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Solvability of some third-order boundary value problems with asymmetric unbounded nonlinearities

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Abstract

In this paper, we present existence and location results for the third-order separated boundary value problems

$$u'''(t) = f(t, u(t), u'(t), u''(t)),$$

with the boundary conditions

$$u(a) = A, \quad u''(a) = B, \quad u''(b) = C$$

or

$$u(a) = A, \quad c_1 u'(a) - c_2 u''(a) = B, \quad c_3 u'(b) + c_4 u''(b) = C,$$

with $c_1, c_2, c_3, c_4 \in \mathbb{R}^+$ and $A, B, C \in \mathbb{R}$.

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We assume $f : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function satisfying one-sided Nagumo-type condition which allows an asymmetric unbounded behaviour. The arguments used concern Leray–Schauder degree and lower and upper solution techniques.

MSC: 34B15

Keywords: Third-order separated boundary value problems; One-sided Nagumo condition; Lower and upper solutions; A priori estimates; Leray–Schauder degree

1. Introduction

The purpose of this paper is to study the third-order differential equation

$$u'''(t) = f(t, u(t), u'(t), u''(t)) \quad \text{for } t \in [a, b], \tag{1}$$

with the following types of boundary conditions

$$u(a) = A, \quad u''(a) = B, \quad u''(b) = C \tag{2}$$

or

$$\begin{aligned} &u(a) = A, \\ &c_1 u'(a) - c_2 u''(a) = B, \\ &c_3 u'(b) + c_4 u''(b) = C, \end{aligned} \tag{3}$$

where $A, B, C \in \mathbb{R}$, $c_i > 0$, $i = 1, \ldots, 4$. The function $f : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and satisfies a growth condition from above but no restriction from below. This asymmetric type of unboundedness can appear since *f* is assumed to satisfy an one-sided Nagumo growth condition which creates some control from above but none from below. More precisely, we assume that there is a positive continuous function φ such that

$$f(t, x, y, z) \leqslant \varphi(|z|), \quad \forall (t, x, y, z) \in E, \tag{4}$$

on some given subset $E \subset [a, b] \times \mathbb{R}^3$, and

$$\int_0^{+\infty} \frac{\xi}{\varphi(\xi)} \, \mathrm{d}\xi = +\infty.$$

Some boundedness of Nagumo-type [15] seem to play a key role in this sort of studies. In fact in [11] it is shown for some second-order problems that existence results are not guaranteed if no Nagumo condition is assumed.

On the other hand in most of the available literature, for example, [1–5,7–9,13,14] existence results are established when the nonlinear term *f* verifies a growth condition of two-sided Nagumo-type, that is, such that

$$|f(t, x, y, z)| \leqslant \varphi(|z|), \quad \forall (t, x, y, z) \in E,$$

which is obviously more restrictive than (4). So the existence and location results obtained in this work for problems (1)–(2) and (1)–(3) improve the existent ones, because they can be applied for unbounded nonlinearities, as it can be seen in the examples containedd in Section 5.

The arguments used rely on degree theory, [12], and lower and upper solutions method and an a priori estimate on u'' is established in Lemma 2, depending on the boundary values $u''(a)$ and $u''(b)$. More precisely, for every $\rho > 0$ it is possible to establish an estimate for the second derivative of the solutions u of Eq. (1) that satisfy

$$u''(a) \leqslant \rho \quad \text{and} \quad u''(b) \geqslant -\rho.$$

The above referred one-sided Nagumo-type condition plays there an important role.

A priori estimates depending on the values of $u''(a)$ and $u''(b)$ can also be found in [6,10] for second and third-order differential equations, respectively. Those estimates can be used since the bounds for $u''(a)$ and $u''(b)$ are trivially satisfied in the boundary problems there considered. However, Lemma 2 of [10] cannot be applied to the problems we present in this paper. In fact, the respective boundary conditions imply that some more care has to be taken. Therefore, we prove Lemma 2 that generalizes the result contained in Lemma 2 of [10] and Theorem 4 improves the main result presented in [10]. Theorem 6 contains an existence result for some Sturm–Liouville-type boundary value problems.

Observe that problem (1)–(2) is not a particular case of (1)–(3) since the constants c_i, $i = 1, \ldots, 4$, in the boundary conditions (3) are positive.

An analogous existence result still holds if it is assumed an one-sided Nagumo condition with reversed inequality in (4), i.e.

$$f(t, x, y, z) \geqslant -\varphi(|z|), \quad \forall (t, x, y, z) \in E.$$

2. A priori estimate

The one-sided Nagumo condition plays an important role in the arguments. We observe that it does not depend on the boundary values.

Definition 1. Given a subset $E \subset [a, b] \times \mathbb{R}^3$, a function $f : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ is said to satisfy one-sided Nagumo-type condition in E if there exists $\varphi \in C(\mathbb{R}_0^+, [k, +\infty[)$, with $k > 0$, such that

$$f(t, x, y, z) \leqslant \varphi(|z|), \tag{5}$$

for all $t \in [a, b]$ and all $(t, x, y, z) \in E$, and

$$\int_0^{+\infty} \frac{s}{\varphi(s)}\, \mathrm{d}s = +\infty. \tag{6}$$

In the next result we establish, under adequate conditions, an a priori bound for the second derivative u'' of the solutions of Eq. (1).

