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# A new model for explaining long-range correlations in human time interval production

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ABSTRACT

Time series displaying long-range correlations have been observed in numerous fields, such as biology, psychology, hydrology, and economics, among others. For rhythmic movements such as tapping tasks, the Wing–Kristofferson model offers a decomposition of the inter-response intervals based on a cognitive component and on a motor component. It has been suggested that the cognitive component should be modeled as a long-memory process and the motor component should be treated as a white noise process. Some probabilistic explanations for long-range dependences have been proposed, such as the aggregation of short-memory processes, the renewal-reward processes, and the error-duration processes. A new approach to the Wing–Kristofferson model which provides insights into the origin of long memory based on regime-switching processes is proposed. Under some assumptions, the autocorrelation function and the spectral density function of the model are obtained. Furthermore, an estimator of the parameters based on the maximization of the frequency-domain representation of the likelihood function is proposed. A simulation study evaluating the sample properties of this estimator is performed. Finally, an experimental study involving tapping tasks with two target frequencies is presented.

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## 1. Introduction

Time series exhibiting long-range correlations have been observed in various fields, such as biology, psychology, hydrology, and economics, among others. Intuitively, this kind of pattern reflects the fact that the current value of the series depends on all the preceding values of the series. The presence of this particular type of structure in a wide range of phenomena seems to reveal its ubiquity. In biology, long-range dependences were found for instance in heartbeat series (Hausdorff and Peng, 1996; Peng et al., 1993) and in stride series (Hausdorff et al., 1996, 1999). In psychology, such fluctuations were observed in cognitive performances, including mental rotation or visual search (Gilden, 2001; Gilden et al., 1995). Other studies reported long-term memory in simple reaction times (Van Orden et al., 2003; Wagenmakers et al., 2004), tapping tasks (Chen et al., 1997; Ding et al., 2002), forearm oscillations (Delignières et al., 2004, 2008), or force production (Pressing, 1999; Wing et al., 2004).

The wide occurrence of long-range correlations in a number of biological and non-biological systems poses some challenging questions, in particular the origin of this type of phenomenon. In the human movement research, some physical

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mechanisms have been proffered to explain these fluctuations, namely from the so-called nomothetic and mechanistic perspectives (Kello et al., 2007; Torre and Wagenmakers, 2009). In the theoretical statistical literature, the list of probabilistic explanations for long-memory processes includes essentially the aggregation of short-memory processes (Granger, 1980), the renewal-reward processes (Taqqu and Levy, 1986), the error-duration processes (Parke, 1999), and the regime-switching processes (Liu, 2000). Some of these statistical solutions have been used in recent papers (Beran et al., 2010; Lux and Morales-Arias, 2010).

Among all these lines of research, the production of rhythmic movements such as repetitive finger tapping tasks and the corresponding long-range correlations have been an issue of great interest. Tapping tasks are useful motor tasks to investigate the perception of time in humans. More precisely, these tasks open a window to the regulation processes of voluntary rhythmic behavior, with important implications for the understanding of rhythm problems, coordination disorders, musical performances, etc. Tapping tasks are very attractive because of the tasks' simplicity and the instrumental parsimony. In fact, these tasks can be performed by children, adults, or subjects with impairments, and they allow for obtaining precise measures with simple equipments. Tapping tasks lead to the production of serial inter-response intervals, which follow long-range dependence patterns. However, the existing studies on this theme are mainly of empirical nature and the corresponding methodologies still need major refinements.

This study is based on tapping tasks and set in a framework inspired by the Wing–Kristofferson model, which offers a decomposition of the inter-response intervals based on a cognitive component and on a motor component. We present a new model with a new long-memory cognitive component and a white noise motor component. The proposed cognitive component is a regime-switching process with added noise, which is capable of producing long memory and providing a biological interpretation of the phenomenon. The starting point for our model is the successive alternation of cognitive strategies to establish the link between the behavioral and the theoretical properties. We also propose an estimator of the parameters based on the maximization of the frequency-domain representation of the likelihood function. In the end, we illustrate the model with a set of experimental series.

The remainder of the paper is organized as follows. Section 2 describes the Wing–Kristofferson model for tapping tasks and introduces two approaches to this model. Section 3 discusses the origin of long memory in cognitive-motor systems. Section 4 proposes a new representation of the Wing–Kristofferson model, as well as an estimator of the parameters. Section 5 presents an experimental study with tapping tasks. Section 6 shows some comments on the results and some ideas for future research.

## 2. Tapping tasks and the Wing–Kristofferson model

In the human movement science, long-range correlations are typically observed in repeated performances of a given system, facing the same task in stable conditions for a prolonged period. An interesting problem within this general scenario is the realization of repetitive finger tapping tasks with a fixed target time interval. In fact, tapping tasks are widely used motor tasks as an experimental solution to understand the temporal structure of human behavior. This kind of task leads to serial time interval production, which exhibits inherent variability.

In his early experimental research on repetitive movements, Stevens (1886) developed a basic experimental design called synchronization–continuation in which the participant has to tap continuously at a given rate. In the first phase (synchronization), the subject has to synchronize his or her taps with a sequence of acoustic periodic signals; in the second phase (continuation), the subject has to try to continue to tap regularly at the same rate but without the information from the external pacer. He also proposed two factors to explain the inter-response intervals variability—long-term fluctuations as a possible consequence of cognitive processes and short-term fluctuations related to motor limitations. Later, the additive structure of two components for repetitive movements was formally described by Wing and Kristofferson (1973a,b), who offered a model repeatedly investigated from that time on. More precisely, the Wing–Kristofferson model is a stochastic hierarchical two-level model that explains the variability of inter-response intervals. This model is based on a cognitive component generating time intervals $C_t$ and on a motor component, responsible for the execution of the task at the end of $C_t$, providing delay intervals $M_t$. In terms of these components, the inter-response intervals $I_t$ are written as

$$I_t = C_t + M_t - M_{t-1}, \quad t \in N.$$

In this two-level formulation, the ratio of the motor standard deviation to the cognitive standard deviation is very important. From a theoretical point of view, this ratio represents a 'noise-to-signal' ratio and, from an empirical point of view, the ratio expresses the numeric relation between the variability of the two processes. Note that, in the study of stochastic signals with added noise, a key measure is the 'signal-to-noise' ratio, which equals the ratio of the signal variance (or standard deviation) to the noise variance (or standard deviation). In this work, the 'noise-to-signal' ratio (i.e., the inverse of that quantity) was used, as it has been proposed in other works (Crato and Ray, 2002).

