE. Together with Corollary B this implies that Lagrangian subalgebras $W \subset L(n, \alpha)$, satisfying $W \dotplus \Delta = L(n, \alpha)$, are in bijection with $(n, 1/(x^n\alpha(x)))$-type series solving the classical Yang-Baxter equation. These correspondences are schematically depicted in Fig. 1.

***Remark 1.1*** Let $r$ be a series of type $(n, s)$. Applying the projection $(a, b)\otimes c \mapsto a\otimes c$ onto the left component to $r$ we obtain the series

$$r_{\text{proj}} = \frac{s(x)y^n\Omega}{x-y} + g(x, y) \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]. \tag{7}$$

Conversely, starting with a series $r_{\text{proj}}$ of the form Eq. (7), we can obtain an $(n, s)$-type series $r$ by taking two Taylor series expansions of $r_{\text{proj}}$ at $x = 0$ and $y = 0$ respectively and then constructing $r$ by combining the coefficients of $b_i y^k$, $k \geq 0$, in these expansions. These two constructions are inverse to each other and hence both $r$ and its projection $r_{\text{proj}}$ contain exactly the same information. Consequently, all the statements made for $(n, s)$-type series can be stated in terms of their projections onto the left component and vice versa. In cotrast with [1], in this paper we give preference to series of type $(n, s)$ rather than to their projections, because the statement that series of type $(n, s)$ generate subspaces of $L(n, \alpha)$ becomes transparent.

Reinterpreting the results of [1] in terms of $(n, s)$-type series we see that skew-symmetric series of type $(n, 1/(x^n\alpha(x)))$, that also solve the GCYBE, exist only for $n = 0, 1$ and $n = 2$ with $\alpha_0 = 0$.

Lagrangian subalgebras of $L(n, \alpha)$, complementary to $\Delta$, correspond to topological Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$. If we instead consider Lagrangian subspaces (not necessarily subalgebras) of $L(n, \alpha)$, we get so called (non-degenerate) topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$. A topological quasi-Lie bialgebra structure on $\mathfrak{g}[\![x]\!]$ consists of

- a skew-symmetric continuous linear map $\delta\colon \mathfrak{g}[\![x]\!] \to (\mathfrak{g}\otimes\mathfrak{g})[\![x, y]\!]$ and
- a skew-symmetric element $\varphi \in (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x, y, z]\!]$,

which are subject to the following three conditions

- $\delta([a, b]) = [a\otimes 1 + 1\otimes a, \delta(b)] - [b\otimes 1 + 1\otimes b, \delta(a)]$, i.e. $\delta$ is a 1-cocycle;
- $\frac{1}{2}\text{Alt}((\delta\otimes 1)\delta(a)) = [a\otimes 1\otimes 1 + 1\otimes a\otimes 1 + 1\otimes 1\otimes a, \varphi]$;
- $\text{Alt}((\delta\otimes 1\otimes 1)\varphi) = 0$,

where $\text{Alt}(x_1\otimes\ldots\otimes x_n) := \sum_{\sigma\in S_n}\text{sgn}(\sigma)x_{\sigma(1)}\otimes\ldots\otimes x_{\sigma(n)}$.

Following [5] we prove the following direct relation between $\delta$, $\varphi$ and skew-symmetric $(n, s)$-type series $r$.

**Proposition C** *There is a bijection between topological quasi-Lie bialgebras and skew-symmetric $(n, s)$-type series. Let $r$ be the $(n, s)$-type series corresponding to $(\mathfrak{g}[\![x]\!], \delta, \varphi)$, then, under the identification $\mathfrak{g}[\![x]\!] \cong \Delta$, we have the following identities:*

- $[a\otimes 1 + 1\otimes a, r] = -\delta(a)$ *for any* $a \in \mathfrak{g}[\![x]\!]$ *and*
- $\text{CYB}(r) = -\varphi$.

*The same is true if $r$ is interpreted as an element in $(\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$.*

In view of this result we call skew-symmetric $(n, s)$-type series quasi-$r$-matrices.

Repeating the ideas from [7] and [5] we show that topological quasi-Lie bialgebras can be twisted similar to topological Lie bialgebras. More precisely, if $\delta$ is a quasi-Lie bialgebra structure on $\mathfrak{g}[\![x]\!]$, given by the Lagrangian subspace $W$, and $s :=$

$\sum_i a_i \otimes b^i \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$ is an arbitrary skew-symmetric tensor, then

$$W_s := \left\{ \sum_i B(b^i, w)a_i - w \mid w \in W \right\} \tag{8}$$

is another (twisted) Lagrangian subspace complementary to the diagonal. This observation implies, that in order to classify all topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ up to twisting it is enough to find one single Lagrangian subspace within each $L(n, \alpha)$. Moreover, it was shown in [1] that substitutions of the form $x \mapsto x + a_2x^2 + a_3x^3 + \ldots, a_i \in F$, allow us to assume that $\alpha$ has the form

$$\alpha = (\ldots, 0, \alpha_0, 0, \ldots, 0).$$

Lagrangian subspaces for such $L(n, \alpha)$ are constructed in Sect. 4.1.

Using Theorem A and Proposition C we explain how twisting of a Lagrangian subspace $W \subset L(n, \alpha)$ is seen at the level of $\delta$ and the corresponding quasi-$r$-matrix $r$.

**Corollary D** *Let* $(\mathfrak{g}[\![x]\!], \delta, \varphi)$ *be a topological quasi-Lie bialgebra structure corresponding to the quasi-r-matrix r. If we twist* $W(r)$ *with a skew-symmetric tensor s we obtain another topological quasi-Lie bialgebra* $(\mathfrak{g}[\![x]\!], \delta_s, \varphi_s)$*, such that*

1. $W(r)_s = W(r - s)$;
2. $\delta_s = \delta + ds$;
3. $\varphi_s = \varphi + \mathrm{CYB}(s) - \frac{1}{2}\mathrm{Alt}((\delta \otimes 1)s)$.

Therefore, to describe all quasi-$r$-matrices up to twisting it is enough to find one single quasi-$r$-matrix for each $L(n, \alpha)$. We achieve that goal in Sect. 4.2 by writing out explicitly series of type $(n, s)$ for subspaces from Sect. 4.1.

The results above, in particular, show that if $r$ is a quasi-$r$-matrix and $\delta(a) := [a \otimes 1 + 1 \otimes a, r]$, then the condition

$$\mathrm{Alt}((\delta \otimes 1 \otimes 1)\mathrm{CYB}(r)) = 0 \tag{9}$$

is trivially satisfied.

We conclude the paper by using Theorem A for construction of Lie algebra splittings $\Delta \dotplus W = L(n, \alpha)$ and the corresponding $(n, s)$-type series, which we call generalized $r$-matrices. These constructions are important in the theory of integrable systems because of their use in the Adler-Konstant-Symes (AKS) scheme and the so-called $r$-matrix approach; see [4, 6]. The subalgebra splittings of $L(0, 0)$ as well as their physical applications were considered in e.g. [9, 10].

Our first result shows that in order to obtain new generalized $r$-matrices from subalgebra splittings $L(n, \alpha) = \Delta \dotplus W$ with $n > 2$, the subalgebra $W$ must be unbounded. Otherwise the situation can be reduced to the splitting of $L(2, \alpha)$.

**Proposition E** *Let* $L(n, \alpha) = \Delta \dotplus W$ *for some subalgebra* $W \subset L(n, \alpha)$ *and* $n > 2$. *Assume W is bounded, i.e. there is an integer* $N > 0$ *such that*

$$x^{-N}\mathfrak{g}[x^{-1}] \subseteq W_+ \subseteq x^N\mathfrak{g}[x^{-1}],$$

*where* $W_+$ *is the projection of* $W \subset L(n,\alpha) = \mathfrak{g}((x)) \oplus \mathfrak{g}[x]/x^n\mathfrak{g}[x]$ *on the first component* $\mathfrak{g}((x))$. *Then we have the inclusion*

$$\{0\} \times [x^2]\mathfrak{g}[x]/x^n\mathfrak{g}[x] \subseteq W$$

*and the image* $\widetilde{W}$ *under the canonical projection* $L(n,\alpha) \to L(2,\alpha)$ *is a subalgebra satisfying* $L(2,\alpha) = \Delta \dotplus \widetilde{W}$.

Despite this result we think that bounded subalgebras $W \subset L(n,\alpha)$ complementary to $\Delta$ are still interesting, because in the case $\alpha \neq 0$ they lead to unbounded orthogonal complements $W^\perp$ which are also important in view of the AKS scheme. We give examples of subalgebras of $L(n,\alpha)$ with unbounded orthogonal complements.

## 2 Topological Manin pairs

Let $F$ be an algebraically closed field of characteristic 0, $\mathfrak{g}$ be a finite-dimensional simple $F$-Lie algebra and $\mathfrak{g}[\![x]\!] := \mathfrak{g} \otimes F[\![x]\!]$ be the Lie algebra with the bracket defined by

$$[a \otimes f, b \otimes g] := [a,b] \otimes fg,$$

for all $a, b \in \mathfrak{g}$ and $f, g \in F[\![x]\!]$. From now on, we always endow $F$ with the discrete topology and view $\mathfrak{g}[\![x]\!]$ as a topological Lie algebra with the $(x)$-adic topology.

A *topological Manin pair* is a pair $(L, \mathfrak{g}[\![x]\!])$, where $L$ is a Lie algebra equipped with an invariant non-degenerate symmetric bilinear form $B$, such that

1. $\mathfrak{g}[\![x]\!] \subseteq L$ is a Lagrangian Lie subalgebra with respect to $B$;
2. for any continuous functional $T: \mathfrak{g}[\![x]\!] \to F$ there exists an element $f \in L$ such that $T = B(f, -)$.

The statements of [8, Proposition 2.9] and [1, Proposition 3.12] give a description of all topological Manin pairs. For precise formulation we need to repeat the definitions of some specific Lie algebras with forms from [1, Section 3.2] and [8, Section 2].

