ELSEVIER
Insurance: Mathematics and Economics 31 (2002) 415–427
INSURANCE
MATHEMATICS & ECONOMICS
www.elsevier.com/locate/econbase

Excess of loss reinsurance and Gerber's inequality in the Sparre Anderson model☆

Maria de Lourdes Centeno

CEMAPRE/ISEG, Technical University of Lisbon, Rua do Quelhas 6, 1200-781 Lisbon, Portugal

Received May 2001; received in revised form September 2002; accepted 24 September 2002

Abstract

Assuming that the reinsurance premium is calculated according to the expected value principle we study an upper bound for the probability of ruin in finite horizon, as function of the excess of loss retention limit. The upper bound used is an extension proved by Grandell [Aspects of Risk Theory, Springer, New York, 1991] of Gerber's bound, see Gerber [Martingales in risk theory, Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker, 1973, pp. 205–216], for the Sparre Anderson model [On the collective theory of risk in the case of contagious between the claims, in: Proceedings of the Transactions on XV International Congress of Actuaries, New York, 1957].

Subj. Class: IM52; IB90; IB92

Keywords: Sparre Anderson model; Reinsurance; Excess of loss; Adjustment coefficient

1. Introduction

Centeno (2002) studied the insurer's adjustment coefficient as function of retention levels for combinations of quota-share with excess of loss reinsurance in the Sparre Anderson model, generalizing some of the results of Centeno (1986). It was shown that the insurer's adjustment coefficient is a unimodal function of the retention levels when the quota-share reinsurance premium is calculated on original terms and the excess of loss premium is calculated according to the expected value principle.

In this paper we confine the study to excess of loss reinsurance, again when the number of claims is described by an ordinary renewal process. An upper bound for finite time ruin probability is considered as a function of the retention, generalizing the study by Centeno (1987) to the non-classical situation.

2. Assumptions and preliminaries

We assume that the number of claims $\{N(t)\}_{t\geq 0}$ follows an ordinary renewal process, i.e. the number of claims, $N(t)$, that occur in the time interval $(0, t]$ can be written as

$$N(t) = \sup\{n : S_n \leq t\} \tag{1}$$

☆ This research was supported by Fundação para a Ciência e a Tecnologia—FCT/POCTI.
E-mail address: lcenteno@iseg.utl.pt (M.L. Centeno).

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PII: S0167-6687(02)00187-7

with $S_0 = 0$, $S_n = T_1 + T_2 + \cdots + T_n$ for $n \geq 1$, where $\{T_i\}_{i=1}^{\infty}$ are independent and identically distributed non-negative random variables. S_n denotes the epoch of the nth claim and T_i is the time between the $(i-1)$th and the ith claim. Let the expected value of T_i be $1/\gamma$.

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed random variables, independent of $\{T_i\}_{i=1}^{\infty}$, where X_i denotes the amount of the ith claim. We assume that F, the distribution function of X_i, is such that $F(0) = 0$, so that negative claims are not possible; that $0 < F(x) < 1$ for $0 < x < +\infty$ (this assumption could be relaxed); that $\mathrm{d}F(x)/\mathrm{d}x$ exists and is continuous; that the moment generating function of $F(x)$, $M_X(r)$, exists for $r \in (-\infty, \tau)$ for some $0 < \tau \leq +\infty$ and that

$$\lim_{r\to\tau} M_X(r) = \lim_{r\to\tau} E[\mathrm{e}^{rX}] = +\infty. \tag{2}$$

Let μ be the expected value of X_i.

The risk process $\{Y(t)\}_{t\geq 0}$, is defined by

$$Y(t) = ct - \sum_{i=1}^{N(t)} X_i, \quad \left(\sum_{i=1}^{0} X_i \stackrel{\text{def}}{=} 0\right), \tag{3}$$

where c—the insurer's premium income per unit of time—is a positive constant. The loss between two claims is $Y_i = X_i - cT_i$, and the relative safety loading is

$$\rho = \frac{(c/\gamma) - \mu}{\mu} = \frac{c}{\gamma\mu} - 1. \tag{4}$$

We assume that $\rho > 0$.

Let

$$g(r) = M_{Y_i}(r) = E[\mathrm{e}^{rY_i}] = E[\mathrm{e}^{rX}]E[\mathrm{e}^{-rcT}], \tag{5}$$

where X and T have the same distribution than X_i and T_i, respectively. The adjustment coefficient R is, in the renewal case, the unique positive solution of

$$g(r) = 1, \tag{6}$$

when such a root exists, or zero otherwise, and the Lundberg's inequality

$$\psi(u) \leq \mathrm{e}^{-Ru} \tag{7}$$

is still valid (this inequality has to be modified for the stationary renewal case). Lundberg's inequality in the ordinary renewal case was first proved by Sparre Anderson (1957) and can be found using a martingale approach in Grandell (1991).

