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PERIODIC SOLUTIONS OF PARABOLIC AND TELEGRAPH EQUATIONS WITH ASYMMETRIC NONLINEARITIES†

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1. INTRODUCTION

This paper is devoted to existence results for one-dimensional parabolic and telegraph equations

$$\begin{aligned} u_t - u_{xx} &= n^2 u + g(x, u) + h(x, t) && \text{a.e. in } Q,\\ u(0, t) &= u(\pi, t) = 0 && \text{on } [0, 2\pi],\\ u(x, 0) &= u(x, 2\pi) && \text{on } [0, \pi], \end{aligned}$$

and

$$\begin{aligned} au_t + u_{tt} - u_{xx} &= n^2 u + g(x, u) + h(x, t) && \text{a.e. in } Q,\\ u(0, t) &= u(\pi, t) = 0 && \text{on } [0, 2\pi],\\ u(x, 0) = u(x, 2\pi), &\quad u_t(x, 0) = u_t(x, 2\pi) && \text{on } [0, \pi], \end{aligned}$$

with asymmetric nonlinearities $g(x, u)$ and forcing term $h(x, t)$, where $a \in \mathbb{R}\backslash\{0\}$, $n \in \mathbb{N}$, $Q = [0, \pi] \times [0, 2\pi]$. Second-order and higher-order *multi-dimensional* equations also will be considered.

By asymmetric nonlinearities we mean that the asymptotic behavior of $u^{-1}g(x, u)$ when $u \to \infty$ may be different from what it is when $u \to -\infty$. Moreover, by jumping nonlinearities, we mean that one of the afore-mentioned quantities may lie above one or more (real) eigenvalues of the corresponding linear part, while the other lies below those same eigenvalues.

In recent years much work has been devoted to existence results for second-order scalar ordinary differential equations, in the nonresonance or resonance case, when the non-linear term is a jumping nonlinearity in the sense described above; we refer to the papers [1–11] and references therein.

As for partial differential equations we refer, e.g. to the papers by Dancer [2], Fučik and Mawhin [12], Gallouët and Kavian [13], Šťastnová and Fučik [14], which are more or

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less concerned with existence results, in the nonresonance or *resonance* situation, for problems with jumping nonlinearities in the framework of partial differential equations. (A telegraph equation without jumping is considered in [15].)

Nevertheless, in [13], the linear part of the partial differential equations must be self-adjoint and the range of $u^{-1}g(x, u)$ must contain only one eigenvalue of *simple* multiplicity. Moreover, in [2], there is no explicit crossing of eigenvalues at higher eigenvalues in the multi-dimensional case. In [16–18] and references therein, elliptic problems with jumping nonlinearities at consecutive (possibly multiple) eigenvalues are studied. The authors take advantage of the *variational* structure of the problems and apply critical point theory to obtain more or less multiplicity results.

The evolution equations considered in this paper have nonself-adjoint linear parts, and the range of $u^{-1}g(x, u)$ may include one or more (not necessarily simple) eigenvalues of the corresponding linear parts. Moreover, we do not require that the quantities $\lim_{u\to\infty} u^{-1}g(x, u)$ and $\lim_{u\to-\infty} u^{-1}g(x, u)$ exist, as is required in [2, 12–14]. We work, instead, with the quantities $\liminf_{u\to\pm\infty} u^{-1}g(x, u)$ and $\limsup_{u\to\pm\infty} u^{-1}g(x, u)$. This allows us to consider oscillatory nonlinearities with possibly different and (asymptotically) large amplitudes. In this paper, we shall concentrate on the situation concerning resonance results; the nonresonance case follows along similar lines (see e.g. [5, 7, 19–23] for additional references on nonresonance and resonance problems).

Let, us also mention that the evolution equations considered need not be dissipative (see e.g. [24]) in the sense that the corresponding initial-boundary value problems do not necessarily have (local) attractors. Indeed, the initial-boundary value problem

$$
\begin{aligned}
u_t(x, t) - u_{xx}(x, t) &= (n^2 + \mu)u(x, t) && \text{in } (0, \pi) \times (0, \infty), \\
u(0, t) = u(\pi, t) &= 0 && \text{on } [0, \infty), \\
u(x, 0) &= r \sin nx && \text{on } [0, \pi],
\end{aligned}
$$

(resp.

$$
\begin{aligned}
au_t + u_{tt} - u_{xx} &= (n^2 + a\mu + \mu^2)u && \text{in } (0, \pi) \times (0, \infty),\ a > 0, \\
u(0, t) = u(\pi, t) &= 0 && \text{on } [0, \infty), \\
u(x, 0) &= r \sin nx && \text{on } [0, \pi], \\
u_t(x, 0) &= \mu r \sin nx && \text{on } [0, \pi])
\end{aligned}
$$

has a unique unbounded solution given by $u(x, t) = r \sin nx\, e^{\mu t}$ for $\mu, r \in \mathbb{R}$ with $\mu > 0$ and $r \neq 0$. Note however that, for $0 < \mu < 2n + 1$ (resp. $0 < a\mu + \mu^2 < 2n + 1$), the only (bounded) time-periodic solution to the above equation(s) is the trivial solution $u = 0$. Thus, the zero solution is not (asymptotically) stable as $t \to \infty$.

This paper is organized as follows. In Section 2, we collect the notation and basic assumptions that we shall suppose fulfilled throughout this paper. Section 3 is devoted to second order nonlinear one-dimensional parabolic and (linearly) damped hyperbolic equations. We compare, in some sense, the nonlinearity $g(x, u)$ with the Fučik spectrum of the corresponding piecewise linear differential equations with homogeneous Dirichlet boundary conditions, and a resonance condition of Landesman–Lazer type with respect to the forcing term $h(x, t)$. More specifically, we assume that (the asymptotic behavior of) $u^{-1}g(x, u)$ lies in a rectangle located in what we should call the Fučik–Landesman–Lazer "resolvent" set. In Section 4, we take up the case of second-order multi-dimensional equations, and we prove results on crossing at not necessarily simple (higher) eigenvalues. Finally, in Section 5 we indicate the conditions under which one can extend our results to higher-order multi-dimensional equations.

2. PRELIMINARIES

Let $Q = \Omega \times [0, 2\pi]$ where $\Omega \subset \mathbb{R}^N$ is a bounded domain whose boundary $\partial\Omega$ is of class C^2. Throughout this paper we shall make use of the anisotropic Sobolev spaces $H^{p,q}(Q) = H^q([0, 2\pi]; H^p(\Omega))$ where p and q are nonnegative integers (see e.g. [6, 12, 25–27] for definitions and properties). Here H^k are the classical Sobolev spaces with the usual Hilbert space structure. (Of course, $H^0 = L^2$ the classical Lebesgue space, and $H^{p,p}(Q) = H^p(Q)$.) $H_0^1(\Omega)$ denotes the subspace of all functions in $H^1(\Omega)$ which vanish on $\partial\Omega$ in the sense of trace.

Let $u \in L^2([0, 2\pi]; H_0^1(0, \pi))$. If $u(x, t) = \sum_{k=1}^{\infty} b_k(t) \sin kx$ is an eigenfunction expansion of $u(x, t)$, then we shall set

$$\bar{u}(x, t) = \sum_{k=1}^{n-1} b_k(t) \sin kx, \qquad u^0(x, t) = b_n(t) \sin nx, \qquad \tilde{u}(x, t) = \sum_{k=n+1}^{\infty} b_k(t) \sin kx$$

and $u^\perp = \bar{u} + \tilde{u}$. For a.e. $t \in [0, 2\pi]$, the notation $\bar{u}(\cdot, t)$, $u^0(\cdot, t)$ and $\tilde{u}(\cdot, t)$ has an obvious meaning. Similar notation will also be used for multi-dimensional expansions and we shall make it more precise in Section 4.

We shall always assume that the (nonlinear) function $g: \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and that it grows at most linearly, that is, $g(\cdot, u)$ is measurable for all $u \in \mathbb{R}$, $g(x, \cdot)$ is continuous for a.e. $x \in \bar{\Omega}$, and there exist a constant $c > 0$ and a function $b \in L^2(\Omega)$ such that

$$|g(x, u)| \leq c|u| + b(x) \tag{1}$$

for a.e. $x \in \bar{\Omega}$ and all $u \in \mathbb{R}$. The expressions $g_\pm(x)$ will denote the quantities

$$g_+(x) = \liminf_{u \to \infty} g(x, u) \qquad \text{and} \qquad g_-(x) = \limsup_{u \to -\infty} g(x, u). \tag{2}$$

Finally, the forcing term $h(x, t)$ will be assumed to be such that $h \in L^2(Q)$.

However, in the case of the *telegraph equation*, slightly stronger regularity conditions on the forcing term h and the (nonlinear) function g will be assumed throughout this paper. More precisely, we shall assume that, in addition to the above conditions, $h \in H^{0,1}(Q)$ and (for a.e. $x \in \bar{\Omega}$) the function $g(x, \cdot): \mathbb{R} \to \mathbb{R}$ is differentiable (a.e.) in u with bounded and measurable (partial) derivative; that is, there exists a constant $C > 0$ such that for a.e. $x \in \bar{\Omega}$,

$$\left| \frac{\partial g(x, \cdot)}{\partial u} \right| \leq C. \tag{3}$$

Note that inequalities (1) and (3) ensure that $g(\cdot, u(\cdot, \cdot)) \in H^{0,1}(Q)$ if $u \in H^{0,1}(Q)$; which implies that generalized solutions to the telegraph equation belong to $H^2(Q)$. We refer to Vejvoda *et al.* [27, Chapter II, Section 2 and Chapter IV]. (It is known that generalized solutions to the telegraph equation belong to $H^1(Q)$, see e.g. Brézis and Nirenberg [28, pp. 308–309].)

