REM WORKING PAPER SERIES

Accumulation of capital, production functions and models of economic growth

João Ferreira do Amaral

REM Working Paper 0270-2023

April 2023

REM – Research in Economics and Mathematics
Rua Miguel Lúpi 20,
1249-078 Lisboa,
Portugal

ISSN 2184-108X

REM – Research in Economics and Mathematics

Rua Miguel Lupi, 20
1249-078 LISBOA
Portugal

Telephone: +351 - 213 925 912
E-mail: rem@iseg.ulisboa.pt

https://rem.rc.iseg.ulisboa.pt/

REM
RESEARCH IN
ECONOMICS AND
MATHEMATICS

https://twitter.com/ResearchRem

https://www.linkedin.com/company/researchrem/

https://www.facebook.com/researchrem/

Accumulation of capital, production functions and models of economic growth

João Ferreira do Amaral[1]

April 2023

Abstract

This paper is about the possible interdependence of the assumptions related to the properties of the process of accumulation of capital and the properties of the production function as they are used in models of economic growth. The case of a semi-bounded substitution production function exemplifies this kind of interdependence and the consequent restrictions that may condition economic growth.

Keywords: economic growth; technological change; innovation; production function

JEL codes : E10, E11,E22,N10,O30

Introduction

Theoretical models of economic growth include at least two kinds of mathematical relations: an equation of accumulation of physical capital and a production function that connects productive capacity to stocks of factors including capital.

It is not usually considered in these types of models the possibility of interference between the properties assumed for the production function and the properties assumed for the process of accumulation of capital.

In this paper we study an example of such interdependence.

The production function we use for that purpose is of a type that we studied in Amaral (1983). It is mainly characterized by the fact that it allows for the possibility of substitution of capital for labour but in a limited way. It is not just a matter of the value of the elasticity of substitution for different combinations of capital and labour but the circumstance that there

[1] REM – Research in Economics and Mathematics/ UECE – Research Unit on Complexity and Economics. e-mail: joaomfamaral@hotmail.com. *REM/UECE - ISEG, Universidade de Lisboa is financially supported by FCT (Fundação para a Ciência e a Tecnologia), Portugal. Grant number_UIDB/05069/2020*

are combinations of capital and labour that are not feasible either for technological reasons or for social motives.

Since as far as I am aware the production function we choose is rarely used if ever in growth theory we begin (section 1) by describing in static terms the production function and its main properties. The source for this section is Amaral (1983). As in Wibe (1981) we call this type of functions Bounded Substitution Functions (BSF, or rather in our case semi-bounded, SBSF as it is shown below).

In section 2 we identify the problem and the assumptions needed to analyse it.

In section 3 we determine the conditions that given a specific SBSF function and some additional assumptions imply the existence of interference between the process of accumulation of capital and the properties of the production function.

1. A SBSF function

The two basic assumptions that allow us to obtain the analytical expression of the function $F(K,L)$ where K is the capital stock and L labour are :

a) F is a homogeneous function of degree 1

b) The marginal productivity of labour is a function of the stock of capital per worker and this function is the simplest possible one: a relation of proportionality. That is:

1) $\partial F/\partial L = \vartheta(K/L)$ where $\vartheta > 0$.

Remark. Another simple formulation is the one that stipulates that the dependence of the marginal productivity of labour on capital per worker is well represented by a power function of K/L, that is $\partial F/\partial L = A(K/L)^{\alpha}$ with $0 < \alpha < 1$ which in the case of a homogeneous of degree 1 function F gives the Cobb-Douglas function $Y = [A/(1-\alpha)]\, K^{\alpha} L^{1-\alpha}$.

Before enunciating the properties of the SBSF function is interesting to compare the partial elasticity of output to labour obtained from 1) with the corresponding formula for the Cobb-Douglas function.

From 1) we get

$(\partial F/\partial L)/(Y/L) = \vartheta(K/Y)$

The corresponding equation for the f Cobb-Douglas function is

$(\partial F/\partial L)/(Y/L) = (1- \alpha)$

This means that differently from what happens with the SBSF function, the value of the partial elasticity of output to labour obtained with the Cobb-Douglas function does not depend on the value of the capital/output ratio. It seems more realistic to use the SBSF function because it means that we consider that, other things equal, a higher capital/output ratio is associated with a higher elasticity of output to labour, something that happens with SBSF but not with Cobb-Douglas.

