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A note on moving average forecasts of long memory processes with an application to quality control

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Abstract

Standard quality control chart interpretation assumes that the observed data are uncorrelated. The presence of autocorrelation in process data has adverse effects on the performance of control charts. The objective of this paper is to assess the behavior of moving average forecast-based control charts on data having correlation that is persistent over very long time horizons, i.e., long-range dependent. We show that charts based on exponentially weighted moving average (EWMA) prediction do not perform well at detecting process shifts in long-range dependent data. We then introduce a new type of control chart, the hyperbolically weighted moving average (HWMA) chart, designed specifically for long-range dependent data. The HWMA charts perform better than the EWMA charts at detecting changes in the level of a long-memory process and also provide competitive performance for process data having only short-range dependence.
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1. Introduction

Statistical quality control has been an active field of research since the beginning of the 20th century. Standard statistical process control (SPC) methods assume that the observed data represent independent realizations of the observed process. The presence of autocorrelation has an adverse effect on control charts developed under the assumption of independence. Much research has been conducted concerning the performance of control charts applied to correlated data following stationary autoregressive moving average (ARMA) or nonstationary autoregressive integrated moving average (ARIMA) models. Modified versions of Shewhart, EWMA and CUSUM charts have been found to be very effective for monitoring stationary processes in which the correlation between observations decays to zero in a suitably fast manner. See, for example, Montgomery and Mastrangelo (1991) and Wardell, Moskowitz, and Plante (1992).

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However, many empirically observed time series have been found to have correlation between distant observations that decays to zero very slowly. Such processes are termed *long-range dependent* or *persistent*. These processes may be stationary even while exhibiting apparent nonstationary behavior in short periods. Thus they provide a bridge between ARMA and ARIMA processes. The class of autoregressive fractionally integrated moving average (ARFIMA) processes having fractional differencing parameter d is commonly used to characterize long-range dependent behavior. An ARFIMA process is stationary and invertible for values of d between -0.5 and 0.5. See Beran (1994) for additional details concerning ARFIMA processes.

In the statistical quality control literature, procedures for dealing with stationary correlated processes with short-memory of the ARMA type have been extensively studied (see, e.g., Montgomery & Mastrangelo, 1991; Wardell et al., 1992, 1994). More recently, Vander Wiel (1996) has studied control charts for nonstationary ARIMA processes. He has argued persuasively for the existence of these types of processes in practice. Since ARFIMA processes bridge the gap between ARMA and ARIMA types, they are of natural interest in quality control research.

In this paper, we investigate the performance of control charts based on exponentially weighted moving average (EWMA) forecasts for detecting level shifts in long-range dependent processes. The theoretically optimal smoothing constant, λ_o, for exponentially weighted forecasts of long-range dependent data is derived. The performance of this smoothing constant relative to other values of λ for EWMA-based control charts is studied via simulation. The relative effectiveness of the charts is measured using the average run length (ARL), i.e. the average number of observations required before an out-of-control signal occurs, and the probability run length (PRL), the probability that an out-of-control signal occurs within k time periods. Small ARL values and high PRL values for small k are desirable when a mean shift exists, indicating that the control chart method detects the shift quickly. We find that charts based on errors from EWMA forecasts do not perform well at detecting shifts in the level of a long-range dependent process, even when the theoretically optimal smoothing constant is used.

Given the poor performance of the EWMA error charts, we propose a new type of chart, the hyperbolically weighted moving average (HWMA) chart, designed specifically for process data having long-range dependence. The performance of HWMA charts for detecting level shifts in long-range dependent data is analyzed by simulation. We find that the HWMA error charts perform better than EWMA error charts at detecting changes in the level of a long memory process and also provide competitive performance for process data having only short-range dependence, such as ARMA processes. Additionally, we investigate the impact of constructing HWMA charts based on an incorrect value of the long-range dependence parameter. This provides a more realistic assessment of how the HWMA chart can be expected to perform in practice.

2. Control charts based on EWMA forecasts and prediction errors

2.1. EWMA forecasting

Let $\{X_t\}$ denote a time series. The EWMA forecast of X_{t+1} based on data observed up to time t is given by

$$\hat{X}_{t+1|t} = (1-\lambda)X_t + \lambda\hat{X}_{t|t-1}, \quad 0 < \lambda < 1. \tag{1}$$