3. ONE-DIMENSIONAL EQUATIONS

Let $n \in \mathbb{N}$ and $\Omega = (0, \pi)$. We shall consider the one-dimensional heat and telegraph problems

$$\begin{aligned} u_t(x,t) - u_{xx}(x,t) &= n^2 u(x,t) + g(x, u(x,t)) + h(x,t) && \text{a.e. in } Q, \\ u(0,t) &= u(\pi, t) = 0 && \text{on } [0, 2\pi], \\ u(x,0) &= u(x, 2\pi) && \text{on } [0, \pi], \end{aligned} \tag{4}$$

and

$$\begin{aligned} au_t + u_{tt} - u_{xx} &= n^2 u + g(x,u) + h(x,t) && \text{a.e. in } Q, \\ u(0,t) &= u(\pi,t) = 0 && \text{on } [0, 2\pi], \\ u(x,0) = u(x,2\pi), &\quad u_t(x,0) = u_t(x,2\pi) && \text{on } [0, \pi], \end{aligned} \tag{5}$$

where $a \in \mathbb{R}$ with $a \neq 0$ and compare, in some sense, the nonlinearity $g(x, u)$ with the Fučik spectrum of the corresponding piecewise problem. A resonance condition of Landesman–Lazer type with respect to the forcing term $h(x, t)$ is also assumed.

For $(\mu, \nu) \in \mathbb{R}^2$, let us consider the (positively homogeneous) piecewise linear problems

$$\begin{aligned} u_t - u_{xx} &= \mu u^+ - \nu u^- && \text{in } Q \\ u(0,t) &= u(\pi,t) = 0 && \text{on } [0, 2\pi], \\ u(x,0) &= u(x,2\pi) && \text{on } [0, \pi], \end{aligned}$$

and

$$\begin{aligned} au_t + u_{tt} - u_{xx} &= \mu u^+ - \nu u^- && \text{in } Q, \\ u(0,t) &= u(\pi,t) = 0 && \text{on } [0, 2\pi], \\ u(x,0) = u(x,2\pi), &\quad u_t(x,0) = u_t(x,2\pi) && \text{on } [0, \pi], \end{aligned}$$

where $u^+(x,t) = \max\{u(x,t), 0\}$ and $u^-(x,t) = \max\{-u(x,t), 0\}$. Multiplying both sides of these equations by u_t and using Fubini's theorem, integration by parts and the boundary and periodicity conditions, it follows that each term is equal to zero with the exception of the first one, so that $|u_t|^2_{L^2(Q)} = 0$ (resp. $a|u_t|^2_{L^2(Q)} = 0$).

Therefore, these piecewise linear equations have a nontrivial solution if and only if the piecewise linear scalar second order Dirichlet problem

$$v''(x) + \mu v^+(x) - \nu v^-(x) = 0 \quad \text{in } (0, \pi), \qquad v(0) = v(\pi) = 0, \tag{6}$$

has a nontrivial solution.

It is well known (see e.g. Fučik [6]) that, for $(\mu, \nu) \in \mathbb{R}^2$, the piecewise linear scalar second order Dirichlet problem (6) has a nontrivial solution if and only if the pair $(\mu, v) \in \bigcup_{i=1}^{\infty} C_i$ where

$$C_1 = \{(\mu, \nu) \in \mathbb{R}^2 : (\mu - 1)(\nu - 1) = 0\},$$

$$C_{2j} = \left\{(\mu, \nu) \in \mathbb{R}^2 : j\left(\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}}\right) = 1\right\},$$

$$C_{2j+1} = \left\{(\mu, \nu) \in \mathbb{R}^2 : j\left(\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}}\right) + \frac{1}{\sqrt{\mu}} = 1 \quad \text{or} \quad j\left(\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}}\right) + \frac{1}{\sqrt{\nu}} = 1\right\}$$

for $j \in \mathbb{N}$.

We recall that the set of $(\mu, \nu) \in \mathbb{R}^2$ such that $(\mu, \nu) \in \bigcup_{i=1}^{\infty} C_i$ is called the Fučik spectrum of the above second-order differential equations and that it reduces to the usual spectrum when $\mu = \nu$.

We shall consider nonlinearities $g(x, u)$ such that the asymptotic behaviour of $u^{-1}g(x, u)$ lies in a rectangle located in what we should call the Fučik–Landesman–Lazer "resolvent" set; the eigenvalue pair $(n^2, n^2) \in \mathbb{R}^2$ is a vertex of this rectangle, whereas the opposite vertex is located on the consecutive Fučik eigenvalue curve.

We state the main result of this section.

THEOREM 1. Suppose that there exist a real number $r > 0$ and a function $B \in L^2([0, \pi])$ such that

$$(\operatorname{sgn} u)g(x, u) \geq B(x) \tag{7}$$

for a.e. $x \in [0, \pi]$ and all $u \in \mathbb{R}$ with $|u| \geq r$.

Moreover, assume that

$$\limsup_{u \to \pm\infty} \frac{g(x, u)}{u} \leq \beta_{\pm}(x) \tag{8}$$

uniformly for a.e. $x \in [0, \pi]$ with $\beta_{\pm} \in L^2([0, \pi])$ such that

$$\beta_+(x) \leq \mu_{n+1} - n^2 \qquad \text{and} \qquad \beta_-(x) \leq \nu_{n+1} - n^2 \tag{9}$$

for a.e. $x \in [0, \pi]$ with strict inequalities on subsets of $[0, \pi]$ of positive measure, where either the pair $(\mu_{n+1}, \nu_{n+1}) \in C_{n+1}$ (the upper Fučik eigenvalue curve) if n is odd or the (open) rectangle $]n^2, \mu_{n+1}[\times]n^2, \nu_{n+1}[\subset \mathbb{R}^2 \setminus \bigcup_{m=1}^{\infty} C_m$ if n is even.

Then equation (4) (resp. equation (5)) has at least one solution $u \in H^{2,1}(Q)$ (resp. $u \in H^2(Q)$) provided

$$\int_Q g_+(x)v^+(x)\,\mathrm{d}x\,\mathrm{d}t - \int_Q g_-(x)v^-(x)\,\mathrm{d}x\,\mathrm{d}t + \int_Q h(x, t)v(x)\,\mathrm{d}x\,\mathrm{d}t > 0 \tag{10}$$

for all $v \in \operatorname{Span}\{\sin nx\}\setminus\{0\}$.

In order to prove this theorem, we shall establish some auxiliary results. We define

$$\mathfrak{A}u = u_t - u_{xx} - n^2u \qquad (\text{respectively } \mathfrak{A}u = au_t + u_{tt} - u_{xx} - n^2u)$$

with $\operatorname{Dom}(\mathfrak{A}) = H^1_{2\pi}([0, 2\pi]; H^1_0(0, \pi) \cap H^2(0, \pi))$ (respectively $H^2_{2\pi}([0, 2\pi]; H^1_0(0, \pi) \cap H^2(0, \pi))$) where the subscript 2π indicates that the involved functions are 2π-periodic in the variable t and consider $Gu(t, x) = g(x, u(t, x))$ to be the Nemytskii operator associated with g.

We shall prove the existence of at least one solution to the above problem by applying degree theory arguments to the study of the operator equation

$$\mathfrak{A}u = Gu + h.$$

Throughout this section, we shall set

$$\mathfrak{L}u(\cdot, t) := -u_{xx}(\cdot, t) - n^2u(\cdot, t),$$

for $u \in \operatorname{Dom}(\mathfrak{A})$ and denote by $(\cdot, \cdot)$ the usual inner product in $L^2(Q)$.

LEMMA 1. There exists a constant $\delta_1 > 0$ such that

$$-(\mathcal{A}\bar{u}, \bar{u}) \geq \delta_1 |\bar{u}|^2_{H^{1,0}}.$$

Proof. For a.e. $t \in [0, 2\pi]$, the function $\bar{u}(\cdot, t) \in H^2(0, \pi) \cap H^1_0(0, \pi) \cap \overline{L^2}(0, \pi)$. Then it follows from Lemma 1 in [29] that

$$-(\mathcal{L}\bar{u}(\cdot, t), \bar{u}(\cdot, t))_{L^2(0,\pi)} \geq \delta_1 |\bar{u}(\cdot, t)|^2_{H^1(0,\pi)}$$

for a.e. $t \in [0, 2\pi]$, where the constant $\delta_1 > 0$ is given in [29]. Therefore, taking the inner product of $-\mathcal{A}\bar{u}$ with $\bar{u}$ in $L^2(Q)$, the conclusion of the lemma follows since $H^{1,0}(Q) = L^2([0, 2\pi]; H^1(0, \pi))$. The proof is complete.

LEMMA 2. There exists a constant $\delta_2 > 0$ such that

$$(\mathcal{A}\tilde{u}, \tilde{u}) \geq \delta_2 |\tilde{u}|^2_{H^{1,0}}$$

for the heat operator, and

$$(\mathcal{A}\tilde{u}, \tilde{u}) \geq \delta_2 |\tilde{u}|^2_{H^{1,0}} - |\tilde{u}_t|_{L^2(Q)}$$

for the telegraph operator.