Grandell (1991, pp. 145–148) generalizes Gerber's upper bound for the probability of ruin in finite horizon for the ordinary renewal case (see Gerber (1973) or Gerber (1979, p. 139)). He shows that

$$\psi(u,t) \leq \exp\left(\min_{r\geq R}(-ur + t\theta(r))\right), \tag{8}$$

where $\theta(r)$ is, for each $r \geq 0$, the solution to

$$E[\mathrm{e}^{rX}]E[\mathrm{e}^{-(cr+\theta)T}] = 1. \tag{9}$$

Note that $\theta(R) = 0$.

Let us consider that the insurer is willing to reinsure this risk by means of an excess of loss arrangement, with retention limit M, i.e. when a claim of size X occurs the insurer is responsible for $X_M = \min(X, M)$ and the reinsurer by $X - X_M = \max(0, X - M)$. Hence the insurer net (of reinsurance) risk at time t is

$$Y_M(t) = (c - c_M)t - \sum_{i=1}^{N(t)} \min(X_i, M), \tag{10}$$

where c_M is the excess of loss reinsurer premium. For a given M, the adjustment coefficient, R_M, is now the unique positive root of

$$g_M(r) = 1, \tag{11}$$

when such a root exists, or zero otherwise, with

$$g_M(r) = E[\mathrm{e}^{rX_M}]E[\mathrm{e}^{-(c-c_M)rT}]. \tag{12}$$

The upper bound for the insurer's probability of ruin in finite horizon, after reinsurance, is

$$\psi_M(u,t) \le \exp\left(\min_{r\ge R_M} f_M(r;u,t)\right), \tag{13}$$

where

$$f_M(r;u,t) = -ur + t\theta_M(r) \tag{14}$$

and $\theta_M(r)$ is the only root to

$$E[\mathrm{e}^{rX_M}]E[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}] = 1. \tag{15}$$

The insurer's expected net profit per period of time is $(c-c_M)-\gamma E[X_M]$. Let L be the set of points for which the insurer's net expected profit is positive, i.e.

$$L = \{M : M \ge 0 \ \text{ and } \ (c-c_M)-\gamma E[X_M] > 0\}. \tag{16}$$

Let

$$\chi_M(r) = \ln\ E[\mathrm{e}^{rX_M}] \tag{17}$$

and

$$\kappa(s) = \ln\ E[\mathrm{e}^{-sT}]. \tag{18}$$

Lemma 1.

(i) *The adjustment coefficient is positive if and only if* $M \in L$.
(ii) *For any* $M > 0$, $\theta_M(r)$ *is a convex function of* r, $\theta_M(0) = \theta_M(R_M) = 0$, $\lim_{r\to\infty}\theta_M(r) = +\infty$, $\lim_{r\to\infty}\theta_M(r)/r = +\infty$, $(\partial/\partial r)\theta_M(0) = \gamma E[X_M] - (c-c_M)$ *and for any* $r \ge 0$, $(c-c_M)r + \theta_M(r) \ge 0$.

Proof.

(i) See Lemma 1 in Centeno (2002).
(ii) (15) is equivalent to

$$\chi_M(r) + \kappa((c-c_M)r + \theta_M(r)) = 0. \tag{19}$$

Differentiating (19) with respect to r, and using (15) we get

$$\frac{\partial}{\partial r}\theta_M(r) = E[X_M\,\mathrm{e}^{rX_M}]\frac{E^2[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}{E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]} - (c-c_M) \tag{20}$$

from where (again using (15))

$$\frac{\partial^2}{\partial r^2}\theta_M(r) = E[X_M^2\,e^{rX_M}]\frac{E^2[e^{-((c-c_M)r+\theta_M(r))T}]}{E[T\,e^{-((c-c_M)r+\theta_M(r))T}]} - 2E^2[X_M\,e^{rX_M}]\frac{E^3[e^{-((c-c_M)r+\theta_M(r))T}]}{E[T\,e^{-((c-c_M)r+\theta_M(r))T}]}$$

$$+E^2[X_M\,e^{rX_M}]\frac{E^4[e^{-((c-c_M)r+\theta_M(r))T}]}{E^3[T\,e^{-((c-c_M)r+\theta_M(r))T}]}E[T^2\,e^{-((c-c_M)r+\theta_M(r))T}]$$

$$= \frac{E[e^{-((c-c_M)r+\theta_M(r))T}]}{E[T\,e^{-((c-c_M)r+\theta_M(r))T}]}\left[\frac{E[X_M^2\,e^{rX_M}]}{E[e^{rX_M}]} - \left(\frac{E[X_M\,e^{rX_M}]}{E[e^{rX_M}]}\right)^2\right]$$

$$+E^2[X_M\,e^{rX_M}]\frac{E^5[e^{-((c-c_M)r+\theta_M(r))T}]}{E^3[T\,e^{-((c-c_M)r+\theta_M(r))T}]}$$

$$\times\left[\frac{E[T^2\,e^{-((c-c_M)r+\theta_M(r))T}]}{E[e^{-((c-c_M)r+\theta_M(r))T}]} - \left(\frac{E[T\,e^{-((c-c_M)r+\theta_M(r))T}]}{E[e^{-((c-c_M)r+\theta_M(r))T}]}\right)^2\right] \quad (21)$$

which is positive, i.e. $\theta_M(r)$ is convex, because

$$\chi_M''(r) = \left[\frac{E[X_M^2\,e^{rX_M}]}{E[e^{rX_M}]} - \left(\frac{E[X_M\,e^{rX_M}]}{E[e^{rX_M}]}\right)^2\right] \quad (22)$$

and

$$\kappa''(s) = \frac{E[T^2\,e^{-sT}]}{E[e^{-sT}]} - \left(\frac{E[T\,e^{-sT}]}{E[e^{-sT}]}\right)^2 \quad (23)$$

are both variances of Esscher transforms.

