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HOMOCLINIC AND PERIODIC SOLUTIONS FOR A CLASS OF SECOND ORDER DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

In this paper we are concerned with the existence of positive homoclinic solutions of the second order equation

$$u'' - a(x)\,u + b(x)\,u^2 + c(x)\,u^3 = 0, \quad x \in \mathbf{R}. \tag{I}$$

where the coefficient functions $a(x), b(x)$ and $c(x)$ are continuous and there are positive constants a, A, b, B, c and C verifying

$$0 < a \le a(x) \le A, \ 0 < c \le c(x) \le C, \tag{1}$$
$$0 \le b \le b(x) \le B. \tag{2}$$

We study also the existence of periodic solutions of the equation

$$u'' + a(x)\,u - b(x)\,u^2 + c(x)\,u^3 = 0, \tag{II}$$

where $a(x)$, $b(x)$, $c(x)$ are continuous 2π–periodic functions satisfying (1) and instead of (2)

$$|b(x)| \le B. \tag{3}$$

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At first assuming that $a(x), b(x)$ and $c(x)$ are 2π–periodic, we prove the existence of a nontrivial positive homoclinic solution of equation (I) whenever

$$B^2 - b^2 < 4ac. \tag{4}$$

Next we suppose that $a(x), b(x)$ and $c(x)$ are even differentiable functions. Assuming moreover that

$$xa'(x) > 0, \quad xb'(x) < 0, \quad xc'(x) < 0, \tag{5}$$

we prove the existence of a unique nontrivial even positive homoclinic solution of equation (I).

Existence results are obtained via variational method. In the first existence theorem, we find the homoclinic solution u as the limit, as $n \to +\infty$, of periodic extensions of positive solutions u_n of approximating periodic problems defined in the intervals $[-n\pi, n\pi]$, $n \in \mathbf{N}$. Such approximating solutions are obtained as critical values of associated functionals, by using Mountain Pass Theorem. The variational characterization of the respective critical values allow us to derive some estimates and pass to the limit. This problem was suggested by the paper of Grossinho and Sanchez [4], where existence of periodic solutions of equation (I) is considered.

In the second existence theorem, we use a similar approximating procedure but for a boundary value problem and apply a lemma of Korman and Ouyang [7] (see also Gidas, Ni and Nirenberg [5]). This study was motivated by the paper of Korman and Lazer [6], where equation (I) is considered with $b(x)$ identically 0. Our second result extends the existence theorem established there for that case.

This type of procedure has been used by different authors to study the existence of homoclinics, namely in the case of Hamiltonian systems. We refer to Rabinowitz [9], Ambrosetti and Bertotti [1], Korman and Lazer [6], Ariolli and Szulkin [2]. However those results do not apply to equation (I). Essentially, not only the nonlinearity we consider does not satisfy the hypotheses assumed there, but also [1], [9] and [2] do not concern positive solutions.

The study of the equation (II) was suggested by a paper of Cronin [3]. Existence of periodic solutions of equation (II) is considered by Grossinho and Sanchez [4] using a critical point theorem in finite dimensional spaces due to Rabinowitz [10], Galerkin type procedure and Gagliardo–Nirenberg inequalities. Main assumptions are

$$m^2 < a \leq a(x) \leq A < (m+1)^2, \; B^2 \leq \frac{9}{2}\left(a - m^2\right) c. \tag{6}$$

We prove the result using a saddle point theorem in infinite dimensional spaces due to E.A. Silva [8]. This approach allows to extend the result to equations of the form

$$u'' + a(x)\, u + g(x,u) = 0, \tag{7}$$

under assumptions on $G(x,u) = \int\limits_0^u g(x,t)\, dt$:

1) $a(x), g(x,u)$ are measurable, 2π periodic in x functions,
2) $c|u|^4 - B|u|^3 \leq G(x,u) \leq C|u|^4 + B|u|^3$,
3) $m^2 < a \leq a(x) \leq A < (m+1)^2$,
4) $B^2 < 2c\left(a - m^2\right)$,
5) $0 < \theta G(x,u) \leq ug(x,u)$, if $|u| \geq R$, where $\theta > 2$.

