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Nonlinear Analysis 63 (2005) e247 – e256

Nonlinear Analysis

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A third-order boundary value problem with one-sided Nagumo condition

M.R. Grossinho[a, b], F.M. Minhós[c, *, 1], A.I. Santos[c]

[a] *Department Mathematics, ISEG, Univ. Técnica de Lisboa, R. Quelhas, 6, 1200-781 Lisboa, Portugal*
[b] *CMAF, Univ. de Lisboa, Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal*
[c] *Department Mathematics and CIMA-UE, University de Évora, R. Romão Ramalho, 59, 7000 Évora, Portugal*

Abstract

In this paper we present an existence and location result for the third-order separated boundary value problem

$$\begin{cases} u'''(t) = f(t, u(t), u'(t), u''(t)), \\ u(a) = A, \\ u''(a) = 0, \\ u''(b) = 0, \end{cases}$$

where $f : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function and $A \in \mathbb{R}$.

One-sided Nagumo condition, lower and upper solutions, a priori estimates and Leray–Schauder degree play an important role in the arguments.

Keywords: Third-order separated boundary value problem; One-sided Nagumo condition; Lower and upper solutions; A priori estimates; Leray–Schauder degree

* Corresponding author.
E-mail addresses: mrg@iseg.utl.pt (M.R. Grossinho), fminhos@uevora.pt (F.M. Minhós), aims@uevora.pt (A.I. Santos).

1 Partially supported by FLAD (Project 210/2004) and FCG (Project 429934/2004).

0362-546X/$ - see front matter
doi:10.1016/j.na.2004.09.035

1. Introduction

In this work we consider the third-order separated boundary value problem denoted by (P)

$$u'''(t) = f(t, u(t), u'(t), u''(t)), \tag{1}$$

$$u(a) = A, \quad u''(a) = 0, \quad u''(b) = 0, \tag{2}$$

where $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and $A \in \mathbb{R}$.

An existence and location result is obtained by degree theory, [6], and lower and upper solutions method. An a priori estimate on u'', established by using a one-sided Nagumo type condition, plays an important role in the arguments.

We observe that commonly the above techniques use two-sided Nagumo conditions, [8]. For example, we refer for separated boundary value problems the results contained in [2,3] for second order, in [1,4] for third order and in [5,7] for higher order.

More precisely, for a third-order boundary value problem of the above type the usually assumed growth condition states that given a subset $E \subset [a, b] \times \mathbb{R}^3$ there is a positive continuous function φ such that

$$|f(t, x, y, z)| \leqslant \varphi(|z|), \quad \forall (t, x, y, z) \in E \tag{3}$$

and

$$\int_0^{+\infty} \frac{\xi}{\varphi(\xi)} \quad \mathrm{d}\xi = +\infty.$$

In this paper condition (3) is replaced by the one-sided assumption

$$f(t, x, y, z) \leqslant \varphi(|z|), \quad \forall (t, x, y, z) \in E. \tag{4}$$

In view of (4), the functions f considered in (P) are not necessarily controlled from below on the variable z by a function φ that satisfies the above integral condition. In fact, $f(t, x, y, z) = -x^3 + y - z^4$ verifies condition (4), but it does not satisfy (3), as it can be seen in the example concerning problem (24)–(25), that illustrates our main result (see Section 3).

This more general situation of f has the drawback that in Lemma 2, estimates for solutions u of the equation of problem (P) are established only for those solutions that satisfy

$$u''(a) \leqslant r \quad \text{and} \quad u''(b) \geqslant -r$$

for some $r > 0$ adequately defined. This, however can still be applied to problem (P) since by the boundary conditions (2) is trivially satisfied.

An analogous existence result still holds if the one-sided Nagumo condition (4) is replaced by

$$f(t, x, y, z) \geqslant -\varphi(|z|), \quad \forall (t, x, y, z) \in E. \tag{5}$$

2. Definitions and a priori bound

First it is defined that for the one-sided Nagumo condition we use forward.