Noticing that $\theta_M(0) = 0$, then $(\partial/\partial r)\theta_M(0) = \gamma E[X_M] - (c - c_M)$ (which is negative for any $M \in L$). That $\lim_{r\to+\infty}\theta_M(r) = +\infty$ and $\lim_{r\to+\infty}\theta_M(r)/r = +\infty$ follows from the fact that $\theta_M(0) = \theta_M(R_M) = 0$ and that $\theta_M(r)$ is convex.

It follows from (15) that $E[e^{-((c-c_M)r+\theta_M(r))T}] \le 1$. Given the convexity of $\kappa(s)$ and applying Jensen's inequality we have

$$1 \ge E[e^{-((c-c_M)r+\theta_M(r))T}] \ge \exp[E(-((c-c_M)r+\theta_M(r))T)] = \exp\left[-\frac{(c-c_M)r+\theta_M(r)}{\gamma}\right]$$

which implies that $(c - c_M)r + \theta_M(r) \ge 0$. □

As it is well known, see Waters (1983), the adjustment coefficient, for the classical model, is a unimodal function of the excess of loss retention limit, when the reinsurance premium is calculated according to the expected value principle, but this is not necessarily the case with other premium principles. Hence in what follows we will assume that the reinsurance premium is calculated according to the expected value principle, i.e.

$$c_M = (1+\alpha)\gamma\int_M^\infty (1 - F(x))\,dx. \quad (24)$$

We also assume that

$$\alpha > \rho, \quad (25)$$

so that the insurer cannot reinsurer the hole risk with a certain profit. Note that under this assumptions there exists a positive M_0 such that $M \in L$ if and only if $M > M_0$.

Lemma 2. *The adjustment coefficient, R_M, is a unimodal function of the retention limit, for $M > M_0$. Its maximum is attained at the unique (finite) point satisfying*

$$M = \frac{1}{R_M}\left(\ln(1+\alpha) + \ln\left(\gamma \frac{E[T\,\mathrm{e}^{-(c-c_M)R_M T}]}{E^2[\mathrm{e}^{-(c-c_M)R_M T}]}\right)\right). \tag{26}$$

Proof. Consider $a = 1$ in part (i) of Result 1 in Centeno (2002). □

Note that in the classical model, i.e. when T is exponentially distributed

$$\gamma \frac{E[T\,\mathrm{e}^{-(c-c_M)R_M T}]}{E^2[\mathrm{e}^{-(c-c_M)R_M T}]} = 1$$

and (26) is equivalent to $M = (1/R)\ln(1+\alpha)$, as it is well known.

3. Gerber's upper bound as a function of the retention

We shall now concentrate on the upper bound given by (13) as a function of the retention limit.

Result 1.

(i) For each $M > 0$, $f_M(r; u, t)$, defined, for $r > 0$, by (14), has a local minimum if and only if the expected surplus at time t is positive. In that case the minimizer is unique, let it be $\hat{r}_M$.

(ii) Suppose that the expected surplus at time t is positive. Then $\hat{r}_M > R_M$—where R_M is the unique positive root of (11) if $M > M_0$ or zero otherwise—if and only if

$$\frac{u}{t} > E[X_M\,\mathrm{e}^{R_M X_M}]\frac{E^2[\mathrm{e}^{-((c-c_M)R_M)T}]}{E[T\,\mathrm{e}^{-((c-c_M)R_M)T}]} - (c - c_M). \tag{27}$$

Proof.

(i) For $M > 0$, it is clear that $f_M(r; u, t)$ is a convex function of r, for $r > 0$ (see Lemma 1). On the other hand, also by Lemma 1

$$\lim_{r\to 0} f_M(r; u, t) = 0$$

and

$$\lim_{r\to +\infty} f_M(r; u, t) = +\infty.$$

Then $f_M(r; u, t)$ will have a minimum (in r) if and only if $(\partial/\partial r) f_M(r; u, t)$ is negative as $r \to 0$. But

$$\frac{\partial}{\partial r} f_M(r; u, t) = -u + t\frac{\partial}{\partial r}\theta_M(r). \tag{28}$$

Hence

$$\lim_{r\to 0}\frac{\partial}{\partial r} f_M(r; u, t) = -u + t(\gamma E[X_M] - (c - c_M))$$

from where the result follows.