**Definition 2.1** We define the Lie algebra $L(\infty) := \mathfrak{g} \otimes A(\infty)$, where $A(\infty)$ is the unital commutative algebra with underlying space $\sum_{i \geq 0} Fa_i \dotplus F[\![x]\!]$ and multiplication given by

$$a_i a_j := 0,\ a_i x^j := a_{i-j} \text{ for } i \geq j \text{ and } a_i x^j := 0 \text{ otherwise.}$$

Let $\mathrm{t}: A \to F$ be the functional, given by $\mathrm{t}(a_0) := 1, \mathrm{t}(a_i) := 0, i \geq 1$ and $\mathrm{t}(F[\![x]\!]) := 0$. We equip $L(\infty)$ with the symmetric non-degenerate invariant bilinear form

$$B\left(a \otimes \left(\sum_{i \geq 0} c_i a_i, f(x)\right), b \otimes \left(\sum_{i \geq 0} t_i a_i, g(x)\right)\right)$$

$$:= \kappa(a,b)\,\mathrm{t}\left(g(x)\sum_{i\geq 0} c_i a_i + f(x)\sum_{i\geq 0} t_i a_i\right). \tag{10}$$

**Definition 2.2** Let $n \geq 1$ and $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$ be an arbitrary sequence. Consider the algebra

$$A(n,\alpha) := F((x)) \oplus F[x]/(x^n).$$

Abusing the notation we denote the element $x^{-n} + \alpha_{n-2}x^{-n+1} + \dots + \alpha_0 x^{-1} + \dots \in F((x))$ with the same letter $\alpha$. Define the functional $\mathrm{t}\colon A(n,\alpha) \to F$ by

$$\mathrm{t}(f,[p]) := \mathrm{res}_0\{\alpha(f-p)\}.$$

Taking the tensor product of $A(n,\alpha)$ with $\mathfrak{g}$ we get the Lie algebra

$$L(n,\alpha) := \mathfrak{g} \otimes A(n,\alpha) \tag{11}$$

which we equip with the form

$$B(a \otimes (f,[p]), b \otimes (g,[q])) := \kappa(a,b)\,\mathrm{t}(fg,[pq]). \tag{12}$$

It is known that the bilinear form $B$ is symmetric non-degenerate and invariant.

**Definition 2.3** Take an arbitrary sequence $\alpha = (\alpha_i \in F \mid -\infty < i \leq -2)$ and let $A(0,\alpha) := F((x))$. We define the functional $\mathrm{t}\colon A(0,\alpha) \to F$ by

$$\mathrm{t}(f) := \mathrm{res}_0\{\alpha f\},$$

where $\alpha = 1 + \alpha_{-2}x + \dots \in F((x))$. We equip the Lie algebra $L(0,\alpha) := \mathfrak{g} \otimes A(0,\alpha)$ with the bilinear form

$$B(a \otimes f, b \otimes g) := \kappa(a,b)\,\mathrm{t}(fg), \tag{13}$$

which is again symmetric non-degenerate and invariant. From now on we identify $F((x))$ with $F((x)) \times \{0\}$ and write $(f,0)$ for elements in $A(0,\alpha)$.

**Definition 2.4** A series of the form $\varphi = x + a_2x^2 + a_3x^3 + \dots \in F[[x]]$ is called a *coordinate transformation*. Coordinate transformations form a group $\mathrm{Aut}_0 F[[x]]$ under substitution which we view as a subgroup of automorphisms of $F[[x]]$.

An element $\varphi \in \mathrm{Aut}_0 F[[x]]$ induces an automorphism of $A(n,\alpha)$ by $f/g \mapsto \varphi(f)/\varphi(g)$ and $[p] \mapsto [\varphi(p)]$ that changes the functional $\mathrm{t}$ to $\mathrm{t} \circ \varphi$. We write $A(n,\alpha)^{(\varphi)}$ for the algebra $A(n,\alpha)$ with the functional $\mathrm{t} \circ \varphi$. It is not hard to see that for any $\varphi \in \mathrm{Aut}_0 F[[x]]$ there is a sequence $\beta$ such that $A(n,\alpha)^{(\varphi)} = A(n,\beta)$.

Let $(L, \mathfrak{g}[\![x]\!])$ be a topological Manin pair. According to [8, Proposition 2.9] as a Lie algebra with form $L \cong L(\infty)$ or $L \cong L(n, \alpha)$, for some $n \geq 0$ and some sequence $\alpha$. Here we identify $\mathfrak{g}[\![x]\!]$ with the diagonal

$$\Delta := \{(f, [f]) \mid f \in \mathfrak{g}[\![x]\!]\} \subset L(n, \alpha).$$

Moreover, we can assume that all the elements $\alpha_i$ in the sequence $\alpha$, except maybe $\alpha_0$, are 0 by virtue of the following result.

**Proposition 2.5** *[1,* Proposition 3.12] *Let $n \geq 0$ and $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$ be a sequence. There exists a $\varphi \in \mathrm{Aut}_0 F[\![x]\!]$ such that $A(n, \alpha) \cong A(n, \beta)^{(\varphi)}$, where $\beta$ is the sequence satisfying $\beta_i = 0$ for all $i \neq 0$ and $\beta_0 = \alpha_0$.*

***Remark 2.6*** Observe that the result of Proposition 2.5 can be interpreted in terms of a formal differential equation. Consider an arbitrary $\alpha(x) = x^{-n} + \alpha_{n-2}x^{-n+1} + \cdots + \alpha_0 x^{-1} + \cdots \in F(\!(x)\!)$ and $\beta(x) = x^{-n} + \alpha_0 x^{-1}$. Then the functionals $\mathrm{t}_\alpha$ and $\mathrm{t}_\beta$ defined on $A(n, \alpha)$ and $A(n, \beta)$ respectively are given by

$$\mathrm{t}_\alpha(f, [p]) = \mathrm{res}_0\{\alpha(f - p)\} \quad \text{and} \quad \mathrm{t}_\beta(f, [p]) = \mathrm{res}_0\{\beta(f - p)\})$$

The equality $A(n, \alpha)^{(\varphi)} = A(n, \beta)$ can be expressed as

$$\mathrm{res}_0\{\beta(x) f(x)\} = \mathrm{res}_0\{\alpha(x) f(\varphi(x))\} = \mathrm{res}_0\{\alpha(\psi(x)) f(x) \psi'(x)\}, \tag{14}$$

where $\psi \in \mathrm{Aut}_0(F[\![x]\!])$ is the compositional inverse of $\varphi$, i.e. $\varphi(\psi(x)) = x$. Since the residue pairing is non-degenerate on $F(\!(x)\!)$, we obtain

$$\alpha(\psi(x))\psi'(x) = \beta(x). \tag{15}$$

In particular, the transformation $\varphi$ is the compositional inverse of the solution to Eq. (15).

## 3 Series of type $(n, s)$ and subspaces of $L(n, \alpha)$

Let $\{b_i\}_{i=1}^d$ be an othonormal basis of $\mathfrak{g}$ with respect to the Killing form $\kappa$. We write $\Omega$ for the quadratic Casimir element $\sum_{i=1}^d b_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$. It satisfies the identity $[a \otimes 1 + 1 \otimes a, \Omega] = 0$ for all $a \in \mathfrak{g}$.

In this section we describe a bijection between subspaces $W \subset L(n, \alpha)$ complementary to $\Delta$ and certain series. The following definition introduces convenient spaces containing these series.

**Definition 3.1** We put $A_1(n, \alpha) := A(n, \alpha) = F(\!(x_1)\!) \oplus F[x_1]/(x_1^n)$ and then define inductively the algebras

$$A_m(n, \alpha) := A_{m-1}(n, \alpha)(\!(x_m)\!) \oplus A_{m-1}(n, \alpha)[x_m]/x_m^n A_{m-1}(n, \alpha), \; m > 1. \tag{16}$$

The functional $\mathrm{t}$ defined on $A(n,\alpha)$ extends inductively to a functional on $A_m(n,\alpha)$. More precisely,

$$\mathrm{t}\left(\sum_{k\geq -N} f_k x_m^k, \sum_{\ell=0}^{n-1}[g_\ell x_m^\ell]\right) := \sum_{k\geq -N} \mathrm{t}(f_k)\,\mathrm{t}(x_m^k,0) + \sum_{\ell=0}^{n-1}\mathrm{t}(g_\ell)\,\mathrm{t}(0,[x_m]^\ell), \tag{17}$$

where $f_k, g_\ell \in A_{m-1}(n,\alpha)$. Since $\mathrm{t}(x^n F[\![x]\!]) = 0$, the sum on the right-hand side of Eq. (17) is finite and well-defined. This allows us to extend the form $B$ on $L(n,\alpha)$ to a symmetric non-degenerate bilinear form on the $\mathfrak{g}$-module

$$L_m(n,\alpha) := \mathfrak{g}^{\otimes m} \otimes A_m(n,\alpha) \tag{18}$$

by letting

$$B((a_1 \otimes \ldots \otimes a_m) \otimes f, (b_1 \otimes \ldots \otimes b_m) \otimes g) := \mathrm{t}(fg) \prod_{k=1}^{m} \kappa(a_k, b_k), \tag{19}$$

for all $a_1, \ldots, a_m, b_1, \ldots, b_m \in \mathfrak{g}$ and $f, g \in A_m(n,\alpha)$.

Fix some integer $n \geq 0$. We interpret the quotient $y^n\Omega/(x-y)$ in the following way

$$\begin{aligned}\frac{y^n\Omega}{x-y} &= \sum_{k=0}^{n-1}\sum_{i=1}^{d} b_i(0, -[x]^{(n-1)-k}) \otimes b_i(y^k, [y]^k)\\ &+ \sum_{k=n}^{\infty}\sum_{i=1}^{d} b_i(x^{(n-1)-k}, 0) \otimes b_i(y^k, 0)\\ &= \sum_{k=0}^{\infty}\sum_{i=1}^{d} w_{k,i} \otimes b_i(y^k, [y]^k) \in (L(n,\alpha)\otimes\mathfrak{g})\,[\![(y,[y])]\!] \subset L_2(n,\alpha),\end{aligned} \tag{20}$$

where $\alpha$ is an arbitrary sequence and we write $b_i(x^\ell, [x]^m)$ meaning $b_i \otimes (x^\ell, [x]^m)$.

