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On the solvability of a boundary value problem for a fourth-order ordinary differential equation

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Abstract

We study the existence and multiplicity of nontrivial periodic solutions for a semilinear fourth-order ordinary differential equation arising in the study of spatial patterns for bistable systems. Variational tools such as the Brezis–Nirenberg theorem and Clark theorem are used in the proofs of the main results.

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1. Introduction

In this paper we study the existence and multiplicity of periodic solutions of a fourth-order ordinary differential equation of the form

$$u^{iv} + Au'' + Bu + f(x, u) = 0, \tag{1}$$

where A and B are constants and $f(x, u)$ is a continuous function, defined in $\mathbf{R}^2$, whose potential $F(x, u) = \int_0^u f(x, t)\,\mathrm{d}t$ satisfies suitable assumptions. The problem is motivated by the study of

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formation of spatial periodic patterns in bistable systems. Recently, interest has turned to the fourth-order parabolic differential equation, involving bistable dynamics, such as the extended Fisher–Kolmogorov (EFK) equation proposed by Coullet, Elphick & Repaux in 1987 and Dee & VanSaarlos in 1988, and the Swift–Hohenberg (SH) equation proposed in 1977. With appropriate changes of variables, stationary solutions of these equations lead to the equation

$$u^{iv} - pu'' - u + u^3 = 0, \tag{2}$$

in which $p > 0$ corresponds to the EFK equation and $p < 0$ to the SH equation. In this note we are interested in the existence of $2L$ periodic solutions of Eq. (1) which is a generalization of Eq. (2). We consider the solvability of the boundary value problem (P) for Eq. (1) with boundary conditions

$$u(0) = u(L) = u''(0) = u''(L) = 0. \tag{3}$$

The solvability of (P) for some extension of Eq. (2) was studied in [1,3,4,7–11] by variational methods. We suppose that $f(x, 0) = 0, \forall x \in \mathbf{R}$ and the potential $F(x, u) = \int_0^u f(x, s)\,\mathrm{d}s$ satisfies the following assumptions:

(H_1) There is a number $p > 2$ and for each bounded interval I there is a constant $c > 0$ such that

$$F(x, u) \geq c|u|^p, \qquad \forall x \in I, \forall u \in \mathbf{R},$$

and

(H_2) $F(x, u) = o(u^2)$ as $u \to 0$, uniformly with respect to x in bounded intervals.

A typical example which satisfies (H_1) and (H_2) is $f(x, u) = b(x)u|u|^{p-2}$, $p > 2$, where $b(x)$ is a continuous, positive function.

The problem (P) has a variational structure and its solutions can be found as critical points of the functional

$$I(u; L) := \frac{1}{2}\int_0^L (u''^2 - Au'^2 + Bu^2)\,\mathrm{d}x + \int_0^L F(x, u)\,\mathrm{d}x \tag{4}$$

in the Sobolev space $X(L) := H^2(0, L) \cap H_0^1(0, L)$. In this work we obtain nontrivial critical points of the functional I using Brezis–Nirenberg's linking theorem and Clark's theorem (see [2,5,6]).

It is easy to see that if $4B \geq A^2$ and $f(x, u)u > 0$ for $x \geq 0$ and $u \neq 0$ the problem (P) has only the trivial solution. We shall assume $4B < A^2$ and study separately the cases $A \leq 0$ (EFK equation) and $A > 0$ (SH equation). Our main results are:

Theorem 1. *Let* $4B < A^2$, $A \leq 0$, *set* $L_1 := \frac{\pi\sqrt{2}}{\sqrt{A+\sqrt{A^2-4B}}}$ *for* $B < 0$ *and let the function* $F(x, u)$ *satisfy the assumptions* (H_1) *and* (H_2).

(a) *If* $B < 0$ *and* $L > L_1$ *the problem* (P) *has at least two nontrivial solutions. If moreover* $F(x, \cdot)$ *is even for each fixed x and* $L > nL_1$ *there exist n distinct pairs of nontrivial solutions of* (P).

(b) *Let* $F(x, \cdot)$ *be convex for each fixed* x. *Then the problem* (P) *has only the trivial solution provided that either* (i) $B \geq 0$ *or* (ii) $B < 0$ *and* $0 < L \leq L_1$.

Theorem 2. *Let* $4B < A^2$, $A > 0$, *set* $L_1 := \frac{\pi\sqrt{2}}{\sqrt{A+\sqrt{A^2-4B}}}$ *and* $h_n = \left(\frac{(n^2+n)A}{2n^2+2n+1}\right)^2$, $n \in \mathbf{N} \cup \{0\}$, *and let the function* $F(x, u)$ *satisfy the assumptions* (H_1) *and* (H_2).