Lemma 2. *Let $\Gamma_1, \Gamma_2, \gamma_1, \gamma_2 \in C([a,b],\mathbb{R})$ satisfy*

$$\Gamma_1(t) \leqslant \Gamma_2(t) \quad \text{and} \quad \gamma_1(t) \leqslant \gamma_2(t), \quad \forall t \in [a,b]$$

and consider the set

$$E = \{(t,x,y,z) \in [a,b] \times \mathbb{R}^3 : \Gamma_1(t) \leqslant x \leqslant \Gamma_2(t),\ \gamma_1(t) \leqslant y \leqslant \gamma_2(t)\}.$$

Let $f : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function that satisfies one-sided Nagumo-type condition in E.

Then, for every $\rho > 0$ there exists $R > 0$ (depending on γ_1, γ_2, φ, ρ) such that for every solution $u(t)$ of

$$u'''(t) = f(t, u(t), u'(t), u''(t)) \tag{7}$$

with

$$u''(a) \leqslant \rho, \quad u''(b) \geqslant -\rho \tag{8}$$

and

$$\Gamma_1(t) \leqslant u(t) \leqslant \Gamma_2(t), \quad \gamma_1(t) \leqslant u'(t) \leqslant \gamma_2(t), \quad \forall t \in [a,b], \tag{9}$$

we have

$$\|u''\|_\infty < R.$$

Proof. Consider $\rho > 0$.

Let u be a solution of (7) such that (8) and (9) hold. Define the non-negative number

$$r := \max\left\{\frac{\gamma_2(b) - \gamma_1(a)}{b-a}, \frac{\gamma_2(a) - \gamma_1(b)}{b-a}\right\}.$$

Assume that $\rho \geqslant r$ and suppose, by contradiction, that $|u''(t)| > \rho$ for every $t \in]a,b[$. If $u''(t) > \rho$, for every $t \in]a,b[$, then we obtain the following contradiction:

$$\begin{aligned}\gamma_2(b) - \gamma_1(a) \geqslant u'(b) - u'(a) &= \int_a^b u''(\tau)\,\mathrm{d}\tau > \int_a^b \rho\,\mathrm{d}\tau \\ &\geqslant \int_a^b r\,\mathrm{d}\tau \geqslant \gamma_2(b) - \gamma_1(a).\end{aligned}$$

If $u''(t) < -\rho$, for every $t \in]a,b[$, a similar contradiction can be derived. So, there is $t \in]a,b[$ such that $|u''(t)| \leqslant \rho$. By (6) we can take $R_1 > \rho$ such that

$$\int_\rho^{R_1} \frac{s}{\varphi(s)}\,\mathrm{d}s > \max_{t\in[a,b]} \gamma_2(t) - \min_{t\in[a,b]} \gamma_1(t). \tag{10}$$

If $|u''(t)| \leqslant \rho$, for every $t \in [a,b]$, then we have trivially $|u''(t)| < R_1$. If not, we can take $t_1 \in [a,b[$ such that $u''(t_1) < -\rho$ or $t_1 \in]a,b]$ such that $u''(t_1) > \rho$. Suppose the first case holds. By (8) we can consider $t_1 < \widehat{t_1} \leqslant b$ such that

$$u''(\widehat{t_1}) = -\rho, \quad u''(t) < -\rho, \quad \forall t \in [t_1, \widehat{t_1}[.$$

Applying a convenient change of variable we have, by (5) and (10),

$$\begin{aligned}\int_{-u''(\widehat{t_1})}^{-u''(t_1)} \frac{s}{\varphi(s)}\,\mathrm{d}s &= \int_{\widehat{t_1}}^{t_1} \frac{-u''(t)}{\varphi(-u''(t))}[-u'''(t)]\,\mathrm{d}t \\ &= \int_{t_1}^{\widehat{t_1}} \frac{f(t,u(t),u'(t),u''(t))}{\varphi(-u''(t))}[-u''(t)]\,\mathrm{d}t \\ &\leqslant \int_{t_1}^{\widehat{t_1}} [-u''(t)]\,\mathrm{d}t = u'(t_1) - u'(\widehat{t_1}) \\ &\leqslant \max_{t\in[a,b]} \gamma_2(t) - \min_{t\in[a,b]} \gamma_1(t) < \int_\rho^{R_1} \frac{s}{\varphi(s)}\,\mathrm{d}s. \end{aligned} \tag{11}$$

Hence $u''(t_1) > -R_1$. Since t_1 can be taken arbitrarily as long as $u''(t_1) < -\rho$ we can conclude that we have, for every $t \in [a, b[$ such that $u''(t) < -\rho$,

$$u''(t) > -R_1.$$

By a similar way, it can be proved that $u''(t) < R_1$, for every $t \in]a, b]$ such that $u''(t) > \rho$. Therefore,

$$|u''(t)| < R_1, \quad \forall t \in [a, b]. \tag{12}$$

Consider now the case $r > \rho$ and take R_2 such that

$$\int_r^{R_2} \frac{s}{\varphi(s)}\,\mathrm{d}s > \max_{t\in[a,b]} \gamma_2(t) - \min_{t\in[a,b]} \gamma_1(t).$$

By (8) we cannot have $|u''(t)| > r$, for every $t \in [a, b]$. So, there is $t \in [a, b]$ such that $|u''(t)| \leqslant r$.