In the original approach, based on experiments with short continuation phases, namely no more than 50 continuous taps, the cognitive and the motor components were regarded as independent white noise sources (e.g., Wing and Kristofferson, 1973a,b; Vorberg and Wing, 1996). More recently, experiments with long continuation phases, namely around 1000 continuous taps, revealed that fluctuations typical of long-memory processes may be embedded in tapping series (e.g., Madison, 2004; Lemoine et al., 2006). In fact, long series of inter-response intervals seem to display long non-periodic oscillations around a mean value reasonably close to the target interval. Diniz et al. (2010) suggested a theoretical and fully parametric approach, based on experiments with long continuation phases, in which the cognitive component is modeled as a long-memory process (namely a fractionally integrated noise) and the motor component is treated as a white noise process, mutually independent. They also provided the autocorrelation function and the spectral density function of the model, as well as an estimator of the parameters.

Formally, a stationary process is said to have long memory if its autocorrelation function $\rho(.)$ satisfies the power law

$$\rho(k) \sim c\, k^{-(1-2d)}, \quad k \to \infty,$$

where $c$ and $d$ are two constants such that $c \neq 0, d \neq 0$, and $d < 0.5$, and $k$ is the lag. This means that the function $\rho(.)$ decays to zero very slowly with a hyperbolic decay. Moreover, the process is said to have persistent long memory if $0 < d < 0.5$, so that $\sum_k \rho(k) = \infty$, reflecting the fact that the remote past has a strong influence into the present. In the frequency domain, a long-memory process can be defined as a process whose spectral density function $f(.)$ satisfies the power law

$$f(\lambda) \sim c\lambda^{-2d}, \quad \lambda \to 0,$$

where $c$ and $d$ are two constants such that $c \neq 0, d \neq 0$, and $d < 0.5$, and $\lambda$ is the frequency. This means that the function $f(.)$ has a pole at zero if $0 < d < 0.5$, that is $f(0) = \infty$, signifying that low frequencies predominate and long-term oscillations are expected.

A basic long-memory process $\{X_t\}$ is the ARFIMA$(p, d, q)$ process with $d \in (-0.5, 0.5)$, which is defined as the unique stationary solution of the difference equations

$$\phi(B)(1-B)^d(X_t - \mu) = \theta(B)Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma_Z^2),$$

where $B$ is the backshift operator given by $B^jX_t = X_{t-j}$, $j = 0, 1, \ldots$, $(1-B)^d$ is a differencing operator defined through the gamma function $\Gamma(.)$, $\phi(B)$ and $\theta(B)$ are two polynomials of degrees $p$ and $q$, respectively (e.g., Brockwell and Davis, 1991).

In the above mentioned study and with the particular time series recorded, an ARFIMA$(0, d, 0)$ process (fractionally integrated noise process) proved to be advisable and sufficient. So, if the inter-response intervals $\{I_t\}$ are corrected for the mean, the cognitive component $\{C_t\}$ is a fractionally integrated noise process, and the motor component $\{M_t\}$ is a white noise process, independent of each other, viz.

$$\{C_t : (1-B)^d\, C_t = Z_t\}, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma_C^2),$$
$$\text{and} \quad \{M_t\} \sim \mathrm{WN}(0, \sigma_M^2),$$

it can be proven that the process $\{I_t\}$ has an autocovariance function $\gamma_I(.)$ of the form

$$\gamma_I(k) = \begin{cases} \dfrac{\Gamma(1-2d)}{\Gamma^2(1-d)}\sigma_C^2 + 2\sigma_M^2, & k = 0, \\ \dfrac{(-1)\Gamma(1-2d)}{\Gamma(2-d)\Gamma(-d)}\sigma_C^2 - \sigma_M^2, & |k| = 1, \\ \dfrac{(-1)^{|k|}\Gamma(1-2d)}{\Gamma(1+|k|-d)\Gamma(1-|k|-d)}\sigma_C^2, & |k| \geq 2, \end{cases}$$

and a spectral density function $f_I(.)$ of the form

$$f_I(\lambda) = |1-e^{-i\lambda}|^{-2d}\frac{\sigma_C^2}{2\pi} + |1-e^{-i\lambda}|^2\frac{\sigma_M^2}{2\pi}, \quad |\lambda| \leq \pi.$$

Note that the process $\{I_t\}$ has long memory and the autocovariance function at lags zero and one depends on both the cognitive and the motor processes; at lag one the values can be negative or positive (Diniz et al., 2010).

## 3. Where does long memory come from?

The source of long memory in a variety of systems has been an issue of intense debate and interdisciplinary interest. In the human movement, long-memory fluctuations have been linked to some physical mechanisms, both from the nomothetic and the mechanistic perspectives. The nomothetic perspective focuses on the ubiquity of long-memory processes and searches for general principles that explain their presence. The mechanistic perspective looks at the singularity of each particular system and proposes a different explanation for each situation (Diniz et al., 2011). In the statistical literature, long-memory fluctuations have been understood to occur, for example, from the aggregation of short-memory processes, the renewal-reward processes, the error-duration processes, and the regime-switching processes. Each of these processes displays long memory as a theoretical result of specific properties of its structural components. Moreover, these processes can provide valuable help in the search for physical interpretations of the empirical fluctuations.

For rhythmic movements such as tapping tasks, Diniz et al. (2010) suggested a theoretical representation of the Wing–Kristofferson model, in which the cognitive component is a fractionally integrated noise process and the motor component is a white noise process. This parametric model exhibits long memory and seems to reproduce relatively well the observed fluctuations. However, this model does not provide an easy explanation for the phenomenon in biological terms. A parsimonious model that could be sustained by physical mechanisms that generate the empirical values would surely be preferable. Such a model would give more useful information to the applied scientists and would make it easier to analyze the observed values. Therefore, it is vital to find alternative representations of the proposed model that generate long memory in a way that may plausibly be part of a biological process. In the past decade, a few interesting solutions have been proposed in terms of physical underlying mechanisms.