Proof. For a.e. $t \in [0, 2\pi]$ the function $\tilde{u}(\cdot, t) \in H^2(0, \pi) \cap H^1_0(0, \pi) \cap \widetilde{L^2}(0, \pi)$. It follows from Lemma 2 in [29] that

$$(\mathcal{L}\tilde{u}(\cdot, t), \tilde{u}(\cdot, t))_{L^2(0,\pi)} \geq \delta_2 |\tilde{u}(\cdot, t)|^2_{H^1(0,\pi)}$$

for a.e. $t \in [0, 2\pi]$, where the constant $\delta_2 > 0$ is given in [29]. Therefore, taking the inner product of $\mathcal{A}\tilde{u}$ with $\tilde{u}$ in $L^2(Q)$, the conclusion of the lemma follows since $H^{1,0}(Q) = L^2([0, 2\pi]; H^1(0, \pi))$. The proof is complete. ■

LEMMA 3. Let $(u_m) \subset \mathrm{Dom}(\mathcal{A})$ and $(p_m) \subset L^2(Q)$ be sequences, and let $C \in L^2(Q)$ be a nonnegative function such that

$$0 \leq p_m(x, t) \leq C(x, t) \qquad \text{a.e. on } Q \text{ for all } m \in \mathbb{N},$$

$$p_m \rightharpoonup 0 \qquad \text{in } L^2(Q) \text{ as } m \to \infty,$$

$$|(\tilde{u}_m)_t|_{L^2(Q)} \leq \rho_1 \qquad \text{for all } m \in \mathbb{N},$$

where $\rho_1 > 0$ is a constant.

Then, there are (common) subsequences, relabeled (u_m), (p_m), satisfying the above conditions such that one has

$$(\mathcal{A}u_m - p_m u_m, \tilde{u}_m - u_m^0 - \bar{u}_m) \geq \delta |u_m^\perp|^2_{H^{1,0}} - \rho_1$$

for the heat operator and

$$(\mathcal{A}u_m - p_m u_m, \tilde{u}_m - u_m^0 - \bar{u}_m) \geq \delta |u_m^\perp|^2_{H^{1,0}} - 2\rho_1^2$$

for the telegraph operator, where $\delta = \min\{\delta_1, \delta_2/2\} > 0$.

Proof. By the definition of u^0, we have $(\mathcal{A}u^0, u^0) = 0$. Therefore, applying Lemmas 1 and 2, we easily derive, in the case of the parabolic operator, that

$$(\mathcal{A}u_m - p_m u_m, \tilde{u}_m - u_m^0 - \bar{u}_m) \geq \delta_1 |\bar{u}_m|^2_{H^{1,0}} + \delta_2 |\tilde{u}_m|^2_{H^{1,0}} - (p_m \tilde{u}_m, \tilde{u}_m)$$

If $|\tilde{u}_m|_{H^{1,0}}$ is bounded (independently of m), it follows from our assumptions that $|\tilde{u}_m|_{H^{1,1}}$ is also bounded and (for a subsequence) $\tilde{u}_m \to \tilde{u}$ in $L^0(Q)$. Then, since $p_m \rightharpoonup 0$, we have that, for m sufficiently large,

$$(p_m \tilde{u}_m, \tilde{u}_m) \leq \rho_1,$$

given that $(p_m \tilde{u}_m, \tilde{u}_m) \to 0$ as $m \to \infty$. This implies that, for m sufficiently large,

$$(\mathcal{A}u_m - p_m u_m, \tilde{u}_m - u_m^0 - \bar{u}_m) \geq \delta_1 |\bar{u}_m|^2_{H^{1,0}} + \delta_2 |\tilde{u}_m|^2_{H^{1,0}} - \rho_1.$$

Thus, the conclusion of the lemma holds.

If $|\tilde{u}_m|_{H^{1,0}}$ is not bounded, then there is a subsequence similarly relabeled such that $|\tilde{u}_m|_{H^{1,0}} \to \infty$. Setting

$$\tilde{y}_m = \frac{\tilde{u}_m}{|\tilde{u}_m|_{H^{1,0}}},$$

it follows that $|\tilde{y}_m|_{H^{1,1}}$ is bounded. Therefore, one can proceed as before and obtain

$$(p_m \tilde{y}_m, \tilde{y}_m) \leq \frac{\delta_2}{2}.$$

Multiplying by $|\tilde{u}_m|^2_{H^{1,0}}$ we get

$$(p_m \tilde{u}_m, \tilde{u}_m) \leq \frac{\delta_2}{2} |\tilde{u}_m|^2_{H^{1,0}},$$

which, as above, clearly implies the conclusion of the lemma. The case of the telegraph operator is analogous. The proof is complete.

Proof of Theorem 1. To prove the existence of at least one solution to the above problem(s) is equivalent to solving the respective operator equation(s)

$$\mathcal{A}u = Gu + h$$

or equivalently

$$u - (\mathcal{A} - \bar{\alpha}I)^{-1}(Gu - \bar{\alpha}u + h) = 0,$$

with $\bar{\alpha} \in \mathbb{R}$ such that $0 < \bar{\alpha} < \min\{\mu_{n+1} - n^2, \nu_{n+1} - n^2\}/2$.

The linear operator $(\mathcal{A} - \bar{\alpha}I)^{-1}: H^0(Q) \to H^0(Q)$ is compact (see [27, 28] for the regularity of the heat and telegraph operators, respectively). Therefore, if we show that there is a constant $\rho > 0$ such that for every possible solution $u \in \mathrm{Dom}(\mathcal{A})$ of the homotopy of problems

$$H(\tau, u) := u - \tau(\mathcal{A} - \bar{\alpha}I)^{-1}(Gu - \bar{\alpha}u + h) = 0, \qquad \tau \in [0, 1],$$

we have

$$|u|_{H^{1,1}} < \rho, \tag{11}$$

then the result follows directly from topological degree theory (see e.g. [20]). Rewriting the homotopy

$$H(\tau, u) = 0$$

in the equivalent form

$$\mathfrak{A}u = \tau Gu + (1 - \tau)\bar{\alpha}u + \tau h,$$

and multiplying both sides by u_t and using Fubini's theorem, integration by parts and the boundary and periodicity conditions, it follows that each term on the left-hand side is equal to zero except the first one, so that $(\mathfrak{A}u, u_t) = |u_t|^2_{L^2(Q)}$ (resp. $(\mathfrak{A}u, u_t) = a|u_t|^2_{L^2(Q)}$). Moreover, by using the same argument in the right-hand side and the fact that

$$\begin{aligned}(g(x, u(x, t)), u_t) &= \int_0^\pi \bar{g}(x, u(x, t))|_0^{2\pi} \, dx \\ &= \int_0^\pi [\bar{g}(x, u(x, 2\pi)) - \bar{g}(x, u(x, 0))] \, dx = 0\end{aligned}$$

by 2π-periodicity, where $\bar{g}(x, u) = \int_\cdot^u g(x, s) \, ds$ is an antiderivative (the potential) of $g(x, u)$ with respect to u, it follows that

$$(\mathfrak{A}u, u_t) = |u_t|^2_{L^2(Q)} = \tau(h, u_t). \tag{12}$$

Therefore, by Cauchy-Schwarz inequality, one has that there is a constant $\rho_1 > 0$, depending only on h (resp. on h and $|a|$), such that

$$|u_t|_{L^2(Q)} \leq \rho_1 \tag{13}$$

for all possible solutions $u \in \mathrm{Dom}(\mathfrak{A})$ to the above homotopy of problems. Hence, it remains to show that there exists a constant $\rho_2 > 0$ such that

$$|u|_{H^{1,0}} < \rho_2. \tag{14}$$

Now, suppose that the claim (14) does not hold. Then, one can find sequences $(\tau_m) \subset (0, 1]$ and $(u_m) \subset \mathrm{Dom}(\mathfrak{A})$ with $|u_m|_{H^{1,0}} \geq m \in \mathbb{N}$ such that

$$H(\tau_m, u_m) = 0.$$

For each $m \in \mathbb{N}$, let us set

$$v_m = \frac{u_m}{|u_m|_{H^{1,0}}}.$$

Then $|v_m|_{H^{1,0}} = 1$. Moreover, by (13), $|v_m|_{H^{1,1}}$ is bounded and $|v_{m_t}|_{L^2} \to 0$ as $m \to \infty$. Dividing the equation $H(\tau_m, u_m) = 0$ by $|u_m|_{H^{1,0}}$, one gets

$$v_m - \tau_m(\mathfrak{A} - \bar{\alpha}I)^{-1}\left(\frac{Gu_m}{|u_m|_{H^{1,0}}} - \bar{\alpha}v_m + \frac{h}{|u_m|_{H^{1,0}}}\right) = 0.$$

Since (v_m) is bounded in $H^{1,1}$ (for a subsequence), (v_m) converges strongly to some $v \in L^2(Q)$. Since g has linear growth and $(\mathcal{A} - \bar{\alpha}I)^{-1}$ is compact, passing to the limit in the above equation and applying $(A - \bar{\alpha}I)$ to both sides, we get

$$\mathcal{A}v = \tau^*\chi + (1 - \tau^*)\bar{\alpha}v,$$

where $\tau^* \in [0, 1]$ and χ denotes the weak limit in $L^2(Q)$ of the sequence $Gu_m/|u_m|_{H^{1,0}}$. An analysis of this equation will show that

$$v \in \operatorname{Span}\{\sin nx\}\backslash\{0\} \qquad \text{and} \qquad \chi = 0 \text{ a.e. in } Q;$$

that is, $v \in N(\mathcal{A})\backslash\{0\}$, where $N(\mathcal{A})$ denotes the nullspace of $\mathcal{A}$.