Moreover the result can extend to equations of the form

$$u'' + a(x)\, u - b(x)\, u^2 + c(x)\, u^{2k+1} = 0,$$

changing (6) by a natural way.

Also we prove existence of periodic solutions of the equation (II) with period $2n\pi$ for sufficiently large n.

We suppose $0 \notin \sigma(L)$ where $L : H^1(\mathbf{R}) \to H^1(\mathbf{R})$ is the operator defined by

$$(Lu, v) = \int \left(u'v' - a(x)\, uv\right) dx,$$

and $B^2 \leq 9c\lambda_0$, where λ_0 is a positive constant independent of n.

2. POSITIVE HOMOCLINIC SOLUTIONS OF EQUATION (I)

In this section we consider the existence of positive solutions of the problem

$$\begin{cases} u'' - a(x)\, u + b(x)\, u^2 + c(x)\, u^3 = 0, & x \in \mathbf{R}, \\ u(\pm\infty) = u'(\pm\infty) = 0, & \end{cases} \tag{P}$$

known as homoclinic solutions.

Suppose $a(x), b(x), c(x)$ are continuous 2π periodic functions such that there are positive constants a, b, B, c and C verifying

$$0 < a \leq a(x),\ 0 \leq b \leq b(x) \leq B,\ 0 < c \leq c(x) \leq C, \tag{8}$$

and

$$4ac > B^2 - b^2. \tag{9}$$

For every $n \in \mathbf{N}$, we consider the following periodic problem

$$\begin{cases} u'' - a(x)u + b(x)u^2 + c(x)u^3 = 0, & x \in (-n\pi, n\pi), \\ u(-n\pi) = u(n\pi). \end{cases} \tag{P_n}$$

Set $I_n = [-n\pi, n\pi]$ and consider the Sobolev space

$$H_n = \left\{ u \in H^1(I_n) : u(-n\pi) = u(n\pi) \right\},$$

with the norm

$$||u||_n = \left(\int_{-n\pi}^{n\pi} \left(u'^2(x) + u^2(x) \right) dx \right)^{\frac{1}{2}}.$$

We solve (P_n) using mountain pass theorem and obtain uniform estimates for their solutions.We need the following technical proposition of [9].

Proposition 1 *Let $u \in H^1_{loc}(\mathbf{R})$. Then:*
(i) If $T \geq 1$, for $x \in [T-1, T+1]$

$$\max_{x \in [T-1,T+1]} |u(x)| \leq 2 \left(\int_{T-1}^{T+1} \left(u'^2(t) + u^2(t) \right) dt \right)^{1/2}. \tag{10}$$

(ii) For every $u \in H_n$

$$||u||_{L^\infty(-n\pi, n\pi)} \leq 2||u||_n. \tag{11}$$

We have

Lemma 2 *Let assumptions (8), (9) hold. Then, for every $n \in N$, the problem (P_n) has a positive solution $u_n(x)$. Moreover, there is a constant $K > 0$, independently of n, such that*

$$||u||_n \leq K. \tag{12}$$

Theorem 3 *Let assumptions (8), (9) hold. Then the problem (P) has a positive homoclinic solution.*

Sketch of proof. For every $n \in \mathbf{N}$, consider the solution u_n of problem (P_n), given by Lemma 2. By (12) and the embedding of H_n in $C[-n\pi, n\pi]$, there is K_1 such that $||u_n||_{C[-n\pi,n\pi]} \leq K_1$. By the equation of (P_n), it follows that there is K_2 such that $||u_n||_{C^2[-n\pi,n\pi]} \leq K_2$, where K_1, K_2 are positive constants independent of n. Consider the periodic extension of u_n to $\mathbf{R}$ and denote it by the same symbol. Then u_n a $2n\pi$-periodic solution of equation (I). There exists a subsequence of (u_n) which converges in $C^2_{loc}(\mathbf{R})$ to a solution u of (P) that satisfies

$$\int_{-\infty}^{\infty} \left(u'^2 + u^2\right) dx < \infty. \tag{13}$$

It remains to show that u is nonzero and $u(\pm\infty) = u'(\pm\infty) = 0$. Let $x_n \in [-n\pi, n\pi]$ be a point of where u_n attains its maximum value. Since $u_n(x_n) > 0$ and $u_n''(x_n) \leq 0$, it follows by the equation (I)

$$\begin{aligned} & u_n(x_n)\left(-a(x_n) + b(x_n)u_n(x_n) + c(x_n)u_n^2(x_n)\right) \\ = \; & -u_n''(x_n) \geq 0. \end{aligned}$$