Two of the most relevant explanations for long memory in cognitive systems are the multiscaled randomness and the regime switching (Wagenmakers et al., 2004; Delignières et al., 2008). The multiscaled-randomness solution is based on the idea that the sum of short-memory processes with different time scales can generate long memory. For example, consider a behavioral time series $Y_t$ given by $Y_t = S_{1t} + \cdots + S_{kt}$, where each series $S_{it}$ is a switching series with probability of switch $p_i = 1 - e^{-(1/\tau_i)}$. The switching series may correspond to fluctuations in attention, variations in motivation, etc. Performing simulation studies for $Y_t$ with sample size $n = 1024$, parcels number $k = 3$, and relaxation rates $\tau_1 = 1$, $\tau_2 = 10$, $\tau_3 = 100$, it can be seen that the simulated series, the autocorrelation functions, and the normalized periodograms seem to have long-memory patterns (Wagenmakers et al., 2004). The regime-switching solution is based on the idea that regime-switching processes with specific properties can produce long memory. For instance, consider a time series $Y_t$ related to a time estimation task and assume that different cognitive strategies are used throughout the task. Suppose further that each strategy is used for a time period $P_t$ sampled from a uniform distribution on $S = \{a, \ldots, b\}$, each strategy has a temporal threshold $L_t$ sampled from a uniform distribution on $I = [c, d]$, and the process reaches the threshold with a speed $V_t = V_1 + \phi_1(V_{t-1} - V_1) + Z_t$, $\{Z_t\} \sim \mathrm{WN}(0, \sigma_Z^2)$. The time series $Y_t$ is then given by $Y_t = L_t/V_t$. Through simulation studies for $Y_t$ with sample size $n = 1024$, distribution supports $S = \{1, \ldots, 100\}$, $I = [250, 350]$, and parameter values $V_1 = 2$, $\phi_1 = 0.5$, $\sigma_Z = 0.1$, among others, it can be observed that the simulated series, the autocorrelation functions, and the normalized periodograms suggest the presence of long-memory processes (Wagenmakers et al., 2004; Delignières et al., 2008). The main disadvantage of the referred solutions is the large number of parameters involved, leading to difficult-to-handle models. Moreover, these solutions still lack a full treatment in mathematical terms and do not include an estimation procedure for observed series.

## 4. A new representation of the Wing–Kristofferson model

In tapping tasks, long series of inter-response intervals exhibit fluctuations typical of long memory, namely long non-periodic waves around a mean value reasonably close to the target interval. Intuitively, this implies that the subjects successively decrease and increase the rhythm of execution, keeping the inter-tap intervals between some stationary bounds. However, how to model and explain this intriguing phenomenon with a biological meaning? As previously referred, a few solutions have been suggested in cognitive systems. Nevertheless, these solutions still need major refinements. Next, a new representation of the Wing–Kristofferson model is presented, in which the cognitive component is a regime-switching process with added noise and the motor component is a white noise process. Under some assumptions, namely a heavy-tail stationary distribution for the regimes' durations, it is shown that this new model can generate long memory and may provide a biological interpretation of the phenomenon. Furthermore, an estimator of the parameters is proposed, based on the maximization of the frequency-domain representation of the likelihood function.

### *4.1. Equation of the model*

Any simple coordinated movement pattern requires the regulation of a large number of components (e.g., neural, muscular). In cognitive experiments, a possible source of stability patterns in observed responses is the successive adoption of cognitive strategies (such as counting numbers or pronouncing words). For tapping tasks, consider that such a solution is plausible and suppose that each cognitive strategy is employed in a time period $T_k$ (number of taps) and is associated to a time level $L_k$ (latent variable). Recall that, in the Wing–Kristofferson model, the inter-response intervals $I_t$ are written as

$$I_t = C_t + M_t - M_{t-1}, \quad t \in N. \tag{1}$$

Suppose that the time intervals $\{I_t\}$ are corrected for the mean, the cognitive component $\{C_t\}$ is a regime-switching process with added noise, and the motor component $\{M_t\}$ is a white noise process, independent of each other, viz.

$$\{C_t : C_t = W_t + Z_t\}, \quad \{W_t\} \sim \mathrm{RS}, \quad \{Z_t\} \sim \mathrm{WN}(0, 1), \tag{2}$$

$$\text{and} \quad \{M_t\} \sim \mathrm{WN}(0, \sigma_M^2). \tag{3}$$

The above process $\{W_t\}$ is a regime-switching process given by

$$W_t = L_k, \quad t \in (S_{k-1}, S_k] \quad \text{and} \quad T_k = S_k - S_{k-1}, \quad t, k \in N, \tag{4}$$

where the sequences $\{T_k\}$ and $\{L_k\}$ represent the regimes' durations and the corresponding levels, respectively. Suppose that each of these sequences is composed of independent and identically distributed random variables such that

(i) $P(T_k \in N) = 1,\ E(T_k) = \mu_T < \infty,\ k \in N$;
(ii) $P(L_k \in R) = 1,\ E(L_k) = \mu_L = 0,\ E(L_k^2) = \sigma_L^2 < \infty,\ k \in N$;
(iii) $\{T_k\}$ and $\{L_k\}$ are independent of each other.

In the following text, $T$ will denote a generic variable $T_k$. According to these conditions, it can be seen that the process $\{W_t\}$ is stationary with a mean value $\mu_W$ equal to 0 and an autocovariance function $\gamma_W(.)$ given by

$$\gamma_W(k) = \Sigma_{t=|k|}^{\infty} P(T > t)\, \mu_T^{-1}\sigma_L^2, \quad |k| = 0, 1, \ldots, \tag{5}$$

where the last expression comes from a result obtained by Liu (2000). Suppose further that the variables $T_k$ have a heavy-tail stationary distribution of the form

$$P(T > t) = P(T \geq t + 1) = (t + 1)^{-\alpha}, \quad t \in N,\ 1 < \alpha < 2. \tag{6}$$

Note that, according to this condition, the mean value $\mu_T$ exists and is finite, since

$$\mu_T = \Sigma_{t=1}^{\infty} P(T \geq t) = \Sigma_{t=1}^{\infty} t^{-\alpha} = \zeta(\alpha), \quad 1 < \alpha < 2, \tag{7}$$

where $\zeta(.)$ denotes the Riemann's zeta function. From Eq. (6), it follows that

$$P(T > t) \sim t^{-\alpha}, \quad t \to \infty,\ 1 < \alpha < 2.$$

By Eq. (5) and Karamata's Theorem, this implies that

$$\gamma_W(k) \sim \mu_T^{-1}\sigma_L^2(\alpha - 1)^{-1}k^{-(\alpha-1)}, \quad k \to \infty,\ 1 < \alpha < 2,$$

which means that the process $\{W_t\}$ has long memory with parameter

$$d = 1 - \alpha/2.$$

This relation implies that the memory parameter $(d)$ increases as the tail parameter $(\alpha)$ decreases.