**Definition 3.2** Since $(L(n,\alpha)\otimes\mathfrak{g})\,[\![(y,[y])]\!]$ is an $F[\![x]\!] \cong F[\![(x,[x])]\!]$-module and

$$(\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!] \cong (\Delta\otimes\mathfrak{g})[\![(y,[y])]\!] \subset (L(n,\alpha)\otimes\mathfrak{g})\,[\![(y,[y])]\!]$$

the series

$$r(x,y) = \frac{s(x)y^n\Omega}{x-y} + g(x,y), \tag{21}$$

where $g \in (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$ and $s \in F[\![x]\!]^\times$, is also inside $(L(n,\alpha)\otimes\mathfrak{g})\,[\![(y,[y])]\!]$. Series of the form Eq. (21) are called *series oftype* $(n,s)$.

***Remark 3.3*** Every series

$$r(x, y) = \frac{h(x, y)\Omega}{x - y} + g(x, y) \in L_2(n, \alpha),$$

where $h \in F[\![x, y]\!]$, $h(x, x) \neq 0$ and $g \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$, has a unique representation as a series of type $(n, s)$. Indeed, write $h(x, x) = x^n s(x)$ for some $s \in F[\![x]\!]^\times$. Then $h(x, y) - y^n s(x) = (x - y) f(x, y)$ for some $f \in F[\![x, y]\!]$. This implies that we can rewrite $r$ in the $(n, s)$ form

$$r(x, y) = \frac{s(x) y^n \Omega}{x - y} + f(x, y)\Omega + g(x, y). \tag{22}$$

In the construction of $f$ we are using the fact that for any $F$-vector space $V$ and any element $h \in V[\![x, y]\!]$

$$h(z, z) = 0 \implies h(x, y) = (x - y) f(x, y) \tag{23}$$

for some $f \in V[\![x, y]\!]$.

**Definition 3.4** For each series $r$ of type $(n, s)$ we define another series $\overline{r}$ of the same type $(n, s)$ by

$$\overline{r}(x, y) := \frac{s(y) x^n \Omega}{x - y} - \tau(g(y, x)) \in (L(n, \alpha) \otimes \mathfrak{g})\, [\![(y, [y])]\!], \tag{24}$$

where $\tau$ is the $F[\![x, y]\!]$-linear extension of the map $a \otimes b \mapsto b \otimes a$. To see that this is an $(n, s)$-type series its enough to apply the argument from Remark 3.3. Series of type $(n, s)$, satisfying $r = \overline{r}$, are called *skew-symmetric*.

**Definition 3.5** *The generalized classical Yang-Baxter equation (GCYBE)* is the equation for an $(n, s)$-type series of the form

$$\begin{aligned} \mathrm{GCYB}(r) := [r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] \\ +[r^{13}(x_1, x_3), \overline{r}^{23}(x_2, x_3)] = 0. \end{aligned} \tag{25}$$

Here $(-)^{13} \colon L_2(n, \alpha) \to (U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g})) \otimes A_3(n, \alpha)$ is the inclusion map given by

$$\begin{aligned} a \otimes b \otimes \left( \sum_{k \geq -N} F(x_1, [x_1]) x_2^k, \sum_{m=0}^{n-1} G(x_1, [x_1])[x_2]^m \right) \\ \mapsto a \otimes 1 \otimes b \otimes \left( \sum_{k \geq -N} F(x_1, [x_1]) x_3^k, \sum_{m=0}^{n-1} G(x_1, [x_1])[x_3]^m \right). \end{aligned}$$

Other inclusions are defined in a similar manner. The commutators are then taken in the associative $A_3(n, \alpha)$- algebra $(U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g})) \otimes A_3(n, \alpha)$.

Before formulating the main theorem of the section we note that if $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$ is an arbitrary sequence and $\alpha(x) = x^{-n} + \alpha_{n-2}x^{-n+1} + \cdots + \alpha_0 x^{-1} + \cdots \in F(\!(x)\!)$ is the corresponding series, then $x^n\alpha(x) \in F[\![x]\!]^\times$.

**Theorem 3.6** *Let $n \in \mathbb{Z}_{\geq 0}$ and $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$ be an arbitrary sequence with the corresponding series $\alpha(x) \in F(\!(x)\!)$. Consider the map*

$$W\colon L_2(n,\alpha) \longrightarrow \{V \subset L(n,\alpha) \mid V \text{ is a subspace}\}$$

*given by*

$$\sum_{i,j} b_i \otimes b_j \otimes \left( \sum_{k \geq -N_i} (f_k^{ij}, [p_k^{ij}])x^k, \sum_{m=0}^{n-1} (g_m^{ij}, [q_m^{ij}])[x]^m \right)$$
$$\mapsto \operatorname{span}_F \left\{ b_i(f_k^{ij}, [p_k^{ij}]) \mid k \geq -N, 1 \leq i, j \leq d \right\}.$$

*The following results are true:*

1. *$W$ defines a bijection between series of type $\left(n, \frac{1}{x^n\alpha(x)}\right)$ and subspaces $V \subseteq L(n,\alpha)$ complementary to the diagonal $\Delta$, i.e. $L(n,\alpha) = \Delta \dotplus V$;*
2. *For any series $r$ of type $\left(n, \frac{1}{x^n\alpha(x)}\right)$ we have $W(r)^\perp = W(\overline{r})$ inside $L(n,\alpha)$;*
3. *Any series $r$ of type $\left(n, \frac{1}{x^n\alpha(x)}\right)$ satisfies $\mathrm{GCYB}(r) = \psi$, where*

$$\psi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[\![(x_1,[x_1]),(x_2,[x_2]),(x_3,[x_3])]\!]$$

*is defined by*

$$B(v_1 \otimes v_2 \otimes v_3, \psi) = B(v_1, [v_2, v_3])$$

*for all $v_1 \in W(\overline{r})$, $v_2, v_3 \in W(r)$.*

***Proof*** Fix an $\left(n, \frac{1}{x^n\alpha(x)}\right)$-type series inside $(L(n,\alpha) \otimes \mathfrak{g})[\![(y,[y])]\!]$

$$\begin{aligned} r(x,y) &= \frac{1}{x^n\alpha(x)} \frac{y^n\Omega}{x-y} + g(x,y) \\ &= \sum_{k=0}^{\infty}\sum_{i=1}^{d} s_{k,i} \otimes b_i(y^k,[y]^k) + \sum_{k=0}^{\infty}\sum_{i=1}^{d} g_{k,i} \otimes b_i(y^k,[y]^k). \end{aligned}$$

It is easy to see that

$$U := \operatorname{span}_F\{w_{k,i} \mid k \geq 0, 1 \leq k \leq d\} \subset L(n,\alpha),$$

where $w_{k,i}$ are defined in Eq. (20), satisfies the condition $\Delta \dotplus U = L(n,\alpha)$. Since $s := \frac{1}{x^n\alpha(x)}$ is invertible, we have $sU \dotplus s\Delta = sU \dotplus \Delta = L(n,\alpha)$. In other words, the space

$$sU = \operatorname{span}_F\{s_{k,i} = sw_{k,i} \mid k \geq 0, 1 \leq k \leq d\} \subset L(n,\alpha) \tag{26}$$

is also complementary to the diagonal. Finally, since $g_{k,i} \in \Delta$ the space

$$W(r) = \operatorname{span}_F\{sw_{k,i} + g_{k,i} \mid k \geq 0, 1 \leq k \leq d\} \subset L(n,\alpha)$$

is complementary to the diagonal. Conversely, if $V \subset L(n,\alpha)$ satisfies $V \dotplus \Delta = L(n,\alpha)$, then for each $k \geq 0$ and $1 \leq i \leq d$ we can find a unique $g_{k,i} \in \Delta$ such that $sw_{k,i} + g_{k,i} \in V$. Define the $(n,s)$ series $r_V$ by

$$r_V(x,y) = \sum_{k\geq 0}\sum_{i=1}^{d}(sw_{k,i} + g_{k,i}) \otimes b_i(y^k, [y]^k).$$

It is now clear, that $W(r_V) = V$. These constructions establish the bijection in part 1.

To prove the second statement, observe that

$$B(sw_{k,i}, b_j(y^\ell, [y]^\ell)) = \delta_{i,j}\delta_{k,\ell}. \tag{27}$$

Furthermore, the straightforward calculation shows that

$$\begin{aligned} B(sw_{k,i}, sw_{\ell,j}) &= \begin{cases} -\operatorname{res}_0\left\{sx^{(n-1)-k-\ell-1}\right\} & \text{if } i=j \text{ and } 0 \leq k,\ell \leq n-1, \\ \operatorname{res}_0\left\{sx^{(n-1)-k-\ell-1}\right\} & \text{if } i=j \text{ and } k,\ell \geq n, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} -s_{k+\ell-n+1} & \text{if } i=j,\ 0 \leq k,\ell \leq n-1 \text{ and } k+\ell \geq n-1, \\ s_{k+\ell-n+1} & \text{if } i=j \text{ and } k\,,\ell \geq n, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $s(x) = \sum_{k=0}^{\infty} s_k x^k$. We write

$$\begin{aligned} \overline{r}(x,y) &= \frac{s(y)x^n\Omega}{x-y} - \tau(g(y,x)) = \frac{s(x)y^n\Omega}{x-y} - \frac{(s(x)y^n - s(y)x^n)\Omega}{x-y} - \tau(g(y,x)) \\ &= \sum_{k\geq 0}\sum_{i=1}^{d}(sw_{k,i} + \overline{g}_{k,i}) \otimes b_i(y^k, [y]^k). \end{aligned}$$