Then, by assumptions (1), (2) and (4),

$$u_n(x_n) \geq \frac{-B + \sqrt{b^2 + 4ac}}{2C} > 0, \tag{14}$$

independently of n.

Let us denote by H_n^* the space that consists of the periodic extensions of the functions of H_n and by f_n^* the functional defined on H_n^* by $f_n^*(u) = f_n(u)$. Observe that as the coefficient functions $a(x), b(x), c(x)$ are 2π periodic functions, if $u_n(x)$ is $2n\pi$ periodic solution of (I), then $u_n(x + 2j\pi)$ is also

$2n\pi$ periodic solution for every integer j. Therefore, replacing $u_n(x)$ by some $u_n(x+2j_n\pi)$ if necessary, we still obtain $2n\pi$-solutions of (I), satisfying the above bounds derived by using variational arguments, such that, moreover, have maximum points x_n appearing in the interval $[-\pi,\pi]$. Therefore we can assume $x_n \to x_0$ in $[-\pi,\pi]$. By the uniform convergence of $\{u_n\}$ on $[-\pi,\pi]$ and by (14), it follows that $u(x_0) > 0$. So, the solution $u(x)$ of (I) is nontrivial and nonnegative.

By (13) and Proposition 1 it follows $u(\pm\infty) = 0$. By assumption (8) there exists $M > 0$ such that $|u''(x)| \leq M$ in $\mathbf{R}$. Suppose, by contradiction, that $u'(+\infty) \neq 0$ (the case $u'(-\infty) \neq 0$) is analogous). Then, there exists $\varepsilon > 0$ and a sequence $y_n \to +\infty$ such that $|u'(y_n)| \geq \varepsilon$, for all n. By Lagrange theorem $|u'(x)| \geq \frac{\varepsilon}{2}$ for $x \in (y_n - \delta, y_n + \delta)$, where $\delta \in \left(0, \frac{\varepsilon}{2M}\right)$ .Therefore

$$\int_{y_n-\delta}^{y_n+\delta} u'^2(x)\,dx \geq \frac{\delta\varepsilon^2}{2},$$

which is a contradiction with (13). ■

Note that if $a(x), b(x)$ and $c(x)$ are positive constants, the assumption (9) is satisfied and there exists a positive homoclinic solution of the equation

$$u'' - au + bu^2 + cu^3 = 0.$$

Further, using the above method, we can prove the existence of symmetric positive homoclinic solution of (I) under adequate assumptions on the coefficient functions. Our result is based on a Lemma proved by Korman and Ouyang [7], and is an extension of a result due to Korman and Lazer [6], where equation (I) is considered with $b(x) \equiv 0$.

Let us consider the problem

$$\begin{cases} u'' + f(x,u) = 0, \quad x \in (-T,T), \\ u(-T) = u(T) = 0. \end{cases} \tag{15}$$

Assuming $f \in C^1([-T,T] \times \mathbf{R}^+)$ and

$$\begin{gathered} f(-x,u) = f(x,u), \quad x \in (-T,T), u > 0, \\ f(x,0) = 0, \quad x \in (-T,T), \\ xf_x(x,u) < 0, \quad x \in (-T,T), u > 0. \end{gathered} \tag{16}$$

Recall the result of Korman and Ouyang [7].

Lemma 4 *Assume that* $f \in C^1([-T,T] \times \mathbf{R}^+)$ *satisfies (16). Then any positive solution of (15) is an even function such that* $u'(x) < 0$ *for* $x \in (0,T)$.