The following proposition shows the analytical expressions of the autocovariance function $\gamma_W(.)$ and of the spectral density function $f_W(.)$.

**Proposition.** *Under the above hypothesis, the process $\{W_t\}$ given by Eq. (4) has an autocovariance function $\gamma_W(.)$ given by*

$$\gamma_W(k) = \frac{\zeta(\alpha) - \sum_{t=1}^{|k|} t^{-\alpha}}{\zeta(\alpha)}\sigma_L^2, \quad |k| = 0, 1, \ldots,$$

*where $\zeta(.)$ denotes the Riemann's zeta function defined by $\zeta(\alpha) = \Sigma_{t=1}^{\infty} 1/t^{\alpha}$ and a sum $\Sigma_{t=1}^{0} x_t$ is considered to be equal to 0; the spectral density function $f_W(.)$ is given by*

$$f_W(\lambda) = \frac{(1 - e^{-i\lambda})^{-1}}{\zeta(\alpha)}[e^{-i\lambda}Li_\alpha(e^{i\lambda}) - Li_\alpha(e^{-i\lambda})]\frac{\sigma_L^2}{2\pi}, \quad |\lambda| \leq \pi,$$

*where $Li_\alpha(.)$ denotes the Jonquière's function defined by $Li_\alpha(z) = \Sigma_{t=1}^{\infty} z^t/t^{\alpha}$.*

*The proof of this proposition is presented in Appendix after Section 6.*

*Now, from Eqs. (1)–(3), it can be proven that the process $\{I_t\}$ has an autocovariance function $\gamma_I(.)$ of the form*

$$\gamma_I(k) = \begin{cases} \sigma_L^2 + 1 + 2\sigma_M^2, & k = 0, \\ \dfrac{\zeta(\alpha) - 1}{\zeta(\alpha)}\sigma_L^2 - \sigma_M^2, & |k| = 1, \\ \dfrac{\zeta(\alpha) - \sum_{t=1}^{|k|} t^{-\alpha}}{\zeta(\alpha)}\sigma_L^2, & |k| \geq 2, \end{cases}$$

*and a spectral density function $f_I(.)$ of the form*

$$f_I(\lambda) = \frac{(1 - e^{-i\lambda})^{-1}}{\zeta(\alpha)}[e^{-i\lambda}Li_\alpha(e^{i\lambda}) - Li_\alpha(e^{-i\lambda})]\frac{\sigma_L^2}{2\pi} + \frac{1}{2\pi} + |1 - e^{-i\lambda}|^2\frac{\sigma_M^2}{2\pi}, \quad |\lambda| \leq \pi.$$

*Note that the process $\{I_t\}$ has long memory with parameter $d = 1 - \alpha/2$, since its autocovariance function $\gamma_I(.)$ satisfies*

$$\gamma_I(k) \sim \gamma_W(k), \quad k \to \infty \quad \text{and} \quad \gamma_W(k) \sim ck^{-(\alpha-1)},\ k \to \infty;$$

*therefore, its spectral density function $f_I(.)$, which is the Fourier transform of the autocovariance function, satisfies*

$$f_I(\lambda) \sim f_W(\lambda), \lambda \to 0 \quad \textit{and} \quad f_W(\lambda) \sim c\lambda^{-2(1-\alpha/2)}, \ \lambda \to 0.$$

*Finally, observe that the autocovariance function at lags zero and one depends on both the cognitive and the motor processes; at lag one the values can be negative or positive.*

### 4.2. Estimation of the model

The estimation of the parameters of the proposed model can be quite complex, since the model is defined as the sum of two processes (mutually independent). A well-known method for estimating the parameters of time series models is to maximize the likelihood or quasi-likelihood function of the parameter vector. In the context of long-memory processes, and especially for long time series, the exact evaluation of the likelihood function poses convergence problems. An efficient alternative is to maximize the frequency-domain representation of the likelihood function.

Assume that $\{Y_t\}$ is a Gaussian process with mean $\mu = 0$ and autocovariance function $\gamma(.)$. Suppose that $\boldsymbol{Y}_n = (Y_1, \ldots, Y_n)'$ is a realization of the process. Let $f(.; \boldsymbol{\beta})$ be the spectral density function of the process, where $\boldsymbol{\beta}$ is the parameter vector, and let $I_n(.)$ be the normalized periodogram of the realization, viz.

$$I_n(\lambda_j) = \frac{1}{2\pi n}\left|\sum_{t=1}^{n} Y_t\, e^{-it\lambda_j}\right|^2, \quad \lambda_j = \frac{2\pi j}{n},$$

where $j = 1 \ldots, [n/2]$ and $[.]$ represents the integer part. Using two approximations due to Whittle (1953) and some simple Riemann's sums, it follows that the negative of the log-likelihood function can be approximated by the function $\mathscr{L}_n(.)$ defined by

$$\mathscr{L}_n(\boldsymbol{\beta}) = \frac{1}{\pi}\left[\sum_{j=1}^{[n/2]} \log f(\lambda_j; \boldsymbol{\beta})\frac{2\pi}{n} + \sum_{j=1}^{[n/2]} \frac{I_n(\lambda_j)}{f(\lambda_j; \boldsymbol{\beta})}\frac{2\pi}{n}\right].$$

An estimator for $\boldsymbol{\beta}$, usually denoted by $\hat{\boldsymbol{\beta}}$, is obtained by minimizing $\mathscr{L}_n(.)$ with respect to $\boldsymbol{\beta}$ (e.g., Beran, 1994). This estimator is consistent and asymptotically normal with covariance matrix $4\pi n^{-1} W^{-1}(\boldsymbol{\beta})$, where $W$ is a matrix whose $(j, k)$ entry is given by

$$w_{jk}(\boldsymbol{\beta}) = \int_{-\pi}^{\pi} f(\lambda; \boldsymbol{\beta}) \frac{\partial^2}{\partial \boldsymbol{\beta}_j \partial \boldsymbol{\beta}_k} f^{-1}(\lambda; \boldsymbol{\beta})\, d\lambda.$$

The standard errors of the estimator are given by the square root of the variances on the main diagonal of the covariance matrix (e.g., Fox and Taqqu, 1986).