Definition 1. Given a subset $E \subset [a, b] \times \mathbb{R}^3$, a continuous function $f : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ is said to satisfy one-sided Nagumo-type condition in E if there exists $\varphi \in C(\mathbb{R}_0^+, [k, +\infty[)$, with $k > 0$, such that

$$f(t, x, y, z) \leqslant \varphi(|z|), \tag{6}$$

for every $(t, x, y, z) \in E$ and

$$\int_0^{+\infty} \frac{s}{\varphi(s)}\, ds = +\infty. \tag{7}$$

Next, we establish an a priori bound for the second derivative of solutions of Eq. (1) under adequate conditions.

We remark that by a solution of problem (P) we mean a function $u \in C^3([a, b])$ satisfying the differential equation (1) and the boundary conditions (2).

Lemma 2. *Let $\Gamma_1, \Gamma_2, \gamma_1, \gamma_2 \in C([a, b], \mathbb{R})$ satisfy*

$$\Gamma_1(t) \leqslant \Gamma_2(t) \text{ and } \gamma_1(t) \leqslant \gamma_2(t), \quad \forall t \in [a, b],$$

and consider the set

$$E = \left\{(t, x, y, z) \in [a, b] \times \mathbb{R}^3 : \Gamma_1(t) \leqslant x \leqslant \Gamma_2(t), \quad \gamma_1(t) \leqslant y \leqslant \gamma_2(t)\right\}.$$

Let $\varphi \in C\left(\mathbb{R}_0^+, [k, +\infty[\right)$, with $k > 0$, such that

$$\int_r^{+\infty} \frac{s}{\varphi(s)}\, ds > \max_{t \in [a,b]} \gamma_2(t) - \min_{t \in [a,b]} \gamma_1(t), \tag{8}$$

where $r \geqslant 0$ is given by

$$r = \max\left\{\frac{\gamma_2(b) - \gamma_1(a)}{b - a}, \frac{\gamma_2(a) - \gamma_1(b)}{b - a}\right\}.$$

Then there exists $R > 0$ (depending on γ_1, γ_2, φ) such that for every continuous function $f : E \to \mathbb{R}$ that satisfies (6) *and every solution $u(t)$ of* (1) *such that*

$$u''(a) \leqslant r, u''(b) \geqslant -r \tag{9}$$

and

$$\Gamma_1(t) \leqslant u(t) \leqslant \Gamma_2(t), \quad \gamma_1(t) \leqslant u'(t) \leqslant \gamma_2(t), \ \forall t \in [a, b], \tag{10}$$

we have

$$\|u''\|_\infty < R.$$

Proof. Let u be a solution of (1) verifying (9) and (10).

Suppose, by contradiction, that $\left|u''(t)\right| > r$, for every $t \in\,]a, b[$. If $u''(t) > r$ for every $t \in\,]a, b[$, then the following contradiction is obtained

$$\gamma_2(b) - \gamma_1(a) \geqslant u'(b) - u'(a) = \int_a^b u''(\tau)\,\mathrm{d}\tau > \int_a^b r\,\mathrm{d}\tau \geqslant \gamma_2(b) - \gamma_1(a).$$

If $u''(t) < -r$, for every $t \in\,]a, b[$, a similar contradiction can be obtained. So, there is $t \in\,]a, b[$ such that $\left|u''(t)\right| \leqslant r$.

Let $R > r$ be such that

$$\int_r^R \frac{s}{\varphi(s)}\,\mathrm{d}s > \max_{t\in[a,b]} \gamma_2(t) - \min_{t\in[a,b]} \gamma_1(t). \tag{11}$$

If $\left|u''(t)\right| \leqslant r$, for every $t \in [a, b]$ then we have $\left|u''(t)\right| < R$. If not, we can take $t_1 \in\,]a, b]$ such that $u''(t_1) > r$ or $t_1 \in [a, b[$ such that $u''(t_1) < -r$. Suppose that the first case holds. By (9) we can consider $I = \left[\widehat{t_1}, t_1\right]$ such that

$$u''(\widehat{t_1}) = r, \quad u''(t) > r, \ \forall\, t \in \left]\widehat{t_1}, t_1\right].$$