We prove

Lemma 5 *Let assumptions (1), (3) and (5) hold. Then, for any* $T \geq 1$, *the problem*

$$\begin{cases} u'' - a(x)\,u + b(x)\,u^2 + c(x)\,u^3 = 0, \quad x \in (-T,T), \\ \qquad\qquad u(-T) = u(T) = 0, \end{cases} \tag{P_T}$$

has a unique positive solution $u_T(x)$. *Moreover,* $u'_T(x) < 0$, *for* $x \in [0,T]$ *and there exists a constant* $K > 0$, *such that*

$$\int_{-T}^{T} \left(u_T'^2(x) + u_T^2(x)\right) dx \leq K, \tag{17}$$

independently of T.

Theorem 6 *Let assumptions (1), (3) and (5) hold. Then problem* (P) *has exactly one positive solution. This solution is an even function with* $u'(x) < 0$ *if* $x \geq 0$.

Sketch of proof. Take $T_n \to \infty$ and let u_n be the solution of problem (P_{T_n}). By Lemma 5

$$\int_{-T_n}^{T_n} \left(u_n'^2(x) + u_n^2(x)\right) dx \leq K.$$

independently of n. Consider the extension of u_n to $\mathbf{R}$ that takes the value 0 in $\mathbf{R} \backslash [-T_n, T_n]$ and denote it by the same symbol. Arguing as in the proof of Theorem 3 we can derive that $u_n \to u$ in $C^2_{loc}(\mathbf{R})$. By Lemma 4, u_n attains its maximum at 0. Then, since

$$u_n(0)\left(a(0) - b(0)\,u_n(0) - c(0)\,u_n^2(0)\right) = u_n''(0) \leq 0,$$

it follows

$$u_n(0) \geq \frac{-b(0) + \left(b^2(0) + 4a(0)\,c(0)\right)^{1/2}}{2c(0)} = \rho_1 > 0,$$

and, therefore $u(0) \geq \rho_1 > 0$. Moreover, u is an even function that attains its only maximum at 0, since the same holds for the functions u_n. Arguing as in the proof of Lemma 4 of Korman and Ouyang [7], we easily obtain $u'(x) < 0$ if $x > 0$, by differentiating the equation (I).

In order to prove uniqueness, observe that if $u,\ v$ are two solutions it follows that

$$\int_{-\infty}^{\infty} uv\left(b(x)(u-v) + c(x)\left(u^2 - v^2\right)\right) dx = 0.$$

Exsitence-uniqueness theorem of Cauchy problem and last indentity imply that $u(x)$ and $v(x)$ cannot be ordered and so they must intersect. Two cases are possible: either $u(x)$ and $v(x)$ have at least two positive points of intersections or only one positive point of intersection. In both cases the proof continues as in the proof of Theorem 2.1 of Laser and Korman [6]. To prove $u(\pm\infty) = u'(\pm\infty) = 0$, we proceed as in the proof of Theorem 3. ■

2. PERIODIC SOLUTIONS OF EQUATION (II)

Existence of 2π periodic solutions of the equation (II) under assumptions (1), (3) and (6) was studied in Grossinho and Sanchez [4] using a critical point theorem in finite dimensional spaces due to Rabinowitz [10], Galerkin type procedure and Gagliardo–Nirenberg inequalities. We prove Theorem 0.2 of [4] using a saddle point theorem in infinite dimensional spaces due to E.A. Silva [8], [11]

Theorem 7 *Let $E = X_1 \oplus X_2$ be a real Banach space, with X_1 finite dimensional. Suppose $f \in C^1(E, \mathbf{R})$ satisfies (PS) condition and*
i) $f(u) \leq 0$, for every $u \in X_1$.
ii) There exists $\rho > 0$ such that $f(u) \geq 0$, for every $u \in X_2$ with $||u|| = \rho$.
iii) There exists $\xi \in X_2, ||\xi|| = 1$ and $\beta \in \mathbf{R}$ such that $f(u) \leq \beta$, for every $u \in X_1 \oplus \mathbf{R}^+\xi$.
Then f possesses a critical point in E other than zero.