If $|u''(t)| \leqslant r$, for every $t \in [a, b]$, then it is trivial that $|u''(t)| < R_2$. If not, we can take $t_1 \in [a, b[$ such that $u''(t_1) < -r$ or $t_1 \in]a, b]$ such that $u''(t_1) > r$. Suppose the first case holds. By (8) we can consider $t_1 < \widehat{t_1} < b$ such that

$$u''(\widehat{t_1}) = -r, \quad u''(t) < -r, \ \forall t \in [t_1, \widehat{t_1}[.$$

Then computations similar to (11) with ρ and R_1 replaced by r and R_2, respectively, yield

$$\int_{-u''(\widehat{t_1})}^{-u''(t_1)} \frac{s}{\varphi(s)}\,\mathrm{d}s < \int_r^{R_2} \frac{s}{\varphi(s)}\,\mathrm{d}s$$

and so $u''(t_1) > -R_2$. Arguing in a similar way we derive as in (12):

$$|u''(t)| < R_2, \quad \forall t \in [a, b].$$

Taking $R = \max\{R_1, R_2\}$ we have $\|u''\|_\infty < R$. $\square$

3. Two-point boundary value problem

Consider problem (1)–(2) where $f : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function.

By a solution of the above problem we mean a function $u \in C^3([a, b])$ satisfying the differential equation (1) and the boundary conditions (2).

Upper and lower solutions will be an important tool to obtain a priori bounds on u and u'. For this problem we define them as follows.

Definition 3. A function $\alpha \in C^3(]a, b[) \cap C^2([a, b])$ is said to be a lower solution of problem (1)–(2) if

$$\alpha'''(t) \geqslant f(t, \alpha(t), \alpha'(t), \alpha''(t)),$$

for $t \in]a, b[$ and

$$\alpha(a) \leqslant A, \quad \alpha''(a) \geqslant B, \quad \alpha''(b) \leqslant C.$$

A function $\beta \in C^3(]a, b[) \cap C^2([a, b])$ is said to be an upper solution of problem (1)–(2) if it satisfies the reversed inequalities.

Next theorem provides an existence and localization result for problem (1)–(2), since we prove the existence of at least a solution and give some information about the strip where the solution and the first derivative lie.

Theorem 4. *Assume that there exist* $\alpha, \beta \in C^3(]a, b[) \cap C^2([a, b])$ *lower and upper solutions of problem* (1)–(2), *respectively, such that*

$$\alpha'(t) \leqslant \beta'(t), \quad \forall t \in [a, b]. \tag{13}$$

Define the set

$$E_* = \{(t, x, y, z) \in [a, b] \times \mathbb{R}^3 : \alpha(t) \leqslant x \leqslant \beta(t), \ \alpha'(t) \leqslant y \leqslant \beta'(t)\}. \tag{14}$$

Let $f : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ *be a continuous function that satisfies the one-sided Nagumo-type condition in* E_* *and such that*

$$f(t, \alpha(t), y, z) \geqslant f(t, x, y, z) \geqslant f(t, \beta(t), y, z), \tag{15}$$

for $(t, y, z) \in [a, b] \times \mathbb{R}^2$ *and* $\alpha(t) \leqslant x \leqslant \beta(t)$.

Then problem (1)–(2) *has at least a solution* $u \in C^3([a, b])$ *such that*

$$\alpha(t) \leqslant u(t) \leqslant \beta(t) \quad \textit{and} \quad \alpha'(t) \leqslant u'(t) \leqslant \beta'(t), \ \ \forall t \in [a, b].$$

Remark 1. The relation $\alpha(t) \leqslant \beta(t)$ is obtained by integrating (13) and using the boundary conditions of Definition 3.

Proof. For $\lambda \in [0, 1]$, consider the modified problem

$$u'''(t) = \lambda f(t, \xi(t, u(t)), \xi_*(t, u'(t)), u''(t)) + u'(t) - \lambda \xi_*(t, u'(t)), \tag{16}$$

with the boundary conditions

$$u(a) = \lambda A,$$
$$u''(a) = \lambda[B + u'(a) - \xi_*(a, u'(a))],$$
$$u''(b) = \lambda[C - u'(b) + \xi_*(b, u'(b))], \tag{17}$$

where the functions $\xi, \xi_* : \mathbb{R}^2 \to \mathbb{R}$ are given by

$$\xi(t, x) = \begin{cases} \beta(t) & \text{if } x > \beta(t), \\ x & \text{if } \alpha(t) \leqslant x \leqslant \beta(t), \\ \alpha(t) & \text{if } x < \alpha(t), \end{cases}$$

and

$$\xi_*(t, y) = \begin{cases} \beta'(t) & \text{if } y > \beta'(t), \\ y & \text{if } \alpha'(t) \leqslant y \leqslant \beta'(t), \\ \alpha'(t) & \text{if } y < \alpha'(t). \end{cases} \tag{18}$$