First of all we decompose the (nonlinear) function g. Let $\delta > 0$. By (8), we can deduce (see the proof of Theorem 1 of [30]) that there exists $R_1 = R_1(\delta) > 0$ such that for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$ with $|u| \geq R_1$

$$|g(x, u| \leq [\beta_u(x) + (\delta/2)]|u| \tag{15}$$

where

$$\beta_u(x) := \begin{cases} \beta_+(x) & \text{if } u > 0 \\ \beta_-(x) & \text{if } u < 0 \end{cases}$$

Without loss of generality we may assume that the constant $r > 0$ given in assumption (7) is such that $r \geq \max\{1, R_1\}$. By Lemma 2 in [30] (also see Lemmas 3 and 4 in [31]), it follows that we can write

$$g(x, u) = q_1(x, u) + g_1(x, u)$$

where q_1 and g_1 are Carathéodory functions such that, for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$,

$$0 \leq uq_1(x, u)$$

and, for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$ with $|u| \geq r$

$$|q_1(x, u)| \leq [\beta_u(x) + (\delta/2)]|u| + 1.$$

Therefore, if we put $\bar{r} > \max\{r, 2/\delta\}$, it follows that, for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$ with $|u| > \bar{r}$

$$0 \leq \frac{q_1(x, u)}{u} \leq \beta_u(x) + \delta.$$

Moreover, there is a function $\sigma_1 \in L^2(\Omega)$ such that

$$|g_1(x, u)| \leq \sigma_1(x)$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$. Therefore, by setting

$$\gamma(x, u) := \begin{cases} \dfrac{q_1(x, u)}{u} & \text{for } |u| \geq r, \\ (\operatorname{sgn} u)q_1(x, (\operatorname{sgn} u)r)\dfrac{u^2}{r^3} & \text{for } 0 < |u| < r, \\ 0 & \text{for } u = 0, \end{cases}$$

it follows that γ is a Carathéodory function such that

$$0 \leq \gamma(x, u) \leq \beta_u(x) + \delta \qquad (\in L^\infty([0, \pi])$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$. Also

$$\lim_{u \to 0} \gamma(x, u) = 0$$

for a.e. $x \in \Omega$.

Then we can write the function g as

$$g(x, u) = \gamma(x, u)u + f(x, u), \tag{16}$$

where $f(x, u) = g(x, u) - \gamma(x, u)u$. It is clear that f is a Carathéodory function such that there exists a function $\sigma \in L^2(\Omega)$ with

$$|f(x, u)| \leq \sigma(x)$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$. Note that, by the above decomposition of g, the function χ, which is the weak limit of the sequence $((Gu_m)/|u_m|_{H^{1,0}(Q)})$ in $L^2(Q)$, is also the weak limit of the sequence $(\gamma(\cdot, u_n(\cdot, \cdot))v_n)$ and, if we denote by χ_v the weak limit of $(\gamma(\cdot, u_n(\cdot, \cdot)))$, then

$$\chi(x, t) = \chi_v(x, t)v(x, t) \qquad \text{a.e. in } Q.$$

Therefore, by using the growth conditions derived for the function $\gamma(x, u)$ on $[0, \pi] \times \mathbb{R}$, the properties of lim inf and lim sup and Fatou's lemma as for instance in [7], it follows that

$$0 \leq \chi_v(x, t) \leq \limsup_{u \to +\infty} \frac{g(x, u)}{u} \leq \beta_+(x) \qquad \text{a.e. on } \{(x, t) : v(x, t) > 0\},$$

$$0 \leq \chi_v(x, t) \leq \limsup_{u \to -\infty} \frac{g(x, u)}{u} \leq \beta_-(x) \qquad \text{a.e. on } \{(x, t) : v(x, t) < 0\},$$

As observed above, $|v_m|_{H^{1,1}}$ is bounded and (v_{m_t}) converges to zero in $L^2(Q)$. Therefore $(v_t) = 0$ (and hence $(v_{tt}) = 0$), which shows that v is independent of t.

Setting $v(x) := v(x, t)$, we have that the function v is a solution to the second order ODE (with "parameter" t)

$$v''(x) + n^2 v(x) + \tau^* \chi_v(x, t)v(x) + (1 - \tau^*)\bar{\alpha} v(x) = 0 \qquad \text{a.e. in } Q,$$
$$v(0) = v(\pi) = 0.$$

For a.e. $t \in [0, 2\pi]$, let us set

$$\chi_+(x, t) := \begin{cases} \tau^* \chi_v(x, t) + (1 - \tau^*)\bar{\alpha} & \text{on } \{x : v(x) > 0\}, \\ \bar{\alpha} & \text{otherwise,} \end{cases}$$

and

$$\chi_-(x, t) := \begin{cases} \tau^* \chi_v(x, t) + (1 - \tau^*)\bar{\alpha} & \text{on } \{x : v(x) < 0\}, \\ \bar{\alpha} & \text{otherwise,} \end{cases}$$

It follows that v is also a solution to

$$v''(x) + n^2 v(x) + \chi_+(x, t)v^+(x) - \chi_-(x, t)v^-(x) = 0 \qquad \text{a.e. in } Q,$$
$$v(0) = v(\pi) = 0.$$

If v does not change sign in $(0, \pi)$, then it follows from the Fredholm alternative and the unique continuation property (of v) that $n = 1$, $\tau^* = 1$ and $\chi_v = 0$ a.e. in $[0, \pi]$; that is, $v \in \mathrm{Span}\{\sin x\}\backslash\{0\}$. (In this case, one can finish the proof of the theorem as in [32] (also see [33]), provided $b, h \in L^p(Q)$ with $p > 3$. However, the arguments developed below allow to treat the weaker case $h \in L^2(Q)$ for the heat equation.)

Otherwise, it follows from the unique continuation property (or oscillatory properties) of second-order ODEs that $v(x) > 0$ and $v(x) < 0$ on subsets of $[0, \pi]$ positive measure which are complement of each other. (Actually, v has only finitely many zeros with nonzero derivative in $(0, \pi)$.)

Note that if $0 \leq \tau^* < 1$, then the functions $[n^2 + \chi_\pm(\cdot, t)]$ satisfy the assumptions of a lemma due to Invernizzi (see e.g. [3, p. 198; 4, p. 645 or 10, p. 287]) for a.e. $t \in [0, 2\pi]$, which implies that $v = 0$ on $[0, \pi]$. This is a contradiction.

Therefore, $\tau^* = 1$. In this case, if $\chi_v(\cdot, t) \neq 0$ on a subset of $(0, \pi)$ of positive measure for some $t \in [0, 2\pi]$ (note that the ODE holds for at least all t in a subset of $[0, 2\pi]$ of positive measure), then it again follows that the functions $[n^2 + \chi_\pm(\cdot, t)]$ satisfy the assumptions of the aforementioned lemma (see e.g. [3, 4, 10]), which implies that $v = 0$ on $[0, \pi]$. This is also a contradiction.

Hence, we have $\tau^* = 1$ and $\chi_v(\cdot, t) = 0$ a.e. on $[0, \pi]$ for all $t \in [0, 2\pi]$ for which the ODE holds; that is for a.e. $t \in [0, 2\pi]$. This, of course, means that $\tau^* = 1$ and $\chi_v = 0$ a.e. in Q, which implies that $v \in \mathrm{Span}\{\sin nx\}\backslash\{0\}$ and $p_m := \tau_m \gamma(\cdot, u_m) + (1 - \tau_m)\bar{\alpha} \rightharpoonup 0$ in $L^2(Q)$.

We shall reach a contradiction by applying Fatou's lemma. Let us show that we can use that argument. Observe that in the eigenfunction expansion of u_m we have $u_m^0(x, t) = b_m(t) \sin mx$. Since $b_m(t)$ can be written as $b_m(t) = a_m + c_m(t)$, where $a_m \in \mathbb{R}$ and $c_m(t)$ has mean value zero, that is, $\int_0^{2\pi} c_m(t)\,\mathrm{d}t = 0$, we can rewrite u_m^0 as

$$u_m^0(x, t) = \varphi_m(x) + \psi_m(x, t)$$

where $\varphi_m(x) = a_m \sin nx$ and $\psi_m(x, t) = c_m(t) \sin nx$. Set

$$z_m(x) := \frac{\varphi_m(x)}{|\varphi_m|_{H^{1,0}}} \left(= \frac{\varphi_m(x)}{2\pi |\varphi_m|_{H^1(0,\pi)}} \right).$$

Taking the inner product in $L^2(Q)$ of

$$\mathfrak{A} u_m = \tau_m G u_m + (1 - \tau_m)\bar{\alpha} u_m + \tau_m h \tag{17}$$

with z_m and using the fact that $\tau_m \neq 0$ and $(u_m, z_m) = |\varphi_m|_{L^2}^2 |\varphi_m|_{H^{1,0}}^{-1}$, we get

$$0 \geq \int_\Omega g(x, u_m(x, t)) z_m(x) + h(x, t) z_m(x)\,\mathrm{d}x\,\mathrm{d}t. \tag{18}$$

We claim that there exists a function $l \in L^1(Q)$ such that, for m sufficiently large,

$$g(x, u_m(x, t)) z_m(x) \geq l(x, t).$$

Let us assume for the moment that this claim holds and finish the proof. Since $u_m(x, t) = v_m(x, t)|u_m|_{H^{1,0}}$ and $(v_m(x, t))$ converges to $v(x, t)$ a.e. in Q, then as $m \to \infty$,

$$u_m(x, t) \to \infty \qquad \text{if } v(x) > 0,$$

$$u_m(x, t) \to -\infty \qquad \text{if } v(x) < 0.$$

So, passing to the lim inf as $m \to \infty$ in (18), by Fatou's lemma we get

$$0 \geq \int_Q g_+(x)v^+(x) - g_-(x)v^+(x) + h(x, t)v(x)\,\mathrm{d}x\,\mathrm{d}t.$$

This is a contradiction with the condition (10).