Consider the quotient

$$\frac{(s(x)y^n - s(y)x^n)\Omega}{x-y} = \frac{y^n(s(x)-s(y))\Omega}{x-y} - \frac{s(y)(x^n-y^n)\Omega}{x-y}$$
$$= \sum_{k\geq 0}\sum_{i=1}^{d} s_k \left(\sum_{\ell=1}^{k} b_i(x^{k-\ell},[x]^{k-\ell})\otimes b_i(y^{(n-1)+\ell},[y]^{(n-1)+\ell})\right.$$
$$\left. - \sum_{\ell=1}^{n} b_i(x^{n-\ell},[x]^{n-\ell})\otimes b_i(y^{k+\ell-1},[y]^{k+\ell-1})\right).$$

The coefficient of $b_i(x^k,[x]^k)\otimes b_i(y^\ell,[y]^\ell)$ in the expression above is

$$\begin{aligned} -s_{k+\ell-(n-1)} &\ \text{ if } 0\leq k,\ell\leq n-1 \text{ and } k+\ell\geq n-1,\\ s_{k+\ell-(n-1)} &\ \text{ if } k,\ell\geq n, \end{aligned}$$

which coincides with $B(sw_{k,i}, sw_{\ell,i})$. If we now expand the coefficients $g_{k,i}$ in the following way

$$g_{k,i} = \sum_{\ell\geq 0}\sum_{j=1}^{d} g_{k,i}^{\ell,j} b_j(x^\ell,[x]^\ell),$$

the coefficients $\overline{g}_{k,i}$ can be rewritten as

$$\overline{g}_{k,i} = -\sum_{\ell\geq 0}\sum_{j=1}^{d}(g_{\ell,j}^{k,i} + B(sw_{k,i}, sw_{\ell,j}))b_i(x^k,[x]^k)\otimes b_j(y^\ell,[y]^\ell).$$

Combining all the results above we obtain the desired equality

$$\begin{aligned} B(sw_{k,i}+g_{k,i}, sw_{\ell,j}+\overline{g}_{\ell,j}) &= B(sw_{k,i}, sw_{\ell,j}) + B(sw_{k,i}, \overline{g}_{\ell,j})\\ &\quad + B(g_{k,i}, sw_{\ell,j}) + B(g_{k,i}, \overline{g}_{\ell,j})\\ &= B(sw_{k,i}, sw_{\ell,j}) + (-g_{k,i}^{\ell,j} - B(sw_{k,i}, sw_{\ell,j})) + g_{k,i}^{\ell,j} + 0\\ &= 0 \end{aligned}$$

which completes the proof of the second statement.

Using the same technique as in [2, Section 1], one can prove that

$$\psi := \mathrm{GCYB}(r) \in (\Delta\otimes\mathfrak{g}\otimes\mathfrak{g})[\![(x_2,[x_2]),(x_3,[x_3])]\!]$$

for any series $r$ of type $(n, s)$. Define $r_{k,i} := sw_{k,i} + g_{k,i}$ and $\overline{r}_{k,i} := sw_{k,i} + \overline{g}_{k,i}$ and rewrite $\mathrm{GCYB}(r)$ as

$$\begin{aligned}\psi = &\sum_{k,\ell\geq 0}\sum_{i,j=1}^{d}[r_{k,i}, r_{\ell,j}] \otimes b_i(x_2^k, [x_2]^k) \otimes b_j(x_3^\ell, [x_3]^\ell)\\ &+\sum_{k\geq 0}\sum_{i=1}^{d} r_{k,i} \otimes \Big([b_i(x_2^k, [x_2]^k) \otimes (1,1), r(x_2, x_3)]\\ &+[(1,1) \otimes b_i(x_3^k, [x_3]^k), \overline{r}(x_2, x_3)]\Big). \end{aligned} \tag{28}$$

Applying $B(\overline{r}_{k_1,i_1} \otimes r_{k_2,i_2} \otimes r_{k_3,i_3}, -)$ to the equation above, we get

$$B(\overline{r}_{k_1,i_1} \otimes r_{k_2,i_2} \otimes r_{k_3,i_3}, \psi) = B(\overline{r}_{k_1,i_1}, [r_{k_2,i_2}, r_{k_3,i_3}]). \tag{29}$$

This gives the last statement because $W(r)$ and $W(\overline{r})$ are generated by $r_{k,i}$ and $\overline{r}_{k,i}$ respectively. ◻

**Corollary 3.7** *Let $n \in \mathbb{Z}_{\geq 0}$, $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$ and $W$ be as in Theorem 3.6. Then*

1. *$W$ defines a bijection between skew-symmtric $\left(n, \frac{1}{x^n\alpha(x)}\right)$-type series and Lagrangian subspaces $V \subseteq L(n, \alpha)$ complementary to the diagonal $\Delta$;*
2. *$W$ defines a bijection between $\left(n, \frac{1}{x^n\alpha(x)}\right)$-type series solving GCYBE and subalgebras $V \subseteq L(n, \alpha)$ complementary to the diagonal $\Delta$.*

As we can see from the proof of Theorem 3.6 the element $\psi$ in $\mathrm{GCYB}(r) = \psi$ represents the obstruction for $W(r)$ from being a Lie subalgebra. This observation raises an interesting question that we do not consider in this paper: what elements $\psi$ can appear on the right-hand side of the above-mentioned equation.

Observe that if $r$ is a series of type $(n, s)$ and it satisfies

$$\begin{aligned}\mathrm{CYB}(r) := &[r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)]\\ &+ [r^{13}(x_1, x_3), r^{23}(x_2, x_3)] = \psi\end{aligned} \tag{30}$$

for some $\psi \in (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x, y, z]\!]$, then $r$ is automatically skew-symmetric and hence solves the first equation as well. To prove that one can e.g. repeat the argument from [1, Lemma 5.2]. In other words, for a fixed $\psi$ solutions to $\mathrm{CYB}(r) = \psi$ form a subclass of solutions to $\mathrm{GCYB}(r) = \psi$. In particular, solutions to $\mathrm{CYB}(r) = 0$. are exactly the skew-symmetric solutions to $\mathrm{GCYB}(r) = 0$. We call the equation $\mathrm{CYB}(r) = \psi$ *Manin-Yang-Baxter equation*.

***Remark 3.8*** As our notation suggest, we could have interpreted $y^n\Omega/(x - y)$ as

$$\frac{y^n\Omega}{x-y} = \sum_{k\geq 0}\sum_{i=1}^{d} b_i x^{-k-1} \otimes b_i y^{n+k} \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$$

and performed all the arithmetic calculations in this form. To restore an $(n, s)$-type series from

$$\frac{s(x)y^n\Omega}{x-y} + g(x, y) \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!] \tag{31}$$

we can simply view $s(x) \in F[\![x]\!]^\times$ and $g(x, y) \in (\mathfrak{g}\otimes\mathfrak{g})[\![x, y]\!]$ as elements in $F[\![(x, [x])]\!]^\times$ and $(\mathfrak{g}\otimes\mathfrak{g})[\![(x, [x]), (y, [y])]\!]$ respectively and reinterpret the singular part $y^n\Omega/(x-y)$ as it was done in Eq. (20).

Conversely, to get a series of the form Eq. (31) from a series of type $(n, s)$ we can just project the latter onto the first component.

In other words, we have a bijection between $(n, s)$-type series in $L_2(n, \alpha)$ and their projections Eq. (31) onto the first component given by different interpretations of the singular part $y^n\Omega/(x-y)$.

Although, all arithmetic operations can be performed in the form Eq. (31), the construction of $W(r)$ and statements like $\Delta \cap W(r) = 0$ require us to pass to the interpretation Eq. (20). This is our main motivation to work directly with $(n, s)$-type series in $L_2(n, \alpha)$ instead of their projections.

In view of Remark 3.8, we have a new proof of [1, Corollary 5.5].

**Corollary 3.9** *Classical (formal) r-matrices, i.e. skew-symmetric elements*

$$\frac{s(x)y^n\Omega}{x-y} + g(x, y) = \frac{1}{x^n\alpha(x)}\frac{y^n\Omega}{x-y} + g(x, y) \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!], \tag{32}$$

*solving GCYBE, are in bijection with skew-symmetric series of type $(n, s)$ solving GCYBE and hence in bijection with Lagrangian Lie subalgebras of $L(n, \alpha)$ complementary to the diagonal $\Delta$.*

The result of [1, Theorem 5.6] can be now formulated in the following way.

**Corollary 3.10** *Skew-symmetric series of type $\left(n, \frac{1}{x^n\alpha(x)}\right)$ that also solve GCYBE exist only for $n = 0, 1$ and $n = 2$ with $\alpha_0 = 0$.*

## 4 Quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$

We remind that $F$ is a discrete algebraically closed field of characteristic 0 and $\mathfrak{g}[\![x]\!]$ is an $F$-Lie algebra equipped with the $(x)$-adic topology.

As we now know, series of type $(n, 1/(x^n\alpha(x)))$ solving CYBE Eq. (30) are in bijection with Lagrangian subalgebras $W \subset L(n, \alpha)$ complementary to the diagonal. On the other hand, such Lagrangian subalgebras are in bijection with non-degenerate topological Lie bialgebra structures. See [1] for their definition and classification.

It turns out, that if we drop the condition on $W$ being a subalgebra, we get so called (non-degenerate) topological quasi-Lie bialgebras. This section is devoted to their classification up to topological twists and coordinate transformations.

**Definition 4.1** A *topological quasi-Lie bialgebra* structure on $\mathfrak{g}[\![x]\!]$ consists of

- a skew-symmetric continuous linear map $\delta\colon \mathfrak{g}[\![x]\!] \to (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$ and
- a skew-symmetric element $\varphi \in (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x,y,z]\!]$,

which are subject to the following conditions

1. $\delta([a,b]) = [a\otimes 1 + 1\otimes a, \delta(b)] - [b\otimes 1 + 1\otimes b, \delta(a)]$, i.e. $\delta$ is a 1-cocycle;
2. $\frac{1}{2}\mathrm{Alt}((\delta\otimes 1)\delta(a)) = [a\otimes 1\otimes 1 + 1\otimes a\otimes 1 + 1\otimes 1\otimes a, \varphi]$;
3. $\mathrm{Alt}((\delta\otimes 1\otimes 1)\varphi) = 0$,

where $\mathrm{Alt}(x_1\otimes\ldots\otimes x_n) := \sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)x_{\sigma(1)}\otimes\ldots\otimes x_{\sigma(n)}$.