(ii) $\hat{r}_M$ is the solution of

$$\frac{\partial}{\partial r} f_M(r; u, t) = 0$$

with $(\partial/\partial r) f_M(r; u, t)$ defined by (28). It is clear that $\hat{r}_M$ will be greater than R_M if and only if $(\partial/\partial r) f_M(r; u, t)$ is negative at $r = R_M$. Substituting (20) into (28) and that $\theta_M(R_M) = 0$, the result follows. □

Let M_1 be the minimum of the values for which the expected surplus at time t is non-negative, i.e.

$$M_1 = \min\{M : M \geq 0 \text{ and } u + t((c - c_M) - \gamma E[X_M]) \geq 0\}. \tag{29}$$

Note that M_1 is zero if and only if $u/t \geq \gamma\mu(\alpha - \rho)$. The following corollary follows from the previous proof.

Corollary 1. *For each $M > M_1$*

$$\psi_M(u,t) \leq \begin{cases} e^{f_M(u,t,\hat{r}_M)} & \text{if } \dfrac{u}{t} > E[X_M\, e^{R_M X_M}] \dfrac{E^2[e^{-((c-c_M)R_M)T}]}{E[T\, e^{-((c-c_M)R_M)T}]} - (c - c_M), \\ e^{f_M(u,t,R_M)} & \text{if } \dfrac{u}{t} \leq E[X_M\, e^{R_M X_M}] \dfrac{E^2[e^{-((c-c_M)R_M)T}]}{E[T\, e^{-((c-c_M)R_M)T}]} - (c - c_M), \end{cases} \tag{30}$$

where R_M is the only positive solution to (11) if $M > M_0$, or zero otherwise, and $r = \hat{r}_M$ is such that $(r, \theta_M(r))$ is the solution to

$$\frac{E[X_M\, e^{rX_M}]E[e^{-((c-c_M)r+\theta_M(r))T}]}{E[e^{rX_M}]E[T\, e^{-((c-c_M)r+\theta_M(r))T}]} - (c - c_M) = \frac{u}{t}, \qquad E[e^{rX_M}]E[e^{-((c-c_M)r+\theta_M(r))T}] = 1. \tag{31}$$

Hence we can conclude that for some values of M it is possible to improve Lundberg's inequality, which implies that in some cases the value of M that minimizes the upper bound provided by Gerber's inequality is different from the value of M that maximizes the adjustment coefficient. That will be the case if

$$\frac{u}{t} > (1+\alpha)\gamma\, e^{-\hat{R}_{\hat{M}}\hat{M}} E[X_{\hat{M}}\, e^{\hat{R}_{\hat{M}} X_{\hat{M}}}] - (c - c_{\hat{M}}), \tag{32}$$

where $(\hat{M}, \hat{R}_{\hat{M}})$ is the solution of

$$E[e^{rX_M}]E[e^{-(c-c_M)rT}] = 1, \qquad e^{rM} E^2[e^{-(c-c_M)rT}] = (1+\alpha)\gamma E[T\, e^{-(c-c_M)rT}] \tag{33}$$

(see (26)). Let us study the behavior of Gerber's bound as a function of the retention limit.

Result 2. If $u/t \geq \gamma\mu(\alpha - \rho)$ then the upper bound to the probability of ruin before time t attains its minimum at $M = 0$.

If $u/t < \gamma\mu(\alpha - \rho)$ then the upper bound, considered as function of M, has an absolute minimum which is attained at the unique point satisfying

$$M = \frac{1}{r^*}\left(\ln(1+\alpha) + \ln\left(\gamma \frac{E[T\, e^{-((c-c_M)r^*+\theta_M(r^*))T}]}{E^2[e^{-((c-c_M)r^*+\theta_M(r^*))T}]}\right)\right),$$

where $\theta_M(r^*)$ is the only solution to (15), and $r^* = \max(\hat{r}, \hat{R})$, where $\hat{r}$ is the solution to

$$(1+\alpha)\gamma\, e^{-rM} \int_0^M (1 + rx)\, e^{rx}(1 - F(x))\, dx - (c - c_M) = \frac{u}{t}$$

and $\hat{R}$ is the adjustment coefficient.

Proof. Notice that

$$\min_{M\geq M_1}\psi_M(u,t)\leq \exp\left(\min_{M\geq M_1}\min_{r\geq R(M)} f_M(r;u,t)\right)=\exp\left(\min_{r\geq R(M)}\min_{M\geq M_1} f_M(r;u,t)\right) \tag{34}$$

with

$$f_M(r;u,t)=-ur+t\theta_M(r), \tag{35}$$

where $\theta_M(r)$ is the solution to (15).