(a) *If* $B \leq 0$ *and* $L > L_1$ *the problem* (P) *has at least two nontrivial solutions. If in addition* $F(x, \cdot)$ *is even for each fixed x and* $L > nL_1$ *there exist n distinct pairs of nontrivial solutions of* (P).

(b) *If* $B > 0$, $M_1 := \frac{\pi\sqrt{2}}{\sqrt{A-\sqrt{A^2-4B}}}$ *and* $L \in]nL_1, nM_1[$ *the problem* (P) *has at least two nontrivial solutions. If in addition* $F(x,\cdot)$ *is even for each fixed* x, $k \in \mathbf{N}$ *and* $n > \frac{k}{2}\left(\frac{A+\sqrt{4B}}{\sqrt{A^2-4B}} - 1\right)$, *the problem* (P) *has* $k+1$ *pairs of nontrivial solutions if* $L \in](n+k)L_1, nM_1[$.

(c) *If* $F(x,\cdot)$ *is convex for each fixed* x *the problem* (P) *has only the trivial solution provided that one of the following holds:* (i) $B \le 0$ *and* $0 < L < L_1$, *or* (ii) $h_n < B \le h_{n+1}$ *and* $L \in T_{n+1}$ *where* T_{n+1} *is a finite union of bounded intervals.*

2. Sketch of proofs

A weak solution of the problem (P) is a function $u \in X(L)$, such that

$$\int_0^L (u''v'' - Au'v' + Buv + f(x,u)v)\,\mathrm{d}x = 0, \qquad \forall v \in X(L).$$

One can prove that a weak solution of (P) is a classical solution of (P) (see [11], Proposition 1). Weak solutions of (P) are critical points of the functional $I : X(L) \to \mathbf{R}$

$$I(u;L) := \frac{1}{2}\int_0^L (u''^2 - Au'^2 + Bu^2)\,\mathrm{d}x + \int_0^L F(x,u)\,\mathrm{d}x. \tag{5}$$

The following lemmas play an important role in further considerations.

Lemma 1. *The scalar product*

$$\langle u, v\rangle = \int_0^L u''v''\,\mathrm{d}x, \qquad u \in X(L), v \in X(L)$$

induces an equivalent norm in $X(L)$. *The set of functions* $\left\{\sin\left(\frac{n\pi x}{L}\right) : n \in \mathbf{N}\right\}$ *is a complete orthogonal basis in* $X(L)$.

Lemma 2. *Let* A, B *be constants and* $f(x,u)$ *be a continuous function such that* (H_1) *holds. Then the functional* I *is bounded from below, coercive and it satisfies the* (PS) *condition.*

Proof of Theorem 1. The polynomial $p(\xi) = \xi^4 - A\xi^2$ and the real functions $p_n(L) = p\left(\frac{n\pi}{L}\right)$ play an important role in the following. Let $A \le 0$. The polynomial $p(\xi)$ is a positive increasing and convex function for $\xi > 0$. The functions $p_n(L)$ are positive decreasing functions for every $n \in \mathbf{N}$ and $p_n(L) \to +\infty$, as $L \to 0$, and $p_n(L) \to 0$, as $L \to +\infty$. They are ordered $0 < p_1(L) < p_2(L) < \cdots < p_n(L) < \cdots$ for every $L > 0$. The graphs of functions $p_n(L)$ with $A = -1$ and $n = 1, 2, 3$ are presented in Fig. 1. Let $B < 0$. The equation $p_n(L) + B = 0$ has the unique solution $L_n = nL_1$, $L_1 := \frac{\pi\sqrt{2}}{\sqrt{A+\sqrt{A^2-4B}}}$ and

$$p_n(L) + B \ge 0 \qquad \text{if } L \le nL_1, \tag{6}$$

$$p_n(L) + B < 0 \qquad \text{if } L > nL_1. \tag{7}$$

Step 1. Nontrivial solutions.