Thus $1-\lambda$ denotes the weight given to the most recently observed data point, with more distant observations receiving exponentially decreasing weight. The smaller the value of λ, the more weight is given to more recent observations, as might be desirable for a strongly positively correlated process. The *one-step-ahead prediction error* is given by

$$e_{t+1} = X_{t+1} - \hat{X}_{t+1|t}.$$

2.2. Construction of EWMA charts

To construct an EWMA chart, the original observations and the EWMA forecasts are charted on the same graph, along with appropriate control limits. Alternatively, the chart can be constructed using the EWMA prediction errors. Assuming that the one-step ahead prediction errors are normally distributed, the control chart limits on these errors are given by

$$\Pr[-z_{\alpha/2}\sigma_e \leq e_{t+1} \leq z_{\alpha/2}\sigma_e] = 1 - \alpha, \tag{2}$$

where $z_{\alpha/2}$ is the upper $1-\alpha/2$ quantile of the unit normal distribution and the estimator for σ_e is usually taken from $\hat{\sigma}_e^2 = (\Sigma_{t=1}^{n} e_t^2)/n$. We can rewrite (2) as

$$\Pr\left[-z_{\alpha/2}\sigma_e \leq X_{t+1} - \hat{X}_{t+1/t} \leq z_{\alpha/2}\sigma_e\right] = 1 - \alpha.$$

Then, if the EWMA is a suitable one-step-ahead predictor, the centerline on the control chart on predictions for period $t+1$ is $\hat{X}_{t+1|t}$ and the upper and lower control limits are

$$\begin{aligned} UCL_{t+1} &= \hat{X}_{t+1|t} + z_{\alpha/2}\sigma_e \\ LCL_{t+1} &= \hat{X}_{t+1|t} - z_{\alpha/2}\sigma_e. \end{aligned} \tag{3}$$

See Montgomery and Mastrangelo (1991) for additional discussion.

2.3. Simulation study

We conducted a simulation study to assess the behavior of EWMA charts for a long-range dependent process in the presence of mean shifts. Two thousand data points were generated from an ARFIMA (0, d, 0) process with mean zero and standard deviation one using the method of Davies and Hart (1987). Mean shifts were introduced at time period $t = 501$. The mean shift values were set at 0.5, 1.0, 1.5, 2.0, 2.5 and 3.0. Shewhart methods were applied to the EWMA forecasts and to the prediction errors from the EWMA forecasts, as discussed above.

The EWMA forecasts were first constructed using $\lambda = 0.9$. This value of λ is commonly used for EWMA charts applied to short-range dependent processes. See, for instance, Crowder (1989) and Lucas and Saccucci (1990). The ARL and PRL values were recorded for each repetition and the average of 100 repetitions was obtained. As an example, the ARL values when $d = 0.4$ are shown in columns 2 and 4 of Table 1. From the table, we observe that the EWMA chart on the forecasts performs much better than the EWMA chart on the errors, i.e. the ARL values are smaller.

Wardell et al. (1992) show that for an AR(1) process, the best conditions for quick detection of shifts in the mean using the EWMA chart are when the process is highly negatively autocorrelated. Columns 6 and 8 of Table 1 give the ARL and PRL values for an ARFIMA process having $d = -0.3$, which has negative correlations. Again, the EWMA chart on the forecasts performs better than the chart based on EWMA errors for all size mean shifts and detects a shift more quickly than in the positively correlated $d = 0.4$ case.

Given that $\lambda = 0.9$ is not a good choice for all types of correlation, we investigate an alternative method for selecting an optimal value of λ in the next section.

3. Optimal value of λ for a long memory process

It is well known that the EWMA is a minimum mean squared error predictor when

Table 1
ARL values for EWMA and HWMA charts applied to an ARFIMA(0, d, 0) process[a]

Mean shift	$d = 0.4$				$d = -0.3$			
	EWMA forecasts	HWMA forecasts	EWMA errors	HWMA errors	EWMA forecasts	HWMA forecasts	EWMA errors	HWMA errors
0.0	369.96	368.37	369.57	380.02	369.73	372.19	377.42	363.21
0.5	56.41	69.97	363.59	194.93	9.19	13.69	363.43	28.07
1.0	15.03	20.41	335.92	107.85	3.99	6.79	350.75	8.36
1.5	6.90	8.16	318.00	53.20	2.26	4.20	320.84	4.18
2.0	4.41	5.03	282.62	22.68	1.57	3.42	298.36	2.33
2.5	3.12	3.28	188.02	5.08	1.01	2.91	217.89	1.42
3.0	2.39	2.50	112.03	0.76	0.85	2.66	86.27	0.67

[a] EWMA charts constructed using $\lambda = 0.9$. HWMA charts constructed using true value of d.

the true data generating process is an ARIMA(0, 1, 1). However, the EWMA can also provide simple, yet useful, forecasts for processes of other types. Cox (1961) derived the optimal value of λ, i.e., that value which minimizes the one-step ahead mean squared prediction errors (MSPE) from EWMA forecasts, for series following an AR(1) model. By means of simulation, Montgomery and Mastrangelo (1991) found the optimal value of λ for general correlated data. Following Cox's method, we derive the optimal value of λ for an ARFIMA process.