For the proposed model, the spectral density function $f(.; \boldsymbol{\beta})$ is established in the preceding section and the parameter vector $\boldsymbol{\beta}$ is given by $\boldsymbol{\beta} = (\alpha, \sigma_L, \sigma_M)$.

Next, a simulation study is presented to evaluate the sample properties of the spectral estimator in the proposed model. The sample size considered was $n = 1000$ because it resembles the maximum size of the experimental series. The parameter values selected were $\alpha = 1.2, 1.4, 1.6$ (which imply long memory with $d = 0.4, 0.3, 0.2$, respectively), $\sigma_L = 1.0, 2.0, 3.0$, and $\sigma_M = 1.0$ (which lead to ratios $\sigma_M/\sigma_L = 1.0, 0.5, 0.3$, respectively). Fig. 1 presents an example of a simulated process with $\alpha = 1.2$, $\sigma_L = 1.0$, and $\sigma_M = 1.0$. It also shows the sample and the theoretical autocorrelation functions, as well as the normalized periodogram and the spectral density function in log–log scale. It is clear that the time series has long non-periodic waves, as was postulated in the proposed theoretical model. The autocorrelation function and the periodogram exhibit behaviors similar to those of the proposed theoretical functions.

The estimation results for each model were obtained from 1000 replications. Table 1 provides simulation means and standard deviations for the estimates of the parameters. The overall performance of the spectral-likelihood estimator seems to be very good. Some general observations are (i) the tail parameter ($\alpha$), the scale parameters ($\sigma_L$ and $\sigma_M$), and the corresponding ratios ($\sigma_M/\sigma_L$) are estimated with high accuracy; (ii) the observed standard deviations of the estimator (within parentheses) are relatively small compared with the corresponding parameter values; (iii) when the tail parameter increases and approaches the non-stationarity barrier ($\alpha = 1.0$), the bias of the estimator increases slightly, as it was expected and has been observed in other works (Crato and Ray, 2002). These results seem reassuring for the possibility of reliably estimating the parameters and testing for the parameters.

## 5. An experimental study

Ten students (two males and eight females, aged 19–20 years) from the Faculty of Human Kinetics of the Technical University of Lisbon participated in two tapping experiments. None of the subjects had extensive practice in rhythmical

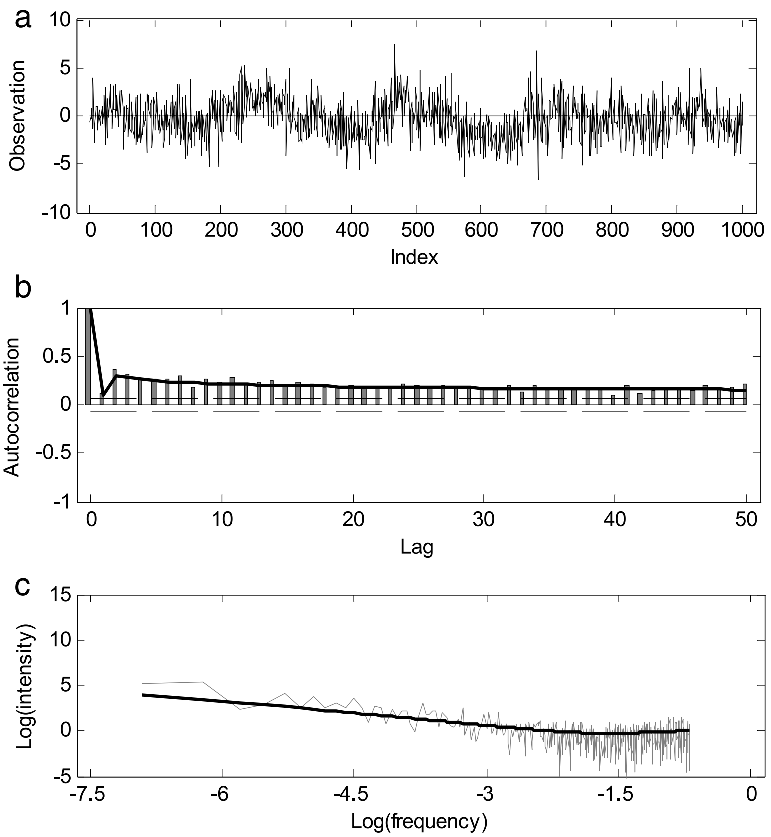


**Fig. 1.** Graphical representations of simulated series. (a) Simulated series of regime switching process with added noise + differenced white noise with $\alpha = 1.2, \sigma_L = 1.0, \sigma_M = 1.0$. (b) Sample autocorrelation function (gray bars) and theoretical autocorrelation function (black line). (c) Normalized periodogram (gray line) and spectral density function (black line) in log–log scale.

**Table 1**
Results of the spectral-likelihood estimator in 1000 simulated series. The values in the first column are the model parameters. The values in the other columns are the means and standard deviations (in parentheses) of the estimated parameters.

| $(\alpha, \sigma_L, \sigma_M, \sigma_M/\sigma_L)$ | $\hat{\alpha}$ | $\hat{\sigma}_L$ | $\hat{\sigma}_M$ | $\hat{\sigma}_M/\hat{\sigma}_L$ |
|---|---|---|---|---|
| (1.2, 1.0, 1.0, 1.0) | 1.228 | 1.100 | 1.080 | 1.088 |
| | (0.112) | (0.109) | (0.102) | (0.125) |
| (1.2, 2.0, 1.0, 0.5) | 1.220 | 1.964 | 1.086 | 0.515 |
| | (0.119) | (0.115) | (0.108) | (0.126) |
| (1.2, 3.0, 1.0, 0.3) | 1.226 | 2.986 | 0.968 | 0.333 |
| | (0.093) | (0.148) | (0.170) | (0.114) |
| (1.4, 1.0, 1.0, 1.0) | 1.418 | 1.004 | 1.006 | 1.055 |
| | (0.061) | (0.102) | (0.103) | (0.130) |
| (1.4, 2.0, 1.0, 0.5) | 1.415 | 1.986 | 1.002 | 0.508 |
| | (0.073) | (0.105) | (0.099) | (0.096) |
| (1.4, 3.0, 1.0, 0.3) | 1.412 | 2.982 | 0.986 | 0.336 |
| | (0.082) | (0.091) | (0.114) | (0.124) |
| (1.6, 1.0, 1.0, 1.0) | 1.608 | 0.999 | 1.003 | 1.004 |
| | (0.042) | (0.099) | (0.095) | (0.112) |
| (1.6, 2.0, 1.0, 0.5) | 1.605 | 1.988 | 0.985 | 0.495 |
| | (0.080) | (0.085) | (0.052) | (0.042) |
| (1.6, 3.0, 1.0, 0.3) | 1.609 | 2.986 | 0.959 | 0.312 |
| | (0.041) | (0.050) | (0.040) | (0.048) |

