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The dual variational principle and equilibria for a beam resting on a discontinuous nonlinear elastic foundation

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1. Introduction

In this paper we study the existence of solutions of the nonlinear fourth-order equation

$$u^{(iv)}(x) + g(u(x)) = 0, \quad x \in (0,1), \tag{1}$$

under the asymmetric nonlinear boundary conditions

$$\begin{aligned} u''(0) &= -f(-u'(0)),\\ u'''(0) &= -h(u(0)), \end{aligned} \tag{2}$$

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$$u''(1)=0,$$
$$u'''(1)=0, \tag{3}$$

where g is a strictly monotonous function that may have some "one-sided" discontinuities and f and h exhibit some singularities.

To fix ideas, let $a\in\mathbf{R}$ and assume that $g:\mathbf{R}\to\mathbf{R}$ is strictly increasing and satisfies

(g_1) g is continuous in $\mathbf{R}\backslash\{a\}$, for some $a\in\mathbf{R}$,
(g_2) $g(a)=0=g(a^-):=\lim_{u\to a^-} g(u)<g(a^+):=\lim_{u\to a^+} g(u)$,

and also that, if $-\infty<a_0<0<b_0<+\infty$ and $-\infty<c_0<0<d_0<+\infty$, the functions f and h satisfy

(f) $f:(a_0,b_0)\to\mathbf{R}$ and $h:(c_0,d_0)\to\mathbf{R}$ are continuous, strictly increasing, $f(0)=h(0)=0$ and $\lim_{s\to a_0,b_0}|f(s)|=+\infty$ and $\lim_{s\to c_0,d_0}|h(s)|=+\infty$.

Problems of this kind appear in the classical bending theory of elastic beams. In fact, it concerns the behaviour of an elastic beam of length 1, when a force is exerted on it by a nonlinear elastic foundation given by the function $g(u)$ when indented by the displacement field u. The beam exhibits an asymmetric behaviour at the end points. In fact, at the end point $x=0$, it rests on elastic supports, namely a vertical spring, where the force $u'''(0)$ is a function of the displacement $u(0)$ through function h and a torsional spring whose constitutive law relates the bending moment $u''(0)$ to the rotation $u'(0)$ through function f. In the case where f and h are constants we have the classical linear spring-type supports. But, in the present case, this dependence is nonlinear and is characterized by the functions f and h which have a singular behaviour with respect to the displacement u at $x=0$. The condition $f(0)=h(0)=0$ means that the only situation where there is no elastic response on the supports is when the displacement is zero, i.e., $u=0$. At the other end point $x=1$, the beam is free (condition (3)). The conditions assumed on g state that the force exerted on the beam depends in a monotonous way on the displacement field u but may have an abrupt behaviour when it attains a certain value a.

From the analytical point of view, there are various approaches to the problems with discontinuous nonlinearities. One of them is the critical point theory for locally Lipschitz functionals (see [5, 7]). Another one is the dual variational method, applied by Ambrosetti and Badiale [3] to elliptic boundary value problems.

In this paper, we follow the idea of [3]. Problem (1)–(3) is solved by using Clarke's dual action principle. This method enables us to associate, in a sense to be precised later, solutions of Eqs. (1)–(3) to critical points of a functional that is well defined and is of class C^1, in spite of the singular behaviour of f and h and of the discontinuity of g.

Fourth-order O.D.E. have been considered by several authors and a large literature on the subject is available. We refer to, for example [1, 11, 13, 15] and their references. In those papers, discontinuities are not considered and the boundary conditions are always linear. In [8], Feireisl studies a linear fourth-order time periodic equation with nonlinear boundary conditions using a Rayleigh–Ritz approximation method to analyse a problem that concerns the slow oscillations of beams on elastic bearings. In [10], the authors use the dual action principal to study the existence of symmetric solutions of a fourth-order O.D.E. with symmetric nonlinear boundary conditions.

The paper is organized as follows. In Section 2, we consider the linear problem

$$\begin{aligned} &w^{iv} = v \in L^p(0,1),\\ &w''(0) = \alpha \in \mathbf{R}, \quad w''(1) = 0,\\ &w'''(0) = \beta \in \mathbf{R}, \quad w'''(1) = 0 \end{aligned} \tag{4}$$

and introduce the linear operator K, such that $w = Kv$ is the solution of (4) that vanishes at 0 and 1. In Section 3, we introduce a dual action functional and find the relation between its critical points and solutions of Eqs. (1)–(3). In Section 4, we apply the results of Section 3 and prove existence results for the problem (1)–(3) for different types of nonlinearities.

