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Numerical Approximation of Multidimensional Parabolic Partial Differential Equations Arising in Financial Mathematics

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In many cases, financial option pricing models give rise to PDEs which turn out to be very difficult to solve by classical analytic tools. In this article, we study the numerical approximation in space of the solution of the Cauchy problem for a multidimensional linear parabolic PDE of second order, with time and space-dependent coefficients. Making use of the L^2 theory of solvability in Sobolev spaces, the solution of the PDE problem is approximated in space, with finite-difference methods. The rate of convergence is estimated.

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1. Introduction

This paper concerns the numerical study of linear parabolic PDEs of second-order on the strip $[0,T] \times \mathbb{R}^d$, with T a positive constant.

Multidimensional PDE problems arise in Financial Mathematics, and also in Mathematical Physics. We are mainly motivated by the application to a class of stochastic models in Financial Mathematics, comprising the non path-dependent options, with fixed exercise, written on multiple assets (*basket or rainbow options*, *exchange options*, *compound options*, European options on

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future contracts and foreign-exchange, and others), and also, a particular type path-dependent options, the *Asian options* (see, e.g., Lamberton and Lapeyre [14]).

Let us assume the modelling of a multi-asset financial option of European type, within a multidimensional version of Black-Scholes model, where the asset price vector's drift and volatility are taken time and space-dependent. Pricing this option can be reduced, making use of a Feynman-Kac type formula, to solving the Cauchy problem (with terminal condition) for a multidimensional parabolic PDE, with null term (see, e.g., Lamberton and Lapeyre [14]). Therefore, instead of approximating directly the option price with probabilistic methods (e.g., Monte-Carlo method), there is the alternative of approximating the solution of the correspondent PDE problem.

In this article, we study the numerical approximation, in the space variables, of the solution of the PDE problem

$$(1) \qquad Lu - u_t + f = 0 \text{ in } Q, \quad u(0,x) = g(x) \text{ in } \mathbb{R}^d,$$

where

$$L(t,x) = a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t,x)\frac{\partial}{\partial x^i} + c(t,x)$$

is a uniformly elliptic operator with respect to the space variables, $Q = [0,T] \times \mathbb{R}^d$, with $T \in (0,\infty)$, and f and g are given functions.

The approximation study is pursued using finite-difference methods.

The numerical methods and possible approximation results are strongly linked to the theory on the solvability of the PDEs. In the present article, we make use of the L^2 theory of solvability of linear PDEs in Sobolev spaces.

The finite-difference method for approximating PDE is a well developed area, which has been extensively researched since the first half of the last century. We refer to Thomée [20] for a brief summary of the method's history, and also for the references of the seminal work by R. Courant, K. O. Friedrichs and H. Lewy, and of further major contributions by other authors.

In particular, a general approach of the numerical approximation, making use of finite differences, of the Cauchy problem for a multidimensional linear parabolic PDE of order $m \geq 2$, with bounded time and space-dependent coefficients, can be found in Thomée [20]. This approach is pursued under a strong setting, where the PDE problem has a classical solution.

The finite-difference method was also early applied to financial option pricing, the pioneering work being due to M. Brennan and E. S. Schwartz in 1978, and was, since then, widely researched in the context of the financial application, and extensively used by practitioners. For the references of the original publications and further major research, we refer to the review paper by Broadie and Detemple [3].

Most studies concerning the numerical approximation of PDE problems in Finance consider the particular case where the PDE coefficients are constant (see, e.g., Barles et al. [1], Boyle and Tian [2], Fusai et al. [9], and Gilli et al. [10]). This occurs, namely, in option pricing under the Black-Scholes model (in one or several dimensions), when the the asset price vector's drift and volatility are taken constant. The simpler PDE, with constant coefficients, is obtained after a standard change of variables (see, e.g., Lamberton and Detemple [14] for the one-dimensional case, and Gonçalves [11] for the multidimensional case).

In Lötstedt et al. [16], a space-time adaptive finite-difference method is developed for the approximation of a more general multidimensional PDE problem, corresponding to a version of Black-Scholes model where the asset price vector's drift and volatility are taken variable but only with respect to the time variable.

Some other studies develop approximation procedures for more complex models, but restricting the analysis to the case of one spatial dimension (see, e.g., Cont and Voltchkova [4], and Düring et al. [5,6]). In in't Hout and Foulon [13] finite-difference methods are used to approximate the solution of a PDE problem, with two spatial dimensions, arising in the Heston model, but the analysis is restricted to the case where the PDE coefficients depend only on the spacial variables.

The PDEs arising from the generalized option pricing model pose three challenges to the numerical approximation: the degeneracy of the equation, the coefficients being time and space-dependent and also unbounded in the space variables. In the present article, we assume that the equation is nondegenerate and the coefficients are bounded, and deal only with the coefficient time and space-dependency. We investigate the approximation in space of the solution of problem (1), pursuing the study under a weak setting, where the PDE problem has a generalized solution, and weak regularity is imposed over the operator's coefficients and data f and g.