Theorem 8 *Assume $a(x), b(x)$ and $c(x)$ are measurable 2π periodic functions such that there are positive constants a, A, B, c and C such that:*

$$\begin{aligned} |b(x)| \leq B,\ 0 < c \leq c(x) \leq C, \\ m^2 < a \leq a(x) \leq A < (m+1)^2, \\ B^2 \leq \frac{9}{2}\left(a - m^2\right) c. \end{aligned}$$

Then the equation (II) has a nontrivial 2π periodic solution.

Sketch of proof. Consider the Sobolev space

$$E = H^1(0, 2\pi) = \left\{ u = \sum_{k \in \mathbf{Z}} c_k e^{ikx}, c_{-k} = \bar{c}_k : \sum_{k \in \mathbf{Z}} \left(1 + k^2\right) c_k^2 < \infty \right\}$$

with usual norm and the functional over E

$$g(u) = \int_0^{2\pi} \left(\frac{1}{2}\left(u'^2 - a(x) u^2\right) + \frac{1}{3} b(x) u^3 - \frac{1}{4} c(x) u^4 \right) dx.$$

It is clear that $g \in C^1(E, \mathbf{R})$ and critical points of f are weak solutions of (II). They are classical solutions if $a(x),\ b(x),\ c(x)$ are continuous functions. Let

$$X_1 = \left\{ u \in E : u = \sum_{k^2 \leq m^2} c_k e^{ikx}, c_{-k} = \bar{c}_k \right\}$$

and

$$X_2 = \left\{ u \in E : u = \sum_{k^2 \geq (m+1)^2} c_k e^{ikx}, c_{-k} = \bar{c}_k \right\}.$$

Let $u \in X_1$. By assumptions of theorem

$$\begin{aligned} g(u) &\leq \pi \sum_{k^2 \leq m^2} c_k^2 \left(k^2 - a\right) + \int_0^{2\pi} \left(\frac{1}{3} b(x) u^3 - \frac{1}{4} c(x) u^4 \right) dx \\ &\leq \int_0^{2\pi} u^2 \left(\frac{m^2 - a}{2} + \frac{1}{3} B|u| - \frac{1}{4} c|u|^2 \right) dx \leq 0. \end{aligned}$$

Let $u \in X_2$. Then

$$\begin{aligned} g(u) &\geq \pi \sum_{k^2 \leq m^2} c_k^2 \left(k^2 - A\right) + \int_0^{2\pi} \left(\frac{1}{3} b(x) u^3 - \frac{1}{4} c(x) u^4\right) dx \qquad (18) \\ &\geq \int_0^{2\pi} u^2 \left(\frac{(m+1)^2 - A}{2} - \frac{1}{3} B|u| - \frac{1}{4} C|u|^2\right) dx. \end{aligned}$$

By (18) it follows $g(u) \geq 0$ for $u \in X_2$ and $||u|| \leq \rho$, where ρ is sufficiently small.

Let $\xi(x) = \frac{1}{\sqrt{\pi\left(1+(m+2)^2\right)}} \cos(m+2) x \in X_2$. For $\bar{u} = \sum\limits_{k^2 \leq m^2} c_k e^{ikx} + \lambda \xi$, with $\lambda \geq 0$,we obtain

$$\begin{aligned} g(\bar{u}) &\leq \pi t^2 (m+1)^2 + \frac{B}{3} \int_0^{2\pi} |\bar{u}|^3 dx - \frac{c}{4} \int_0^{2\pi} \bar{u}^4 dx \\ &\leq \frac{(m+1)^2}{2} ||\bar{u}||_2^2 + \frac{B}{3} ||\bar{u}||_3^3 - \frac{c}{4} ||u||_4^4 \qquad (19) \\ &\leq \frac{(m+1)^2}{2} ||\bar{u}||_2^2 + q_1 \frac{B}{3} ||\bar{u}||_2^3 - q_2 \frac{c}{4} ||u||_2^4, \end{aligned}$$

with $q_1, q_2 \in \mathbf{R}^+$. The function $\varphi(t) = \frac{(m+1)^2}{2} t^2 + q_1 \frac{B}{3} t^3 - q_2 \frac{c}{4} t^4$ is bounded from above by a constant $\beta > 0$. By (19) it follows $g(\bar{u}) \leq \beta$.