2. Preliminary results

Consider the Sobolev space $W^{4,p}(0,1)$, $p \geq 1$, with the usual norm, and let W^0 denote its subspace defined by

$$W^0 := \{w \in W^{4,p}(0,1)\colon\ w''(1) = 0,\ w'''(1) = 0\} \subset L^p(0,1).$$

Lemma 1. *Let $L : W^0 \to L^p(0,1) \times \mathbf{R} \times \mathbf{R}$ be the linear operator defined by*

$$L(w) = (w^{(iv)}, w''(0), w'''(0)).$$

Then

(i) *$Ker(L)$ is a two-dimensional space such that $Ker(L) = span\{1, x\}$,*

(ii) *$Im(L) = \{(v, \alpha, \beta) \in L^p(0,1) \times \mathbf{R} \times \mathbf{R}\colon\ \alpha = \int_0^1 xv(x)\,\mathrm{d}x,\ \beta = -\int_0^1 v(x)\,\mathrm{d}x\}$,*

(iii) *$Ker(L)^0 = \{v \in L^p(0,1)\colon (v,0,0) \in Im(L)\}$, where A^0 denotes the annihilator of the set A in the duality $\langle L^p, L^{p'} \rangle$.*

Proof. (i) It is easy to see that $v \in W^0$ satisfies $w^{(iv)} = 0$, $w''(0) = 0$ and $w'''(0) = 0$ if $w = ax + b$ for some $a, b \in \mathbf{R}$, which implies the result.

(ii) By definition, $(v, \alpha, \beta) \in Im(L)$ if there exists $w \in W^0$ such that

$$\begin{aligned} &w^{(iv)} = v,\\ &w''(0) = \alpha, \quad w''(1) = 0,\\ &w'''(0) = \beta, \quad w'''(1) = 0. \end{aligned}$$

So, integrating the equation, it follows that if $(v,\alpha,\beta)\in Im(L)$ then $\alpha=\int_0^1 xv(x)\,\mathrm{d}x$ and $\beta=-\int_0^1 v(x)\,\mathrm{d}x$. As for the converse statement, given $(v,\alpha,\beta)\in L^p(0,1)\times\mathbf{R}\times\mathbf{R}$ such that $\alpha=\int_0^1 xv(x)\,\mathrm{d}x$, $\beta=-\int_0^1 v(x)\,\mathrm{d}x$, it is easy to see that the function

$$w=\frac{1}{6}\int_0^x (x-t)^3 v(t)\,\mathrm{d}t+\alpha\frac{x^2}{2}+\beta\frac{x^3}{6}$$

satisfies the above problem, and so, $(v,\alpha,\beta)\in Im(L)$.

(iii) It follows easily from (i) and (ii). □

From Lemma 1, we have

Proposition 1. *Given $v\in L^p(0,1)$, α, $\beta\in\mathbf{R}$, consider the linear problem*

$$\begin{aligned} &w^{(iv)}=v,\\ &w''(0)=\alpha,\quad w''(1)=0,\\ &w'''(0)=\beta,\quad w'''(1)=0. \end{aligned}\tag{5}$$

Then

(i) *Problem* (5) *has a solution if*

$$\alpha=\int_0^1 xv(x)\,\mathrm{d}x,\qquad \beta=-\int_0^1 v(x)\,\mathrm{d}x.\tag{6}$$

(ii) *In the affirmative case, w is a solution of problem* (5) *if there are $a,b\in\mathbf{R}$ such that*

$$w(x)=w_0(x)+ax+b$$

where $w_0(x)=\frac{1}{6}\int_0^x (x-t)^3 v(t)\,\mathrm{d}t+(x^2/2)\int_0^1 tv(t)\,\mathrm{d}t-(x^3/6)\int_0^1 v(t)\,\mathrm{d}t$.

(iii) *If problem* (5) *is solvable and $w(0)$ and $w'(0)$ are prescribed, the solution is unique and satisfies*

$$w(x)=w_0(x)+w'(0)x+w(0).$$

Proof. Associate to problem (5) the linear operator $L:W^0\to L^p(0,1)$ defined before as

$$L(w)=(w^{(iv)},w''(0),w'''(0)).$$

By Lemma 1, $Ker(L)$ consists of the linear functions $ax+b$, with $a,b\in\mathbf{R}$. So, the results follow easily from standard computations, Lemma 1 and arguments contained in its proof. □

Define the linear operator $K:L^p(0,1)\to C[0,1]$

$$Kv=\frac{1}{6}\int_0^x (x-t)^3 v(t)\,\mathrm{d}t+\frac{x^2}{2}\int_0^1 tv(t)\,\mathrm{d}t-\frac{x^3}{6}\int_0^1 v(t)\,\mathrm{d}t,\tag{7}$$

which associates to every $v \in L^p(0,1)$ the unique solution of (5) that satisfies the additional conditions $w(0)=w'(0)=0$. Using the definition of K and standard arguments of functional analysis, we can easily derive the following result, as in [9].