We summarize the content of the article. In Section 2, we establish some well-known facts on the solvability of linear PDEs under a general framework. In Section 3, we discretize in space problem (1), with the use of finite-difference methods. We set a discrete framework and, by showing that it is a particular case of the general framework presented in the previous Section, we deduce an existence and uniqueness result for the solution of the discretized problem. In Section 4, we prove that the solution of the discretized problem approximates the solution of the continuous problem (1), and determine a rate of convergence. In the final Section 5, we establish a convergence result, under weaker conditions, for the special case of one dimension in space.

2. Preliminaries and classical results

We establish some facts on the solvability of linear PDEs under a general framework.

Let V be a reflexive separable Banach space embedded continuously and densely into a Hilbert space H with inner product $(\ ,\ )$. Then H^*, the dual space of H, is also continuously and densely embedded into V^*, the dual of V. Let us use the notation $\langle\ ,\ \rangle$ for the duality. Let H^* be identified with H in the usual way, by the help of the inner product. Then we have the so called normal triple $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$, with continuous and dense embeddings.

Let us consider the Cauchy problem for an evolution equation

$$L(t)u(t) - \frac{\partial u(t)}{\partial t} + f(t) = 0, \quad u(0) = g, \tag{2}$$

where $L(t)$ and $\partial/\partial t$ are linear operators from V to V^* for every $t \geq 0$, $f \in L^2([0,T];V^*)$ with $T \in (0,\infty)$ and $g \in H$.

Notation. The notation $L^2([0,T];W)$, with W a function space, stands for the set of all L^2 W-valued functions on $[0,T]$.

We assume the operator $L(t)$ is continuous and coercive, and impose some regularity over the data f and g:

Assumption 1. There exist constants $\lambda > 0$, K, M and N such that

1. $\langle L(t)v, v\rangle + \lambda|v|_V^2 \leq K|v|_H^2, \quad \forall v \in V$ and $\forall t \in [0,T]$;
2. $|L(t)v|_{V^*} \leq M|v|_V, \quad \forall v \in V$ and $\forall t \in [0,T]$;
3. $\int_0^T |f(t)|_{V^*}^2 dt \leq N$ and $|g|_H \leq N$.

We define the generalized solution of problem (2).

Definition 1. We say that $u \in C([0,T];H)$ is a generalized solution of problem (2) on $[0,T]$ if

1. $u \in L^2([0,T];V)$;
2. For all $t \in [0,T]$

$$(u(t), v) = (g, v) + \int_0^t \langle L(s)u(s), v\rangle ds + \int_0^t \langle f(s), v\rangle ds$$

holds for all $v \in V$.

N o t a t i o n. We denote by $C([0,T];W)$, with W a function space, the set of all continuous W-valued functions on $[0,T]$.

We state a well-known existence and uniqueness result for the solution of problem (2). This result is a special case of a more general one proved in Lions [15] for nonlinear evolution equations.

Theorem 1. *Under (1)–(3) in Assumption 1, problem (2) has a unique generalized solution on* $[0,T]$. *Moreover,*

$$\sup_{t\in[0,T]} |u(t)|_H^2 + \int_0^T |u(t)|_V^2 dt \le N\Big(|g|_H^2 + \int_0^T |f(t)|_{V^*}^2 dt\Big),$$

where N *is a constant.*

Let us now consider the particular Cauchy problem

$$Lu - u_t + f = 0 \ \text{ in } \ Q, \quad u(0,x) = g(x) \ \text{ in } \ \mathbb{R}^d, \tag{3}$$

where L is the second-order operator

$$L(t,x) = a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t,x)\frac{\partial}{\partial x^i} + c(t,x), \tag{4}$$

with real coefficients, $Q = [0,T]\times\mathbb{R}^d$, with $T\in(0,\infty)$, and f and g are given functions.

N o t a t i o n. The above operator L is written using the usual summation convention. In the sequel, we will use this notation whenever it simplifies the writing.

We set a particular framework for problem (3). Consider the Sobolev space $W^{m,2}(U)$,[3] with U be a domain in $\mathbb{R}^d$, i.e., an open subset of $\mathbb{R}^d$, and $m \ge 0$ an integer, consisting of all locally summable functions $v : U \to \mathbb{R}$ such that for each α with $|\alpha| \le m$, $D^\alpha v$ exists in the weak sense and the norm $|v|_{W^{m,2}(U)} := \big(\sum_{|\alpha|\le m}\int_U |D^\alpha v|^2 dx\big)^{1/2}$ is finite. $W^{m,2}(U)$ is a complete normed linear space. Endowed with the inner product

$(v,w)_{W^{m,2}(U)} := \sum_{|\alpha|\le m}\int_U D^\alpha v D^\alpha w\, dx$, for all $v,w\in W^{m,2}(U)$,

which generates the norm, $W^{m,2}(U)$ is a Hilbert space.

N o t a t i o n. In the sequel, when $U=\mathbb{R}^d$ we drop the argument in the function space notation. For instance, we denote $W^{m,2}(\mathbb{R}^d) =: W^{m,2}$.