In order to complete the proof of Theorem 1, it remains to prove the above claim. In what follows we shall denote by the same symbol C several constants independent of u_m. Multiplying equation (17) by $\tilde{u}_m - u_m^0 - \bar{u}_m$ and using the decomposition of g given in (16), we get

$$(\mathfrak{A}u_m - \tau_m\gamma(\cdot, u_m)u_m - (1 - \tau_m)\bar{a}u_m, \tilde{u}_m - u_m^0 - \bar{u}_m)$$
$$= (\tau_m f(\cdot, u_m) + \tau_m h, \tilde{u}_m - u_m^0 - \bar{u}_m).$$

Therefore, by Lemma 3 with $p_m := \tau_m\gamma(\cdot, u_m) + (1 - \tau_m)\bar{\alpha} \rightharpoonup 0$ in $L^2(Q)$, we have

$$\delta|u_m^{\perp}|^2_{H^{1,0}} \leq (|f|_{L^2(Q)} + |h|_{L^2(Q)})(|\tilde{u}_m|_{L^2(Q)} + |u_m^0|_{L^2(Q)} + |\bar{u}_m|_{L^2(Q)}) + \rho_1,$$

where $\rho_1 > 0$ is given in (13). Hence,

$$\delta|u_m^{\perp}|^2_{H^{1,0}} \leq C(|u_m^{\perp}|_{H^{1,0}} + |u_m^0|_{H^{1,0}}),$$

which obviously implies that

$$|u_m^{\perp}|_{H^{1,0}} \leq C + (C + C|u_m^0|_{H^{1,0}})^{1/2}. \tag{19}$$

Furthermore, from the equality $u_m^0(x, t) = \varphi_m(x) + \psi_m(x, t)$ and the fact that, by Wirtinger's inequality,

$$|\psi_m|_{L^2} \leq C|\psi_{m_t}|_{L^2} \leq C|u_{m_t}|_{L^2} \leq C,$$

it follows that

$$|\psi_m|_{H^{1,1}} \leq C. \tag{20}$$

Hence, by (19)

$$|\psi_m|_{H^{1,1}} + |u_m^{\perp}|_{H^{1,0}} \leq C + (C + C|\varphi_m|_{H^{1,0}})^{1/2},$$

which immediately implies that

$$|\varphi_m|_{H^{1,0}} \to \infty,$$

and

$$\frac{|\psi_m|_{H^{1,1}} + |u_m^{\perp}|_{H^{1,0}}}{|\varphi_m|_{H^{1,0}}} \to 0, \qquad \text{as } m \to \infty.$$

Therefore, taking the limit in $L^2(Q)$, we easily derive that

$$v = \lim_{m\to\infty} v_m = \lim_{m\to\infty} \frac{u_m}{|u_m|_{H^{1,0}}} = \lim_{m\to\infty} \frac{\varphi_m}{|\varphi_m|_{H^{1,0}}} = \lim_{m\to\infty} z_m.$$

Now, observe that, by (19) and (20) and the fact that $|u_{m_t}|_{L^2}$ is bounded, we also have

$$\frac{|\psi_m|_{H^{1,1}} + |u_m^{\perp}|_{H^{1,1}}}{|\varphi_m|_{H^{1,0}}^{1/2}} \leq C.$$

Thus, by the Sobolev imbedding theorem (see e.g. [26, 27]), there is a function $d \in L^2(Q)$ such that a.e. $(x, t) \in Q$,

$$\frac{|\psi_m(x, t) + u_m^{\perp}(x, t)|}{|\varphi_m|_{H^1(0,\pi)}^{1/2}} \leq d(x, t).$$

Therefore,

$$\begin{aligned} \gamma(\cdot, u_m)u_m z_m &= \frac{1}{2}\gamma(\cdot, u_m)\frac{(u_m^2 + \varphi_m^2 - (u_m - \varphi_m)^2)}{|\varphi_m|_{H^{1,1}}} \\ &\geq -\frac{1}{2}\gamma(\cdot, u_m)\frac{(u_m^{\perp} + \psi_m)^2}{|\varphi_m|_{H^{1,1}}} \geq -\frac{1}{2}(c + b(x) + 1)(d(x, t))^2. \end{aligned}$$

Moreover, there is a function $e \in L^2([0, \pi])$ such that, for a.e. $x \in [0, \pi]$,

$$|z_m(x)| \leq e(x),$$

which implies that

$$f(\cdot, u_m)z_m \geq -\sigma(x)e(x) \qquad \text{a.e. on } Q.$$

Therefore, defining

$$l(x, t) := -\frac{1}{2}(c + b(x) + 1)(d(x, t))^2 - \sigma(x)e(x),$$

and using the decomposition of g given in (16), it follows that

$$\begin{aligned} g(x, u_m(x, t))z_m(x) &= \gamma(x, u_m(x, t))u_m(x, t)z_m(x, t) + f(x, u_m(x, t))z_m(x) \\ &\geq l(x, t), \end{aligned}$$

which concludes the proof of the above claim. The proof of Theorem 1 is complete.

We derive many results in [1, 3, 14, 31, 34] as special cases.

Corollary 1. Assume that conditions (7), (8) and (9) of Theorem 1 hold, and let $h \in L^2(0, \pi)$. Then the second order ordinary differential equation

$$\begin{aligned} -u_{xx}(x) &= n^2u(x) + g(x, u(x)) + h(x) \qquad \text{a.e. in } (0, \pi), \\ u(0) &= u(\pi) = 0, \end{aligned} \tag{21}$$

has a solution $u \in H^2(0, \pi)$ provided

$$\int_0^{\pi} g_+(x)v^+(x)\,dx - \int_0^{\pi} g_-(x)v^-(x)\,dx + \int_0^{\pi} h(x)v(x)\,dx > 0 \tag{22}$$

for all $v \in \text{Span}\{\sin nx\}\backslash\{0\}$.

Proof. Let us consider equation (4) with $h(x, t) = h(x)$, independent of t. Since (22) implies (10), it follows from Theorem 1 that equation (4) has a solution $u \in H^{2,1}(Q)$. If we multiply both sides of the equation

$$u_t(x, t) - u_{xx}(x, t) = n^2 u(x, t) + g(x, u(x, t)) + h(x)$$

by u_t and integrate by parts, we get, as in (12), that $|u_t|^2_{L^2(Q)} = (h, u_t)$. Since h is independent of t, it follows that $(h, u_t) = 0$. Hence $u_t = 0$; that is u is also independent of t. Thus, u is a solution of equation (21). The proof is complete.

4. MULTI-DIMENSIONAL EQUATIONS

Next, we shall look into extending our results to the more general multi-dimensional parabolic and telegraph equations

$$\begin{aligned} u_t - \operatorname{div}(A(x)\nabla u) &= \lambda_n u + g(x, u) + h(x, t) && \text{a.e. in } Q, \\ u(x, t) &= 0 && \text{on } \partial\Omega \times [0, 2\pi], \\ u(x, 0) &= u(x, 2\pi) && \text{on } \bar{\Omega}, \end{aligned} \tag{23}$$

and

$$\begin{aligned} au_t + u_{tt} - \operatorname{div}(A(x)\nabla u) &= \lambda_n u + g(x, u) + h(x, t) && \text{a.e. in } Q \\ u(x, t) &= 0 && \text{on } \partial\Omega \times [0, 2\pi], \\ u(x, 0) = u(x, 2\pi),\ u_t(x, 0) &= u_t(x, 2\pi) && \text{on } \bar{\Omega}. \end{aligned} \tag{24}$$

First we compare the nonlinearity (at double resonance) with two consecutive eigenvalues, and then we present results on crossing of not necessarily simple (higher) eigenvalue(s); it should be remembered that a complete description of the Fučik spectrum is not available for multi-dimensional equations.

Here the differential operator

$$\mathcal{L}(x, D)u = -\operatorname{div}(A(x)\nabla u)$$

is uniformly (strongly) elliptic and symmetric, with Lipschitz continuous entries on $\bar{\Omega}$; that is, the matrix $A(x)$ is symmetric and

$$\langle \xi, A(x)\xi \rangle > 0 \text{ for all } x \in \bar{\Omega} \text{ and all } \xi \in \mathbb{R}^N \setminus \{0\}.$$

Throughout this paper ∇u denotes the gradient of u with respect to the space variable $x \in \mathbb{R}^N$ only. As notation, let

$$\mathcal{A}u := u_t + \mathcal{L}(x, D)u,$$

and, respectively,

$$\mathcal{A}u := au_t + u_{tt} + \mathcal{L}(x, D)u,$$

with the following domains $\operatorname{Dom}(\mathcal{A}) = H^1_{2\pi}([0, 2\pi]; H^1_0(\Omega) \cap H^2(\Omega))$ (respectively $\operatorname{Dom}(\mathcal{A}) = H^2_{2\pi}([0, 2\pi]; H^1_0(\Omega) \cap H^2(\Omega)))$, where the subscript 2π indicates that the involved functions are 2π-periodic in the variable t. As before, $Gu(t, x) = g(x, u(t, x))$ is the Nemytskii operator associated with g.