It is now time to describe the properties of the chosen SBSF.

The integration of equation 1) for a homogeneous function of degree 1 gives

$Y = K [\vartheta \log(L/K) + \vartheta_1]$, with $\vartheta > 0$

Some of the properties of this function are (Amaral 1983, p. 126):

a) From all the quasi–homogeneous two-variable functions of degree N given (M,P) the only ones that verify the equation $\partial Y/\partial L = \vartheta(K/L)$ are those where $M=N$ (a two-variable function $f(x,y)$ is quasi-homogeneous of degree N given M and P if and only if for each number $\lambda > 0$ we have $f(\lambda^M x, \lambda^P y) = \lambda^N f(x,y)$. If $M = P$ the function is homogeneous of degree N/M)

b) (Corollary) The only homogeneous functions that verify the equation $\partial Y/\partial L = \vartheta(K/L)$ are those of degree $N = 1$

c) We have necessarily $\vartheta \log(L/K) + \vartheta_1 > 0$, that is $K/L < e^{\vartheta_1/\vartheta}$

d) More realistically we have $\partial Y/\partial K > 0$ that is

2) $K/L < e^{(\vartheta_1/\vartheta) -1}$

a harder constraint than the previous one and one that allows us to obtain the following property:

e) The elasticity of substitution is given by $\sigma = 1- \vartheta(K/Y)$ and as $Y/K > \vartheta > 0$ we have $0 < \sigma < 1$

Remarks

1. Conditions c) and d) restrict the substitution of capital for labour, but correspond to a situation where K/L is bounded from above only (excluding the trivial inequality $K/L > 0$) and that is the reason why we call this situation *semi-bounded* factor substitution.

2. Condition d) is not logically binding as condition c) is. However it is not easy to imagine a situation in real life where $\partial Y/\partial K < 0$ for certain values of L and K.

After this static approach it is time to identify our problem which is a problem of dynamic modelling.

2. The problem: additional assumptions

Frequently for a given economic growth model the assumptions that support respectively the equation of accumulation of capital and the properties of the production function are not mutual independent assumptions.

The importance of this possibility is high not only from a theoretical perspective but also from the point of view of econometric estimation of empirical models.

We studied this type of situation in 1977 (Amaral, 1977) for the case of an accumulation process such that the rate of growth of GDP is a weighted average of the rate of growth of population (assumed constant and equal to $m > 0$) and the rate of growth of net investment. That is

3) $Y'(t)/Y(t) = \sigma K''(t)/K'(t) + (1-\sigma)m$ with $0 < \sigma < 1$ and $Y'(0)/Y(0) > m$ [2]

which can be written as

4) $s(t) = s(0)[K'(t)/K'(0)]^{(1-\sigma)}e^{(\sigma-1)mt}$

where $s(t) \equiv K'(t)/Y(t)$ is the net investment ratio.

This kind of growth where, for a certain time period the rate of growth of net investment is higher than the rate of growth of GDP and this one higher than the rate of growth of the population was a frequent characteristic of growth in the fifties and sixties of the XX century. It was even a condition for the *take-off* according to Rostow (1999 p.39).

[2] For any function $x(t)$, $x'(t)$ represents the first derivative of $x(t)$.

In the 1977 paper we showed that if the capacity utilization rate doesn´t change the process 3) implies that the marginal capital/output ratio, that is $K'(t)/Y'(t)$ cannot be strictly decreasing and that it may be constant but not indefinitely so that is at the most till the moment $T`*$ such that

$$T^* = - \log \{1- m/[Y'(0)/Y(0)]\}/[m(1-\sigma)/\sigma].$$

Assuming a constant capacity utilization rate the properties of the marginal capital/output ratio are a consequence of the properties of the production function so that we have here an example of interdependence of the assumptions related to the accumulation process and those related to the production function.

As a side note we stress the fact that de evolution of the fifties and sixties was far from equilibrium because it could not be sustained for ever. This shows the inanity of the theory of economic growth when its focus is on the quest for equilibrium, that is the quest for something that never happens .More of this later.