[3]For the theory of linear PDE in Sobolev spaces see, e.g., Evans [8], pp. 241-289.

Now, switch point of view and consider the functions $w : Q \to \mathbb{R}$ as functions in $[0,T]$ with values in $\mathbb{R}^\infty$ such that, for all $t \in [0,T]$, $w(t) := \{w(t,x) : \ x \in \mathbb{R}^d\}$.

We impose a coercivity condition over the operator (4), and make some assumptions on the regularity of the operator's coefficients and on the free data f and g:

Assumption 2. Let $m \geq 0$ be an integer.

1. There exists a constant $\lambda > 0$ such that
$$\sum_{i,j=1}^{d} a^{ij}(t,x)\xi^i\xi^j \geq \lambda \sum_{i=1}^{d} |\xi^i|^2,$$
for all $t \geq 0$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$;

2. The coefficients in L and their derivatives in x up to the order m are measurable functions in $[0,T] \times \mathbb{R}^d$ such that
$$|D_x^\alpha a^{ij}| \leq K \ \ \forall |\alpha| \leq m \vee 1, \quad |D_x^\alpha b^i| \leq K, \ \ |D_x^\alpha c| \leq K \ \ \forall |\alpha| \leq m,$$
for any $t \in [0,T]$, $x \in \mathbb{R}^d$, with K a constant, and D_x^α denoting the α^{th} partial derivative operator with respect to x;

3. $f \in L^2([0,T]; W^{m-1,2})$, $\ g \in W^{m,2}$.

N o t a t i o n. For $m = 0$, we use the notation $W^{m-1,2} = W^{-1,2} := (W^{1,2})^*$, where $(W^{1,2})^*$ is the dual of $W^{1,2}$.

We define the generalized solution of problem (3).

Definition 2. We say that $u \in C([0,T]; L^2)$ is a generalized solution of problem (3) on $[0,T]$ if

1. $u \in L^2([0,T]; W^{1,2})$;

2. For all $t \in [0,T]$
$$(u(t), \varphi) = (g, \varphi) + \int_0^t \big\{ -(a^{ij}(s) D_i u(s), D_j \varphi)$$
$$+ (b(s) D_i u(s) - D_j a^{ij}(s) D_i u(s), \varphi) + (c(s) u(s), \varphi) + \langle f(s), \varphi \rangle \big\} ds$$
holds for all $\varphi \in C_0^\infty$.

N o t a t i o n. The notation (,) in the above Definition 2 stands for the inner product in L^2. C_0^∞ denotes the set of all infinitely differentiable functions on $\mathbb{R}^d$ with compact support.

R e m a r k 1. In Definition 2, alternatively to the infinite differentiability of φ required in (2), it can be required that $\varphi \in W^{1,2}$.

Finally, we state a well-known result on the existence and uniqueness of the solution of (3). This result can be obtained from the general result in abstract spaces (Theorem 1), using the appropriate triples of spaces (see, e.g., Gyöngy [12], p. 67, for a more general case of SPDEs).

Theorem 2. *Under (1)–(3) in Assumption 2, problem (3) admits a unique generalized solution u on $[0,T]$. Moreover,*

$$u \in C([0,T]; W^{m,2}) \cap L^2([0,T]; W^{m+1,2})$$

and

$$\sup_{0\le t\le T} |u(t)|^2_{W^{m,2}} + \int_0^T |u(t)|^2_{W^{m+1,2}} dt \le N\Big(|g|^2_{W^{m,2}} + \int_0^T |f(t)|^2_{W^{m-1,2}} dt\Big)$$

with N a constant.

3. Numerical approximation in space: The discrete framework

In this section, we discretize in space problem (3), with the use of a finite-difference scheme. By considering suitable discrete function spaces, we can show that the discrete framework we set is a particular case of the general framework presented in Section 2, therefore holding an existence and uniqueness result for the solution of the discretized problem.

We define the h-grid on $\mathbb{R}^d$, with $h \in (0,1]$

$$Z_h^d = \Big\{x \in \mathbb{R}^d : x = h\sum_{i=1}^d e_i n_i, \quad n_i = 0, \pm 1, \pm 2, \dots\Big\}.$$

Denote

$$\partial_i^+ u = \partial_i^+ u(t,x) = h^{-1}(u(t,x+he_i) - u(t,x))$$

and

$$\partial_i^- u = \partial_i^- u(t,x) = h^{-1}(u(t,x) - u(t,x-he_i)),$$

the forward and backward discrete differences in space, respectively. Define the discrete operator

$$L_h(t,x) = a^{ij}(t,x)\partial_j^-\partial_i^+ + b^i(t,x)\partial_i^+ + c\,(t,x).$$

We consider the discrete problem

$$L_h u - u_t + f_h = 0 \ \text{ in } \ Q(h), \quad u(0,x) = g_h(x) \ \text{ in } \ Z_h^d, \tag{5}$$

where $Q(h) = [0,T] \times Z_h^d$, with $T \in (0,\infty)$, and f_h and g_h are functions such that $f_h : Q(h) \to \mathbb{R}$ and $g_h : Z_h^d \to \mathbb{R}$.