activities. Each subject was instructed to press a finger switch with his or her index finger in synchrony with periodic auditory signals emitted by a metronome. After 10 signals, the metronome was turned off and the subject tried to continue to tap regularly at the same rate. The task was pursued up to the recording of around 1000 continuous taps. Two target frequencies, $F_1 = 1.250$ Hz (i.e., $T_1 = 800$ ms) and $F_2 = 0.625$ Hz (i.e., $T_2 = 1600$ ms), were studied. Each student performed the task successfully under the two conditions, in a random order, and in separate days.

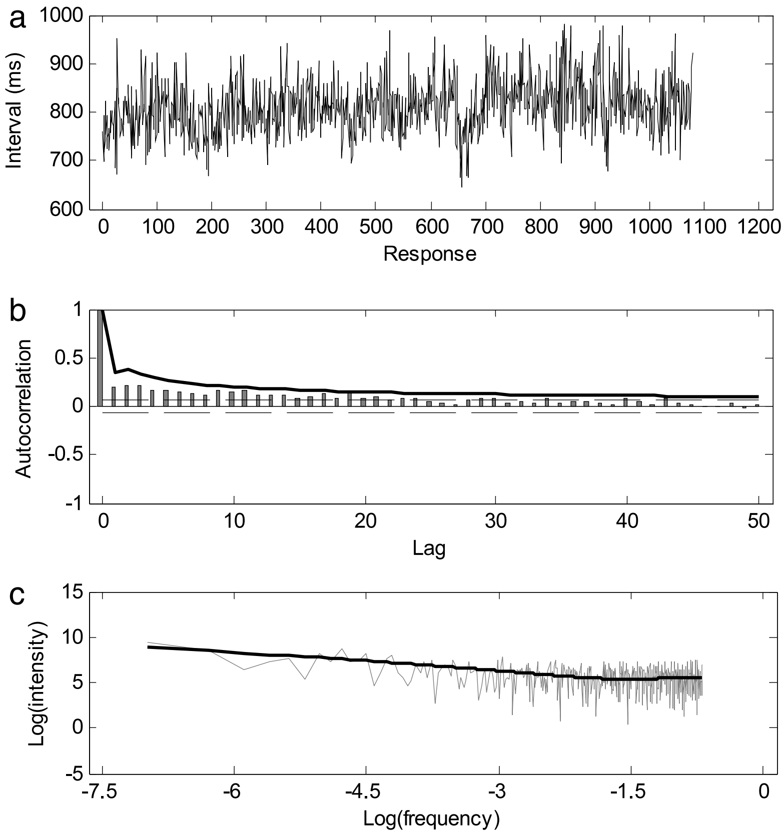


**Fig. 2.** Graphical representations of experimental series with target 800 ms. (a) Time series of inter-response intervals for Subject A and target interval $T_1 = 800$ ms. (b) Sample autocorrelation function (gray bars) and theoretical autocorrelation function (black line). (c) Normalized periodogram (gray line) and spectral density function (black line) in log–log scale.

The computer program AcqKnowledge by BIOPAC Systems was used to identify the time $R_t$ of each tap and to determine the time intervals $I_t$ between successive taps

$$I_t = R_t - R_{t-1}, \quad t = 1, \dots, n.$$

In order to avoid the initial transient, the first 30 points of each time series were eliminated (Delignières et al., 2004). Figs. 2 and 3 present examples of two time series of inter-response intervals with target intervals of 800 ms and 1600 ms, respectively. They also show the sample and the theoretical autocorrelation functions, as well as the normalized periodograms and the spectral density functions in log–log scale. It is evident that the time series have non-periodic waves, which are more visible in the series with the larger target interval. The autocorrelation functions and the periodograms bear a close resemblance to those of the simulated processes.

In order to estimate the proposed model, each time series was subjected to some operations. First, the series was submitted to a method for detecting and removing the outliers (mainly observational errors). Second, the series was corrected for the mean. The proposed model was then fitted to the series. Table 2 provides the results for the estimates of the parameters and the standard errors. Some interesting remarks are (i) the estimates of the tail parameter ($\alpha$) are in the range specified for persistent processes ($\alpha \in (1.0, 2.0)$); the estimates of this parameter decrease as the target intervals increase, for most of the subjects; (ii) the estimates of the cognitive standard deviation ($\sigma_L$) are larger than the corresponding estimates of the motor standard deviation ($\sigma_M$); both estimates increase as the target intervals increase, for most of the subjects; (iii) the estimates of the ratio of the components' standard deviations ($\sigma_M/\sigma_L$) are all smaller than one; the estimates decrease as the target intervals increase, except for subject H; (iv) the observed standard errors of the estimator (within parentheses) are relatively small compared with the parameter estimates, which reinforces the significance of the estimates. These results show that the memory and the variance tend to increase as the target intervals increase, as it has been stated in other works (Madison, 2004; Stevens, 1886). The $t$-tests for the difference between the mean values with targets 800 ms and 1600 ms were inconclusive for remark (i) and consistent with remarks (ii) and (iii). As the number of observations is small and the normality assumption is questionable, the non-parametric Wilcoxon tests were also performed and the outputs were similar.