Then, by a convenient change of variable and applying assumptions (6) and (11), we have

$$\begin{aligned}\int_{u''(\widehat{t_1})}^{u''(t_1)} \frac{s}{\varphi(s)}\,\mathrm{d}s &= \int_{\widehat{t_1}}^{t_1} \frac{u''(t)}{\varphi(u''(t))} u'''(t)\,\mathrm{d}t = \int_{\widehat{t_1}}^{t_1} \frac{f(t, u(t), u'(t), u''(t))}{\varphi(u''(t))} u''(t)\,\mathrm{d}t \\ &\leqslant \int_{\widehat{t_1}}^{t_1} u''(t)\,\mathrm{d}t = u'(t_1) - u'(\widehat{t_1}) \leqslant \max_{t\in[a,b]} \gamma_2(t) - \min_{t\in[a,b]} \gamma_1(t) \\ &< \int_r^R \frac{s}{\varphi(s)}\,\mathrm{d}s.\end{aligned}$$

Hence $u''(t_1) < R$. Since t_1 can be taken arbitrarily as long as $u''(t) > r$ we can conclude that, for every $t \in\,]a, b]$ such that $u''(t) > r$, we have

$$u''(t) < R.$$

By a similar way, it can be proved that $u''(t) > -R$, for every $t \in [a, b[$ such that $u''(t) < -r$. □

Remark 1. Observe that condition (7) implies (8).

For problem (1)–(2) lower and upper solutions are defined as it follows.

Definition 3. A function $\alpha \in C^3([a, b])$ is said to be a lower solution of problem (1)–(2) if

$$\alpha'''(t) \geqslant f(t, \alpha(t), \alpha'(t), \alpha''(t)),$$

and

$$\alpha(a) \leqslant A, \quad \alpha''(a) \geqslant 0, \quad \alpha''(b) \leqslant 0.$$

A function $\beta \in C^3([a,b])$ is said to be an upper solution of problem (1)–(2) if it verifies the reversed inequalities.

Upper and lower solutions will be an important tool to define the referred set E and to obtain a priori bounds on u and u'.

3. Existence and location result

Next theorem contains an existence and location result for problem (1)–(2).

Theorem 4. *Assume that there exist* $\alpha, \beta \in C^3([a,b])$ *lower and upper solutions of problem* (1)–(2), *respectively, such that*

$$\alpha'(t) \leqslant \beta'(t), \quad \forall t \in [a,b]. \tag{12}$$

Define the set

$$E_* = \left\{(t,x,y,z) \in [a,b] \times \mathbb{R}^3 : \alpha(t) \leqslant x \leqslant \beta(t),\ \alpha'(t) \leqslant y \leqslant \beta'(t)\right\}.$$

Let $f : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ *be a continuous function that satisfies the one-sided Nagumo condition in* E_* *and such that*

$$f(t,\alpha(t),y,z) \geqslant f(t,x,y,z) \geqslant \quad f(t,\beta(t),y,z) \tag{13}$$

for $(t,y,z) \in [a,b] \times \mathbb{R}^2$ *and* $\alpha(t) \leqslant x \leqslant \beta(t)$.

Then problem (1)–(2) *has at least a solution* $u \in C^3([a,b])$ *such that*

$$\alpha(t) \leqslant u(t) \leqslant \beta(t) \quad \text{and } \alpha'(t) \leqslant u'(t) \leqslant \beta'(t), \quad \forall t \in [a,b].$$

Remark 2. The relation $\alpha(t) \leqslant \beta(t)$ is obtained by integration of (12) and using the boundary conditions of Definition 3.