Consider functions $v : Z_h^d \to \mathbb{R}$. We introduce the 0-order discrete Sobolev space

$$l^{0,2} = \{v : Z_h^d \to \mathbb{R} : |v|_{l^{0,2}} < \infty\},$$

where the norm $|v|_{l^{0,2}}$ is defined by

$$|v|_{l^{0,2}} = \Big(\sum_{x \in Z_h^d} |v(x)|^2 h^d \Big)^{1/2}.$$

Define the inner product

$$(v,w)_{l^{0,2}} = \sum_{x \in Z_h^d} v(x) w(x) h^d,$$

for any $v, w \in l^{0,2}$, which induces the above norm.

It could be checked trivially that $(\ , \ )_{l^{0,2}}$ and $| \ \ |_{l^{0,2}}$, as defined above, are an inner product and a norm, respectively. We show next the good structure of $l^{0,2}$.

Proposition 1. *The function space $l^{0,2}$ is a Hilbert space.*

P r o o f. To prove that $l^{0,2}$ is a Hilbert space we have to prove that the inner product space $l^{0,2}$ is complete, i.e., that any Cauchy sequence in $l^{0,2}$ is convergent in the space norm.

Let (v_n) be a Cauchy sequence in $l^{0,2}$, i.e, for all $\varepsilon > 0$ exists N such that for $m, n > N$

$$|v_m - v_n|_{l^{0,2}} = \Big(\sum_{x \in Z_h^d} |v_m(x) - v_n(x)|^2 h^d \Big)^{1/2} < \varepsilon. \tag{6}$$

Therefore, for every $x \in Z_h^d$

$$|v_m(x) - v_n(x)|^2 h^d < \varepsilon^2, \qquad \text{for } m, n > N. \tag{7}$$

Let us fix $x = x_0$. From (7), we see that $(v_1(x_0),\, v_2(x_0),\, \ldots)$ is a Cauchy sequence of real numbers, therefore convergent. Write $v_m(x_0) \to v(x_0)$. Using these limits, we define $v = v(x)$, for each $x \in Z_h^d$.

Let B be a ball in Z_h^d. From (6), for $m, n > N$

$$\sum_{x \in B} |v_m(x) - v_n(x)|^2 h^d < \varepsilon^2.$$

Letting $n \to \infty$, for $m > N$

$$\sum_{x \in B} |v_m(x) - v(x)|^2 h^d \leq \varepsilon^2.$$

Letting now the diameter of B go to ∞, for $m > N$

$$\sum_{x \in Z_h^d} |v_m(x) - v(x)|^2 h^d \leq \varepsilon^2. \tag{8}$$

Inequality (8) implies that $v_m - v \in l^{0,2}$. As $v_m \in l^{0,2}$, it follows, owing to the Minkowski inequality for sums, that

$$v = v_m + (v - v_m) \in l^{0,2}.$$

Finally, (8) also implies that $v_m \to v$, and the result is proved. ∎

For functions $v : Z_h^d \to \mathbb{R}$ we introduce also the discrete Sobolev space of order 1

$$l^{1,2} = \{v : Z_h^d \to \mathbb{R} : \ |v|_{l^{1,2}} < \infty\},$$

with the norm $|v|_{l^{1,2}}$ defined by

$$|v|_{l^{1,2}} = \Big(|v|^2_{l^{0,2}} + \sum_{i=1}^{d} |\partial_i^+ v|^2_{l^{0,2}} \Big)^{1/2}.$$

Let us endow this function space with the inner product, generating the above norm,

$$(v, w)_{l^{1,2}} = (v, w)_{l^{0,2}} + \sum_{i=1}^{d} (\partial_i^+ v, \partial_i^+ w)_{l^{0,2}},$$

where v, w are any functions in $l^{1,2}$.

To show that the discrete framework we set is a particular case of the general framework considered in Section 2, we begin by checking that $l^{1,2}$ is a reflexive and separable Banach space, continuously and densely embedded into the Hilbert space $l^{0,2}$.

Following the same steps as in the proof of Proposition 1, it could be easily proved that $l^{1,2}$ is a complete inner product space. Therefore $l^{1,2}$ is reflexive. We prove next that $l^{1,2}$ is separable.

Proposition 2. *The function space $l^{1,2}$ is separable.*

Proof. We have to prove that $l^{1,2}$ has a countable dense subset.

Consider the set $S = B \cup \{x + he_i : x \in B,\ i = 1, 2, \ldots, d\}$, with B a ball in Z_h^d. Consider the set of all functions $w(x) \in l^{1,2}$ taking rational values if $x \in S$ and vanishing outside S, and denote it by l. The set l is countable.