The function $q(x, u) = c(x) u^3 - b(x) u^2$ under assumptions (1),(2) satisfies the Rabinowitz's condition $0 < \theta Q(x, u) \leq u q(x, u)$, if $|u| \geq R$,where $Q(x, u) = \int\limits_0^u q(x, t)\, dt$, $2 < \theta$ and R is sufficiently large. By a standard way it follows that there are constants C_1, C_2 such that $Q(x, u) \geq C_1 |u|^\theta - C_2$. The last allows to prove that the functioniol g satisfies (PS) condition. The assertion follows by Theorem 7 ■

Considered approach allows to extend the result to equations of the form

$$u'' + a(x) u + g(x, u) = 0,$$

under assumptions on $G(x, u) = \int\limits_0^u g(x, t)\, dt$:

1) $a(x), g(x,u)$ are measurable, 2π periodic in x functions,
2) $c|u|^4 - B|u|^3 \leq G(x,u) \leq C|u|^4 + B|u|^3$,
3) $m^2 < a \leq a(x) \leq A < (m+1)^2$,
4) $B^2 < 2c(a - m^2)$,
5) $0 < \theta G(x,u) \leq ug(x,u)$, if $|u| \geq R$, where $\theta > 2$.

Moreover the result can extend to equations of the form

$$u'' + a(x)u - b(x)u^2 + c(x)u^{2k+1} = 0.$$

In this case assumption (10) changes, with respect to B, into

$$B^{2k} \leq \frac{c(3k)^{2k}}{k+1}\left(\frac{a - m^2}{2k-1}\right)^{2k-1}. \tag{20}$$

We can prove also existence of periodic solutions with period $2n\pi$ for $n \geq n_0$ to the equation (II). Let $H := H^1(\mathbf{R})$ equipped with the norm

$$||u||^2 = \int_{-\infty}^{\infty} \left(u'^2(x) + u^2(x)\right) dx.$$

Suppose $L : H \to H$ the operator defined by

$$(Lu, v) = \int_{-\infty}^{\infty} (u'v' - a(x)uv)\, dx, \tag{21}$$

and $B^2 \leq 9c\lambda_0$, where λ_0 is a positive constant independent of n. We assume that

$$0 \notin \sigma(L).$$

Let $H_n = \{u \in H^1(-n\pi, n\pi) : u(-n\pi) = u(n\pi)\}$ with the norm $||u||_n^2 = ||u||_{n,2}^2 + ||u||_{n,2}^2$, where $||u||_{n,p}^p = \int_{I_n} |u|^p dx$, $p \geq 1$ is the usual L^p norm. Let $L_n : H_n \to H_n$ be the self- adjoint operator defined by

$$(L_n u, v) = \int_{I_n} (u'v' - a(x)uv)\, dx.$$

Critical points of the functional $g_n : H_n \to \mathbf{R}$ defined by

$$g_n(u) = \frac{1}{2}(L_n u, u) + \int_{I_n} \left(\frac{1}{3} b(x) u^3 - \frac{1}{4} c(x) u^4 \right) dx,$$

are weak $2n\pi$ periodic solutions of equation (II). To prove existence of critical points to g_n we use Theorem 7 and two technical lemmas appearing in the work of Arioli and Szulkin [2]. We prove

Theorem 9 *Assume $a(x)$, $b(x)$, $c(x)$ are measurable 2π periodic functions such that there are positive constants A, B, c and C such that*

$$0 \leq a(x) \leq A, \ |b(x)| \leq B, \ 0 < c \leq c(x) \leq C.$$

Consider the operator L, defined in (21) and suppose

$$0 \notin \sigma(L).$$

Then there exists $\lambda_0 > 0$ and $n_0 \in \mathbf{N}$ such that if

$$B^2 \leq 9\lambda_0 c \tag{22}$$

the equation (II) has a nontrivial $2n\pi$ periodic solution for $n \geq n_0$.

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