Take $r_1 > 0$ such that, for every $t \in [a, b]$,

$$-r_1 < \alpha'(t) \leqslant \beta'(t) < r_1, \tag{19}$$

$$f(t, \alpha(t), \alpha'(t), 0) - r_1 - \alpha'(t) < 0, \tag{20}$$

$$f(t, \beta(t), \beta'(t), 0) + r_1 - \beta'(t) > 0, \tag{21}$$

$$B - \alpha'(a) < r_1, \quad -C - \alpha'(b) < r_1,$$
$$\beta'(a) - B < r_1, \quad C + \beta'(b) < r_1. \tag{22}$$

Step 1: Every solution u of problem (16)–(17) satisfies in $[a, b]$:

$$|u'(t)| < r_1 \quad \text{and} \quad |u(t)| < r_0,$$

with r_1 given above and $r_0 = |A| + r_1(b - a)$, independent of $\lambda \in [0, 1]$.

Let u be a solution of problem (16)–(17). Assume, by contradiction, that there exists $t \in [a, b]$ such that either $u'(t) \geqslant r_1$ or $u'(t) \leqslant -r_1$. Suppose that the first case holds. Define

$$\max_{t\in[a,b]} u'(t) := u'(t_0) \quad (\geqslant r_1 > 0).$$

If $t_0 \in]a, b[$, then $u''(t_0) = 0$ and $u'''(t_0) \leqslant 0$. Hence, for $\lambda \in]0, 1]$, by (15), (19) and (21) we have the following contradiction:

$$\begin{aligned} 0 &\geqslant u'''(t_0) \\ &= \lambda f(t_0, \xi(t_0, u(t_0)), \xi_*(t_0, u'(t_0)), u''(t_0)) + u'(t_0) - \lambda\xi_*(t_0, u'(t_0)) \\ &= \lambda f(t_0, \xi(t_0, u(t_0)), \beta'(t_0), 0) + u'(t_0) - \lambda\beta'(t_0) \\ &\geqslant \lambda f(t_0, \xi(t_0, u(t_0)), \beta'(t_0), 0) + r_1 - \lambda\beta'(t_0) \\ &\geqslant \lambda[f(t_0, \beta(t_0), \beta'(t_0), 0) + r_1 - \beta'(t_0)] > 0 \end{aligned}$$

and, for $\lambda = 0$,

$$0 \geqslant u'''(t_0) = u'(t_0) \geqslant r_1 > 0.$$

If $t_0 = a$ then

$$\max_{t \in [a,b]} u'(t) := u'(a) \quad (\geqslant r_1 > 0),$$

and $u''(a^+) = u''(a) \leqslant 0$. If $\lambda = 0$ then $u''(a) = 0$ and so $u'''(a) \leqslant 0$. Therefore, the above computations with t_0 replaced by a yield a contradiction. For $\lambda \in]0, 1]$, by (18) and (22), we get the following contradiction:

$$\begin{aligned} 0 \geqslant u''(a) &= \lambda[B + u'(a) - \xi_*(a, u'(a))] = \lambda[B + u'(a) - \beta'(a)] \\ &> \lambda[u'(a) - r_1] \geqslant 0. \end{aligned}$$

The case $t_0 = b$ is analogous. Thus, $u'(t) < r_1$, for every $t \in [a, b]$. In a similar way, we prove that $u'(t) > -r_1$, for every $t \in [a, b]$.

Furthermore, since $u(a) = \lambda A$, the estimate $|u(t)| < r_0$, where $r_0 := |A| + r_1(b - a)$, is easily obtained by integration.

Step 2: There exists $R > 0$ such that every solution u of problem (16)–(17) satisfies in $[a, b]$

$$|u''(t)| < R$$

independent of $\lambda \in [0, 1]$.

Consider the set

$$E_{**} := \{(t, x, y, z) \in [a, b] \times \mathbb{R}^3 : -r_0 \leqslant x \leqslant r_0, -r_1 \leqslant y \leqslant r_1\},$$

and the function $F_\lambda : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$F_\lambda(t, x, y, z) = \lambda f(t, \xi(t, x), \xi_*(t, y), z) + y - \lambda \xi_*(t, y). \tag{23}$$

Since f satisfies one-sided Nagumo-type condition in E_*, consider the function $\varphi \in C(\mathbb{R}_0^+, [k, +\infty[)$ such that (5) and (6) hold with E replaced by E_*. Thus, for $(t, x, y, z) \in E_{**}$, we have, by (18) and (19),

$$\begin{aligned} F_\lambda(t, x, y, z) &= \lambda f(t, \xi(t, x), \xi_*(t, y), z) + y - \lambda \xi_*(t, y) \\ &\leqslant \lambda \varphi(|z|) + r_1 - \alpha'(t) \leqslant \varphi(|z|) + 2r_1. \end{aligned}$$

Take $\overline{\varphi}(z) := \varphi(|z|) + 2r_1$ then

$$\int_0^{+\infty} \frac{s}{\overline{\varphi}(s)}\, \mathrm{d}s = \int_0^{+\infty} \frac{s}{\varphi(|s|) + 2r_1}\, \mathrm{d}s \geqslant \frac{1}{1 + 2r_1/k} \int_0^{+\infty} \frac{s}{\varphi(|s|)}\, \mathrm{d}s,$$

and so $\overline{\varphi}(z)$ satisfies (6). Therefore, F_λ satisfies the one-sided Nagumo-type condition in E_{**} with $\varphi(z)$ replaced by $\overline{\varphi}(z)$, independent of λ.