We are now considering these functions as functions of both r and M. Calculating the derivative of (35) with respect to M, we get

$$\frac{\partial}{\partial M}f_M(r;u,t)=t\frac{\partial}{\partial M}\theta_M(r) \tag{36}$$

and

$$\frac{\partial}{\partial M^2}f_M(r;u,t)=t\frac{\partial}{\partial M^2}\theta_M(r). \tag{37}$$

Differentiating (15) with respect to M, we get

$$r\,\mathrm{e}^{rM}(1-F(M))E[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]-\left(\gamma(1+\alpha)(1-F(M))r+\frac{\partial}{\partial M}\theta_M(r)\right)\times E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]E[\mathrm{e}^{rX_M}]=0$$

from where it follows that

$$\begin{aligned}\frac{\partial}{\partial M}\theta_M(r)&=\frac{r(1-F(M))\,\mathrm{e}^{rM}E[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}{E[\mathrm{e}^{rX_M}]E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}-(1+\alpha)\gamma r(1-F(M))\\&=\frac{r(1-F(M))}{E[\mathrm{e}^{rX_M}]E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}[\mathrm{e}^{rM}E[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]\\&\quad-(1+\alpha)\gamma E[\mathrm{e}^{rX_M}]E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]]\end{aligned} \tag{38}$$

and considering (15), we can conclude that $\partial\theta_M(r)/\partial M=0$ if and only if $P_M(r)=0$, with

$$P_M(r)=\mathrm{e}^{rM}E^2[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]-(1+\alpha)\gamma E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}], \tag{39}$$

i.e.

$$M=\frac{1}{r}\left(\ln(1+\alpha)+\ln\left(\gamma\frac{E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}{E^2[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}\right)\right). \tag{40}$$

Note that (40) is equivalent to (26) for $r=R_M$.

Calculating the second derivative of $\theta_M(r)$, with respect to M, in the points where the first is null we obtain

$$\begin{aligned}&\left.\frac{\partial^2}{\partial M^2}\theta_M(r)\right|_{(\partial/\partial M)\theta_M(r)=0}\\&\quad=\frac{r(1-F(M))}{E[\mathrm{e}^{rX_M}]E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}\{r\,\mathrm{e}^{rM}E[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]\\&\quad-2\,\mathrm{e}^{rM}(1+\alpha)\gamma r(1-F(M))E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]\\&\quad+(1+\alpha)^2\gamma^2 r(1-F(M))E[\mathrm{e}^{rX_M}]E[T^2\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]\}\end{aligned}$$

which is, having in consideration (15), equivalent to

$$\left.\frac{\partial^2}{\partial M^2}\theta_M(r)\right|_{(\partial/\partial M)\theta_M(r)=0} = \frac{r(1-F(M))}{E[\mathrm{e}^{rX_M}]E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}B_M(r), \tag{41}$$

where

$$B_M(r) = r(1+\alpha)\gamma \times \left\{E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]\int_0^M \mathrm{e}^{rx}\,\mathrm{d}F(x)+(1-F(M))(1+\alpha)\gamma\kappa''((c-c_M)r+\theta_M(r))\right\} \tag{42}$$

which is positive. On the other hand, when M is zero, $X_0 \equiv 0$, the equation defining $\theta_0(r)$ is $E[\mathrm{e}^{-((c-c_0)r+\theta_0(r))T}] = 1$, which implies that $\theta_0(r) = -(c-c_0)r$. Hence

$$\lim_{M\to 0} P_M(r) = -\alpha$$

and for any positive r,

$$\lim_{M\to +\infty} P_M(r) = +\infty.$$

Hence for fixed r, u and t, $f_M(r;u,t)$ has a local minimum, which is unique and attained at the point $\hat{M}(r)$ such that $(M, \theta_M(r)) = (\hat{M}(r), \theta_{\hat{M}(r)}(r))$ is the solution to

$$\begin{aligned} &\mathrm{e}^{rM}E^2[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}] - (1+\alpha)\gamma E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}] = 0, \\ &E[\mathrm{e}^{rX_M}]E[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}] = 1. \end{aligned} \tag{43}$$

Let us now study the function $f_{\hat{M}(r)}(r;u,t)$. Using the implicit function theorem we can see that

$$\frac{\mathrm{d}}{\mathrm{d}r}f_{\hat{M}(r)}(r;u,t) = -u + t\left.\frac{\partial}{\partial r}\theta_M(r)\right|_{(\partial/\partial M)\theta_M(r)=0} \tag{44}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}f_{\hat{M}(r)}(r;u,t) = t\left.\frac{(\partial^2/\partial r^2)\theta_M(r)\times(\partial^2/\partial M^2)\theta_M(r) - ((\partial^2/\partial r\partial M)\theta_M(r))^2}{(\partial^2/\partial M^2)\theta_M(r)}\right|_{(\partial/\partial M)\theta_M(r)=0},$$

which has the same sign as the numerator.

Differentiating (38) with respect to r we get

$$\begin{aligned} &\left.\frac{\partial^2}{\partial r\partial M}\theta_M(r)\right|_{(\partial/\partial M)\theta_M(r)=0} \\ &= r(1+\alpha)\gamma(1-F(M)) \\ &\quad\times\left[M + E[X_M\,\mathrm{e}^{rX_M}]E[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]\left(\frac{E[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]E[T^2\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}{E^2[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]} - 2\right)\right]. \end{aligned}$$