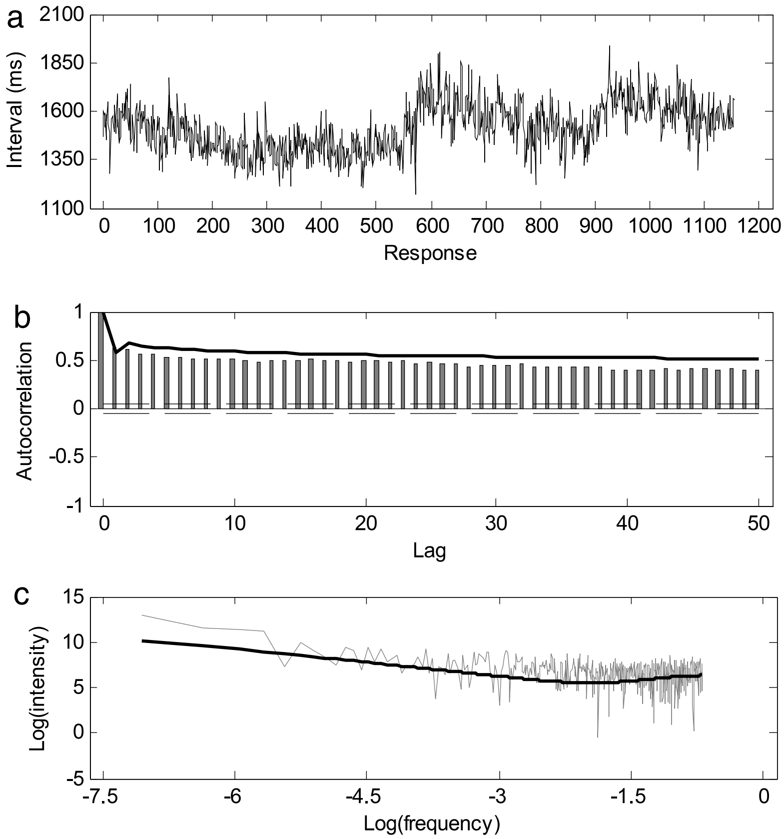


**Fig. 3.** Graphical representations of experimental series with target 1600 ms. (a) Time series of inter-response intervals for Subject A and target interval $T_2 = 1600$ ms. (b) Sample autocorrelation function (gray bars) and theoretical autocorrelation function (black line). (c) Normalized periodogram (gray line) and spectral density function (black line) in log–log scale.

## 6. Conclusions

The Wing–Kristofferson model provides an understanding of the inter-response intervals in tapping tasks, based on a cognitive component and on a motor component. Previous works revealed that the first component can be regarded as a long-memory process and the second component as a white noise process. We suggest a new parametric model in which the cognitive component is a regime-switching process with added noise. The autocorrelation function of this model exhibits a hyperbolic decay and the spectral density function has a pole at the zero frequency. This supports the hypothesis of long-term oscillations in the series. The spectral-likelihood estimator proposed for this model is a consistent estimator. The simulation results show small biases and small variances.

The proposed model seems to offer a suitable explanation for the variability of the inter-response intervals in tapping tasks. In spite of the natural differences, the estimated parameters reveal some stability across individuals. The estimates of the tail parameter are always larger than one and, in some cases, close to one, which provides strong evidence for a long-memory cognitive process. The estimates of the ratio of the standard deviations are always smaller than one, which stresses the predominance of the cognitive part. The estimates of the tail parameter are usually smaller in the series of the larger target interval, while the estimates of the scale parameters are usually larger in those series. This offers strong support for the hypothesis that the memory and the variance increase when the difficulty of the task increases. Cognitive and motor processes seem to objectively reflect the complexity of the task, exhibiting additional variation in the realization of more difficult tasks.

The present results may be very helpful for studying the timing structures in this sort of task. The proposed model, based on a regime-switching process, provides a way of searching for an intuitively appealing biological interpretation. The latent level switches may be understood as level switches of a cognitive or neurological variable responsible for maintaining the rhythm in these tasks. This may be very useful for neuroscience analysis of time control, particularly in conditions of progressive deterioration of nervous structures (Ivry and Keele, 1989; Ivry and Spencer, 2004).

Further research is required to understand the underlying mechanisms of the variability of inter-response intervals and its dependence on interval duration. It is important to find alternative representations that generate long memory in a way

**Table 2**
Results of the spectral-likelihood estimator in experimental series. The letters in the first column represent the subjects. The values in the second column are the target intervals. The values in the other columns are the estimated parameters and standard errors (in parentheses).

| Subject | Target | $\hat{\alpha}$ | $\hat{\sigma}_L$ | $\hat{\sigma}_M$ | $\hat{\sigma}_M/\hat{\sigma}_L$ |
|---|---|---|---|---|---|
| A | 800 | 1.447 | 42.357 | 18.162 | 0.429 |
| | | (0.107) | (3.230) | (0.962) | |
| | 1600 | 1.089 | 72.294 | 28.308 | 0.392 |
| | | (0.037) | (12.981) | (0.931) | |
| B | 800 | 1.172 | 79.262 | 20.757 | 0.262 |
| | | (0.059) | (10.312) | (1.357) | |
| | 1600 | 1.315 | 138.040 | 16.277 | 0.118 |
| | | (0.065) | (11.548) | (3.887) | |
| C | 800 | 1.081 | 58.194 | 34.788 | 0.598 |
| | | (0.022) | (6.696) | (1.199) | |
| | 1600 | 1.515 | 163.683 | 0.359 | 0.002 |
| | | (0.100) | (11.217) | (8.339) | |
| D | 800 | 1.656 | 110.259 | 37.852 | 0.342 |
| | | (0.099) | (5.411) | (2.660) | |
| | 1600 | 1.334 | 109.419 | 37.030 | 0.338 |
| | | (0.077) | (8.518) | (1.998) | |
| E | 800 | 1.119 | 28.402 | 16.718 | 0.589 |
| | | (0.040) | (3.768) | (0.443) | |
| | 1600 | 1.102 | 172.803 | 88.374 | 0.511 |
| | | (0.049) | (33.820) | (3.132) | |
| F | 800 | 1.039 | 36.170 | 22.730 | 0.620 |
| | | (0.014) | (5.475) | (0.542) | |
| | 1600 | 1.182 | 179.111 | 61.077 | 0.341 |
| | | (0.065) | (26.706) | (3.430) | |
| G | 800 | 1.471 | 67.061 | 16.110 | 0.240 |
| | | (0.104) | (5.607) | (1.968) | |
| | 1600 | 1.372 | 136.677 | 31.734 | 0.230 |
| | | (0.100) | (14.370) | (3.303) | |
| H | 800 | 1.348 | 37.192 | 13.462 | 0.363 |
| | | (0.074) | (2.584) | (0.610) | |
| | 1600 | 1.172 | 112.996 | 53.644 | 0.475 |
| | | (0.044) | (11.112) | (1.756) | |
| I | 800 | 1.227 | 111.520 | 38.343 | 0.344 |
| | | (0.109) | (20.723) | (2.289) | |
| | 1600 | 1.174 | 202.051 | 53.447 | 0.265 |
| | | (0.044) | (21.325) | (3.533) | |
| J | 800 | 1.202 | 26.884 | 23.675 | 0.881 |
| | | (0.082) | (3.773) | (0.673) | |
| | 1600 | 1.170 | 179.925 | 54.342 | 0.302 |
| | | (0.098) | (43.705) | (3.255) | |
| Mean | 800 | 1.276 | 59.730 | 24.260 | 0.467 |
| | 1600 | 1.243 | 146.700 | 42.459 | 0.297 |
| *t*-stat[a] | – | 0.455 | −5.401 | −1.977 | 2.211 |
| *p*-value[b] | – | 0.330 | 0.000 | 0.039 | 0.027 |