Moreover, for $\rho := 2r_1$ every solution u of (16)–(17) satisfies

$$u''(a) = \lambda[B + u'(a) - \xi_*(a, u'(a))] \leqslant \lambda[B + u'(a) - \alpha'(a)] \leqslant 2r_1 = \rho,$$

$$u''(b) = \lambda[C - u'(b) + \xi_*(b, u'(b))] \geqslant \lambda[C - u'(b) + \alpha'(b)] \geqslant -2r_1 = -\rho.$$

Defining

$$\Gamma_1(t) := -r_0 = -|A| - r_1(b-a),$$

$$\Gamma_2(t) := r_0 = |A| + r_1(b-a),$$

$$\gamma_1(t) := -r_1 \quad \text{and} \quad \gamma_2(t) := r_1,$$

the hypotheses of Lemma 2 are satisfied with E replaced by E_{**} so there exists $R > 0$, depending on r_1 and φ, such that $|u''(t)| < R$, for every $t \in [a, b]$. As r_1 and φ do not depend on λ, we conclude that the estimate $|u''(t)| < R$ is also independent of λ.

Step 3: For $\lambda = 1$, there exists at least one solution $u_1(t)$ for problem (16)–(17).

Define the operators

$$\mathscr{L} : C^3([a, b]) \subset C^2([a, b]) \longmapsto C([a, b]) \times \mathbb{R}^3$$

and

$$\mathscr{N}_\lambda : C^2([a, b]) \longmapsto C([a, b]) \times \mathbb{R}^3$$

by

$$\mathscr{L}u = (u''' - u', u(a), u''(a), u''(b)),$$

and

$$\mathscr{N}_\lambda\, u = (\lambda f(t, \xi(t, u(t)), \xi_*(t, u'(t)), u''(t)) - \lambda\xi_*(t, u'(t)), A_\lambda, B_\lambda, C_\lambda),$$

with

$$A_\lambda = \lambda A,$$

$$B_\lambda = \lambda[B + u'(a) - \xi_*(a, u'(a))],$$

$$C_\lambda = \lambda[C - u'(b) + \xi_*(b, u'(b))].$$

Observe that $\mathscr{L}$ has a compact inverse. Therefore, we can consider the completely continuous operator

$$\mathscr{T}_\lambda : (C^2([a, b]), \mathbb{R}) \longmapsto (C^2([a, b]), \mathbb{R})$$

defined by

$$\mathscr{T}_\lambda(u) = \mathscr{L}^{-1}\mathscr{N}_\lambda(u).$$

For R given by Step 2, take the set

$$\Omega = \{x \in C^2([a, b]) : \|x\|_\infty < r_0, \|x'\|_\infty < r_1, \|x''\|_\infty < R\}.$$

By Steps 1 and 2, for every u solution of (16)–(17), $u \notin \partial\Omega$ and so the degree $d(I-\mathscr{T}_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0, 1]$ and due to the invariance under homotopy

$$d(I - \mathscr{T}_0, \Omega, 0) = d(I - \mathscr{T}_1, \Omega, 0).$$

Since the equation $\mathscr{T}_0(x) = x$, equivalent to the problem

$$\begin{cases} u'''(t) - u'(t) = 0, \\ u(a) = 0, \quad u''(a) = 0, \quad u''(b) = 0, \end{cases}$$

has only the trivial solution then, by the degree theory,

$$d(I - \mathscr{T}_0, \Omega, 0) = \pm 1.$$

So, the equation $\mathscr{T}_1(x) = x$ has at least a solution and therefore the equivalent problem

$$u'''(t) = f(t, \xi(t, u(t)), \xi_*(t, u'(t)), u''(t)) + u'(t) - \xi_*(t, u'(t)),$$

$$u(a) = A,$$

$$u''(a) = B + u'(a) - \xi_*(a, u'(a)),$$

$$u''(b) = C - u'(b) + \xi_*(b, u'(b)),$$

has at least a solution $u_1(t)$ in Ω.

Step 4: The function $u_1(t)$ is a solution of (1)–(2), too.