Lemma 2. *Consider the operator K. Then*
(i) *There exists* $k>0$ *such that* $|Kv(x)| \le k\|v\|_p$, $\forall v \in L^p(0,1)$, $\forall x \in [0,1]$,
(ii) $0 \le \int_0^1 Kv.v\,\mathrm{d}x \le k\|v\|_p^2$,
(iii) *K is completely continuous.*

3. Variational formulation

Using the notation of condition (g_2) put

$$I_a := [0, g(a^+)]$$

and define the multi-valued function

$$\hat{g}(t) := g(t) \quad \text{if } t \neq a,$$

$$\hat{g}(t) := I_a \quad \text{if } t = a.$$

Considering the function $\hat{g}^*$ defined by

$$\hat{g}^*(s) := a \quad \text{if } s \in I_a,$$

$$\hat{g}^*(s) := t \quad \text{with } g(t) = s \text{ if } s \notin I_a$$

we can say that the multi-valued function $\hat{g}$ admits an inverse function, $\hat{g}^*$, in the following sense:

$$\hat{g}^*(s) = t \quad \text{iff } s \in \hat{g}(t).$$

It is clear that $\hat{g}^* \in C(\mathbf{R})$ is increasing and its primitive $G^*(v) = \int_0^v \hat{g}^*(t)\,\mathrm{d}t$ is a convex function.

Consider the primitives of the strictly increasing functions f and h, respectively,

$$F(t) = \int_0^t f(s)\,\mathrm{d}s, \qquad H(t) = \int_0^t h(s)\,\mathrm{d}s.$$

The functions F and H are strictly convex. Let F^* and H^* denote their respective Fenchel–Legendre transforms [12]. Then

$$\begin{aligned} F^*(s) &= st - F(t), \quad s = f(t), \\ H^*(s) &= st - H(t), \quad s = h(t) \end{aligned} \tag{8}$$

and $F^*(s)$ and $H^*(s)$ are convex functions. By condition (f), the functions f and h are invertible. Let f^* and h^* denote their respective inverse functions. It follows by Eq. (8), and since $f(0) = 0 = h(0)$, that

$$F^*(t) = \int_0^t f^*(s)\,\mathrm{d}s, \qquad H^*(t) = \int_0^t h^*(s)\,\mathrm{d}s.$$

Consider the linear functions $\alpha, \beta : L^p(0,1) \to \mathbf{R}$ that to each $v \in L^p(0,1)$ associate, respectively,

$$\alpha_v := \alpha(v) = \int_0^1 xv(x)\,dx, \qquad \beta_v := \beta(v) = -\int_0^1 v(x)\,dx$$

and let J^* be the functional defined in $L^p(0,1)$ as follows:

$$J^*(v) = \frac{1}{2}\int_0^1 Kv.v\,dx + \int_0^1 G^*(v)\,dx + F^*(\alpha_v) + H^*(\beta_v).$$

From the definition of K, G^*, F^*, H^* it follows that J^* is a C^1 functional, weakly lower semi-continuous and

$$J^{*\prime}(v)\phi = \int_0^1 Kv\phi\,dx + \int_0^1 \hat{g}^*(v)\phi\,dx + f^*(\alpha_v)\alpha_\phi + h^*(\beta_v)\beta_\phi$$

for all $\phi \in L^p(0,1)$. Then we can state the following result which relates the critical points of J^* and the solutions of problem (1)–(3). By a solution of (1)–(3), we mean a function $u \in W^{4,p}(0,1)$ that satisfies Eq. (1) a.e. in $(0,1)$ and the boundary conditions (2) and (3).

Theorem 1. *Let $v \in L^p(0,1)$ be a critical point of J^*. Then there is $l \in Ker(L)$ such that $u = l - Kv$ is a solution of problem* (1)–(3).

Proof. Let v be a critical point of J^*. Take $\phi \in Ker(L)^\perp$ arbitrarily. By Lemma 1, $\alpha_\phi = \beta_\phi = 0$, and, so,

$$J^{*\prime}(v)\phi = \int_0^1 (Kv + \hat{g}^*(v))\phi\,dx = 0,$$

which shows that $Kv + \hat{g}^*(v) \in Ker(L)$. Then, by Lemma 1, there is a linear function l such that

$$Kv(x) + \hat{g}^*(v(x)) = l(x).$$

Put

$$u(x) := l(x) - Kv(x) = \hat{g}^*(v(x)). \tag{9}$$

Then we have

$$v(x) \in \hat{g}(u(x)). \tag{10}$$

Let $\Omega_a = \{x \in [0,1]:\ u(x) = a\}$. If $x \in [0,1]\backslash\Omega_a$, then $u(x) \neq a$ and by (10) one has $\hat{g}(u(x)) = g(u(x))$. This implies