[a] *t*-statistic for the difference between the mean values.
[b] *p*-value for the one-sided test.

that may plausibly be part of a biological process. These models may provide other insights into the nature and the origin of the observed patterns. More specifically, this may shed some light on how current movements strongly depend on previous movements, which is a key feature in structured sequences of rhythmical movements.

## Appendix. Proof of the proposition of Section 4

**Proof.** From Eqs. (5)–(7), and using some mathematical simplifications, it follows that the autocovariance function $\gamma_W(.)$ is given by

$$\gamma_W(k) = \frac{\sum_{t=|k|+1}^{\infty} P(T \geq t)}{\sum_{t=1}^{\infty} P(T \geq t)} \sigma_L^2 = \frac{\zeta(\alpha) - \sum_{t=1}^{|k|} t^{-\alpha}}{\zeta(\alpha)} \sigma_L^2, \quad |k| = 0, 1, \ldots,$$

where $\zeta(.)$ denotes the Riemann's zeta function defined by $\zeta(\alpha) = \Sigma_{t=1}^{\infty} 1/t^{\alpha}$ and a sum $\Sigma_{t=1}^{0} x_t$ is considered to be equal to 0. Thus, the spectral density function $f_W(.)$ satisfies

$$f_W(\lambda) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma_W(k)\,e^{-ik\lambda} = \sum_{k=-\infty}^{\infty}\frac{\zeta(\alpha)-\sum_{t=1}^{|k|}t^{-\alpha}}{\zeta(\alpha)}e^{-ik\lambda}\frac{\sigma_L^2}{2\pi}, \quad |\lambda| \le \pi.$$

Through some simple mathematical operations, it can be seen that

$$f_W(\lambda) = -\frac{\sigma_L^2}{2\pi} + \sum_{k=0}^{\infty}\frac{\zeta(\alpha)-\sum_{t=1}^{k}t^{-\alpha}}{\zeta(\alpha)}e^{ik\lambda}\frac{\sigma_L^2}{2\pi} + \sum_{k=0}^{\infty}\frac{\zeta(\alpha)-\sum_{t=1}^{k}t^{-\alpha}}{\zeta(\alpha)}e^{-ik\lambda}\frac{\sigma_L^2}{2\pi}. \tag{8}$$

In the above expression, let $S$ be the series given by

$$S = \sum_{k=0}^{\infty}\frac{\zeta(\alpha)-\sum_{t=1}^{k}t^{-\alpha}}{\zeta(\alpha)}e^{-ik\lambda}.$$

From some properties of numerical series, it follows that

$$\begin{aligned} S &= \sum_{k=0}^{\infty}e^{-ik\lambda} - \frac{1}{\zeta(\alpha)}\sum_{k=0}^{\infty}\left(e^{-ik\lambda}\sum_{t=1}^{k}t^{-\alpha}\right) = \sum_{k=0}^{\infty}e^{-ik\lambda} - \frac{1}{\zeta(\alpha)}\sum_{t=1}^{\infty}\left(t^{-\alpha}\sum_{k=t}^{\infty}e^{-ik\lambda}\right) \\ &= \frac{1}{1-e^{-i\lambda}} - \frac{1}{\zeta(\alpha)}\sum_{t=1}^{\infty}\left(t^{-\alpha}\frac{e^{-it\lambda}}{1-e^{-i\lambda}}\right) = \frac{1}{1-e^{-i\lambda}} - \frac{1}{1-e^{-i\lambda}}\frac{1}{\zeta(\alpha)}\sum_{t=1}^{\infty}\frac{e^{-it\lambda}}{t^{\alpha}}. \end{aligned}$$

In sum, the series $S$ can be written as

$$S = \frac{1}{1-e^{-i\lambda}}\left[1 - \frac{Li_\alpha(e^{-i\lambda})}{\zeta(\alpha)}\right], \tag{9}$$

where $Li_\alpha(.)$ denotes the Jonquière's function defined by $Li_\alpha(z) = \Sigma_{t=1}^{\infty} z^t/t^\alpha$.

From Eqs. (8) and (9), it follows that the spectral density function $f_W(.)$ satisfies

$$\begin{aligned} f_W(\lambda) &= -\frac{\sigma_L^2}{2\pi} + \frac{1}{1-e^{i\lambda}}\left[1-\frac{Li_\alpha(e^{i\lambda})}{\zeta(\alpha)}\right]\frac{\sigma_L^2}{2\pi} + \frac{1}{1-e^{-i\lambda}}\left[1-\frac{Li_\alpha(e^{-i\lambda})}{\zeta(\alpha)}\right]\frac{\sigma_L^2}{2\pi} \\ &= \frac{(1-e^{-i\lambda})^{-1}}{\zeta(\alpha)}\left[\,e^{-i\lambda}Li_\alpha(e^{i\lambda}) - Li_\alpha(e^{-i\lambda})\right]\frac{\sigma_L^2}{2\pi}. \quad \square \end{aligned}$$

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