Let $\{X(t)\}$ be a stationary process with

$$\mathrm{E}\{X(t)\} = \mu,$$

$$\mathrm{Var}\{X(t)\} = \sigma^2, \qquad \mathrm{Cov}\{X(t), X(t+k)\} = \rho_k \sigma^2.$$

For a long-range dependent process, which has an AR(∞) representation, the h-step ahead MSPE is given by

$$\Delta_e(h; \lambda) = \sigma^2 + (1-\lambda)^2 \mathrm{E}\left(\sum_{r=0}^{\infty} \lambda^r X_{t-r}\right)^2 - 2(1-\lambda)\sigma^2 \sum_{r=0}^{\infty} \lambda^r \rho_{r+h}.$$

Simplifying further gives

$$\Delta_e(h; \lambda) = \frac{2\sigma^2}{1+\lambda} + \frac{2\sigma^2(1-\lambda)}{(1+\lambda)} \sum_{r=0}^{\infty} \lambda^{r+1} \rho_{r+1} - 2\sigma^2(1-\lambda) \sum_{r=0}^{\infty} \lambda^r \rho_{r+h}. \quad (4)$$

For predicting one-step ahead, substituting $h = 1$ in (4) gives

$$\Delta_e(1; \lambda) = \frac{2\sigma^2}{1+\lambda} + \frac{2\sigma^2(1-\lambda)}{(1+\lambda)} \sum_{r=0}^{\infty} \lambda^{r+1} \rho_{r+1} - 2\sigma^2(1-\lambda) \sum_{r=0}^{\infty} \lambda^r \rho_{r+1}.$$

For an ARFIMA(0, d, 0) process,

$$\rho_{k+1} = \left(\frac{k+d}{k+1-d}\right)\rho_k.$$

Simplifying, we have

$$\Delta_e(1; \lambda) = \frac{2\sigma^2}{1+\lambda} - \frac{2\sigma^2(1-\lambda)}{(1+\lambda)} \sum_{r=0}^{\infty} \lambda^r \rho_r \left(\frac{r+d}{r+1-d}\right).$$

Using the hypergeometric series,

$$F(a, b; c; z) = F(b, a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

we obtain the one-step ahead prediction error as

$$\Delta_e(1;\lambda)=\frac{2\sigma^2}{1+\lambda}-\frac{2\sigma^2(1-\lambda)}{(1+\lambda)}\frac{d}{(1-d)}F(d+1,1;2-d;\lambda),\quad d>0. \tag{5}$$

We differentiate (5) with respect to λ and equate the resulting value to zero to find the optimal value of λ, obtaining

$$(1-\lambda^2)\frac{(d+1)}{(2-d)}F(d+2,2;3-d;\lambda)-2F(d+1,1;2-d;\lambda)+\frac{(1-d)}{d}=0. \tag{6}$$

The optimum value of λ is the root of (6). For example, the optimal values of λ for $d=0.1$ and $d=0.4$ are 0.9871 and 0.6631 respectively. As d increases, the positive autocorrelation in the series increases. Therefore, in forecasting more weight is given to the most recent value, resulting in a smaller value of λ.

To investigate whether the performance of EWMA charts can be improved by forecasting with the optimal λ value, we conducted a second simulation study, analogous to the first. As the above analysis holds only for positive values of d, the ARL and PRL values for negative d were not calculated in the second study. The ARL and PRL values for the EWMA charts on the forecasts and errors were computed just as before. There was no significant improvement in the performance of the EWMA charts using the optimal value of λ. See Ramjee (2000) for more detailed results.

In summary, the EWMA forecast charts are capable of determining out-of-control values when there are moderately large shifts in the mean of a long-range dependent process, but the charts on the EWMA forecast *errors* do not perform well at detecting shifts of any size. These results are analogous to those for processes following ARMA models. In the next section, we investigate a new type of chart designed specifically for processes having long-range dependence.

4. Analysis of long-range dependent data using hyperbolically weighted moving average charts

Given the poor performance of the EWMA error charts for detecting level changes for long-range dependent time series, we propose a new type of chart, the hyperbolic weighted moving average (HWMA) chart. The autocorrelation function of a long memory process follows a hyperbolic decay as opposed to an exponential decay. Thus we propose to forecast future values by applying hyperbolically decaying weights to the process data as opposed to exponentially decaying weights. The hyperbolic weights are a function of the parameter d, which characterizes the rate of decay in the correlation of the process.

4.1. Constructing HWMA charts

Like the EWMA, the HWMA is a moving average of past data where each data point is assigned a weight. Unlike the EWMA, however, where the weights decrease exponentially, the HWMA weights decrease hyperbolically.