**Lemma 4.2** *There is a one-to-one correspondence between triples* $(L, \mathfrak{g}[\![x]\!], W)$*, where* $(L, \mathfrak{g}[\![x]\!])$ *is a topological Manin pair and* $W \subset L$ *is a Lagrangian subspace satisfying* $W \dotplus \mathfrak{g}[\![x]\!] = L$*, and quasi-Lie bialgebra structures on* $\mathfrak{g}[\![x]\!]$.

***Proof*** We start with a topological Manin pair $(L, \mathfrak{g}[\![x]\!])$. If $W \subset L$ is a Lagrangian subspace complementary to $\mathfrak{g}[\![x]\!]$, then it is easy to see that $W \cong \mathfrak{g}[\![x]\!]'$. Therefore, we have an isomorphism of vector spaces

$$L \cong \mathfrak{g}[\![x]\!] \dotplus \mathfrak{g}[\![x]\!]'.$$

The form on $L$ under this isomorphism becomes standard evaluation form $\langle -,-\rangle$ on $\mathfrak{g}[\![x]\!] \dotplus \mathfrak{g}[\![x]\!]'$. We fix such an isomorphism.

Let us define two linear functions

$$p_1\colon \mathfrak{g}[\![x]\!]'\otimes\mathfrak{g}[\![y]\!]' \to \mathfrak{g}[\![x]\!] \ \text{ and } \ p_2\colon \mathfrak{g}[\![x]\!]'\otimes\mathfrak{g}[\![y]\!]' \to \mathfrak{g}[\![x]\!]'$$

by $[f,g] = p_1(f\otimes g) + p_2(f\otimes g)$. We put

$$\delta := p_2^\vee\colon (\mathfrak{g}[\![x]\!]')^\vee \cong \mathfrak{g}[\![x]\!] \to (\mathfrak{g}[\![x]\!]'\otimes\mathfrak{g}[\![y]\!]')^\vee \cong (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!],$$

and let $\psi \in (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x,y,z]\!]$ be the unique element satisfying the condition

$$\langle h, [f,g]\rangle = \langle h, p_1(f\otimes g)\rangle = \langle f\otimes g\otimes h, \psi\rangle \ \text{ for all } f,g,h\in\mathfrak{g}[\![x]\!]'. \tag{33}$$

The skew-symmetry of $p_2$ implies the skew-symmetry of $\delta$, whereas the skew-symmetry of $p_1$ and the invariance of the evaluation form yield the skew-symmetry of $\psi$.

Next, we observe that for all $a,b\in\mathfrak{g}[\![x]\!]$ and $f,g\in\mathfrak{g}[\![x]\!]'$ we have

$$\begin{aligned}&\langle [a,f], g\rangle = \langle a, [f,g]\rangle = \langle a, p_2(f\otimes g)\rangle = \langle \delta(a), f\otimes g\rangle = \langle (f\otimes 1)\delta(a), g\rangle,\\ &\langle [a,f], b\rangle = -\langle f, [a,b]\rangle = -\langle f\circ \mathrm{ad}_a, b\rangle.\end{aligned}$$

In other words, the invariance of the form forces the following equality to hold

$$[a,f] = -f\circ\mathrm{ad}_a + (f\otimes 1)\delta(a). \tag{34}$$

Using Eq. (34) and non-degeneracy of the form we show that $\delta$ is a 1-cocycle:

$$\begin{aligned}\langle\delta([a,b]), f\otimes g\rangle &= \langle [a,b], p_2(f\otimes g)\rangle = \langle [a,b],[f,g]\rangle = \langle [[a,b],f],g\rangle\\ &= \langle -[[b,f],a]-[[f,a],b],g\rangle\\ &= \langle [f\circ \mathrm{ad}_b - (f\otimes 1)\delta(b),a] - [f\circ \mathrm{ad}_a - (f\otimes 1)\delta(a),b],g\rangle\\ &= -\langle a,[f\circ \mathrm{ad}_b,g]\rangle + \langle b,[f\circ \mathrm{ad}_a,g]\rangle + \langle (f\otimes \mathrm{ad}_a)\delta(b),g\rangle\\ &\quad -\langle (f\otimes \mathrm{ad}_b)\delta(a),g\rangle\\ &= \langle [a\otimes 1 + 1\otimes a,\delta(b)] - [b\otimes 1 + 1\otimes b,\delta(a)], f\otimes g\rangle.\end{aligned} \tag{35}$$

The 1-cocycle condition implies that $\delta$ is continuous as it was noted in [1, Remark 3.16].

For conditions 2 and 3 from the definition of a topological quasi-Lie bialgebra consider the Jacobi identity for $f,g,h\in \mathfrak{g}[\![x]\!]'$:

$$\begin{aligned}0 &= [p_1(f\otimes g),h] + [p_1(g\otimes h),f] + [p_1(h\otimes f),g]\\ &\quad + p_1(p_2(f\otimes g)\otimes h) + p_1(p_2(g\otimes h)\otimes f) + p_1(p_2(h\otimes f)\otimes g)\\ &\quad + p_2(p_2(f\otimes g)\otimes h) + p_2(p_2(g\otimes h)\otimes f) + p_2(p_2(h\otimes f)\otimes g).\end{aligned} \tag{36}$$

We denote by $\circlearrowleft$ the summation over circular permutations of symbols $f$, $g$ and $h$, e.g. $\circlearrowleft \langle p_1(f\otimes g),h\rangle = \langle p_1(f\otimes g),h\rangle + \langle p_1(g\otimes h),f\rangle + \langle p_1(h\otimes f),g\rangle$. Applying $\langle -,a\rangle$ to Eq. (36) for an arbitrary $a\in\mathfrak{g}[\![x]\!]$ gives

$$\begin{aligned}\langle p_2(p_2\otimes 1)(\circlearrowleft f\otimes g\otimes h),a\rangle &= -\langle \circlearrowleft [p_1(f\otimes g),h],a\rangle\\ \langle p_2\otimes 1(\circlearrowleft f\otimes g\otimes h),\delta(a)\rangle &= \circlearrowleft \langle -h\circ \mathrm{ad}_a, p_1(f\otimes g)\rangle\\ \langle \circlearrowleft f\otimes g\otimes h,(\delta\otimes 1)\delta(a)\rangle &= \circlearrowleft \langle f\otimes g\otimes(-h\circ \mathrm{ad}_a),\psi\rangle\\ \langle f\otimes g\otimes h, \mathrm{Alt}((\delta\otimes 1)\delta(a))/2\rangle &= -\langle f\otimes g\otimes h,[1\otimes 1\otimes a + 1\otimes a\otimes 1\\ &\qquad + a\otimes 1\otimes 1,\psi]\rangle,\end{aligned}$$

where the very last identity holds because of the skew-symmetry of $\psi$. Multiplying this equality by 2 we get the relation

$$\langle f\otimes g\otimes h, \mathrm{Alt}((\delta\otimes 1)\delta(a)) + 2[1\otimes 1\otimes a + 1\otimes a\otimes 1 + a\otimes 1\otimes 1,\psi]\rangle = 0.$$

Letting $\varphi := -\psi$ we obtain the second identity from the definition of a topological quasi-Lie bialgebra structure. Applying instead $\langle s,-\rangle$, $s\in\mathfrak{g}[\![x]\!]'$ to the Jacobi identity Eq. (36) we get the desired

$$\mathrm{Alt}((\delta\otimes 1\otimes 1)\psi) = 0.$$

Therefore, $(\mathfrak{g}[\![x]\!],\delta,\varphi)$ is a topological quasi-Lie bialgebra.

For the converse direction, we put $L := \mathfrak{g}[\![x]\!]\dotplus\mathfrak{g}[\![x]\!]'$ with the standard evaluation form; we let $p_1$ be the unique element in $\mathrm{Hom}_{F-\mathrm{Vect}}(\mathfrak{g}[\![x]\!]'\otimes\mathfrak{g}[\![x]\!]',\mathfrak{g}[\![x]\!])$ satisfying

Eq. (33) with $\psi := -\varphi$; we define $p_2 := \delta'$, i.e. the dual map of $\delta$. The Lie bracket between two elements in $\mathfrak{g}[\![x]\!]'$ is given by the sum $p_1 + p_2$. Defining $[a, f]$ as in Eq. (34) the evaluation form becomes invariant and we get a topological Manin pair $(L, \mathfrak{g}[\![x]\!])$ with the Lagrangian subspace $\mathfrak{g}[\![x]\!]'$. These constructions are clearly inverse to each other. □

Combining the classification of Manin pairs mentioned in Sect. 2 with Corollary 3.7 and Lemma 4.2 we get the following description of all topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$.

**Lemma 4.3** *There is a bijection between topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ and Lagrangian subspaces $W \subset L(n, \alpha)$ or $L(\infty)$ complementary to the diagonal $\Delta$, where $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$ is an arbitrary sequence and $n \geq 0$. Moreover, such Lagrangian subspaces $W \subset L(n, \alpha)$ are in bijection with skew-symmetric sequences of type $(n, 1/(x^n \alpha(x)))$.*

In view of this result we call skew-symmetric series of type $(n, s)$ as well as their projections onto the first component *quasi-r-matrices*. Quasi-Lie bialgebra structures can also be described using their associated quasi-$r$-matrices in the following way.

**Proposition 4.4** *Assume $(\mathfrak{g}[\![x]\!], \delta, \varphi)$ is a topological quasi-Lie bialgebra and let $r \in L_2(n, \alpha)$ be the corresponding quasi-r-matrix given by the bijection from Lemma 4.3. Under the identification $\mathfrak{g}[\![(x, [x])]\!] \cong \mathfrak{g}[\![x]\!]$ we have the following identities:*

- $[a \otimes 1 + 1 \otimes a, r] = -\delta(a)$ *for any* $a \in \mathfrak{g}[\![x]\!]$ *and*
- $\mathrm{CYB}(r) = -\varphi$.