Indeed, the solution $u_1(t)$ of the above problem will also be a solution of problem (1)–(2) since it satisfies

$$\alpha(t) \leqslant u_1(t) \leqslant \beta(t), \quad \alpha'(t) \leqslant u_1'(t) \leqslant \beta'(t), \quad \forall t \in [a, b].$$

Suppose, by contradiction, that there exists $t \in [a, b]$ such that

$$\alpha'(t) > u_1'(t)$$

and define

$$\min_{t \in [a,b]} [u_1'(t) - \alpha'(t)] := u_1'(t_2) - \alpha'(t_2) < 0.$$

If $t_2 \in]a, b[$ then $u_1''(t_2) = \alpha''(t_2)$ and $u_1'''(t_2) - \alpha'''(t_2) \geqslant 0$. Therefore, by (15) and Definition 3, we have the contradiction:

$$\begin{aligned} 0 &\leqslant u_1'''(t_2) - \alpha'''(t_2) \\ &\leqslant f(t_2, \xi(t_2, u_1(t_2)), \alpha'(t_2), u_1''(t_2)) + u_1'(t_2) - \alpha'(t_2) \\ &\quad - f(t_2, \alpha(t_2), \alpha'(t_2), \alpha''(t_2)) \\ &< f(t_2, \xi(t_2, u_1(t_2)), \alpha'(t_2), \alpha''(t_2)) - f(t_2, \alpha(t_2), \alpha'(t_2), \alpha''(t_2)) \leqslant 0. \end{aligned} \tag{24}$$

If $t_2 = a$ we have

$$\min_{t \in [a,b]} [u_1'(t) - \alpha'(t)] := u_1'(a) - \alpha'(a) < 0$$

and

$$u_1''(a) - \alpha''(a) = u_1''(a^+) - \alpha''(a^+) \geqslant 0.$$

By Definition 3 this yields a contradiction

$$\begin{aligned}\alpha''(a) \leqslant u_1''(a) &= B + u_1'(a) - \xi_*(a, u_1'(a)) \\ &= B + u_1'(a) - \alpha'(a) < B \leqslant \alpha''(a).\end{aligned}$$

Then $t_2 \neq a$ and, in a similar way, we prove that $t_2 \neq b$. Thus

$$\alpha'(t) \leqslant u_1'(t), \quad \forall t \in [a, b].$$

Using an analogous technique, it can be obtained that $u_1'(t) \leqslant \beta'(t)$, for every $t \in [a, b]$. From

$$\alpha'(t) \leqslant u_1'(t) \leqslant \beta'(t), \quad \forall t \in [a, b]$$

by integration we have

$$\alpha(t) \leqslant u_1(t) \leqslant \beta(t), \quad \forall t \in [a, b].$$

Therefore, u_1 is in fact a solution of problem (1)–(2). □

4. Separated boundary value problem

Consider now problem (1)–(3) with $f : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ a continuous function and $c_i \in \mathbb{R}^+$, for $i = 1, \ldots, 4$, $A, B, C \in \mathbb{R}$.

The following lower and upper solutions definition will be an essential tool in the approach that follows.

Definition 5. Consider $c_i \in \mathbb{R}^+$, for $i = 1, \ldots, 4$ and $A, B, C \in \mathbb{R}$.

A function $\alpha \in C^3(]a, b[) \cap C^2([a, b])$ is said to be a lower solution of problem (1)–(3) if

$$\alpha'''(t) \geqslant f(t, \alpha(t), \alpha'(t), \alpha''(t)),$$

for $t \in]a, b[$ and

$$\begin{aligned}&\alpha(a) \leqslant A, \\ &c_1\alpha'(a) - c_2\alpha''(a) \leqslant B, \\ &c_3\alpha'(b) + c_4\alpha''(b) \leqslant C. \end{aligned} \tag{25}$$

A function $\beta \in C^3(]a, b[) \cap C^2([a, b])$ is said to be an upper solution of problem (1)–(3) if it satisfies the reversed inequalities.

We are now in position to state and prove an existence and location result for problem (1)–(3).

Theorem 6. *Assume that there are* $\alpha, \beta \in C^3(]a,b[) \cap C^2([a,b])$ *lower and upper solutions of problem* (1)–(3), *respectively, such that*

$$\alpha'(t) \leqslant \beta'(t), \quad \forall t \in [a,b]$$

and consider the set E_* *defined in* (14).

Let $f : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ *be a continuous function that satisfies the one-sided Nagumo-type condition in* E_* *and* (15).

Then problem (1)–(3) *has at least a solution* $u \in C^3([a,b])$ *such that*

$$\alpha(t) \leqslant u(t) \leqslant \beta(t) \quad \textit{and} \quad \alpha'(t) \leqslant u'(t) \leqslant \beta'(t), \quad \forall t \in [a,b].$$

Proof. For $\lambda \in [0,1]$, consider the modified problem composed by the differential equation (16) with the boundary conditions

$$\begin{aligned} &u(a) = \lambda A, \\ &u''(a) = \frac{\lambda}{c_2}[c_1 u'(a) - B], \\ &u''(b) = \frac{\lambda}{c_4}[C - c_3 u'(b)]. \end{aligned} \tag{26}$$

Take $r_1 > 0$ such that, for every $t \in [a,b]$, (19)–(21) hold and

$$\frac{|B|}{c_1} < r_1, \quad \frac{|C|}{c_3} < r_1. \tag{27}$$

Step 1: Every solution u of problem (16)–(26) satisfies in $[a,b]$

$$|u'(t)| < r_1 \quad \text{and} \quad |u(t)| < r_0,$$

with $r_0 = |A| + r_1(b-a)$, independent of $\lambda \in [0,1]$;

Let u be a solution of problem (16)–(26).