Proof. For $\lambda \in [0,1]$ consider the modified problem

$$u'''(t) = \lambda f(t, \xi(t,u(t)), \xi_*(t,u'(t)), u''(t)) + u'(t) - \lambda \xi_*(t,u'(t)) \tag{14}$$

with the boundary conditions

$$u(a) = \lambda A, \quad u''(a) = 0, \quad u''(b) = 0, \tag{15}$$

where the continuous functions $\xi, \xi_* : \mathbb{R}^2 \to \mathbb{R}$ are given by

$$\xi(t,x) = \begin{cases} \beta(t) & \text{if } x > \beta(t), \\ x & \text{if } \alpha(t) \leqslant x \leqslant \beta(t), \\ \alpha(t) & \text{if } x < \alpha(t) \end{cases} \tag{16}$$

and

$$\xi_*(t,y) = \begin{cases} \beta'(t) & \text{if } y > \beta'(t), \\ y & \text{if } \alpha'(t) \leqslant y \leqslant \beta'(t), \\ \alpha'(t) & \text{if } y < \alpha'(t). \end{cases} \tag{17}$$

Take $r_1 > 0$ such that, for every $t \in [a, b]$,

$$-r_1 < \alpha'(t) \leqslant \beta'(t) < r_1, \tag{18}$$

$$f(t, \alpha(t), \alpha'(t), 0) - r_1 - \alpha'(t) < 0, \tag{19}$$

$$f(t, \beta(t), \beta'(t), 0) + r_1 - \beta'(t) > 0. \tag{20}$$

Step 1: *Every solution u of problem* (14)–(15) *satisfies in* $[a, b]$

$$\left|u'(t)\right| < r_1 \quad \text{and} \quad |u(t)| < r_0 \tag{21}$$

with r_1 *given above and* $r_0 = |A| + r_1(b-a)$, *independently of* $\lambda \in [0, 1]$.

Let u be a solution of problem (14)–(15).

Assume, by contradiction, that there exists $t \in [a, b]$ such that either $u'(t) \geqslant r_1$ or $u'(t) \leqslant -r_1$. Suppose that the first case holds and define

$$u'(t_0) := \max_{t\in[a,b]} \quad u'(t) \quad (\geqslant r_1 > 0).$$

If $t_0 \in\,]a, b[$, then $u''(t_0) = 0$ and $u'''(t_0) \leqslant 0$. For $\lambda \in\,]0, 1]$ and following the technique used in [4], by (13), (18) and (20), we have the contradiction

$$\begin{aligned} 0 &\geqslant u'''(t_0) \\ &= \lambda f(t_0, \xi(t_0, u(t_0)), \xi_*(t_0, u'(t_0)), u''(t_0)) + u'(t_0) - \lambda\xi_*(t_0, u'(t_0)) \\ &= \lambda f(t_0, \xi(t_0, u(t_0)), \beta'(t_0), 0) + u'(t_0) - \lambda\beta'(t_0) \\ &\geqslant \lambda f(t_0, \xi(t_0, u(t_0)), \beta'(t_0), 0) + r_1 - \lambda\beta'(t_0) \\ &\geqslant \lambda \left[f(t_0, \beta(t_0), \beta'(t_0), 0) + r_1 - \beta'(t_0)\right] > 0 \end{aligned}$$

and, for $\lambda = 0$,

$$0 \geqslant u'''(t_0) = u'(t_0) \geqslant r_1 > 0.$$

If $t_0 = a$ then

$$\max_{t\in[a,b]} \quad u'(t) = u'(a) \quad (\geqslant r_1 > 0),$$

which together with the boundary condition of this problem $u''(a)=0$ implies that $u'''(a) \leqslant 0$. Then the above computations with t_0 replaced by a yield a contradiction.

The case $t_0 = b$ is analogous. Thus, $u'(t) < r_1$, for every $t \in [a, b]$.

In a similar way we can prove that $u'(t) > -r_1$, for every $t \in [a, b]$. Furthermore, since $u(a) = \lambda A$, the estimate $|u(t)| < r_0$, where $r_0 = |A| + r_1(b-a)$, is easily obtained by integration.

Step 2: *There exists* $r_2 > 0$ *such that every solution u of problem* (14)–(15) *satisfies in* $[a, b]$

$$\left|u''(t)\right| < r_2 \tag{22}$$

independently of $\lambda \in [0, 1]$.