*The same is true for the projection $r \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$.*

***Proof*** We start, as in the proof of Lemma 4.2, by fixing an identification $L(n, \alpha) = \Delta \dot{+} W(r) \cong \mathfrak{g}[\![x]\!] \dot{+} \mathfrak{g}[\![x]\!]'$. Let $\{v_{k,i}\}$ be a basis for $\mathfrak{g}[\![x]\!]'$ dual to $\{\varepsilon_{k,i} := b_i y^k\}$. Then $r = \sum_{k\geq 0} \sum_{i=1}^{d} v_{k,i} \otimes \varepsilon_{k,i}$ and we have

$$\begin{aligned}[a \otimes 1 + 1 \otimes a, r] &= \sum_{k\geq 0} \sum_{i=1}^{d} [a, v_{k,i}] \otimes \varepsilon_{k,i} + v_{k,i} \otimes [a, \varepsilon_{k,i}] \\ &= \sum_{k\geq 0} \sum_{i=1}^{d} (-v_{k,i} \circ \mathrm{ad}_a + (v_{k,i} \otimes 1)\delta(a)) \otimes \varepsilon_{k,i} + v_{k,i} \otimes [a, \varepsilon_{k,i}].\end{aligned}$$

Applying $\langle v_{\ell,j} \otimes v_{m,t}, -\rangle$ to the equality above we get

$$\begin{aligned}\langle v_{\ell,j} \otimes v_{m,t}, [a \otimes 1 + 1 \otimes a, r]\rangle &= \sum_{k\geq 0} \sum_{i=1}^{d} \langle v_{\ell,j} \otimes v_{m,t}, (v_{k,i} \otimes 1)\delta(a) \otimes \varepsilon_{k,i}\rangle \\ &= \langle v_{\ell,j}, (v_{m,t} \otimes 1)\delta(a)\rangle \\ &= \langle v_{\ell,j} \otimes v_{m,t}, -\delta(a)\rangle.\end{aligned}$$

Applying instead $\langle \varepsilon_{\ell,j} \otimes v_{m,t}, -\rangle$ to the same equality we obtain

$$\begin{aligned}\langle \varepsilon_{\ell,j} \otimes v_{m,t}, [a \otimes 1 + 1 \otimes a, r]\rangle &= \sum_{k\geq 0}\sum_{i=1}^{d}\langle \varepsilon_{\ell,j} \otimes v_{m,t}, (-v_{k,i} \circ \mathrm{ad}_a) \otimes \varepsilon_{k,i} \\ &\quad + v_{k,i} \otimes [a, \varepsilon_{k,i}]\rangle \\ &= -\langle \varepsilon_{\ell,j}, v_{m,t} \circ \mathrm{ad}_a\rangle + \langle v_{m,t}, [a, \varepsilon_{\ell,j}]\rangle \\ &= 0.\end{aligned}$$

This implies the desired equality $[a \otimes 1 + 1 \otimes a, r] = -\delta(a)$. The identity $\mathrm{CYB}(r) = -\varphi$ follows from the skew-symmetry of $r$, Theorem 3.6 and the fact that $\varphi = -\psi$ according to the proof of Lemma 4.2. □

***Remark 4.5*** Assume $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ is a series such that

$$[f(x) \otimes 1 + 1 \otimes f(y), r(x, y)] \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!] \tag{37}$$

for all $f \in \mathfrak{g}[\![x]\!]$. Write $r = s(x^{-1}, y) + g(x, y)$, where $s \in x^{-1}(\mathfrak{g} \otimes \mathfrak{g})[x^{-1}][\![y]\!]$ and $g \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$. Then, because of Eq. (37), we must have

$$[a \otimes 1 + 1 \otimes a, s(x^{-1}, y)] = 0$$

for all $a \in \mathfrak{g}$. Since the $\mathfrak{g}$-invariant elements of $\mathfrak{g} \otimes \mathfrak{g}$ are precisely the multiples of the quadratic Casimir element $\Omega$, we have the identity $s(x^{-1}, y) = p(x^{-1}, y)\Omega$ for some $p \in x^{-1}F[x^{-1}][\![y]\!]$. Furthermore, the condition

$$[ax \otimes 1 + 1 \otimes ay, p(x^{-1}, y)\Omega] = [a(x - y) \otimes 1, p(x^{-1}, y)\Omega] \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$$

implies $(x - y)p(x^{-1}, y) \in F[\![x, y]\!]$, meaning that there exists an $s \in F[\![y]\!]$ such that $p(x^{-1}, y) = s(y)/(x - y)$. In other words, $r$ has the form Eq. (21). This result can be considered as another motivation to study series of type $(n, s)$.

Observe that if we know one Lagrangian subspace $W_0$ inside $L \cong \mathfrak{g}[\![x]\!] \dot{+} \mathfrak{g}[\![x]\!]'$ then any other Lagrangian subspace can be constructed from $W_0$ through twisting. More precisely, if $s = \sum_i a_i \otimes b^i \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$ is a skew-symmetric tensor, then we can associate with it a (twisted) Lagrangian subspace

$$W_s := \left\{\sum_i B(b^i, w)a_i - w \mid w \in W\right\} \subseteq L \tag{38}$$

complementary to $\mathfrak{g}[\![x]\!]$. The converse is also true; for proof see [3]. In other words, the following statement holds.

**Lemma 4.6** *There is a bijection between Lagrangian subspaces $W \subseteq L(n, \alpha)$ or $L(\infty)$ and skew-symmetric tensors in $(\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$.*

Combining Lemma 4.4, Eq. (38) and the algorithm for constructing a quasi-$r$-matrix from a Lagrangian subspace $W \subset L(n,\alpha)$, $W \dotplus \Delta = L(n,\alpha)$, we obtain the following twisting rules for Lagrangian subspaces, quasi-Lie bialgebra structures and quasi-$r$-matrices.

**Lemma 4.7** *Let $(\mathfrak{g}[\![x]\!], \delta, \varphi)$ be a topological quasi-Lie bialgebra structure corresponding to the quasi-r-matrix $r$. If we twist $W(r)$ with a skew-symmetric tensor $s$ as described in Eq. (38) we obtain another topological quasi-Lie bialgebra $(\mathfrak{g}[\![x]\!], \delta_s, \varphi_s)$, such that*

1. $W(r)_s = W(r - s)$;
2. $\delta_s = \delta + ds$;
3. $\varphi_s = \varphi + \mathrm{CYB}(s) - \frac{1}{2}\mathrm{Alt}((\delta \otimes 1)s)$,

*where* $ds(a) := [a \otimes 1 + 1 \otimes a, s]$.

***Remark 4.8*** Since any quasi-$r$-matrix $r$ defines a topological quasi-Lie bialgebra structure $\delta(a) = [a \otimes 1 + 1 \otimes a, r]$ on $\mathfrak{g}[\![x]\!]$, the third condition in Definition 4.1 is trivially satisfied. In other words,

$$\mathrm{Alt}((\delta \otimes 1 \otimes 1)\mathrm{CYB}(r)) = 0$$

for any quasi-$r$-matrix $r$.

Lemma 4.6 and Lemma 4.7 state that, in order to obtain a description of topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ up to twisting it is enough to find a single Lagrangian subspace $W_0$, complementary to $\mathfrak{g}[\![x]\!]$, inside $L(\infty)$ and each $L(n,\alpha)$. The same is true for the associated quasi-$r$-matrices

The case $L(\infty)$ is trivial, because by definition $\mathfrak{g}[\![x]\!]' = \bigoplus_{j \geq 0} \mathfrak{g} \otimes a_j \subseteq L(\infty)$ is a Lagrangian subalgebra (see Lemma 2.1). Similar to the Lie bialgebra case, topological quasi-Lie bialgebras corresponding to the Manin pair $(L(\infty), \mathfrak{g}[\![x]\!])$ are called *degenerate*.

Let us now focus on *non-degenerate* topological quasi-Lie bialgebra structures, i.e. the ones corresponding to the Manin pairs $(L(n,\alpha), \Delta)$. By Proposition 2.5 for each Manin pair $(L(n,\alpha), \Delta)$ there exists an appropriate coordinate transformation that makes it into $(L(n,\beta), \Delta)$, where $\beta_0 = \alpha_0$ and all other $\beta_i = 0$. This means, that to classify all non-degenerate topological quasi-Lie bialgebras on $\mathfrak{g}[\![x]\!]$, up to coordinate transformations and twisting, it is enough to construct a Lagrangian subspace $W_0$ within each $L(n,\alpha_0) := L(n, (\dots, 0, \alpha_0, 0, \dots, 0))$ complementary to $\Delta$. Equivalently, it is enough to find a quasi-$r$-matrix of type $(n,\alpha_0)$ for any $n \geq 0$ and $\alpha_0 \in F$.