$$-u^{(iv)}(x) = (Kv)^{iv}(x) = v(x) = g(u(x)),$$

that is,

$$u^{(iv)}(x) + g(u(x)) = 0 \quad \text{if } x \in [0,1]\backslash\Omega_a.$$

Since $u \in W^{4,p}(0,1)$, by the one-dimensional version of a theorem of Stampacchia [14],

$$u^{(iv)}(x) = 0 \quad \text{a.e. in } \Omega_a \tag{11}$$

and according to the fact that $g(a) = 0$ it follows

$$u^{(iv)}(x) + g(u(x)) = 0 \quad \text{for a.e. } x \in \Omega_a.$$

Hence, u satisfies

$$u^{(iv)}(x) + g(u(x)) = 0 \quad \text{for a.e. } x \in [0,1].$$

Let us see that u also satisfies the boundary conditions (2) and (3). By definition of Kv, $(Kv)''(1) = (Kv)'''(0) = 0$, and so it is clear by (9) that u satisfies (3). As for (2), again by the definition of Kv, $Kv(0) = (Kv)'(0) = 0$, and it follows by (9) that

$$l(x) = u'(0)x + u(0).$$

Take, now, a test function $\phi \in L^p(0,1)$ such that

$$\alpha_\phi = \int_0^1 x\phi \, dx \neq 0, \qquad \beta_\phi = -\int_0^1 \phi \, dx = 0.$$

Then by (9)

$$0 = J^{*\prime}(v)\phi = \int_0^1 l(x)\phi(x)\, dx + f^*(\alpha_v)\alpha_\phi = (u'(0) + f^*(\alpha_v)) \int_0^1 x\phi(x)\, dx$$

which implies $f^*(\alpha_v) = -u'(0)$ and therefore $f(-u'(0)) = \alpha_v$. Since

$$u''(0) = (l - Kv)''(0) = -(Kv)''(0) = -\alpha_v,$$

it follows $u''(0) = -f(-u'(0))$. Finally, take a test function $\phi \in L^p(0,1)$ such that

$$\alpha_\phi = \int_0^1 x\phi \, dx = 0, \qquad \beta_\phi = -\int_0^1 \phi \, dx \neq 0.$$

By

$$0 = J^{*\prime}(v)\phi = \int_0^1 l(x)\phi(x)\, dx + h^*(\beta_v)\beta_\phi = (u(0) - h^*(\beta_v)) \int_0^1 \phi(x)\, dx,$$

it follows $h^*(\beta_v) = u(0)$ and therefore $\beta_v = h(u(0))$. On the other hand,

$$u'''(0) = (l - Kv)'''(0) = -(Kv)'''(0) = -\beta_v.$$

Then $u'''(0) = -h(u(0))$. Hence, u is a solution of (1)–(3). □

4. Existence results

Lemma 3. *Let $g(t)$ be a function verifying (g_1) and (g_2) and let $\hat{g}$ and $\hat{g}^*$ be the functions introduced in Section 2. If $G(t)=\int_0^t g(s)\,\mathrm{d}s$ is such that*

$$\frac{C_1}{p}|t|^p - D_1 \le G(t) \le \frac{C_2}{p}|t|^p + D_2,$$

with D_1, D_2 constants and $p>1$, then $G^*(t)=\int_0^t \hat{g}^*(s)\,\mathrm{d}s$ satisfies the estimates

$$C_2'|t|^{p'} - D_2 + G(a) \le G^*(t) \le C_1'|t|^{p'} + D_1 + G(a). \tag{12}$$

Proof. Let (α_n) be a sequence of positive numbers converging to 0 and such that, for $t\in[a, a+\alpha_n]$,

$$\frac{1}{\alpha_n} g(a+\alpha_n)(t-a) \le g(t).$$

Consider the sequence of increasing continuous functions $(g_n(t))$ defined as

$$g_n(t) = \begin{cases} g(t), & t\le a \text{ or } t\ge a+\alpha_n, \\ \frac{1}{\alpha_n} g(a+\alpha_n)(t-a), & a\le t\le a+\alpha_n \end{cases}$$

and for each $n\in\mathbf{N}$ consider the respective primitive $G_n(t)=\int_0^t g(s)\,\mathrm{d}s$ and the corresponding Legendre–Fenchel transform

$$G_n^*(s) = st - G_n(t), \qquad s = g_n(t).$$

We show first that the following assertions hold:
(A_1) $G_n(t)\ge G(t) - (\alpha_n/2)g(a+1)$, if $a\ge 0$,
(A_2) $G_n(t)\le G(t) + (\alpha_n/2)g(a+1)$, if $a<0$,
(A_3) $G_n^*(s)\to G^*(s) - G(a)$.
In fact, consider the following sets:

$$\Phi_n = \left\{ (x,y):\ \begin{array}{l} a\le x\le a+\alpha_n, \\ \frac{1}{\alpha_n} g(a+\alpha_n)(x-a)\le y\le g(x) \end{array} \right\},$$

$$\Delta_n = \left\{ (x,y):\ \begin{array}{l} a\le x\le a+\alpha_n, \\ \frac{1}{\alpha_n} g(a+\alpha_n)(x-a)\le y\le g(a+\alpha_n) \end{array} \right\}.$$