Assume, by contradiction, that there exists $t \in [a,b]$ such that either $u'(t) \geqslant r_1$ or $u'(t) \leqslant -r_1$. Suppose that the first case holds. Defining

$$\max_{t\in[a,b]} u'(t) := u'(t_0) \quad (\geqslant r_1 > 0)$$

and following the arguments of Theorem 4, Step 1, it can be shown that $t_0 \notin]a,b[$. If $t_0 = a$, then

$$\max_{t\in[a,b]} u'(t) := u'(a) \quad (\geqslant r_1 > 0)$$

and $u''(a^+) = u''(a) \leqslant 0$. For $\lambda = 0$ we have $u''(a) = 0$, $u'''(a) \leqslant 0$ and therefore we obtain the contradiction

$$0 \geqslant u'''(a) = u'(a) \geqslant r_1 > 0.$$

For $\lambda \in]0,1]$, by (27), since $c_1 c_2 > 0$, we have the following contradiction:

$$0 \geqslant u''(a) = \frac{\lambda}{c_2}[c_1 u'(a) - B] \geqslant \frac{1}{c_2}[c_1 r_1 - B] > 0.$$

The arguments are similar to show that $t_0 \neq b$. Then, $u'(t) < r_1$, for every $t \in [a, b]$. Analogously, it can be proved that $u'(t) > -r_1$, for every $t \in [a, b]$.

Thus, $|u'(t)| < r_1$, for every $t \in [a, b]$, and the estimate $|u(t)| < r_0$, where $r_0 := |A| + r_1(b - a)$, is easily obtained by integration.

Step 2: There exists $r_2 > 0$ such that every solution u of problem (16)–(26) satisfies in $[a, b]$

$$|u''(t)| < r_2$$

independent of $\lambda \in [0, 1]$.

Consider the set

$$E_{**} := \{(t, x, y, z) \in [a, b] \times \mathbb{R}^3 : -r_0 \leqslant x \leqslant r_0, -r_1 \leqslant y \leqslant r_1\},$$

and the function $F_\lambda : E_{**} \to \mathbb{R}$ given by (23).

Defining $\overline{\varphi}(z) := \varphi(|z|) + 2r_1$ and following previous arguments it is easy to see that F_λ satisfies the one-sided Nagumo-type condition in E_{**} with $\varphi(z)$ replaced by $\overline{\varphi}(z)$, independent of λ.

Moreover, for

$$\rho := \max\left\{\frac{c_1 r_1 + |B|}{c_2}, \frac{c_3 r_1 + |C|}{c_4}\right\}$$

every solution u of problem (16)–(26) satisfies

$$u''(a) = \frac{\lambda}{c_2}[c_1 u'(a) - B] < \frac{c_1 r_1 + |B|}{c_2} \leqslant \rho,$$
$$u''(b) = \frac{\lambda}{c_4}[C - c_3 u'(b)] > -\frac{c_3 r_1 + |C|}{c_4} \geqslant -\rho.$$

Defining

$$\Gamma_1(t) := -r_0, \quad \Gamma_2(t) := r_0,$$
$$\gamma_1(t) := -r_1 \quad \text{and} \quad \gamma_2(t) := r_1,$$

assumptions of Lemma 2 are satisfied with E replaced by E_{**} and there exists $r_2 > 0$, depending on r_1 and φ but independent of λ, such that

$$|u''(t)| < r_2, \quad \forall t \in [a, b].$$

Step 3: There exists at least a solution $u_1(t)$ for problem (16)–(26), for $\lambda = 1$.

The proof of this statement is parallel to the proof of Step 3 of Theorem 4 with obvious modifications due to the boundary conditions.

Step 4: This function $u_1(t)$ is a solution of (1)–(3) too.

As in Step 4 of Theorem 4 the statement follows from the fact that

$$\alpha(t) \leqslant u_1(t) \leqslant \beta(t), \quad \alpha'(t) \leqslant u_1'(t) \leqslant \beta'(t), \quad \forall t \in [a, b].$$

Assuming by contradiction that there exists $t_2 \in [a, b]$ such that

$$\min_{t\in[a,b]} [u_1'(t) - \alpha'(t)] := u_1'(t_2) - \alpha'(t_2) < 0$$

it can be derived, as in (24), that $t_2 \notin]a, b[$. If $t_2 = a$ we have

$$\min_{t\in[a,b]} [u_1'(t) - \alpha'(t)] := u_1'(a) - \alpha'(a) < 0,$$

$$u_1''(a) - \alpha''(a) = u_1''(a^+) - \alpha''(a^+) \geqslant 0$$

and by (25), since $c_2 > 0$, this yields to the contradiction:

$$\begin{aligned} 0 &\leqslant u_1''(a) - \alpha''(a) = \frac{c_1}{c_2} u_1'(a) - \frac{1}{c_2}[B + c_2\alpha''(a)] \\ &\leqslant \frac{c_1}{c_2}[u_1'(a) - \alpha'(a)] < 0. \end{aligned}$$