### 4.1 Lagrangian subspaces of $L(n, \alpha_0)$

As before we let $\{b_i\}_{i=1}^d$ be an orthonormal basis for $\mathfrak{g}$ with respect to the Killing form $\kappa$. The form $B$ on $L(n,\alpha_0)$ has the following explicit form

$$B(a\otimes(f,[p]),b\otimes(g,[q])) = \begin{cases}\kappa(a,b)\left\{\text{coeff}_{n-1}(fg-pq)-\alpha_0\text{coeff}_0(fg-pq)\right\} & \text{if } n\geq 2,\\ \kappa(a,b)\text{coeff}_{n-1}(fg-pq) & \text{if } n=0,1.\end{cases} \tag{39}$$

We now present an explicit construction for a Lagrangian subspace of $L(n,\alpha_0)$ complementary to $\Delta$ for arbitrary $n \geq 0$ and $\alpha_0 \in F$. Using the twisting procedure from Lemma 4.7, this subspace can be twisted in order to obtain all other Lagrangian subspaces of $L(n,\alpha_0)$ complementary to $\Delta$.

n = 0: When $n = 0$, the subalgebra $W_0 := x^{-1}\mathfrak{g}[\![x^{-1}]\!] \subseteq \mathfrak{g}(\!(x)\!)$ is known to be Lagrangian.

n = 1: For $n = 1$ it is easy to see that the subspace

$$W_0 := \text{span}_F\{b_i(1,-1), b_i(x^{-k},0) \mid k \geq 1,\ 1 \leq i \leq d\} \subset L(1,\alpha_0) \tag{40}$$

is Lagrangian and complementary to the diagonal $\Delta$.

n = 2k: For even $n \geq 2$ and arbitrary $\alpha_0 \in F$ the subspace $W_0 \subset L(n,\alpha_0)$ spanned by the elements

$$b_i\left\{(x^{(n-1)-m},0)-\alpha_0(x^{2(n-1)-m},0)+\alpha_0^2(x^{3(n-1)-m},0)-\alpha_0^3(x^{4(n-1)-m},0)+\dots\right\},$$
$$0 \leq m \leq \frac{n}{2}-1,$$
$$b_i\left(0,-[x]^{(n-1)-\ell}\right),\ \frac{n}{2} \leq \ell < n-1,$$
$$b_i\left(0,-1+\frac{\alpha_0}{2}[x]^{n-1}\right),$$
$$b_i(x^{-k},0),\ k \geq 1,$$

is Lagrangian and complementary to the diagonal.

n = 2k + 1: Modifying slightly the basis for even case we obtain the following basis for $W_0 \subset L(n,\alpha_0)$ with odd $n \geq 3$:

$$b_i\left\{(x^{(n-1)-m},0)-\alpha_0(x^{2(n-1)-m},0)+\alpha_0^2(x^{3(n-1)-m},0)-\alpha_0^3(x^{4(n-1)-m},0)+\dots\right\},$$
$$0 \leq m \leq \frac{n-1}{2}-1,$$
$$b_i\left\{(x^{\frac{n-1}{2}},-[x]^{\frac{n-1}{2}})-\alpha_0(x^{\frac{3(n-1)}{2}},0)+\alpha_0^2(x^{\frac{5(n-1)}{2}},0)-\alpha_0^3(x^{\frac{7(n-1)}{2}},0)+\dots\right\},$$
$$b_i(0,-[x]^{(n-1)-\ell}),\ \frac{n-1}{2}+1 \leq \ell < n-1,$$
$$b_i\left(0,-1+\frac{\alpha_0}{2}[x]^{n-1}\right),$$
$$b_i(x^{-k},0),\ k \geq 1.$$

The subspaces above were constructed by ”guessing”. However, there is an abstract procedure that produces Lagrangian subspaces for arbitrary $n$ and $\alpha$. We present it here for completeness.

The easiest skew-symmetric $(n, s)$-type series is given by

$$r(x, y) := \frac{1}{2}\left(\frac{s(x)y^n\Omega}{x-y} + \frac{s(y)x^n\Omega}{x-y}\right) = \frac{s(x)y^n\Omega}{x-y} + \frac{\Omega}{2}\left(\frac{s(y)x^n - s(x)y^n}{x-y}\right)$$
$$= \frac{s(x)y^n\Omega}{x-y} - \frac{1}{2}\sum_{k,\ell=0}^{\infty}\sum_{i,j=1}^{d} B(sw_{k,i}, sw_{\ell,j}) b_i(x, [x])^k \otimes b_j(y, [y])^\ell,$$

where we recall that

$$B(sw_{k,i}, sw_{\ell,j}) = \begin{cases} -s_{k+\ell-n+1} & \text{if } i = j,\ 0 \le k, \ell \le n-1 \text{ and } k+\ell \ge n-1, \\ s_{k+\ell-n+1} & \text{if } i = j \text{ and } k\,, \ell \ge n, \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 3.7 the subspace

$$W(r) = \operatorname{span}_F\left\{ sw_{k,i} - \frac{1}{2}\sum_{\ell=0}^{\infty} B(sw_{\ell,i}, sw_{k,i}) b_i(x, [x])^\ell \,\middle|\, k \ge 0, 1 \le d \le n \right\}$$
$$= \operatorname{span}_F\left\{ sw_{k,i} + \frac{1}{2}\left(\sum_{\ell=0}^{n-1} s_{k+\ell-n+1} b_i(x, [x])^\ell - \sum_{\ell=n}^{\infty} s_{k+\ell-n+1} b_i(x, [x])^\ell\right) \,\middle|\, k \ge 0, 1 \le d \le n \right\}$$

is Lagrangian and complementary to the diagonal. Here we used the convention that $s_k = 0$ for $k < 0$. Calculating the basis explicitly for some particular $s$ requires some effort and it may not look as friendly as the ones given above.

### 4.2 Quasi-$r$-matrices

The goal of this section is to describe the quasi-$r$-matrices corresponding to the Lagrangian subspaces described in the previous section. The twisting procedure from Lemma 4.7 then yields all other quasi-$r$-matrices.

The proof of Theorem 3.6 gives us an algorithm for constructing a series of type $(n, s(x) := 1/(x^n\alpha(x)))$ from a subspace $W \subset L(n, \alpha)$ complementary to the diagonal. More precisely, the desired series is given by

$$\sum_{k\ge 0}\sum_{i=1}^{d} v_{k,i} \otimes b_i(y^k, [y]^k), \tag{41}$$

where

$$W = \text{span}_F\{v_{k,i} \mid k \geq 0,\ 1 \leq i \leq d\} \text{ and } B(v_{k,i}, b_j(y^\ell, [y]^\ell)) = \delta_{i,j}\delta_{k,\ell},$$

i.e. $\{v_{k,i}\}$ is a basis of $V$ dual to $\{b_i(y^k, [y]^k)\}$. Indeed, non-degeneracy of the form $B$ then implies that $v_{k,i}$ has the desired form $v_{k,i} = sw_{k,i} + g_{k,i}$ for some $g_{k,i} \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$.

Applying this idea to $W_0$'s constructed in the preceding section we get the following series.

n = 0: The classical $r$-matrix (equivalently $(0, 1)$-type series) corresponding to $W_0 := x^{-1}\mathfrak{g}[\![x^{-1}]\!] \subseteq \mathfrak{g}(\!(x)\!)$ is the Yang's matrix $\Omega/(x - y)$.

n = 1: The quasi-$r$-matrix corresponding to $\text{span}_F\{b_i(1, -1), b_i(x^{-k}, 0) \mid k \geq 1,\ 1 \leq i \leq d\} \subset L(1, \alpha_0)$ is

$$\frac{y\Omega}{x - y} + \frac{1}{2}\sum_{i=1}^{d} b_i(1, -1) \otimes b_i(1, 1) \in L_2(1, 1) \text{ with the projection}$$
$$\frac{y\Omega}{x - y} + \frac{1}{2}\Omega \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!].$$

n = 2k: For even $n \geq 2$ and arbitrary $\alpha_0 \in F$ we have the following quasi-$r$-matrix

$$\frac{1}{1 + \alpha_0 x^{n-1}}\frac{y^n\Omega}{x - y} + \frac{\Omega}{1 + \alpha_0 x^{n-1}}\sum_{0 \leq m < \frac{n}{2}} x^{(n-1)-m}y^m$$
$$+ \frac{\alpha_0\Omega}{(1 + \alpha_0 x^{n-1})(1 + \alpha_0 y^{n-1})}\left(y^{2(n-1)} + \sum_{\frac{n}{2} \leq \ell < n-1} x^{(n-1)-\ell}y^{(n-1)+\ell} - \frac{1}{2}x^{n-1}y^{n-1}\right).$$

n = 2k + 1: In the odd case $n \geq 3$ the series corresponding to $W_0 \subset L(n, \alpha_0)$ is

$$\frac{1}{1 + \alpha_0 x^{n-1}}\frac{y^n\Omega}{x - y} + \frac{\Omega}{1 + \alpha_0 x^{n-1}}\left(x^{\frac{n-1}{2}}y^{\frac{n-1}{2}} + \sum_{0 \leq m < \frac{n-1}{2}} x^{(n-1)-m}y^m\right)$$
$$+ \frac{\alpha_0\Omega}{(1 + \alpha_0 x^{n-1})(1 + \alpha_0 y^{n-1})}\left(y^{2(n-1)} + \sum_{\frac{n-1}{2} < \ell < n-1} x^{(n-1)-\ell}y^{(n-1)+\ell} - \frac{1}{2}x^{n-1}y^{n-1}\right).$$

## 5 Lie algebra splittings of $L(n, \alpha)$ and generalized $r$-matrices

By Corollary 3.7 we have a bijection between subalgebras of $L(n, \alpha)$ and series of type $(n, 1/(x^n\alpha(x)))$ solving GCYBE. Therefore, we can construct new solutions to GCYBE by finding subalgebras of $L(n, \alpha)$ complementary to the diagonal. However, as the following result shows, the most interesting new solutions should arise from unbounded subalgebras of $L(n, \alpha)$, $n > 2$.