Let $\{X_t\}$ denote an ARFIMA(0, d, 0) process with mean zero. The minimum mean squared error forecast for X_t is:

$$\begin{aligned}\hat{X}_t&=\frac{d}{1!}X_{t-1}+\frac{d(1-d)}{2!}X_{t-2}+\frac{d(1-d)(2-d)}{3!}X_{t-3}+\cdots\\&=-\frac{\Gamma(1-d)}{\Gamma(2)\Gamma(-d)}X_{t-1}-\frac{\Gamma(2-d)}{\Gamma(3)\Gamma(-d)}X_{t-2}-\frac{\Gamma(3-d)}{\Gamma(4)\Gamma(-d)}X_{t-3}-\cdots,\end{aligned}$$

where $\Gamma(\cdot)$ denotes the gamma function.

By Shepard's formula, which states that

$$\frac{\Gamma(j-d)}{\Gamma(j+1)} \sim j^{-d-1} \text{ as } j \to \infty,$$

we have the following approximation:

$$\hat{X}_t \simeq -\frac{1}{1^{1+d}\Gamma(-d)} X_{t-1} - \frac{1}{2^{1+d}\Gamma(-d)} X_{t-2} - \frac{1}{3^{1+d}\Gamma(-d)} X_{t-3} - \cdots.$$

Thus

$$\hat{X}_t = -\sum_{i=1}^{t-1} \frac{1}{i^{1+d}\Gamma(-d)} X_{t-i}. \tag{7}$$

For computation of the weights, it is useful to use the identity $d\Gamma(-d) = -\Gamma(1-d)$, giving a positive argument for the gamma function. Thus, we propose the HWMA forecast of X_{t+1} given information up to time t:

$$\hat{X}_{t+1|t} = \frac{d}{\Gamma(1-d)} \sum_{i=1}^{t} \frac{1}{i^{1+d}} X_{t+1-i}. \tag{8}$$

The one-step-ahead prediction errors are given by

$$e_{t+1} = X_{t+1} - \hat{X}_{t+1|t}. \tag{9}$$

Control charts based on HWMA forecasts are constructed analogously to those based on EWMA forecasts, but using (8) and (9) with specified d.

4.2. Simulation results for HWMA-forecast and HWMA-error charts

The performance of the HWMA charts for detecting mean shifts is analyzed by calculating the ARL and PRL values. The ARL values for $d = 0.4$ for the HWMA forecast and prediction error charts are given in columns 3, 5, 7, and 9 of Table 1. Comparing the results for the HWMA forecast-based charts with those based on EWMA forecasts, we see that for smaller mean shifts, the two types of charts perform similarly. The charts based on prediction errors from HWMA forecasts perform much better than those based on EWMA prediction errors, however.

4.3. Impact of unknown d

For industrial practice, the true value of d for constructing a HWMA-based chart will not be known. We thus investigated by simulation the impact of using a value d^* different from the true d. The values of d for the data generating process were fixed at $d = 0.1$, 0.3, 0.4 and 0.475. The values of d^* used to construct the hyperbolic weights were set at 0.1, 0.2, 0.3 and 0.4. The ARL values for the HWMA forecast-based charts for all of the above combinations were then computed. When $d^* = 0.3$, the HWMA charts were found to perform well for each of the true d values investigated. Thus, in practice, given a series with unknown d, a robust method for constructing HWMA forecasts is to use $d = 0.3$. Complete simulation results for this study can be found in Ramjee (2000).

4.4. Impact of using HWMA charts on short-range dependent processes

Given that the HWMA-based charts are to be preferred over the EWMA charts for detecting changes in the mean of a long-memory process, it is reasonable to investigate how the HWMA charts perform when applied to short-memory processes. To see whether the HWMA charts provide competitive performance for process data having standard short-range dependence, AR(1) processes with $\phi = 0.9$ and MA(1) processes with $\theta = 0.9$ were generated. Simulation results indicated no significant differences in ARL and PRL values for charts based on EWMA and HWMA forecasts. Hence the HWMA charts provide competitive performance for process data having standard short-range dependence correlation structure. Again, detailed simulation results can be found in Ramjee (2000).

5. Summary

Comparing the performance of control charts based on EWMA and HWMA forecasts and forecast errors, we find that the EWMA-based charts tend to detect larger shifts in the mean slightly faster than the HWMA charts. However, charts based on EWMA forecast errors do not perform well for mean shifts of any size in positively correlated processes. The HWMA-based charts perform fairly well on both the forecasts and errors, although they work best for negatively correlated data. Thus we recommend using HWMA-based charts over EWMA-based charts for long-range dependent data.

Future research will focus on the performance of EWMA and HWMA forecast-based control charts for nonstationary ARFIMA processes.

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