Consider the set

$$E_{**} := \left\{(t,x,y,z) \in [a,b] \times \mathbb{R}^3 : -r_0 \leqslant x \leqslant r_0, -r_1 \leqslant y \leqslant r_1\right\},$$

and the function $F_\lambda : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$F_\lambda(t,x,y,z) = \lambda f(t, \xi(t,x), \xi_*(t,y), z) + y - \lambda \xi_*(t,y).$$

Since f satisfies a one-sided Nagumo condition in E_*, consider the function $\varphi \in C\left(\mathbb{R}_0^+, [k,+\infty[\right)$ such that (6) and (7) hold with E replaced by E_*. Thus, for $(t,x,y,z) \in E_{**}$, we have, by (17) and (18),

$$\begin{aligned} F_\lambda(t,x,y,z) &= \lambda f(t, \xi(t,x), \xi_*(t,y), z) + y - \lambda \xi_*(t,y) \\ &\leqslant \lambda\varphi(|z|) + r_1 - \alpha'(t) \leqslant \varphi(|z|) + 2r_1. \end{aligned} \tag{23}$$

Take $\overline{\varphi}(z) := \varphi(|z|) + 2r_1$. Since

$$\int_0^{+\infty} \frac{s}{\overline{\varphi}(s)}\,\mathrm{d}s = \int_0^{+\infty} \frac{s}{\varphi(|s|)+2r_1}\mathrm{d}s \geqslant \frac{1}{1+2r_1/k}\int_0^{+\infty} \frac{s}{\varphi(|s|)}\,\mathrm{d}s,$$

then $\overline{\varphi}(z)$ satisfies (7) (and thus (8) by Remark 1). Therefore, F_λ satisfies the one-sided Nagumo condition in E_{**} with $\varphi(z)$ replaced by $\overline{\varphi}(z)$, independently of λ.

Moreover, if we put

$$\Gamma_1(t) := -r_0 = -|A| - r_1(b-a), \Gamma_2(t) := r_0 = |A| + r_1(b-a),$$

$$\gamma_1(t) := -r_1, \;\; \gamma_2(t) := r_1 \;\text{ and }\; r_{**} = \frac{2r_1}{b-a}$$

and take Remark 1 into account, assumptions (6) and (8) of Lemma 2 are satisfied with E replaced by E_{**} and r by r_{**}. So, since every solution u of (14)–(15) satisfy

$$u''(a) = 0 \leqslant r_{**}, u''(b) = 0 \geqslant -r_{**}$$

there exists $r_2 > 0$, depending on r_1 and φ, such that $\left|u''(t)\right| < r_2$, for every $t \in [a,b]$. As r_1 and φ do not depend on λ, we can conclude that the estimate $\left|u''(t)\right| < r_2$ is also independent of λ.

Step 3: *For* $\lambda = 1$ *problem* (14)–(15) *has a solution* $u_1(t)$.

Define the operators

$$\mathscr{L} : C^3([a,b]) \subset C^2([a,b]) \longmapsto C([a,b]) \times \mathbb{R}^3,$$

$$\mathscr{N}_\lambda : C^2([a,b]) \longmapsto C([a,b]) \times \mathbb{R}^3$$

by

$$\mathscr{L}u = (u''' - u', u(a), u''(a), u''(b)),$$

$$\mathscr{N}_\lambda u = (\lambda f(t, \xi(t,u(t)), \xi_*(t,u'(t)), u''(t)) - \lambda\xi_*(t,u'(t)), \lambda A, 0, 0).$$

Observe that $\mathscr{L}$ has a compact inverse. Therefore we can consider the completely continuous operator

$$\mathscr{T}_\lambda : \left(C^2\left([a,b]\right), \mathbb{R}\right) \longmapsto \left(C^2\left([a,b]\right), \mathbb{R}\right)$$

defined by

$$\mathscr{T}_\lambda(u) = \mathscr{L}^{-1}\mathscr{N}_\lambda(u).$$

Take the set

$$\Omega = \left\{x \in C^2\left([a,b]\right) : \|x\|_\infty < r_0,\ \left\|x'\right\|_\infty < r_1,\ \left\|x''\right\|_\infty < r_2\right\}.$$

By (21) and (22), for every u solution of (14) and (15), $u \notin \partial\Omega$ and so the degree $d(I - \mathscr{T}_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0,1]$. Then, due to the invariance under homotopy and since $\mathscr{T}_0(x) = x$ has only the trivial solution we have

$$d(I - \mathscr{T}_0, \Omega, 0) = \pm 1 = d(I - \mathscr{T}_1, \Omega, 0).$$

Therefore problem (14)–(15) has at least a solution $u_1(t)$.