After some tedious algebra calculations we get

$$\left.\frac{\partial^2}{\partial r^2}\theta_M(r) \times \frac{\partial^2}{\partial M^2}\theta_M(r) - \left(\frac{\partial^2}{\partial r\partial M}\theta_M(r)\right)^2\right|_{(\partial/\partial M)\theta_M(r)=0}$$

$$= r^2(1-F(M))(1+\alpha)\gamma \frac{E^2[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}{E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}\int_0^M x^2\,\mathrm{e}^{rx}\,\mathrm{d}F(x)$$

$$+ r^2(1-F(M))(1+\alpha)\gamma E[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]\left(\frac{E[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]E[T^2\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}{E^2[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}-2\right)$$

$$\times\left\{(1-F(M))(1+\alpha)\gamma E[X_M^2\,\mathrm{e}^{rX_M}] + E^2[X_M\,\mathrm{e}^{rX_M}]\frac{E^2[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}{E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}\right.$$

$$\left. -2(1-F(M))(1+\alpha)\gamma M E[X_M\,\mathrm{e}^{rX_M}]\right\}$$

$$= r^2(1-F(M))(1+\alpha)\gamma \frac{E^2[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}{E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}\int_0^M x^2\,\mathrm{e}^{rx}\,\mathrm{d}F(x)$$

$$+ r^2(1-F(M))(1+\alpha)\gamma E[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]\left(\frac{E[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]E[T^2\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}{E^2[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}-2\right)$$

$$\times\left((1-F(M))(1+\alpha)\gamma\int_0^M x^2\,\mathrm{e}^{rx}\,\mathrm{d}F(x) + \left(\int_0^M x\,\mathrm{e}^{rx}\,\mathrm{d}F(x)\right)^2\frac{E^2[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}{E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}\right)$$

$$= r^2(1-F(M))(1+\alpha)\gamma \frac{E^3[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}{E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}\kappa''((c-c_M)r+\theta_M(r))$$

$$\times\left((1-F(M))(1+\alpha)\gamma\int_0^M x^2\,\mathrm{e}^{rx}\,\mathrm{d}F(x) + \left(\int_0^M x\,\mathrm{e}^{rx}\,\mathrm{d}F(x)\right)^2\frac{E^2[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}{E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}\right)$$

$$+ r^2(1-F(M))(1+\alpha)\gamma \frac{E^3[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}{E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}$$

$$\times\left\{E[\mathrm{e}^{rX_M}]\int_0^M x^2\,\mathrm{e}^{rx}\,\mathrm{d}F(x) - (1-F(M))\,\mathrm{e}^{-rM}\int_0^M x^2\,\mathrm{e}^{rx}\,\mathrm{d}F(x) - \left(\int_0^M x\,\mathrm{e}^{rx}\,\mathrm{d}F(x)\right)^2\right\}$$

$$= r^2(1-F(M))(1+\alpha)\gamma \frac{E^3[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}{E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}\kappa''((c-c_M)r+\theta_M(r))$$

$$\times\left((1-F(M))(1+\alpha)\gamma\int_0^M x^2\,\mathrm{e}^{rx}\,\mathrm{d}F(x) + \left(\int_0^M x\,\mathrm{e}^{rx}\,\mathrm{d}F(x)\right)^2\frac{E^2[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}{E[T\,\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}\right)$$

$$+ r^2(1-F(M))(1+\alpha)\gamma \frac{E^3[\mathrm{e}^{-((c-c_M)r+\theta_M(r))T}]}{E[T\,\mathrm{e}^{-((c-_M)r+\theta_M(r))T}]}$$

$$\times\left(\int_0^M \mathrm{e}^{rx}\,\mathrm{d}F(x)\right)^2\left(\frac{\int_0^M x^2\,\mathrm{e}^{rx}\,\mathrm{d}F(x)}{\int_0^M \mathrm{e}^{rx}\,\mathrm{d}F(x)} - \left(\frac{\int_0^M x\,\mathrm{e}^{rx}\,\mathrm{d}F(x)}{\int_0^M \mathrm{e}^{rx}\,\mathrm{d}F(x)}\right)^2\right),$$

which is positive, implying that $f_{\hat{M}(r)}(r; u, t)$ is a convex function of r. Hence we can conclude that there is at most one solution to (44) and that when it exists it is the global minimum of $f_{\hat{M}(r)}(r; u, t)$. But we have (see Lemma 1) that

$$\lim_{r\to 0} f_{\hat{M}(r)}(r; u, t) = 0$$

and

$$\lim_{r\to 0} \frac{\mathrm{d}}{\mathrm{d}r} f_{\hat{M}(r)}(r; u, t) = -u - t\mu\rho < 0.$$

If $u/t \geq \gamma\mu(\alpha - \rho)$, then M_1 given by (29) is zero and when M goes to 0, r given by the solution to (40) goes to infinity and

$$\lim_{r\to +\infty} f_{\hat{M}(r)}(r; u, t) = \lim_{r\to +\infty} (-ru - rt\gamma\mu(\alpha - \rho)) = -\infty,$$

and the first part of the result is proved.