Let $L > L_1$. There exists a natural number n such that $nL_1 < L \le (n+1)L_1$. Let $\varphi_n \in E_n = sp\left\{\sin\frac{\pi x}{L}, \ldots, \sin\frac{n\pi x}{L}\right\}$ so that $\varphi_n(x) = \sum_{k=1}^n c_k \sin\left(\frac{k\pi x}{L}\right)$, and set $c_1^2 + \cdots + c_n^2 = \rho^2$. We have $I(\varphi_n; L) < 0$, for sufficiently small $\rho > 0$. Let $u \in E_n^\perp$ and $\|u\| \le \rho$. It follows that $p_{n+1}(L) + B \ge 0$

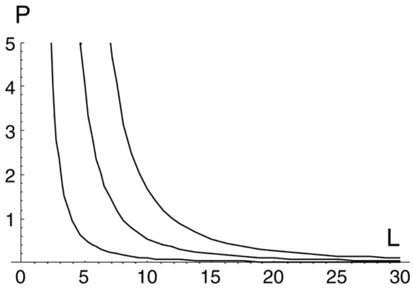

Fig. 1. Graphs of functions $p_n(L) = \left(\frac{n\pi}{L}\right)^4 + \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, 3$.

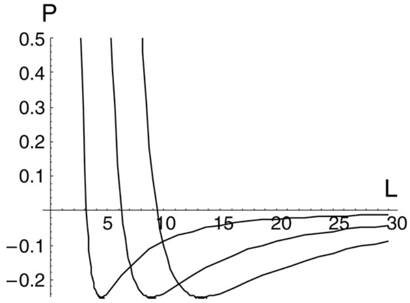

Fig. 2. Graphs of functions $p_n(L) = \left(\frac{n\pi}{L}\right)^4 - \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, 3$.

if $nL_1 < L \leq (n+1)L_1$ by (6). Since $p_{n+1}(L) < p_{n+2}(L) < \cdots$, by assumption (H_1) there exists $C(L) > 0$ such that $I(u; L) \geq \frac{1}{2}(p_{n+1}(L) + B)\|u\|^2_{L^2} + C(L)\|u\|^p_{L^2(0,L)} \geq 0$, if $u \in E_n^{\perp}$. The functional I satisfies the (PS) condition. In view of Brezis–Nirenberg's linking theorem (see [2]), for $L > L_1$ the functional I has at least two nontrivial critical points. Suppose that the function $F(x, u)$ is even with respect to u. Then I is an even functional and by Clark's theorem (see [11]) for $L > nL_1$ there exist at least n pairs of nontrivial critical points of I.

Step 2. Trivial solutions.

Let F be a convex function, $B < 0$ and $0 < L \leq L_1$. We have seen that $p_1(L) + B \geq 0$. As $P\left(\frac{k\pi}{L}\right) \geq p_1(L) + B \geq 0$ we infer that the quadratic summand in $I(u; L)$ is non-negative, hence it is a convex quadratic form. Then, if $B < 0$ and $0 < L \leq L_1$ the functional I is convex, positive for $u \neq 0$, and its only critical point is zero. If $B \geq 0$ the same argument applies for every $L > 0$, which completes the proof of Theorem 1. □

Proof of Theorem 2. *Step 1. Nontrivial and trivial solutions in the case* $B \leq 0$. Let $A > 0$. The polynomial $p(\xi) = \xi^4 - A\xi^2$ is positive for $\xi > \sqrt{A}$ and it has a negative minimum $p_0 = -\frac{A^2}{4}$ at $\xi_0 = \frac{\sqrt{A}}{2}$. The functions $p_n(L)$ are decreasing if $0 < L < n\pi\sqrt{2/A}$ and increasing if $L > n\pi\sqrt{2/A}$, $p_n(L) > 0$ if $0 < L < n\pi/\sqrt{A}$ and $p_n(L) < 0$ if $L > n\pi/\sqrt{A}$. The graphs of functions $p_n(L)$ with $A = 1$ and $n = 1, 2, 3$ are presented in Fig. 2. One can show that $p_k(L) + B < 0$ iff $L > kL_1$. If $L > L_1$ there exists a natural number n such that $nL_1 < L \leq (n+1)L_1$ and $p_k(L) + B \geq 0$, $k = 1, \ldots, n$ and

$p_k(L) + B \geq 0, \quad k \geq n + 1$. In this case the proof is finished exactly as in the proof of Theorem 1, Step 1 for nontrivial solutions and Step 2 for trivial solutions.