Let v be an arbitrary function in $l^{1,2}$. For any given $\varepsilon > 0$, we can choose $w \in l$ such that

$$\begin{aligned}
&\sum_{x\in B} |v(x) - w(x)|^2 h^d + \sum_{i=1}^{d}\sum_{x\in B} |\partial_i^+(v(x) - w(x))|^2 h^d \\
&= \sum_{x\in B} |v(x) - w(x)|^2 h^d \\
&\quad + \sum_{i=1}^{d}\sum_{x\in B} |h^{-1}(v(x+he_i) - w(x+he_i) - (v(x) - w(x)))|^2 h^d \qquad (9)\\
&\le \sum_{x\in B} |v(x) - w(x)|^2 h^d + 2\sum_{i=1}^{d}\sum_{x\in B} |v(x+he_i) - w(x+he_i)|^2 h^{d-2} \\
&\quad + 2\sum_{i=1}^{d}\sum_{x\in B} |v(x) - w(x)|^2 h^{d-2} < \frac{\varepsilon^2}{2}.
\end{aligned}$$

Also, as $|v|^2_{l^{1,2}}$ is an absolutely convergent series, for any given $\varepsilon > 0$ we can choose the diameter of B such that

$$\sum_{x\notin B} |v(x)|^2 h^d + \sum_{i=1}^{d}\sum_{x\notin B} |\partial_i^+ v(x)|^2 h^d < \frac{\varepsilon^2}{2}. \tag{10}$$

From (9) and (10), we obtain

$$|v - w|_{l^{1,2}} < \varepsilon,$$

and the result is proved. ∎

We now check that $l^{1,2}$ is continuously and densely embedded in $l^{0,2}$. The continuity follows immediately from

$$|v|_{l^{0,2}} \le |v|_{l^{1,2}}, \quad \text{for all } v \in l^{1,2}.$$

For the denseness, we prove the following result:

Proposition 3. *The function space $l^{1,2}$ is densely embedded in $l^{0,2}$.*

Proof. We want to prove that $\overline{l^{1,2}} = l^{0,2}$. Let us take an arbitrary function $v \in l^{0,2}$. Let B be a ball in Z_h^d. We consider the function w such that

$$w(x) = \begin{cases} v(x), & x \in B \\ 0, & \text{otherwise.} \end{cases}$$

This function belongs obviously to $l^{1,2}$. Furthermore, for any given $\varepsilon > 0$,

$$|v - w|_{l^{0,2}} < \varepsilon,$$

if the diameter of B is chosen sufficiently large. The result is proved. ∎

Now, we change point of view and consider the functions $w : Q(h) \to \mathbb{R}$ as functions in $[0,T]$ with values in $\mathbb{R}^\infty$, defined by $w(t) := \{w(t,x) : x \in Z_h^d\}$, for all $t \in [0,T]$. For these functions, we consider the subspaces $C([0,T]; l^{0,2})$ and

$$L^2\big([0,T]; l^{1,2}\big) = \big\{w : [0,T] \to l^{1,2} : |w|_{L^2} < \infty\big\},$$

with $|w|_{L^2}^2 = \int_0^T |w(t)|_{l^{1,2}}^2 \, dt$.

We make some assumptions over the regularity of the data f_h and g_h in problem (5).

Assumption 3. We assume

1. $f_h \in L^2([0,T]; l^{0,2})$;
2. $g_h \in l^{0,2}$.

Remark 2. In the above Assumption 3, (1) can be replaced by the weaker assumption $f_h \in L^2([0,T]; (l^{1,2})^*)$, where $(l^{1,2})^*$ denotes the dual space of $l^{1,2}$.

Remark 3. The boundedness of the discrete difference

$$\partial_i^+ a^{ij} = \partial_i^+ a^{ij}(t,x) = h^{-1}(a^{ij}(t, x + he_i) - a^{ij}(t,x))$$

can be obtained from (2) in Assumption 2. In fact,

$$|\partial_i^+ a^{ij}(t,x)| = |h^{-1}(a^{ij}(t, x + he_i) - a^{ij}(t,x))| \leq \Big|\frac{\partial}{\partial x^i}\, a^{ij}(t, x + \tau e_i)\Big|,$$

for some τ such that $0 < \tau < h$. Thus $|(\partial/\partial x^i)\, a^{ij}| \leq K$ implies $|\partial_i^+ a^{ij}| \leq K$.

We define the generalized solution of problem (5).

Definition 3. We say that $u \in C([0,T]; l^{0,2}) \cap L^2([0,T]; l^{1,2})$ is a generalized solution of problem (5) if for all $t \in [0,T]$

$$(u(t),\varphi) = (g_h,\varphi) + \int_0^t \Big\{ -(a^{ij}(s)\partial_i^+ u(s), \partial_j^+ \varphi) + (b^i(s)\partial_i^+ u(s) - \partial_j^+ a^{ij}(s)\partial_i^+ u(s), \varphi) + (c(s)u(s),\varphi) + \langle f_h(s),\varphi\rangle \Big\} ds,$$

holds for all $\varphi \in l^{1,2}$.