If we consider the linear problems

$$\begin{aligned} u_t - \operatorname{div}(A(x)\nabla u) &= \lambda u && \text{a.e. in } Q, \\ u(x,t) &= 0 && \text{on } \partial\Omega \times [0, 2\pi], \\ u(x,0) &= u(x,2\pi) && \text{on } \bar{\Omega}, \end{aligned} \tag{25}$$

and

$$\begin{aligned} au_t + u_{tt} - \operatorname{div}(A(x)\nabla u) &= \lambda u && \text{a.e. in } Q \\ u(x,t) &= 0 && \text{on } \partial\Omega \times [0, 2\pi], \\ u(x,0) &= u(x,2\pi) && \text{on } \bar{\Omega}, \end{aligned} \tag{26}$$

and multiply both sides of the respective equations by u_t, then by Fubini's theorem, integration by parts and boundary and periodicity conditions, it follows, as previously, that $|u_t|^2_{L^2(Q)} = 0$, that is, $u_t = 0$ (respectively, $u_t = 0$ and $u_{tt} = 0$). This means that the eigenvalues and the eigenfunctions related to (25) and (26) are the same as those of the Dirichlet problem

$$\begin{aligned} -\operatorname{div}(A(x)\nabla u) &= \lambda u && \text{a.e. in } \Omega, \\ u(x) &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{27}$$

More precisely, there is a countable set of eigenvalues tending to ∞, and each eigenvalue is positive and has finite multiplicity. Writing the (distinct) eigenvalues in increasing order, we have

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots,$$

and the corresponding orthogonal system of eigenfunctions of (27) is a basis of $L^2(\Omega)$. We recall that these eigenfunctions enjoy the unique continuation property.

Let $u \in L^2([0, 2\pi]; H^1_0(\Omega))$. Then u has an eigenfunction expansion

$$u(\cdot, t) = \sum_{k=1}^{\infty} P_k u(\cdot, t),$$

where, for (a.e.) $t \in [0, 2\pi]$, $P_k u(\cdot, t)$ is the orthogonal projection onto the eigenspace $N(\mathfrak{L} - \lambda_k I)$. As in Section 2, for $u \in L^2(Q)$, we shall set $u(x,t) = \bar{u}(x,t) + u^0(x,t) + \tilde{u}(x,t)$, where

$$\bar{u}(\cdot, t) = \sum_{k=1}^{n-1} P_k u(\cdot, t), u^0(\cdot, t) = P_n u(\cdot, t), \tilde{u}(\cdot, t) = \sum_{k=n+1}^{\infty} P_k u(\cdot, t),$$

and $u^\perp = \bar{u} + \tilde{u}$.

Furthermore, as notation, let

$$\Lambda = \lambda_{n+1} - \lambda_n.$$

LEMMA 4. Let $\beta_\pm(x) \in L^\infty(\Omega)$ be functions such that

$$0 \le \beta_\pm(x) \le \Lambda \qquad \text{a.e. in } \Omega \tag{28}$$

and

$$\int_{w>0} (\Lambda - \beta_+)w^2 + \int_{w<0} (\Lambda - \beta_-)w^2 > 0 \tag{29}$$

for all $w \in N(\mathcal{A} - \lambda_{n+1}I)\setminus\{0\} = N(\mathcal{L} - \lambda_{n+1}I)\setminus\{0\}$.

Then there exist constants $\delta = \delta(\beta_\pm(x)) > 0$ and $\eta = \eta(\beta_\pm(x))$ such that for all $\chi_\pm \in L^\infty(Q)$ satisfying

$$0 \le \chi_\pm(x, t) \le \beta_\pm(x) + \delta \qquad \text{a.e. on } Q \tag{30}$$

and all $u \in \mathrm{Dom}(\mathcal{A})$, we have

$$(\mathcal{A}u - \lambda_n u + \chi_- u^- - \chi_+ u^+, \tilde{u} - \bar{u} - u^0) \ge \eta |u^\perp|^2_{H^{1,0}} \tag{31}$$

for the heat operator and

$$(\mathcal{A}u - \lambda_n u + \chi_- u^- - \chi_+ u^+, \tilde{u} - \bar{u} - u^0) \ge \eta |u^\perp|^2_{H^{1,0}} - |\tilde{u}_t|^2_{L^2} + |\bar{u}_t|^2_{L^2} + |u^0_t|^2_{L^2} \tag{32}$$

for the telegraph equation.

Proof. Let $u \in \mathrm{Dom}(\mathcal{A})$ and $\chi_\pm$ satisfy (30). For a.e. $t \in [0, 2\pi]$, the function $u(\cdot, t) \in H^2(\Omega) \cap H^1_0(\Omega)$ and $\chi_\pm(\cdot, t) \in L^\infty(\Omega)$. Then it follows from Lemma 1 in [30] (with $\Gamma_\pm = \beta_\pm$) that there exist constant $\delta = \delta(\beta_\pm) > 0$, $\eta = \eta(\beta_\pm)$ (independent of t) such that

$$(\mathcal{L}u - \lambda_n u + \chi_-(\cdot, t)u^- - \chi_+(\cdot, t)u^+, \tilde{u} - \bar{u} - u^0)_{L^2(\Omega)} \ge \eta |u^\perp|^2_{H^1(\Omega)}$$

for a.e. $t \in [0, 2\pi]$.

Therefore, taking the inner product of $\mathcal{A}u - \lambda_n u + \chi_- u^- - \chi_+ u^+$ with $\tilde{u} - \bar{u} - u^0$ in $L^2(Q)$, the conclusion follows easily since $(u_t, \tilde{u} - \bar{u} - u^0) = 0$ and $H^{1,0}(Q) = L^2([0, 2\pi]; H^1(\Omega))$. The proof is complete.

LEMMA 5. Assume that the conditions (28) and (29) hold. Then there exists a constant $\delta = \delta(\beta_\pm) > 0$ such that for all $\chi_\pm \in L^\infty(Q)$ and all $v \in \mathrm{Dom}(\mathcal{A})$, with $v_t = 0$, satisfying

$$\begin{gathered} 0 \le \chi_\pm(x, t) \le \beta_\pm(x) + \delta \qquad \text{a.e. on } Q \\ \mathcal{A}v - \lambda_n v + \chi_-(x, t)v^- - \chi_+(x, t)v^+ = 0 \qquad \text{a.e. on } Q, \end{gathered} \tag{33}$$

one has that $v \in N(\mathcal{A} - \lambda_n I) = N(\mathcal{L} - \lambda_n I)$.

Proof. We apply Lemma 4 which implies, by (33) and (31) in the case of the heat equation, and by (33), (32) and the fact that $v_t = 0$ in the case of the telegraph equation, that $v^\perp = 0$, that is, $v = v^0$. Since v is independent of t, then $v(\cdot) = P_n v(\cdot)$, that is, $v \in N(\mathcal{A} - \lambda_n I) = N(\mathcal{L} - \lambda_n I)$. The proof is complete.

THEOREM 2. Suppose that there exist a real number $r > 0$ and a function $B \in L^2(\Omega)$ such that

$$(\operatorname{sgn} u)g(x, u) \geq B(x) \tag{34}$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$ with $|u| \geq r$.

Moreover, assume that

$$0 \leq \liminf_{u \to \pm\infty} \frac{g(x, u)}{u} \leq \limsup_{u \to \pm\infty} \frac{g(x, u)}{u} \leq \beta_{\pm}(x) \leq \Lambda \tag{35}$$

uniformly for a.e. $x \in \Omega$ with $\beta_{\pm} \in L^2(\Omega)$ such that

$$\int_{w>0} (\Lambda - \beta_+)w^2 + \int_{w<0} (\Lambda - \beta_-)w^2 > 0 \tag{36}$$

for all $w \in N(\mathfrak{A} - \lambda_{n+1}I)\backslash\{0\}$. Then equation (23) (resp. equation (24)) has at least one solution $u \in \operatorname{Dom}(\mathfrak{A})$ provided

$$\int_Q g_+(x)v^+(x)\,\mathrm{d}x\,\mathrm{d}t - \int_Q g_-(x)v^-(x)\,\mathrm{d}x\,\mathrm{d}t + \int_Q h(x, t)v(x)\,\mathrm{d}x\,\mathrm{d}t > 0 \tag{37}$$

for all $v \in N(\mathfrak{A} - \lambda_n I)\backslash\{0\}$.

Proof. The proof is very similar to the proof of Theorem 1 of Section 3. Therefore we shall only sketch the main arguments and indicate the differences that appear in this case. We refer to the proof of Theorem 1 for details.