If $u/t < \gamma\mu(\alpha - \rho)$, the solution in r, let it be r_1, of (40) for $M = M_1$ is finite and

$$\begin{aligned}\lim_{r\to r_1} \frac{\mathrm{d}}{\mathrm{d}r} f_{\hat{M}(r)}(r; u, t) &= -u + tE[X_{M_1}\,\mathrm{e}^{r_1 X_{M_1}}]\frac{E^2[\mathrm{e}^{-((c-c_{M_1})r_1+\theta_{M_1}(r_1))T}]}{E[T\,\mathrm{e}^{-((c-c_{M_1})r_1+\theta_{M_1}(r_1))T}]} - t(c - c_{M_1})\\ &= tE[X_{M_1}\,\mathrm{e}^{r_1 X_{M_1}}]\frac{E^2[\mathrm{e}^{-((c-c_{M_1})r_1+\theta_{M_1}(r_1))T}]}{E[T\,\mathrm{e}^{-((c-c_{M_1})r_1+\theta_{M_1}(r_1))T}]} - t\gamma E[X_{M_1}]\\ &= t\frac{E[X_{M_1}\,\mathrm{e}^{r_1 X_{M_1}}]E[\mathrm{e}^{-((c-c_{M_1})r_1+\theta_{M_1}(r_1))T}]}{E[\mathrm{e}^{r_1 X_{M_1}}]E[T\,\mathrm{e}^{-((c-c_{M_1})r_1+\theta_{M_1}(r_1))T}]} - t\gamma E[X_{M_1}] \geq 0.\end{aligned}$$

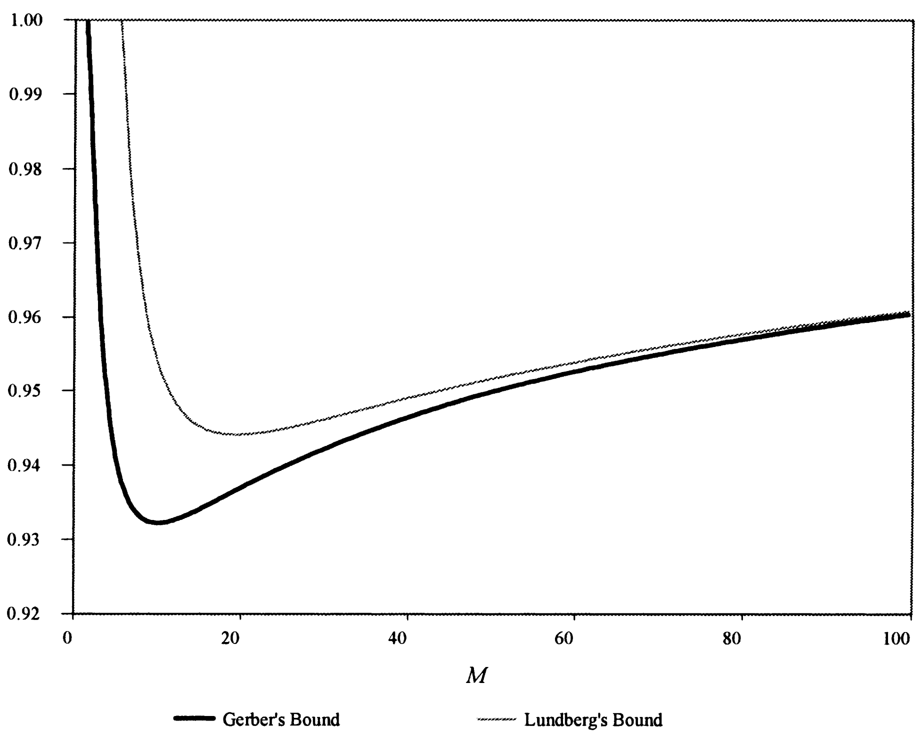

Fig. 1. T is $\gamma(0.5, 0.5)$.

The last inequality follows because X_{M_1} and $e^{r_1 X_{M_1}}$ are positively correlated for any $r > 0$ and T and e^{-sT} are negatively correlated for any $s > 0$, and by Lemma 1 we have that $(c - c_{M_1})r_1 + \theta_{M_1}(r_1)$ is greater or equal to zero.

Hence $\hat{r}$ exists and is smaller than r_1 and the proof is finished. □

4. An example

Let the individual claim amount distribution be Pareto (2, 1), i.e. $F(x) = 1 - 1/(1+x)^2$, $x > 0$. Let $c = 1.12$, $\alpha = 0.8$, $u = 2$ and $t = 10$. We consider that the inter arrival time T is Gamma (n, β) distributed, i.e. with density function given by

$$p(t) = \frac{\beta^n}{\Gamma(n)} e^{-\beta t} t^{n-1}, \quad t > 0. \tag{45}$$

In this case

$$\theta_M(r) = \beta[(E[X_M])^{1/n} - 1] - (c - c_M)r. \tag{46}$$

We shall consider three different situations: (n, β) equal respectively to (0.5, 0.5), (1, 1) (the classical model) and (2, 2), to which Figs. 1–3 correspond, respectively. These figures show Gerber's and Lundberg's upper bounds. Tables 1–3 give the values attained by these functions at the minimum of each of them, in the three inter arrival situations.