Then $t_2 \neq a$ and, analogously, we prove that $t_2 \neq b$. Therefore,

$$\alpha'(t) \leqslant u_1'(t), \quad \forall t \in [a, b].$$

Using a similar technique, it can be deduced that $u_1'(t) \leqslant \beta'(t)$, for all $t \in [a, b]$ and by integration

$$\alpha(t) \leqslant u_1(t) \leqslant \beta(t),$$

for every $t \in [a, b]$. $\square$

Remark 2. Theorems analogous to the existence and localization results presented here can be obtained if the reversed one-sided Nagumo condition is assumed, i.e.

$$f(t, x, y, z) \geqslant -\varphi(|z|), \quad \forall (t, x, y, z) \in E,$$

with adequate modifications.

5. Examples

The following examples not only illustrate the applicability of Theorems 4 and 6, respectively, but also prove that the nonlinearities considered satisfy the one-sided Nagumo condition and do not satisfy the two-sided Nagumo.

Example 1. Let $\sigma(z) = (6 + 4\,\mathrm{sgn}(z))/(5 - \mathrm{sgn}(z))$, where

$$\mathrm{sgn}(z) = \begin{cases} 1, & z > 0, \\ 0, & z = 0, \\ -1, & z < 0, \end{cases}$$

$A, C \in \mathbb{R}$ and consider the boundary value problem

$$u'''(t) = -2[u^3(t) + 1][1 - u'(t)]^2 - [u''(t)]^{\sigma(u''(t))}, \tag{28}$$

$$u(0) = A, \quad u''(0) = 0, \quad u''(1) = C, \tag{29}$$

If $A \in [0, 1]$ and $C \in [-1, 0]$ then the functions $\alpha, \beta : [0, 1] \to \mathbb{R}$ given by

$$\alpha(t) = -\frac{t^3}{6} \quad \text{and} \quad \beta(t) = 1 + t$$

are, respectively, lower and upper solutions of (28)–(29), according to Definition 3. The nonlinearity

$$f(t, x, y, z) = \begin{cases} -2(x^3 + 1)(1 - y)^2 - \sqrt{z^5} & \text{if } z \geqslant 0, \\ -2(x^3 + 1)(1 - y)^2 - \sqrt[3]{z} & \text{if } z < 0 \end{cases}$$

is continuous in $[0, 1] \times \mathbb{R}^3$, satisfies assumptions (15) and the one-sided Nagumo condition with

$$\varphi(z) = 1 + \sqrt[3]{|z|},$$

for every $(t, x, y, z) \in E$, where

$$E = \left\{ (t, x, y, z) \in [0, 1] \times \mathbb{R}^3 : -\frac{t^3}{6} \leqslant x \leqslant 1 + t, -\frac{t^2}{2} \leqslant y \leqslant 1 \right\}.$$

Therefore, by Theorem 4, there is at least a solution $u(t)$ of problem (28)–(29) such that, for every $t \in [0, 1]$,

$$-\frac{t^3}{6} \leqslant u(t) \leqslant 1 + t \quad \text{and} \quad -\frac{t^2}{2} \leqslant u'(t) \leqslant 1.$$

Notice that the function $f(t, x, y, z) = -(x^3 + 1)(1 - y)^2 - z^{\sigma(z)}$ does not satisfy the two-sided Nagumo-type condition of [7].

Example 2. Consider now the problem

$$u'''(t) = -[u(t) + 1][u'(t)]^2 - [u''(t)]^4, \tag{30}$$

$$u(0) = 0, \quad u'(0) - u''(0) = B, \quad u'(1) + u''(1) = C, \tag{31}$$

with $B, C \in \mathbb{R}$. The nonlinearity

$$f(t, x, y, z) = -(x + 1)y^2 - z^4$$

is continuous in $[0, 1] \times \mathbb{R}^3$. If $B, C \in [-1, 0]$ then the functions $\alpha, \beta : [0, 1] \to \mathbb{R}$ defined by

$$\alpha(t) = -t \quad \text{and} \quad \beta(t) = 0$$

are, respectively, lower and upper solutions of (30)–(31). Moreover, defining

$$E = \{(t, x, y, z) \in [0, 1] \times \mathbb{R}^3 : -t \leqslant x \leqslant 0, -1 \leqslant y \leqslant 0\},$$

f satisfies condition (15) and the one-sided Nagumo condition with $\varphi(z) = 1$, in E.

Therefore, by Theorem 6, there is at least a solution $u(t)$ of problem (30)–(31) such that, for every $t \in [0, 1]$,

$$-t \leqslant u(t) \leqslant 0 \quad \text{and} \quad -1 \leqslant u'(t) \leqslant 1.$$

Observe that, like in first example, the function

$$f(t, x, y, z) = -(x + 1)y^2 - z^4$$

does not satisfy the two-sided Nagumo condition.

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