We prove the existence of a solution to the above problem(s) using again topological degree theory. Choose $\bar{\alpha} \in \mathbb{R}$ such that $0 < \bar{\alpha} < \Lambda$. So, since $(\mathfrak{A} - \bar{\alpha}I)^{-1}: H(Q) \to H(Q)$ is a compact operator (see [27]), it suffices to show that there is a constant $\rho > 0$ such that for every possible solution $u \in \operatorname{Dom}(\mathfrak{A})$ of the homotopy of problems

$$H(\tau, u) := u - \tau(\mathfrak{A} - \bar{\alpha}I)^{-1}(Gu - \bar{\alpha}u + h) = 0, \qquad \tau \in [0, 1],$$

we have

$$|u|_{H^{1,0}} < \rho.$$

Arguing by way of contradiction, we can assume that there exist sequences $(\tau_m) \subset (0, 1]$ and $(u_m) \subset \operatorname{Dom}(\mathfrak{A})$ with $|u_m|_{H^{1,0}} \geq m \in \mathbb{N}$ such that $h(\tau_m, u_m) = 0$. Also, if we set $v_m := u_m/|u_m|_{H^{1,0}}$, then (v_m) is bounded in $H^{1,1}(Q)$ and, for a subsequence, it converges in $L^2(Q)$ to some function v independent of t. Using the decomposition of g given in (16), (v_m) satisfies

$$\begin{aligned} &\mathfrak{A}v_m - \lambda_n v_m - (1 - \tau_m)\bar{\alpha}v_m - \tau_m \frac{g(x, u_n)}{|u_m|_{H^{1,0}}} - \tau_m \frac{h}{|u_m|_{H^{1,0}}} \\ &= \mathfrak{A}v_m - \lambda_n v_m - (1 - \tau_m)\bar{\alpha}v_m - \tau_m \gamma(x, u_m)v_m \\ &\quad - \tau_m \frac{f(x, u_m)}{|u_m|_{H^{1,0}}} - \tau_m \frac{h}{|u_m|_{H^{1,0}}} = 0. \end{aligned} \tag{38}$$

Taking the (weak) limit, we have

$$-\mathcal{Q}v(x) + \lambda_n v(x) + \chi_+(x,t)v^+(x) - \chi_-(x,t)v^-(x) = 0 \qquad \text{a.e. in } Q, \tag{39}$$

with

$$\begin{aligned} 0 \le \chi_+(x,t) &\le \limsup_{u\to+\infty} \frac{g(x,u)}{u} \le \beta_+(x) \qquad \text{a.e. on } \{(x,t) : v(x) > 0\} \\ 0 \le \chi_-(x,t) &\le \limsup_{u\to-\infty} \frac{g(x,u)}{u} \le \beta_-(x) \qquad \text{a.e. on } \{(x,t) : v(x) < 0\}. \end{aligned} \tag{40}$$

At this point we apply Lemma 5. In fact, by the above asssumptions, the inequalities (28) and (29) are satisfied. Also, by (40), it follows that (30) is trivially satisfied. Therefore, $v \in N(\mathcal{Q} - \lambda_n I)\backslash\{0\}$. As before, we use the notation

$$u_m^0(x,t) = \varphi_m(x) + \psi_m(x,t), \tag{41}$$

and we set

$$z_m(x) := \frac{\varphi_m(x)}{|\varphi_m|_{H^{1,0}}} \left(= \frac{\varphi_m(x)}{2\pi|\varphi_m|_{H^1(\Omega)}}\right).$$

Taking the inner product in $L^2(Q)$ of (38) with z_m, and using the fact that $\tau_m \ne 0$ and $(u_m, z_m) = |\varphi_m|^2_{L^2}|\varphi_m|^{-1}_{H^{1,0}}$, we get

$$0 \ge \int_Q g(x, u_m(x,t))z_m(x) + h(x,t)z_m(x)\,\mathrm{d}x\,\mathrm{d}t. \tag{42}$$

Proceeding as in the proof of Theorem 1 (with obvious modifications) we can show, using Lemma 4 instead of Lemma 3, that an estimate analogous to (19) holds, that is,

$$|u_m^\perp|_{H^{1,0}} \le C + (C + C|u_m^0|_{H^{1,0}})^{1/2}$$

and we can also prove that there exists a function $l \in L^1(Q)$ such that, for m sufficiently large,

$$g(x, u_m(x,t))z_m(x) \ge l(x,t).$$

Taking the lim inf as $m \to \infty$ in (42), and using Fatou's lemma and (35) we get a contradiction with the condition (37). The proof is complete. ■

We derive results of [33] (also [15] in the autonomous case), [14, 30, 32, 34] as special cases.

Corollary 2. Assume that conditions (34), (35) and (36) of Theorem 2 hold, and let $h \in L^2(\Omega)$. Then the elliptic partial differential equation

$$\begin{aligned} -\mathrm{div}(A(x)\,\nabla u) &= \lambda_n u + g(x,u) + h(x) \qquad &\text{a.e. in } \Omega, \\ u(x) &= 0 &\text{on } \partial\Omega, \end{aligned} \tag{43}$$

has a solution $u \in H^2(\Omega)$ provided

$$\int_\Omega g_+(x)v^+(x)\,\mathrm{d}x - \int_\Omega g_-(x)v^-(x)\,\mathrm{d}x + \int_\Omega h(x)v(x)\mathrm{d}x > 0$$

for all $v \in N(\mathcal{L} - \lambda_n I)\backslash\{0\}$.

Proof. The proof is similar to the proof of Corollary 1 of Section 3, using, at the appropriate step, equation (23) and Theorem 2 instead of equation (4) and Theorem 1. The proof is complete.

Now, we shall prove results on the crossing of (not necessarily *simple*) eigenvalue(s). The following lemma will be useful in obtaining these results.

LEMMA 6. Let $\theta \in L^\infty(\Omega)$ be a function such that $0 \le \theta(x) < \Lambda$ a.e. on Ω. Then, there exists a constant $\kappa = \kappa(\theta, \mathfrak{A}) > \Lambda$ such that for all $\chi_\pm \in L^\infty(Q)$ and all $v \in \operatorname{Dom}\mathfrak{A}$, with $v_t = 0$, satisfying

$$0 \le \chi_-(x, t) \le \theta(x), \quad 0 \le \chi_+(x, t) \le \kappa \qquad \text{a.e. on } \Omega,$$

$$-\mathfrak{A}v + \lambda_n v + \chi_+(x, t)v^+ - \chi_-(x, t)v^- = 0 \qquad \text{a.e. on } Q,$$

one has that $v \in N(\mathfrak{A} - \lambda_n I) = N(\mathcal{L} - \lambda_n I)$.

Proof. Suppose the conclusion of the lemma does not hold. Then, for each $m \in \mathbb{N}$, there exists functions $\chi_\pm^m \in L^\infty(Q)$ and $v_m \in \operatorname{Dom}(\mathfrak{A})$, with $v_{m_t} = 0$, satisfying

$$0 \le \chi_-^m(x, t) \le \theta(x), \quad 0 \le \chi_+^m(x, t) \le \Lambda + \frac{1}{m} \qquad \text{a.e. on } \Omega,$$

$$-\mathfrak{A}v_m + \lambda_n v_m + \chi_+^m(x, t)v_m^+ - \chi_-^m(x, t)v_m^- = 0 \qquad \text{a.e. on } Q,$$

such that $v_m \notin N(\mathfrak{A} - \lambda_n I)$.

Now, taking the inner product (in $L^2(Q)$) of the left-hand side of the above equation with $\tilde{v} - \bar{v} - v^0$, it follows from Lemma 4 (with $\beta_- = \theta$ and $\beta_+ = \Lambda$; recall that all eigenfunctions associated with λ_{n+1} change sign in Ω and enjoy the unique continuation property, so that (29) holds) that, for m sufficiently large, one has $v_m^\perp = 0$. But $v_{m_t} = 0$. Therefore, $v_m(\cdot) = P_n v_m(\cdot)$. This is a contradiction with the fact that $v_m \notin N(\mathfrak{A} - \lambda_n I)$ for each $m \in \mathbb{N}$. The proof is complete.

THEOREM 3. Suppose that there exist a real number $r > 0$ and a function $B \in L^2(\Omega)$ such that

$$(\operatorname{sgn} u)g(x, u) \ge B(x) \tag{44}$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$ with $|u| \ge r$.

Let $\theta \in \mathbb{R}$ be a constant such that $0 < \theta < \Lambda$ and set $\beta_-(x) := \theta$ and $\beta_+(x) := \kappa > \Lambda$, where $\kappa = \kappa(\theta, \mathfrak{A})$ is the constant given by Lemma 6. Assume that

$$0 \le \liminf_{u \to -\infty} \frac{g(x, u)}{u} \le \limsup_{u \to -\infty} \frac{g(x, u)}{u} \le \theta < \Lambda \tag{45}$$

and

$$0 \le \liminf_{u \to +\infty} \frac{g(x, u)}{u} \le \limsup_{u \to +\infty} \frac{g(x, u)}{u} \le \kappa \tag{46}$$

uniformly for a.e. $x \in \Omega$. Then equation (23) (resp. equation (24)) has at least one solution $u \in \mathrm{Dom}(\mathcal{A})$ provided

$$\int_Q g_+(x)v^+(x)\,\mathrm{d}x\,\mathrm{d}t - \int_Q g_-(x)v^-(x)\,\mathrm{d}x\,\mathrm{d}t + \int_Q h(x,t)v(x)\,\mathrm{d}x\,\mathrm{d}t > 0 \tag{47}$$

for all $v \in N(\mathcal{A} - \lambda_n I)\backslash\{0\}. = N(\mathcal{L} - \lambda_n I)\backslash\{0\}$.