It is clear that $\Phi_n\subset\Delta_n$ and, if $S(\Phi)$ denotes the area of $\Phi\subset\mathbf{R}^2$, for n big enough

$$S(\Phi_n)\le S(\Delta_n) = \tfrac{1}{2} g(a+\alpha_n)\alpha_n \le \frac{\alpha_n}{2} g(a+1).$$

Then, if $a\ge 0$,

$$G_n(t) = \int_0^t g_n(s)\,\mathrm{d}s \ge \int_0^t g(s)\,\mathrm{d}s - S(\Phi_n) \ge G(t) - \frac{\alpha_n}{2} g(a+1),$$

and (A_1) holds. If $a<0$, we can argue in an analogous way and prove (A_2). As for (A_3), observe that

$$G_n^*(0) \to -G(a). \tag{13}$$

In fact, $G_n^*(0) = -G_n(a)$. If $a \geq 0$, then $G_n^*(0) = -G(a)$ since $G_n(a) = G(a)$. If $a<0$, as g_n is an increasing sequence such that $g_n(t) \to g(t)$ and $\int_a^0 g_n(t)\,\mathrm{d}s < \int_a^0 g(t)\,\mathrm{d}s$, by Beppo–Levi theorem,

$$G_n(a) = -\int_a^0 g_n(t)\,\mathrm{d}t \to -\int_a^0 g(t)\,\mathrm{d}t = G(a).$$

Therefore, (13) holds. Besides that, it is clear that g_n^* converges to $\hat{g}^*$ uniformly and so

$$\int_0^s g_n^*(\xi)\,\mathrm{d}\xi \to \int_0^s \hat{g}^*(\xi)\,\mathrm{d}\xi. \tag{14}$$

Hence, using the fact that

$$G_n^*(s) = \int_0^s g_n^*(\xi)\,\mathrm{d}\xi + G_n^*(0),$$

we conclude by (13) and (14) that (A_3) holds.

Then, if $a \geq 0$, (A_1) combined with the fact

$$G_n(t) \leq G(t) \leq \frac{C_2}{p}|t|^p + D_2$$

implies

$$\frac{C_1}{p}|t|^p - D_1 - \frac{\alpha_n}{2}g(a+1) \leq G_n(t) \leq \frac{C_2}{p}|t|^p + D_2. \tag{15}$$

In an analogous way, if $a<0$, (A_2) and the fact that

$$\frac{C_1}{p}|t|^p - D_1 \leq G(t) \leq G_n(t)$$

imply that

$$\frac{C_1}{p}|t|^p - D_1 \leq G_n(t) \leq \frac{C_2}{p}|t|^p + D_2 + \frac{\alpha_n}{2}g(a+1). \tag{16}$$

From (15) and (16) and known properties of Fenchel–Legendre transform we derive that

$$C_2'|t|^{p'} - D_2 \leq G_n^*(t) \leq C_1'|t|^{p'} + D_1 + \frac{\alpha_n}{2}g(a+1), \quad a \geq 0$$

or

$$C_2'|t|^{p'} - D_2 - \frac{\alpha_n}{2}g(a+1) \leq G_n^*(t) \leq C_1'|t|^{p'} + D_1, \quad a<0.$$

Then, using (A_3), by passing to the limit we obtain the estimate

$$C_2'|s|^{p'} - D_2 + G(a) \leq G^*(s) \leq C_1'|s|^{p'} + D_1 + G(a)$$

and the proof is completed. □

Theorem 2. *Let g, f and h satisfy* (g_1), (g_2) *and* (f), *respectively. If* $G(t) = \int_0^t g(s)\,\mathrm{d}s$ *is such that*

$$\frac{C_1}{p}|t|^p - D_1 \leq G(t) \leq \frac{C_2}{p}|t|^p + D_2, \quad p > 1,$$

then problem (1)–(3) *has a solution.*

Proof. The result follows by minimization of the functional J^* as in [9]. By Lemma 3,

$$C_2'|v|^{p'} - D_2 + G(a) \leq G^*(v) \leq C_1'|v|^{p'} + D_1 + G(a)$$

and, since $F^* \geq 0$, $H^* \geq 0$ and $\int_0^1 Kv.v\,\mathrm{d}x \geq 0$, we derive

$$J^*(v) \geq C_2'\|v\|_{p'}^{p'} - D_2 + G(a).$$

So J^* is coercive on $L^p(0,1)$.