Notation. In the above definition, and in the sequel, $(\ ,\ )$ denotes the inner product in $l^{0,2}$.

We prove next an existence and uniqueness result for the solution of the discrete problem (5), providing, in addition, an estimate for the solution. With this result, we show that the numerical scheme is stable, i.e., informally, that the discrete problem's solution remains bounded independently of the space-step h. The result is obtained as consequence of Theorem 1, remaining only to show that, within the discrete framework we constructed, (1)–(2) in Assumption 1 hold.

Theorem 3. *Under (1)–(2) in Assumption 2 and (1)–(2) in Assumption 3, problem (5) admits a unique generalized solution U on $[0,T]$. Moreover*

$$\sup_{0\le t\le T} |u(t)|^2_{l^{0,2}} + \int_0^T |u(t)|^2_{l^{1,2}}\, dt \le N\Big(|g_h|^2_{l^{0,2}} + \int_0^T |f_h(t)|^2_{l^{0,2}}\, dt\Big),$$

with N a constant independent of h.

Proof. Let $L_h(s) : l^{1,2} \to (l^{1,2})^*$ and define for all $\varphi, \psi \in l^{1,2}$

$$\langle L_h(s)\psi,\varphi\rangle := -(a^{ij}(s)\partial_i^+\psi, \partial_j^+\varphi) + (b^i(s)\partial_i^+\psi - \partial_j^+ a^{ij}(s)\partial_i^+\psi, \varphi) + (c(s)\psi,\varphi).$$

It suffices to prove the energy estimates:

1. $\exists K, \lambda > 0$ constants : $\langle L_h(s)\psi,\psi\rangle \le K|\psi|^2_{l^{0,2}} - \lambda|\psi|^2_{l^{1,2}} \quad \forall \psi \in l^{1,2}$;
2. $\exists K$ constant : $|\langle L_h(s)\psi,\varphi\rangle| \le K|\psi|_{l^{1,2}} \cdot |\varphi|_{l^{1,2}} \quad \forall \varphi,\psi \in l^{1,2}$.

For the first property, omitting the variable $x \in Z_h^d$ in the writing, we have

$$
\begin{aligned}
\langle L_h(s)\psi, \psi\rangle &= -\sum_{i,j=1}^{d} \sum_{x\in Z_h^d} a^{ij}(s)\partial_i^+\psi\, \partial_j^+\psi\, h^d \\
&\quad + \sum_{i,j=1}^{d} \sum_{x\in Z_h^d} (b^i(s) - \partial_j^+ a^{ij}(s))\partial_i^+\psi\,\psi\, h^d + \sum_{x\in Z_h^d} c(s)\psi\,\psi\, h^d \\
&\le -\lambda \sum_{i=1}^{d} \sum_{x\in Z_h^d} |\partial_i^+\psi|^2 h^d + 2K \sum_{x\in Z_h^d} \sum_{i=1}^{d} |\partial_i^+\psi\,\psi| h^d + K \sum_{x\in Z_h^d} |\psi|^2 h^d \\
&= -\lambda \sum_{i=1}^{d} |\partial_i^+\psi|^2_{l^{0,2}} + 2K \sum_{i=1}^{d} \sum_{x\in Z_h^d} |\partial_i^+\psi\,\psi| h^d + K|\psi|^2_{l^{0,2}},
\end{aligned} \tag{11}
$$

owing to (1) and (2) in Assumption 2. Applying the Cauchy inequality with ε to the second term of last member in (11), we obtain

$$
\begin{aligned}
&\langle L_h(s)\psi, \psi\rangle \\
&\le -\lambda \sum_{i=1}^{d} |\partial_i^+\psi|^2_{l^{0,2}} + \varepsilon K \sum_{i=1}^{d} \sum_{x\in Z_h^d} |\partial_i^+\psi|^2 h^d + \frac{K}{\varepsilon} \sum_{i=1}^{d} \sum_{x\in Z_h^d} |\psi|^2 h^d + K|\psi|^2_{l^{0,2}} \\
&= -\lambda \sum_{i=1}^{d} |\partial_i^+\psi|^2_{l^{0,2}} - \lambda|\psi|^2_{l^{0,2}} + \varepsilon K \sum_{i=1}^{d} |\partial_i^+\psi|^2_{l^{0,2}} + \frac{K}{\varepsilon}|\psi|^2_{l^{0,2}} + (K+\lambda)|\psi|^2_{l^{0,2}} \\
&\le -\lambda|\psi|^2_{l^{1,2}} + K|\psi|^2_{l^{0,2}},
\end{aligned}
$$

with $\lambda > 0$, K constants, by taking ε sufficiently small, and the first property is proved.