Proof. The proof is very similar to the proofs of Theorems 1 and 2 and, therefore, we shall be very brief. So, reasoning as in the proofs of the previous theorems, we derive, by way of contradiction, that there exists $v \neq 0$, independent of t, satisfying, similarly to (39),

$$-\mathcal{A}v(x) + \lambda_n v(x) + \chi_+(x,t)v^+(x) - \chi_-(x,t)v^-(x) = 0 \qquad \text{a.e. in } Q \tag{48}$$

with

$$0 \le \chi_+(x,t) \le \limsup_{u\to+\infty} \frac{g(x,u)}{u} \le \kappa \qquad \text{a.e. on } \{(x,t) : v(x) > 0\}$$

$$0 \le \chi_-(x,t) \le \limsup_{u\to-\infty} \frac{g(x,u)}{u} \le \theta \qquad \text{a.e. on } \{(x,t) : v(x) < 0\}.$$

Here $\bar{\alpha} = \theta/2$. Since all assumptions of Lemma 6 are fulfilled, we derive that the function $v \in N(\mathcal{A} - \lambda_n I)\backslash\{0\}$. Thus, it follows from (48), that $\chi_\pm = 0$ a.e. on Q. This implies that $p_m \rightharpoonup 0$ in $L^2(Q)$ as $m \to \infty$. Therefore, proceeding as in the proof of Theorem 2 from (41) on, we get a contradiction to condition (47). The proof is complete. ■

Even in the case of elliptic partial differential equations, the following corollary on crossing eigenvalues (directly) for higher eigenvalues (i.e. $n \ge 2$) is still new.

COROLLARY 3. Assume that the conditions (44), (45) and (46) of Theorem 2 hold and let $h \in L^2(\Omega)$. Then the elliptic partial differential equation

$$-\mathrm{div}(A(x)\,\nabla u) = \lambda_n u + g(x,u) + h(x) \qquad \text{a.e. in } \Omega,$$

$$u(x) = 0 \qquad \text{on } \partial\Omega,$$

has a solution $u \in H^2(\Omega)$ provided

$$\int_\Omega g_+(x)v^+(x)\,\mathrm{d}x - \int_\Omega g_-(x)v^-(x)\,\mathrm{d}x + \int_\Omega h(x)v(x)\,\mathrm{d}x > 0$$

for all $v \in N(\mathcal{L} - \lambda_n I)\backslash\{0\}$.

Proof. The proof is similar to the proof of Corollary 1 of Section 3, using, at the appropriate step, equation (23) and Theorem 3 instead of equation (4) and Theorem 1. The proof is complete.

5. HIGHER ORDER EQUATIONS

In this section, we shall briefly indicate the conditions under which the results of the previous sections can be extended to higher order equations.

Here $\partial\Omega$ is of class C^{2m} and $\mathcal{L}(x, D)$ is a higher-order differential operator of the form

$$\mathcal{L}(x, D) = \sum_{|i|,|j| \le m} (-1)^{|i|+1} D^i(a_{ij}(x) D^j u)$$

which is assumed to be uniformly (strongly) elliptic and symmetric; i.e.

$$a_{ij} = a_{ji}, \qquad \sum_{|i|=|j|=m} a_{ij}(x)\xi^i \xi^j > c|\xi|^{2m} \qquad \text{for all } x \in \bar{\Omega} \text{ and all } \xi \in \mathbb{R}^N,$$

where $a_{ij} \in C^{|i|+|j|}(\bar{\Omega})$ for $0 \le |i|, |j| \le m$, and $c > 0$ is a constant. Here i and j denote multi-indices with $|i| = \sum_{q=1}^{N} i_q$ where i_q is a nonnegative integer for $1 \le q \le N$.

Throughout this section, we shall assume that the eigenfunctions of the realization of the linear operator $\mathcal{L}(x, D)$ on $H_0^m(\Omega) \cap H^{2m}(\Omega)$ enjoy the open set type unique continuation property; this is particularly true if the coefficients $a_{ij} \in C^\infty(\bar{\Omega})$.

We consider the higher order parabolic and telegraph equations

$$\begin{aligned} u_t - \mathcal{L}(x, D)u &= \lambda_n u + g(x, u) + h(x, t) && \text{a.e. in } Q, \\ u(\cdot, t) &\in H_0^m(\Omega) && \text{on } [0, 2\pi], \\ u(x, 0) &= u(x, 2\pi) && \text{on } \bar{\Omega}, \end{aligned} \tag{49}$$

and, with $0 \ne a \in \mathbb{R}$,

$$\begin{aligned} au_t + u_{tt} - \mathcal{L}(x, D)u &= \lambda_n u + g(x, u) + h(x, t) && \text{a.e. in } Q \\ u(\cdot, t) &\in H_0^m(\Omega) && \text{on } [0, 2\pi], \\ u(x, 0) = u(x, 2\pi),\ u_t(x, 0) &= u_t(x, 2\pi) && \text{on } \bar{\Omega}. \end{aligned} \tag{50}$$

First we compare the nonlinearity (at double resonance) with two consecutive eigenvalues, and then we present results on crossing of not necessarily simple eigenvalue(s); it should be remembered that a complete description of the Fučik spectrum is not available for higher-order (multi-dimensional) equations (see e.g. Fučik [3]). We shall present the results only in the case of equation (49), similar results may be derived for equation (50). As before, the following notations will be used:

$$\mathfrak{A}u := u_t - \mathcal{L}(x, D)u,$$

and, respectively,

$$\mathfrak{A}u := au_t + u_{tt} - \mathcal{L}(x, D)u,$$

with the following domains $\mathrm{Dom}(\mathfrak{A}) = H_{2\pi}^1([0, 2\pi];\ H_0^m(\Omega) \cap H^{2m}(\Omega))$ (respectively $\mathrm{Dom}(\mathfrak{A}) = H_{2\pi}^2([0, 2\pi]; H_0^m(\Omega) \cap H^{2m}(\Omega)))$, where the subscript 2π indicates that the involved functions are 2π-periodic in the variable t. Furthermore, Λ will denote the quantity

$$\Lambda = \lambda_{n+1} - \lambda_n.$$

THEOREM 4. Suppose that there exist a real number $r > 0$ and a function $B \in L^2(\Omega)$ such that

$$(\operatorname{sgn} u) g(x, u) \geq B(x) \tag{51}$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$ with $|u| \geq r$.

Moreover, assume that

$$0 \leq \liminf_{u \to \pm\infty} \frac{g(x, u)}{u} \leq \limsup_{u \to \pm\infty} \frac{g(x, u)}{u} \leq \beta_\pm(x) < \Lambda \tag{52}$$

uniformly for a.e. $x \in \Omega$ with $\beta_\pm \in L^2(\Omega)$ such that

$$\int_{w>0} (\Lambda - \beta_+) w^2 + \int_{w<0} (\Lambda - \beta_-) w^2 > 0 \tag{53}$$

for all $w \in N(\mathcal{A} - \lambda_{n+1} I) \setminus \{0\}$. Then equation (49) has at least one solution $u \in \operatorname{Dom}(\mathcal{A})$ provided

$$\int_Q g_+(x) v^+(x) \, dx \, dt - \int_Q g_-(x) v^-(x) \, dx \, dt + \int_Q h(x, t) v(x) \, dx \, dt > 0 \tag{54}$$

for all $v \in N(\mathcal{A} - \lambda_n I) \setminus \{0\}$.

We also have a result on crossing at (not necessarily *simple*) eigenvalues. In order to obtain this result, we shall assume that, in addition to the conditions at the beginning of this section, all the eigenfunctions associated with the eigenvalue λ_{n+1} change sign in Ω (see e.g. the proof of Lemma 6).

THEOREM 3. Suppose that there exist a real number $r > 0$ and a function $B \in L^2(\Omega)$ such that

$$(\operatorname{sgn} u) g(x, u) \geq B(x) \tag{55}$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$ with $|u| \geq r$.

Let $\theta \in \mathbb{R}$ be a constant such that $0 < \theta < \Lambda$ and set $\beta_-(x) := \theta$ and $\beta_+(x) := \kappa > \Lambda$, where $\kappa = \kappa(\theta, \mathcal{A})$ is the constant given by a result similar to Lemma 6 adapted to the case of higher order equations.

Assume that

$$0 \leq \liminf_{u \to -\infty} \frac{g(x, u)}{u} \leq \limsup_{u \to -\infty} \frac{g(x, u)}{u} \leq \theta < \Lambda \tag{56}$$

and

$$0 \leq \liminf_{u \to +\infty} \frac{g(x, u)}{u} \leq \limsup_{u \to +\infty} \frac{g(x, u)}{u} \leq \kappa \tag{57}$$

uniformly for a.e. $x \in \Omega$. Then equation (49) has at least one solution $u \in \operatorname{Dom}(\mathcal{A})$ provided

$$\int_Q g_+(x) v^+(x) \, dx \, dt - \int_Q g_-(x) v^-(x) \, dx \, dt + \int_Q h(x, t) v(x) \, dx \, dt > 0 \tag{58}$$

for all $v \in N(\mathcal{A} - \lambda_n I) \setminus \{0\} = N(\mathcal{L} - \lambda_n I) \setminus \{0\}$.

We shall omit the proofs of these results since they are very similar to those in the previous section. Of course, several corollaries concerning the corresponding nonlinear higher order elliptic partial differential equations are easy to derive. We refer to Section 4 for similar details.

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