The general problem we deal with in the present paper as in 1977 is the search for interdependence of the assumptions on the accumulation of capital and those on the production functions. However the precise situation is different from the 1977 case. In the present case the specific properties of the production function include a semi-bounded factor substitution.

For a better understanding of the problem consider the following simple equation of accumulation

$$K'(t) + \delta K(t) = sf(t)\, Y(t)$$

where $Y(t)= f(t)\, G(L(t), K(t))$, G is a production function homogeneous of degree 1 and $f(t)$ a function that represents technological progress.

Dividing by $K(t)$ we get

$$K'(t)/K(t) + \delta = sf(t)G^*(L(t)/K(t))$$

If there is perfect substitutability between the two factors, growth of GDP per capita can persist for ever even if the rate of growth of capital is constant and higher that rate of growth of the population provided that the technical change compensates the decline of the quotient $L(t)/K(t)$. The situation is different however when there are limitations to the substitution of capital for labour.

This is the specific problem that we are going to investigate. We use the SBFS production mentioned earlier and the following additional assumptions:

– *Firstly,* an assumption of economic policy: we assume that the authorities control directly or indirectly the ratio of net investment and for each moment t they fix a constant relation r of net investment to GDP so that

a) $K'(t) = rY(t)$

- *Secondly*, six assumptions on the behaviour of the variables that are not necessarily options of economic policy:

b) Gross savings in each moment is bounded according to the following inequality:

$K'(t) + \delta(t)K(t) \leq sY(t)$ s constant, $0 < s < 1$

Remark This economy is not necessarily a closed one. It is total savings (domestic plus foreign) that is bounded and this means that there are limitations of foreign credit as it is usually the case

c) There is technical progress represented by a technological wave (for this concept see Amaral 2022). Formally:

$Y(t) = f[a(t)]K(t)[\vartheta \log(L(t)/K(t)) + \vartheta_1]$

where f[.] is a monotone increasing function and a(t) represents the share of the total stock of capital that embodies the technical change of the wave . We assume that $a(t)$ is an increasing function during an initial interval, that is the time intervals that are considered are the intervals $0 \leq t \leq T^*$ where T^* is the moment where the value of $a(t)$ starts to decline.

Obviously, since $0 < a(t) <1$ $a(t)$ is bounded.

d) Function δ is a proportion of the stock of capital that gives us the value of wear and tear of the stock of capital plus – perhaps more important - the value of the stock of capital that is replaced in each moment because of obsolescence. We assume that obsolescence is predominant when technical change is rapid. The velocity of the spreading of the technical change may be measured by the derivative $a'(t)$ so that we assume $\delta[.]$ to be an increasing function of the values $a'(t)$, that is $\delta[a'(t)]$.

Remark. The role of obsolescence is often underestimated in growth models (but see, for example Salter, 1969, chapter IV for the relation between obsolescence and the delay in adopting new techniques). This is a mistake because when technical change accelerates the impact of obsolescence rises and there is a direct link with Schumpeter creative destruction and all its consequences (see Aghion *et al* 2021 chapter 11). The case of the growing obsolescence of transportation equipment that consumes fossil fuels or the obsolescence of equipment due to the digital revolution are just two examples of obsolescence that is characteristic of present societies. Circular economy may benefit from obsolescence of a given type of equipment but this only emphasizes the importance of the phenomenon.

e) The function $a(t)$ is a logistic function, that is for all the t such that $0 \le t \le T^*$, we have

$a(t) = M/(1+be^{-\lambda t})$ with $M = a(0)(1+b)$, $b > 1$, $a(0) > 0$ and $\lambda > 0$

Remark. Since we are interested only in the behaviour of $a(t)$ till the moment T^* the asymptotic behaviour of $a(t)$ is irrelevant.

f) Till the moment T^* there is only one technological wave in development

g) The capacity utilization rate is constant during the period $0 \le t \le T^*$

- *Thirdly,* a methodological hypothesis

h) Time is supposed to be continuous and this assumption is considered to be an adequate approximation to describe phenomena that are basically discrete

The objective of the analysis is to verify if there are further restrictions to growth aside from those that exist for a SBFS production function.