Moreover, by the compactness of K, the convexity and continuity of G^* and by the continuity of F^* and H^*, it follows that J^* is weakly lower semicontinuous. So J^* has a critical point in $L^p(0,1)$, which minimizes J^*, and by Theorem 1 we obtain a solution of (1)–(3). □

Lemma 4. *Let g(t) be a function verifying* (g_1) *and* (g_2) *and* $p > 2$. *Let* $\hat{g}$, $\hat{g}^*$, $G(t) = \int_0^t g(s)\,\mathrm{d}s$ and $G^*(t) = \int_0^t g^*(s)\,\mathrm{d}s$ *be the functions introduced in Section* 2. *If*

$$pG(u) \leq g(u)u + C,$$

then

$$p'G^*(v) \geq g^*(v)v + G(a) + C',$$

where C, C' *are positive constants and* $p' = p/(p-1) < 2$.

Proof. Consider the sequence of increasing continuous functions $g_n(t)$ defined in the proof of Lemma 3, their respective primitives $G_n(t) = \int_0^t g(s)\,\mathrm{d}s$ and the assertions (A_1) and (A_2) proved there. Suppose that $a \geq 0$. If we put

$$\varepsilon(t) := g(t) - g_n(t),$$

it is clear that for n large enough

$$\varepsilon(t)t \le g(a+\alpha_n)\chi_{[a,a+\alpha_n]}(t)(a+\alpha_n) \le g(a+1)(a+1) := K,$$

where $\chi_{[a,a+\alpha_n]}$ is the characteristic function of the set $[a, a+\alpha_n]$. Then

$$pG_n(t) \le pG(t) \le g(t)t + C = g_n(t)\,t + \varepsilon(t)t + C \le g_n(t)\,t + K + C$$

and using Fenchel–Legendre transform properties, we have

$$p'G_n^*(v) \ge g_n^*(v)v + \frac{K+C}{p-1}, \tag{17}$$

with $p' = p/(p-1) < 2$.

Suppose now that $a<0$. Then, by (A_2),

$$pG_n(t) \le pG(t) + p\frac{\alpha_n}{2}g(a+1) \le g(t)\,t + C + p\frac{\alpha_n}{2}g(a+1).$$

Using again Fenchel–Legendre transform properties

$$p'G_n^*(v) \ge g^*(v)v + \frac{C}{p-1} + \frac{p}{p-1}\frac{\alpha_n}{2}g(a+1), \tag{18}$$

with $p' = p/(p-1) < 2$. Passing to the limit either in (17) or in (18), we easily obtain

$$p'G^*(v) \ge g^*(v)v + \frac{C}{p-1} + G(a), \quad p' < 2,$$

which completes the proof. □

We observe that if $a=0$ then $u=0$ is the unique trivial solution of problem (1)–(3). Our next result, whose assumptions imply that $a=0$, establishes the existence of a nontrivial solution of (1)–(3).

Theorem 3. *Let $g:\mathbf{R}\to\mathbf{R}$ be a decreasing function such that $\gamma := -g$ satisfies the conditions (g_1) and (g_2) (with g replaced by γ). Suppose that*

$$\frac{C_1}{p}|u|^p \le -G(u) \le \frac{C_2}{p}|u|^p, \tag{19}$$

$$pG(u) \ge g(u)u - C_3, \tag{20}$$

where C_1, C_2 and C_3 are positive constants and $p>2$. Assume that functions f and h satisfy condition (f) *and*

$$\lim_{s\to 0}\frac{s}{f(s)} = \lim_{s\to 0}\frac{s}{h(s)} = 0. \tag{21}$$

Then problem (1)–(3) *has a nontrivial solution.*

Proof. We use arguments similar to those of Theorem 11 [9]. Take $\gamma = -g$ and let $\hat{\gamma}$ and $\hat{\gamma}^*$ be the corresponding functions introduced in Section 2 and $\Gamma^*(v) = \int_0^v \hat{\gamma}^*(s)\,\mathrm{d}s$. Then Γ is strictly convex and, by (19) and Lemma 3,

$$C_2'|v|^{p'} + \Gamma(a) \le \Gamma^*(v) \le C_1'|v|^{p'} + \Gamma(a). \tag{22}$$

Consider the functional

$$J_1^*(v) = \frac{1}{2}\int_0^1 Kv.v\,\mathrm{d}x - \int_0^1 \Gamma^*(v)\,\mathrm{d}x + F^*(\alpha_v) + H^*(\beta_v).$$

We will show that $-J_1^*$ satisfies the conditions of mountain pass lemma.