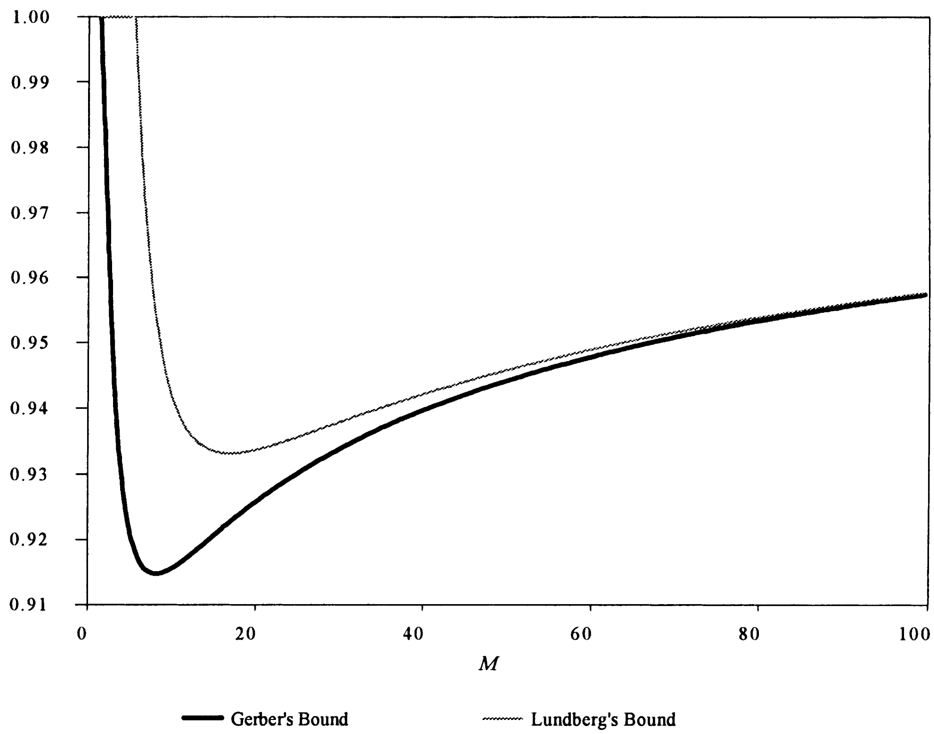

Fig. 2. T is exponential.

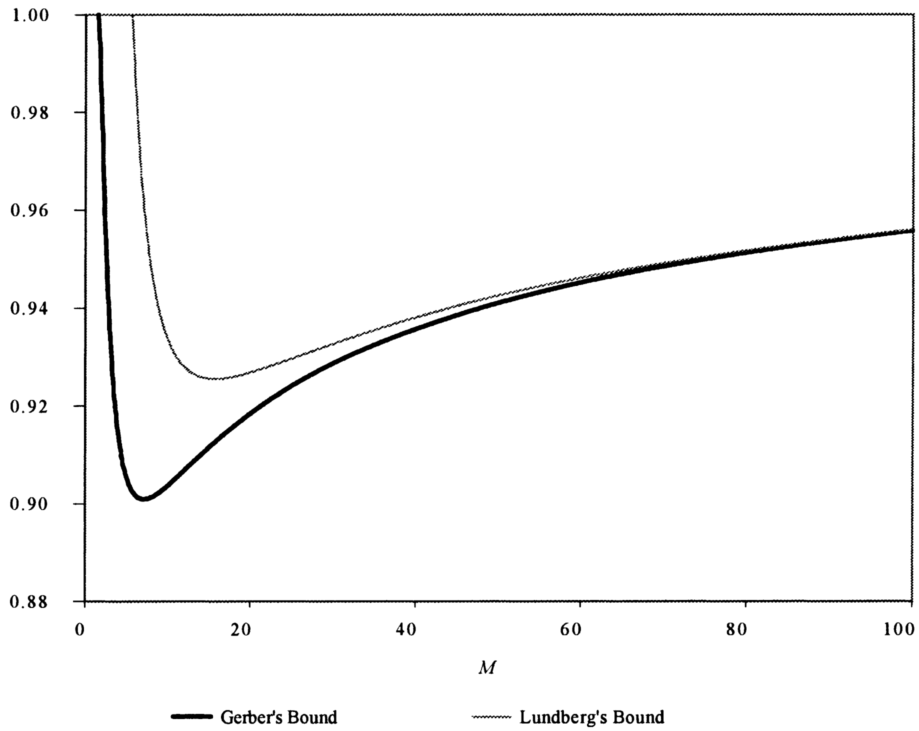

Fig. 3. T is $\gamma(2, 2)$.

Table 1
Optimal XL retentions, interarrival times $\gamma(0.5, 0.5)$

M	Gerber's bound	Lundberg's bound
10.08	0.932221915	0.953787398
19.45	0.936602006	0.944148910

Table 2
Optimal XL retentions, interarrival times exponential

M	Gerber's bound	Lundberg's bound
8.185	0.914732859	0.953562132
16.98	0.922630450	0.933110799

Table 3
Optimal XL retentions, interarrival times $\gamma(2, 2)$

M	Gerber's bound	Lundberg's bound
7.12	0.900886417	0.960125393
15.665	0.912537080	0.925415440

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