**Proposition 5.1** *Let $L(n, \alpha) = \Delta \dotplus W$ for some subalgebra $W \subset L(n, \alpha)$ and $n > 2$. Assume $W$ is bounded, i.e. there is an integer $N > 0$ such that*

$$x^{-N}\mathfrak{g}[x^{-1}] \subseteq W_+ \subseteq x^N\mathfrak{g}[x^{-1}],$$

*where $W_+$ is the projection of $W \subset L(n, \alpha) = \mathfrak{g}(\!(x)\!) \oplus \mathfrak{g}[x]/x^n\mathfrak{g}[x]$ on the first component $\mathfrak{g}(\!(x)\!)$. Then there is an element $\sigma \in \mathrm{Aut}_{F[x]-\mathrm{LieAlg}}(\mathfrak{g}[x])$ such that*

$$\{0\} \times [x^2]\mathfrak{g}[x]/x^n\mathfrak{g}[x] \subseteq (\sigma \times \sigma)W \subseteq x\mathfrak{g}[x^{-1}] \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]$$

*and the image $\widetilde{W}$ under the canonical projection $L(n, \alpha) \to L(2, \alpha)$ is a subalgebra satisfying $L(2, \alpha) = \Delta \dotplus \widetilde{W}$.*

*In the language of $(n, s)$-type series: Let*

$$r = \frac{s(x)y^n\Omega}{x - y} + g(x, y)$$

*be the generalized $r$-matrix corresponding to a bounded $W \subset L(n, \alpha)$, $n \geq 2$. Then there is $p(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})[x, y]$ of degree at most one in $x$ and an element $\sigma \in \mathrm{Aut}_{F[x]-\mathrm{LieAlg}}(\mathfrak{g}[x])$ such that*

$$(\sigma(x) \otimes \sigma(y))r(x, y) = y^{n-2}\underbrace{\left(\frac{s(x)y^2\Omega}{x - y} + p(x, y)\right)}_{r'(x,y)},$$

*where $r'$ is a generalized $r$-matrix in $L_2(2, \alpha)$.*

***Proof*** The condition $x^{-N}\mathfrak{g}[x^{-1}] \subseteq W_+ \subseteq x^N\mathfrak{g}[x^{-1}]$ means exactly that $W_+$ is an order. Moreover, since $W$ is complementary to the diagonal, we have $W_+ + \mathfrak{g}[x] = \mathfrak{g}[x, x^{-1}]$. It was shown in [11] that such orders, up to the action of some $\sigma \in \mathrm{Aut}_{F[x]-\mathrm{LieAlg}}(\mathfrak{g}[x])$, are contained in a maximal order $\mathfrak{M}$ associated to the so called fundamental simplex $\Delta_{\mathrm{st}}$. These maximal orders are explicitly described in [11] and satisfy $\mathfrak{M} \subseteq x\mathfrak{g}[x^{-1}]$. Therefore, we have $\sigma W_+ \subseteq \mathfrak{M} \subseteq x\mathfrak{g}[x^{-1}]$. Moreover, we have the identity

$$(\sigma \times \sigma)W \dotplus \Delta = L(n, \alpha),$$

implying the inclusion $\{0\} \times [x^2]\mathfrak{g}[x]/x^n\mathfrak{g}[x] \subseteq (\sigma \times \sigma)W$. The remaining parts follow straightforward from the construction Theorem 3.6. □

Unfortunately, we have not found a new example of an unbounded subalgebra of $L(n,\alpha)$. However, we present an infinite family of bounded subalgebras. We believe these examples are still interesting because their orthogonal complements, which are important in the view of Adler-Kostant-Symes scheme, are unbounded if $\alpha \neq 0$.

Consider the subspaces of $L(n,\alpha_0)$, $n > 0$:

$$W_0 = \text{span}_F\{b_i(x^{-k},0), b_i(1,0), b_i(0,-[x]^\ell) \mid k \geq 1,\ 1 \leq \ell \leq n-1\},$$
$$W_1 = \text{span}_F\{b_i(x^{-k},0), b_i(0,-1), b_i(0,-[x]^\ell) \mid k \geq 1,\ 1 \leq \ell \leq n-1\}.$$

These are clearly subalgebras. The corresponding generalizerd $r$-matrices are

$$\begin{aligned} r_0 &= \frac{1}{1+\alpha_0 x^{n-1}}\frac{y^n\Omega}{x-y} + \frac{y^{n-1}\Omega}{(1+\alpha_0 x^{n-1})(1+\alpha_0 y^{n-1})} \\ &\quad + \frac{\alpha_0\Omega}{(1+\alpha_0 x^{n-1})(1+\alpha_0 y^{n-1})}\left(y^{2(n-1)} + \sum_{0\leq \ell < n-1} x^{(n-1)-\ell} y^{(n-1)+\ell}\right) \\ &= \frac{y^{n-1}}{1+\alpha_0 y^{n-1}}\left(\frac{y\Omega}{x-y} + \Omega\right), \\ r_1 &= \frac{1}{1+\alpha_0 x^{n-1}}\frac{y^n\Omega}{x-y} + \frac{\alpha_0\Omega}{(1+\alpha_0 x^{n-1})(1+\alpha_0 y^{n-1})} \\ &\quad \times \left(y^{2(n-1)} + \sum_{0<\ell<n-1} x^{(n-1)-\ell} y^{(n-1)+\ell}\right) \\ &= \frac{1}{1+\alpha_0 y^{n-1}}\frac{y^n\Omega}{x-y}. \end{aligned}$$

By considering decompositions $\mathfrak{g} = \mathfrak{s}_1 \dot{+} \mathfrak{s}_2$ of $\mathfrak{g}$ into direct sums of subalgebras we can get an infinite family of generalized $r$-matrices "in between" $r_0$ and $r_1$. More precisely, let $\{s_{1,i}\}_{i=1}^{d_1}$ and $\{s_{2,j}\}_{j=1}^{d_2}$ be bases for $\mathfrak{s}_1$ and $\mathfrak{s}_2$ respectively. Such a decomposition leads to another subalgera of $L(n,\alpha_0)$:

$$W_{01} := \text{span}_F\Big\{b_i(x^{-k},0), s_{1,m}(1,0), s_{2,j}(0,1), b_i(0,-[x]^\ell) \mid k\geq 1,\ 1\leq \ell \leq n-1,\ 1\leq i \leq d,\ 1 \leq m \leq d_1,\ 1 \leq j \leq d_2\Big\}.$$

Rewrite the elements $b_i$ in terms of $s_{1,m}$ and $s_{2,j}$:

$$b_i = \sum_{m=1}^{d_1} \lambda^i_{1,m} s_{1,m} + \sum_{j=1}^{d_2} \lambda^i_{2,j} s_{2,j},$$

where $\lambda^i_{1,m}, \lambda^i_{2,j} \in F$. Finding a basis in $W_{12}$ dual to $\{b_i(y^m,[y]^m)\} \subset \Delta$ and then projecting the generating series for $W_{01}$ onto the first component we obtain the following generalized $r$-matrix

$$
\begin{aligned}
r_{01} &= \frac{1}{1+\alpha_0 x^{n-1}} \frac{y^n \Omega}{x-y} + \frac{\alpha_0 \Omega}{(1+\alpha_0 x^{n-1})(1+\alpha_0 y^{n-1})} \\
&\quad \times \left( y^{2(n-1)} + \sum_{0<\ell<n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} \right) \\
&\quad + \frac{y^{n-1}}{1+\alpha_0 y^{n-1}} \sum_{i=1}^{d} \sum_{m=1}^{d_1} \lambda_{1,m}^{i} s_{1,m} \otimes b_i \\
&= \frac{y^{n-1}}{1+\alpha_0 y^{n-1}} \left( \frac{y\Omega}{x-y} + \sum_{i=1}^{d} \sum_{m=1}^{d_1} \lambda_{1,m}^{i} s_{1,m} \otimes b_i \right). \qquad (42)
\end{aligned}
$$

Clearly $r_{01}$ coincides with $r_0$ when $\mathfrak{s}_1 = \mathfrak{g}$ and $r_1$ if $\mathfrak{s}_2 = \mathfrak{g}$. The corresponding orhogonal complements are

$$W_0^{\perp} = W(\overline{r_0}) = \operatorname{span}_F \left\{ b_i \left(0, [x]^{n-1}\right), b_i \left( \frac{x^{-k(n-1)-m}}{1+\alpha_0 x^{n-1}}, 0 \right) \mid k \geq -1, 0 < m < n-1 \right\},$$

$$W_1^{\perp} = W(\overline{r_1}) = \operatorname{span}_F \left\{ b_i \left( \frac{x^{-k(n-1)-m}}{1+\alpha_0 x^{n-1}}, 0 \right) \mid k \geq -1, 0 \leq m < n-1 \right\},$$

$$
\begin{aligned}
W_{01}^{\perp} = W(\overline{r_{01}}) &= \mathfrak{s}_1^{\perp} \left( \frac{x^{n-1}}{1+\alpha_0 x^{n-1}}, 0 \right) \dotplus \mathfrak{s}_2^{\perp} (0, [x]^{n-1}) \\
&\quad \dotplus \operatorname{span}_F \left\{ b_i \left( \frac{x^{-k(n-1)-m}}{1+\alpha_0 x^{n-1}}, 0 \right) \mid k \geq -1, 0 < m < n-1 \right\}, \qquad (43)
\end{aligned}
$$

which are unbounded because of the factor $1/(1+\alpha_0 x^{n-1})$.

Note that a series of type $(n, s)$ defines a subspace inside $L(n, \alpha)$ for any $\alpha$, because the subalgebra property is not affected by the form. With the previous examples in mind we can prove the following statement.

**Lemma 5.2** *Let $B_0$ and $B_\alpha$ be the bilinear forms on $L(n, 0)$ and $L(n, \alpha)$ respectively. For a series $r$ of type $(n, s)$ we have*

$$W(r)^{\perp_{B_\alpha}} = \frac{1}{x^n \alpha(x)} W(r)^{\perp_{B_0}} \subset L(n, \alpha). \qquad (44)$$

***Proof*** Set $u(x) := 1/(x^n \alpha(x))$. Write

$$r = \sum_{k \geq 0} \sum_{i=1}^{d} (s w_{k,i} + g_{k,i}) \otimes b_i (y^k, [y]^k) \text{ and } \overline{r} = \sum_{k \geq 0} \sum_{i=1}^{d} (s w_{k,i} + \overline{g_{k,i}}) \otimes b_i (y^k, [y]^k).$$

Then by Theorem 3.6 and definition Eq. (12) $B_\alpha(sw_{k,i} + g_{k,i}, u(sw_{\ell,j} + \overline{g_{\ell,j}})) = B_0(sw_{k,i} + g_{k,i}, sw_{\ell,j} + \overline{g_{\ell,j}}) = 0$. □

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## Declarations

**Conflict of interest** The authors have no competing interests to declare that are relevant to the content of this article.



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