3 The problem: results

From a) and b) we get

$K(t) \le (s-r)Y(t)/ \delta[a'(t)]$

Using c) we obtain

$[\vartheta \log(L(t)/K(t)) + \vartheta_1] \ge \delta[a'(t)]/\{(s-r) f[a(t)]\}$

$K(t)/L(t) \le \exp\{(\vartheta_1/\vartheta) - \delta[a'(t)]/\{\vartheta(s-r) f[a(t)]\}\}$

As we have seen in the static analysis we have necessarily the inequality

5) $K(t)/L(t) \leq exp\{(\vartheta_1/\vartheta) - 1\}$

so that the dynamic induced by the accumulation equation introduces a second constrain, that is harder than 5) if

6) $\delta[a'(t)]/\{\vartheta(s-r)f[a(t)]\} > 1$

On the other hand if

6*) $\delta[a'(t)]/\{\vartheta(s-r) f[a(t)]\} \leq 1$

the new constraint is redundant.

Since $a'(t)$ has a maximum $a'(t^*)$ at $t^* = (\log b)/\lambda$, that is $a'(t^*) = M\lambda/4$ (see annex) and since

$f[.]$ is monotone increasing, the following condition 7) implies that the condition 6*) is verified so that there isn't a new binding constraint:

7) $\delta[M\lambda/4]/f[M/(1+b)] \leq \vartheta(s-r)$

Based on these calculations we obtain the following conclusions:

- The higher is λ, the higher is the possibility that inequality 7) is not verified. This doesn't mean that there is a new binding constraint but it doesn't exclude that existence either

- The lower is ϑ, (that is the lower is the marginal productivity of labour for a given ratio K/L) the higher is the possibility of having a second binding constraint. Note that the inequality

$K/L \leq e^{\vartheta 1/\vartheta -1}$ depends on the ratio ϑ_1/ϑ but in what concerns the possible second constraint is only the value of ϑ that matters.

- The smaller is the difference $(s-r)$ the greater is the possibility of existing a second binding constraint. This is a case of the situation that Salter (1969, p. 63) characterizes as "*An economy with a low rate of gross investment is restricted in the rate at which new techniques can be brought in use*." The Portuguese economic evolution since 2000 is an example of an economy where the rate of gross investment declined (from 28% of GDP in 2000 to 20% in 2022) and undoubtedly this has affected negatively technical change and growth.

Conclusion

Economic growth theory has wasted too much time studying an abstract situation that doesn't exist even as a feasible goal, that is dynamic equilibrium. There is no such thing because technical progress fortunately exists and induces permanent shocks on growth variables that make illusory the smoothness of dynamical equilibrium.

Therefore instead of inventing unrealistic assumptions in order to justify the existence of dynamic equilibrium it is more useful to study the restrictions to economic growth that really exist. The objective of this paper was to explore an example of such restrictions that result from the interdependence of the properties of the accumulation process and those of the production function. We obtained constraints that are partially a consequence of assuming a SBSF function. This shows that in our studies we shouldn't be too ready to assume the innocence of perfect substitutability of factors, an assumption that is far from being validated by economic history.

We also emphasize the importance of considering adequately the effects of obsolescence as it occurs in our societies where technical progress reigns.

ANNEX

Maximum of $a'(t)$

$a(t) = M/(1+be^{-\lambda t})$

The condition of maximum for $a'(t)$ is given by $a''(t) = 0$, since second order conditions are met.

After some calculation we obtain the maximising $t^* = (\log b)/\lambda$ which is positive since we assumed $b > 1$ and $\lambda > 0$ (assumption e)) and the maximum is

$a'(t^*) = M\lambda/4.$

REFERENCES

Aghion , Philippe; Antonin, Céline; Bunel, Simon (2021). *O poder da destruição criadora*. Temas e Debates- Círculo de Leitores.

Amaral , João Ferreira do (1977). Um modelo simples de crescimento económico *Economia,* vol. 1, nº 3.

Amaral, João Ferreira do (1983). *Um modelo input/ouput não- linear* [PhD thesis]. ISEG.

Amaral, João Ferreira do (2022). *Technological waves and economic growth: thoughts on the digital revolution*. REM Working Paper 0220-2022.

Rostow, W.W. (1999). *The stages of economic growth*. Cambridge University Press.

Salter,W.E.G. (1969). *Productivity and Technical Change*. Cambridge University Press.

Wibe, Sören (1981). *Some notes on bounded substitution*. Umeå Economic Studies No.83. University of Umeå