The second property follows from (2) in Assumption 2, using the Cauchy-Schwarz inequality

$$
\begin{aligned}
|\langle L_h(s)\psi, \varphi\rangle| = \Big| &- \sum_{i,j=1}^{d} \sum_{x\in Z_h^d} a^{ij}(s)\partial_i^+\psi\; \partial_j^+\varphi\, h^d + \sum_{i=1}^{d} \sum_{x\in Z_h^d} b^i(s)\partial_i^+\psi\,\varphi\, h^d \\
&- \sum_{i,j=1}^{d} \sum_{x\in Z_h^d} \partial_j^+ a^{ij}(s)\partial_i^+\psi\,\varphi\, h^d + \sum_{x\in Z_h^d} c(s)\psi\,\varphi\, h^d \Big|
\end{aligned}
$$

$$
\begin{aligned}
&\leq K\sum_{i,j=1}^{d}\sum_{x\in Z_h^d}|\partial_i^+\psi\,\partial_j^+\varphi|\,h^d+K\sum_{i=1}^{d}\sum_{x\in Z_h^d}|\partial_i^+\psi\,\varphi|\,h^d+K\sum_{x\in Z_h^d}|\psi\,\varphi|h^d\\
&\leq K\sum_{i=1}^{d}|\partial_i^+\psi|_{l^{0,2}}\sum_{j=1}^{d}|\partial_j^+\varphi|_{l^{0,2}}+K\sum_{i=1}^{d}|\partial_i^+\psi|_{l^{0,2}}|\varphi|_{l^{0,2}}+K|\psi|_{l^{0,2}}|\varphi|_{l^{0,2}}\\
&\leq K|\psi|_{l^{1,2}}\cdot|\varphi|_{l^{1,2}},
\end{aligned}
$$

where the above writing convention is kept. Owing to Theorem 1, the result follows. ■

4. Numerical approximation in space: Approximation results

In this section, we study the approximation properties of the numerical scheme (5). We begin by investigating the consistency of the numerical scheme, and prove that the discrete finite differences approximate the partial derivatives (with accuracy of order 1). The result is obtained under stronger regularity assumptions, and using Sobolev embedding.

Theorem 4. *Let m be an integer strictly greater than $d/2$. Let $u(t)\in W^{m+2,2}$, $v(t)\in W^{m+3,2}$, for all $t\in[0,T]$. Then there exists a constant N independent of h such that*

1. $\displaystyle\sum_{x\in Z_h^d}|u_{x^i}(t,x)-\partial_i^+u(t,x)|^2h^d\leq h^2N|u(t)|^2_{W^{m+2,2}}$,

2. $\displaystyle\sum_{x\in Z_h^d}|v_{x^ix^j}(t,x)-\partial_j^-\partial_i^+v(t,x)|^2h^d\leq h^2N|v(t)|^2_{W^{m+3,2}}$,

for all $t\in[0,T]$.

In order to prove Theorem 4, we state two results. We recall a fundamental theorem on the embedding of $W^{m,2}(U)$ into better spaces (see, e.g., Evans [8], p. 270).

Theorem 5 (Sobolev embedding). *Let U be a bounded domain in $\mathbb{R}^d$ with a C^1 boundary. Let $v\in W^{m,2}(U)$. If $m>\frac{d}{2}$ then $v\in C^{(m-[\frac{d}{2}]-1)+\delta}(U)$, where*

$$
\delta=\begin{cases}\left[\frac{d}{2}\right]+1-\frac{d}{2}, & \text{if } \frac{d}{2} \text{ is not an integer}\\ \text{any positive number}<1, & \text{if } \frac{d}{2} \text{ is an integer.}\end{cases}
$$

Moreover $|v|_{(m-[\frac{d}{2}]-1)+\delta;U} \leq N|v|_{W^{m,2}(U)}$, *with* N *a constant depending only on* m, d, δ *and* U.

Notation. We use the notation $|v|_{k+\delta;U}$ for the norm of v in the Hölder space $C^{k+\delta}(U)$ $(0<\delta<1,\ k=0,1,\ldots)$.

We also recall the following propriety of the Sobolev spaces (see, e.g., Evans [8], p. 247):

Proposition 4. *Let* $v \in W^{m,2}(U)$. *If* V *is an open subset of* U, *then* $v \in W^{m,2}(V)$.

We now prove Theorem 4.

Proof of Theorem 4. Let us prove (1). By the mean-value theorem,

$$\partial_i^+ u(t,x) = h^{-1}(u(t,x+he_i)-u(t,x)) = u_{x^i}(t,x+\theta he_i)$$

and

$$u_{x^i}(t,x)-\partial_i^+ u(t,x) = u_{x^i}(t,x)-u_{x^i}(t,x+\theta he_i) = \theta h\, u_{x^ix^i}(t,x+\theta' he_i),$$

for some $0<\theta'<\theta<1$. We consider d-cells

$$R_h = \{(x^1,x^2,\ldots,x^d)\in\mathbb{R}^d:\ \ x_h^i < x^i < x_h^i+h,\ \ i=1,2,\ldots,d\},$$

with $x_h=(x_h^1,x_h^2,\ldots,x_h^d)\in Z_h^d$ fixed.