It is clear that $J_1^*(0) = \Gamma^*(0) = \Gamma(a)$. By (21)

$$\lim_{s\to 0}\frac{f^*(s)}{s} = \lim_{s\to 0}\frac{h^*(s)}{s} = 0.$$

Then, given $\varepsilon > 0$ there exists $\delta > 0$ such that, if $|s| < \delta$,

$$|f^*(s)| < \varepsilon|s|, \qquad |h^*(s)| < \varepsilon|s|,$$

and therefore, if $|\alpha_v| < \delta$, $|\beta_v| < \delta$,

$$F^*(\alpha_v) + H^*(\beta_v) < \frac{\varepsilon}{2}(|\alpha_v|^2 + |\beta_v|^2) < \varepsilon\delta^2.$$

So, as $|\alpha_v| \le \|v\|_1$ and $|\beta_v| \le \|v\|_1$, if we take $\|v\|_1 < \delta$,

$$-J_1^*(v) \ge -k\|v\|_{p'}^2 + C_2'\|v\|_{p'}^{p'} + \Gamma(a) - \varepsilon\delta^2$$

and since $p' < 2$ we conclude that if τ is small enough there is $\rho > 0$ such that if $\|v\|_{p'} = \tau > 0$ then $-J_1^*(v) \ge \Gamma(a) + \rho$.

On the other hand, note that

$$(K1)(x) = \frac{1}{6}\int_0^x (x-t)^3\,\mathrm{d}x + \frac{x^2}{2}\int_0^1 t\,\mathrm{d}t - \frac{x^3}{6}\int_0^1 \mathrm{d}t = -\frac{x^4}{24} + \frac{x^2}{4} - \frac{x^3}{6}$$

and so

$$\int_0^1 (K1)(x)\,\mathrm{d}x = \frac{1}{30} > 0.$$

Then, since $F^* \ge 0$, $H^* \ge 0$, for $m \in \mathbf{R}^+$,

$$-J_1^*(m) \le -\frac{1}{2}m^2\int_0^1 (K1)(x)\,\mathrm{d}x + \int_0^1 \Gamma^*(m)\,\mathrm{d}x - F^*\left(\frac{m}{2}\right) - H^*(-m)$$

$$\le -\frac{m^2}{60} + C_1'|m|^{p'} + \Gamma(a)$$

and therefore $-J_1^* \to -\infty$ as $m \to \infty$.

Hence the geometrical conditions of mountain pass theorem are satisfied.

Now we show that the functional $-J_1^*$ satisfies $(\mathrm{PS})_c$ condition. Suppose that $u_n \in L^{p'}(0,1)$ is such that $-J_1^*(u_n) \to c$ and $(-J_1^*)'(u_n) \to 0$. Denote by the same symbol C several constants independent of n. Then

$$\begin{aligned} C + C\|u_n\|_{p'} &\geq -J_1^*(u_n) + \tfrac{1}{2}J_1^{*\prime}(u_n)u_n \\ &= \int_0^1 (\Gamma^*(u_n) - \tfrac{1}{2}\hat{\gamma}^*(u_n)u_n)\,\mathrm{d}x \\ &\quad - (F^*(\alpha_n) - f^*(\alpha_n)\alpha_n) - (H^*(\alpha_n) - h^*(\beta_n)\beta_n), \end{aligned}$$

where $\alpha_n := \alpha_{u_n}$ and $\beta_n := \beta_{u_n}$.

Since f^* and h^* are bounded and $|\alpha_n| \leq \|u_n\|_1$ and $|\beta_n| \leq \|u_n\|_1$, we have

$$|F^*(\alpha_n)| \leq C\|u_n\|_1, \qquad |H^*(\alpha_n)| \leq C\|u_n\|_1$$

and

$$|f^*(\alpha_n)\alpha_n| \leq C\|u_n\|_1 \quad \text{and} \quad |h^*(\beta_n)\beta_n| \leq C\|u_n\|_1.$$

These facts, together with (20) and Lemma 4 (with G replaced by Γ), imply that

$$\left(1 - \frac{p'}{2}\right)\int_0^1 \Gamma^*(u_n)\,\mathrm{d}x \leq C_a + C\|u_n\|_{p'},$$

where C_a depends on $\Gamma(a)$ but is independent of n, This inequality and (22) show that (u_n) is bounded, since $1 < p' < 2$. Therefore, there is $\bar{u} \in L^{p'}$ such that (for a subsequence, if necessary) $u_n \rightharpoonup \bar{u}$ weakly in $L^{p'}$ and, by Lemma 2,

$$Ku_n \to K\bar{u} \quad \text{in } C^0.$$

Also for subsequences $\alpha_n \to \alpha_{\bar{u}}$ and $\beta_n \to \beta_{\bar{u}}$.

If we prove that $-J_1^*(\bar{u}) = c$ and $(-J_1^*)'(\bar{u}) = 0$, $(\mathrm{PS})_c$ condition will follow. For that we use a standard argument that we include here briefly for completeness.