Step 2. Nontrivial solutions in the case $B > 0$. Let $\Delta_n =]nL_1, nM_1[$. Observe that $L \in \Delta_n \cap \Delta_{n+k}$ iff $(n+k)L_1 < L < nM_1$ which implies

$$n > \frac{k}{2}\left(\frac{A + \sqrt{4B}}{\sqrt{A^2 - 4B}} - 1\right). \tag{8}$$

Hence, given $n \in \mathbf{N}$, if $k \in \mathbf{N}$ is the largest integer satisfying (8) the condition $L \in \Delta_n \cap \Delta_{n+k}$ is equivalent to the set of inequalities $p_j(L) + B < 0$, $j \in \{n, n+1, \dots, n+k\}$ and $p_j(L) + B \geq 0$, $j \notin \{n, n+1, \dots, n+k\}$. Let

$$E_{k+1} := sp\left\{\sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{(n+1)\pi x}{L}\right), \dots, \sin\left(\frac{(n+k)\pi x}{L}\right)\right\}.$$

With a computation similar to the Step 1 in the proof of Theorem 1 one can show that I has a local linking at 0. Then I has at least two nontrivial critical points by the Brezis–Nirenberg theorem. Let F be even with respect to u and $L \in \Delta_n \cap \Delta_{n+k}$. We have $\sup\{I(u; L) : u \in E_{k+1}, \|u\| \leq \rho\} < 0$ for sufficiently small ρ. Then, by Clark's theorem there exist at least $k+1$ nontrivial pairs of critical points of the functional I.

Step 3. Trivial solutions in the case $B > 0$. We consider the solvability of the inequality

$$q(L) + B \geq 0, \tag{9}$$

where $q(L) = \inf\{p_n(L) : n \in \mathbf{N}\}$. Let $0 < B \leq \left(\frac{4}{25}\right) A^2 = h_1$ and

$$T_1 := \begin{cases}]0, L_1], & B < h_1 \\ ]0, L_1] \cup \{l_1\}, & B = h_1. \end{cases} \tag{10}$$

If $B \leq h_1$, the inequality (9) holds iff $L \in T_1$. Let $l_0 = 0$, $h_n < B < h_{n+1}$ and

$$D_{n+1} =]0, L_1] \cup [M_1, 2L_1] \cup \cdots \cup [nM_1, (n+1)L_1].$$

Let

$$T_{n+1} := \begin{cases} D_{n+1}, & h_n < B < h_{n+1}, \\ D_{n+1} \cup \{l_{n+1}\}, & B = h_{n+1}. \end{cases} \tag{11}$$

If $h_n < B \leq h_{n+1}$ the inequality (9) is satisfied iff $L \in T_{n+1}$. Let F be convex in the second variable and $L \in T_n$ if $h_n < B \leq h_{n+1}$. With an argument used in the proof of Theorem 1, we conclude that I is convex, positive for $u \neq 0$. Its only critical point is zero, and this completes the proof of Theorem 2. □

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References

[1] G.J.B. van den Berg, L.A. Peletier, W.C. Troy, Global branches of multy bump periodic solutions of Swift–Hohenberg equation, Arch. Ration. Mech. Anal. 158 (2001) 91–153.
[2] H. Brezis, L. Nirenberg, Remarks on finding critical points, Comm. Pure Appl. Math. XLIV (1991) 939–963.
[3] J.V. Chaparova, Existence and numerical approximations of periodic solutions of semilinear fourth-order differential equations, J. Math. Anal. Appl. 273 (2002) 121–136.
[4] J.V. Chaparova, L.A. Peletier, S.A. Tersian, Existence and nonexistence of nontrivial solutions of fourth and sixth order ODE, Adv. Differential Equations 8 (2003) 1237–1258.
[5] D.C. Clark, A variant of the Lyusternik-Schnirelmann theory, Indiana Univ. Math. J. 22 (1972) 65–74.
[6] M.R. Grossinho, S.A. Tersian, An Introduction to Minimax Theorems and Their Applications Differential Equations, Kluwer Academic Press, 2001.
[7] L.A. Peletier, W.C. Troy, Spatial Patterns: Higher Order Models in Physics and Mechanics, Birkhäuser, Boston, 2001.
[8] L.A. Peletier, W.C. Troy, V. der Vorst, Stationary solutions of a fourth-order nonlinear diffusion equation, Differ. Equ. 31 (2) (1995) 301–314.
[9] L.A. Peletier, V. Rottschäfer, Large time behavior of solutions of the Swift-Hohenberg equation, C. R. Acad. Sci. Paris Sér. I Math. 336 (2003) 225–230.
[10] L.A. Peletier, V. Rottschäfer, Pattern selection of solutions of the Swift-Hohenberg equation, Physica D (in press).
[11] S.A. Tersian, J.V. Chaparova, Periodic and homoclinic solutions of extended Fisher-Kolmogorov equation, J. Math. Anal. Appl. 266 (2001) 490–506.