Step 4: *u_1 is a solution of (P).*

Arguing by contradiction and following the arguments used in [4] it can be derived in a standard way, using the definitions of solution and upper and lower solutions and condition (13), that

$$\alpha'(t) \leqslant u_1'(t) \leqslant \beta'(t),\ \forall t \in [a,b]$$

and by integration

$$\alpha(t) \leqslant u_1(t) \leqslant \beta(t),\ \forall t \in [a,b]\,.$$

Therefore u_1 is in fact a solution of problem (1)–(2), too. □

Remark 3. An analogous existence and location result can be established, with adequate modifications, if the reversed one-sided Nagumo condition (5) is assumed.

Example. Consider the boundary value problem

$$u'''(t) = -u^3(t) + u'(t) - \left[u''(t)\right]^4, \tag{24}$$

$$u(0) = 0,\ \ u''(0) = 0, u''(1) = 0. \tag{25}$$

We note that the boundary conditions (25) are a particular case of (2) and of the problem presented in [4] with $c_1 = c_3 = 0$, $A = B = C = 0$ and $c_2, c_4 > 0$. As we see next the nonlinearity

$$f(t,x,y,z) = -x^3 + y - z^4,$$

satisfies the hypotheses of Theorem 4 but it does not satisfy the assumptions assumed in [4], namely the two-sided Nagumo condition. In fact, f is continuous in $[0,1] \times \mathbb{R}^3$ and verifies Nagumo conditions (6) and (7) with

$$\varphi(z) = 2 + z^2,$$

for every $(t, x, y, z) \in E$, where

$$E = \left\{ (t, x, y, z) \in [0, 1] \times \mathbb{R}^3 : -t \leqslant x \leqslant t, -1 \leqslant y \leqslant 1 \right\},$$

and assumption (13).

Moreover, the functions $\alpha, \beta : [0, 1] \to \mathbb{R}$

$$\alpha(t) = -t \quad \text{and} \quad \beta(t) = t$$

are, respectively, lower and upper solutions of (24)–(25), according to Definition 3.

Therefore, by Theorem 4, there is at least a solution $u(t)$ of problem (24)–(25) such that, for every $t \in [0, 1]$,

$$-t \leqslant u(t) \leqslant t \quad \text{and} \quad -1 \leqslant u'(t) \leqslant 1.$$

Notice, however, that the function $f(t, x, y, z) = -x^3 + y - z^4$ does not verify the two-sided Nagumo-type condition. In fact, suppose by contradiction that there is a positive continuous function φ such that

$$|f(t, x, y, z)| \leqslant \varphi(|z|), \quad \forall (t, x, y, z) \in E$$

and

$$\int_0^{+\infty} \frac{s}{\varphi(s)} \, \mathrm{d}s = +\infty.$$

In particular,

$$-f(t, x, y, z) \leqslant \varphi(|z|), \forall (t, x, y, z) \in E,$$

and, so for $t, x = 1$, $y = -1$ and $z \in \mathbb{R}$

$$-f(1, 1, -1, z) = 2 + z^4 \leqslant \varphi(|z|).$$

As $\int_0^{+\infty} s/(2 + s^4) \, \mathrm{d}s$ is finite, we have the following contradiction:

$$+\infty > \int_0^{+\infty} \frac{s}{2 + s^4} \, \mathrm{d}s \geqslant \int_0^{+\infty} \frac{s}{\varphi(s)} \, \mathrm{d}s = +\infty.$$

Remark 4. The same arguments can be applied for a more general class of differential equations

$$u'''(t) = -u^{2k+1}(t) + \left[u'(t)\right]^{2p+1} - \left[u''(t)\right]^{2m+4},$$

with k, p, m non-negative integer, and with boundary conditions given by (25).

Acknowledgements

The authors are grateful to the referee for his useful remarks.

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