Observe first that as for every $\phi \in L^{p'}(0,1)$

$$\int_0^1 \hat{\gamma}^*(u_n)\phi\,\mathrm{d}x = -J_1^{*\prime}(u_n)\phi + \int_0^1 (Ku_n)\phi\,\mathrm{d}x + f^*(\alpha_n)\alpha_\phi + h^*(\beta_n)\beta_\phi,$$

then $\hat{\gamma}^*(u_n)$ is weakly convergent.

Moreover, as $\hat{\gamma}^*$ is monotonous we have

$$\begin{aligned} &-J_1^{*\prime}(u_n)(u_n - \phi) \\ &\quad = -\int_0^1 Ku_n(u_n - \phi)\,\mathrm{d}x + \int_0^1 \hat{\gamma}^*(u_n)(u_n - \phi)\,\mathrm{d}x - f^*(\alpha_n)(\alpha_n - \alpha_\phi) \\ &\qquad - h^*(\beta_n)(\beta_n - \beta_\phi) \\ &\quad = -\int_0^1 Ku_n(u_n - \phi)\,\mathrm{d}x + \int_0^1 [\hat{\gamma}^*(u_n) - \hat{\gamma}^*(\phi)](u_n - \phi)\,\mathrm{d}x \end{aligned}$$

$$+\int_0^1 \hat{\gamma}^*(\phi)(u_n-\phi)\,\mathrm{d}x - f^*(\alpha_n)(\alpha_n-\alpha_\phi) - h^*(\beta_n)(\beta_n-\beta_\phi)$$
$$\geq -\int_0^1 Ku_n(u_n-\phi)\,\mathrm{d}x + \int_0^1 \hat{\gamma}^*(v)(u_n-\phi)\,\mathrm{d}x - f^*(\alpha_n)(\alpha_n-\alpha_\phi)$$
$$- h^*(\beta_n)(\beta_n-\beta_\phi).$$

As u_n is bounded, $J_1^{*\prime}(u_n)(u_n-\phi)\to 0$. So, passing to the limit,

$$0 \geq -\int_0^1 [K\bar{u}(\bar{u}-\phi) - \hat{\gamma}^*(\phi)(\bar{u}-\phi)]\,\mathrm{d}x - f^*(\alpha_{\bar{u}})(\alpha_{\bar{u}}-\alpha_\phi) - h^*(\beta_{\bar{u}})(\beta_{\bar{u}}-\beta_\phi).$$

Taking arbitrarily $w\in L^{p'}(0,1)$ and $\lambda>0$ and making $\phi=\bar{u}+\lambda w$ in the above inequality, we obtain, after dividing by $-\lambda$,

$$0 \leq -\int_0^1 [(K\bar{u})w - \hat{\gamma}^*(\bar{u}+\lambda w)w]\,\mathrm{d}x - f^*(\alpha_{\bar{u}})\alpha_w - h^*(\beta_{\bar{u}})\beta_w.$$

Letting $\lambda\to 0$ we derive by Lebesgue's theorem

$$-\int_0^1 [(K\bar{u})w - \hat{\gamma}^*(\bar{u})w]\,\mathrm{d}x - f^*(\alpha_{\bar{u}})\alpha_w - h^*(\beta_{\bar{u}})\beta_w w \geq 0$$

for all $w\in L^{p'}(0,1)$. Since w was arbitrary it follows immediately that

$$-\int_0^1 [(K\bar{u})w - \hat{\gamma}^*(\bar{u})w]\,\mathrm{d}x - f^*(\alpha_{\bar{u}})\alpha_w - h^*(\beta_{\bar{u}})\beta_w w = 0,$$

that is,

$$J_1^{*\prime}(\bar{u}) = 0.$$

Observe also that by the convexity of Γ^* we have

$$\int_0^1 \hat{\gamma}^*(\bar{u})(\bar{u}-u_n)\,\mathrm{d}x \geq \int_0^1 [\Gamma^*(\bar{u})-\Gamma^*(u_n)]\,\mathrm{d}x \geq \int_0^1 \hat{\gamma}^*(u_n)(\bar{u}-u_n)\,\mathrm{d}x$$
$$= -J_1^{*\prime}(u_n)(\bar{u}-u_n) + \int_0^1 (Ku_n)(\bar{u}-u_n)\,\mathrm{d}x$$
$$+ f^*(\alpha_n)(\alpha_{\bar{u}}-\alpha_n) + h^*(\beta_n)(\beta_{\bar{u}}-\beta_n).$$

Since the sides of the above inequalities tend to zero we conclude that

$$\int_0^1 \Gamma^*(u_n)\,\mathrm{d}x \to \int_0^1 \Gamma^*(\bar{u})\,\mathrm{d}x$$

and then it follows that $J_1^*(u_n)\to J_1^*(\bar{u})=c$, which completes the proof. □

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