For every $x_h\in Z_h^d$,

$$|u_{x^ix^i}(t,x_h+\theta' he_i)| \leq \sup_{x\in R_h}|u_{x^ix^i}(t,x)|,$$

and then

$$|u_{x^i}(t,x_h)-\partial_i^+ u(t,x_h)|^2 \leq h^2 \sup_{x\in R_h}|u_{x^ix^i}(t,x)|^2. \tag{12}$$

Let us consider the particular d-cell where $h=1$ and $x_h=x_1=(0,0,\ldots,0)$, and denote it by R_1^0. We have

$$\sup_{x\in R_h}|u_{x^ix^i}(t,x)| = \sup_{x\in R_1^0}|u_{x^ix^i}(t,x_h+hx)|. \tag{13}$$

Take open balls $B_h\supset R_h$, such that the vertices $x_h^i, x_h^i+h,\ i=1,2,\ldots,d$, of the d-cell lie on the limiting sphere. Denote B_1^0 the ball containing R_1^0. We have

$$\sup_{x\in R_1^0}|u_{x^ix^i}(t,x_h+hx)|^2 \quad\leq\quad \sup_{x\in B_1^0}|u_{x^ix^i}(t,x_h+hx)|^2 \tag{14}$$

Taking in mind Proposition 4, as $B_1^0 \subset U$ is an bounded domain of class C^1, the hypotheses of Theorem 5 are satisfied and, for $m > d/2$,

$$
\begin{aligned}
\sup_{x \in B_1^0} |u_{x^i x^i}(t, x_h + hx)|^2 \leq & N \sum_{|\alpha| \leq m} \int_{B_1^0} |D_x^\alpha u_{x^i x^i}(t, x_h + hx)|^2 dx \\
\leq & N \sum_{|\alpha| \leq m+2} \int_{B_1^0} |D_x^\alpha u(t, x_h + hx)|^2 dx \\
= & N \sum_{|\alpha| \leq m+2} \int_{B_h} |D_x^\alpha u(t, x)|^2 h^{-d} h^{2|\alpha|} dx \\
\leq & N \sum_{|\alpha| \leq m+2} \int_{B_h} |D_x^\alpha u(t, x)|^2 h^{-d} dx
\end{aligned} \tag{15}
$$

Then, by (12), (13), (14) and (15), owing to the particular geometry of the framework we have set, we finally obtain

$$
\begin{aligned}
\sum_{x_h \in Z_h^d} |u_{x^i}(t, x_h) - \partial_i^+ u(t, x_h)|^2 h^d & \leq h^2 N \sum_{|\alpha| \leq m+2} \sum_{x_h \in Z_h^d} \int_{B_h(x_h)} |D_x^\alpha u(t, x)|^2 dx \\
& \leq 2h^2 N \sum_{|\alpha| \leq m+2} \sum_{x_h \in Z_h^d} \int_{R_h(x_h)} |D_x^\alpha u(t, x)|^2 dx \\
& \leq h^2 N |u(t)|^2_{W^{m+2,2}},
\end{aligned}
$$

where $B_h(x_h) := B_h$, $R_h(x_h) := R_h$, and the proof for (1) is complete. The proof for (2) is similar. ■

Finally, owing to the stability and consistency properties of the numerical scheme, we prove the convergence of the discrete problem's solution to the exact problem's solution, and compute a convergence rate. The accuracy obtained is of order 1.

Theorem 6. *Let the hypotheses of Theorems 2 and 3 be satisfied. Let m be an integer strictly greater than $d/2$, and denote by u the solution of (3) in Theorem 2 and by u_h the solution of (5) in Theorem 3. Assume also that $u \in L^2([0,T]; W^{m+3,2})$. Then*

$$
\begin{aligned}
& \sup_{0 \leq t \leq T} |u(t) - u_h(t)|^2_{l^{0,2}} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}} \, dt \\
& \leq h^2 N \int_0^T |u(t)|^2_{W^{m+3,2}} dt + N \Big(|g - g_h|^2_{l^{0,2}} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}} \, dt \Big),
\end{aligned}
$$

for some constant N independent of h.

P r o o f. From (3) and (5), we have that $u-u_h$ satisfies the problem

$$
(16)\qquad \begin{cases} L_h(u-u_h)-\dfrac{d}{dt}(u-u_h)+(L-L_h)u+(f-f_h)=0 & \text{in } Q(h) \\ (u-u_h)(0,x)=(g-g_h)(x) & \text{in } Z_h^d. \end{cases}
$$

We have that $(f-f_h)\in L^2([0,T];l^{0,2})$ and $(g-g_h)\in l^{0,2}$, obviously. With respect to the term $(L-L_h)u$, note that

$$
\sum_{x\in Z_h^d} |(L-L_h)u(t)|^2h^d
$$
$$
=\sum_{x\in Z_h^d} \Big|a^{ij}(t,x)\big(\frac{\partial^2}{\partial x^i\partial x^j}-\partial_j^-\partial_i^+\big)u(t,x)+b^i(t,x)\big(\frac{\partial}{\partial x^i}-\partial_i^+\big)u(t,x)\Big|^2h^d<\infty,
$$

if $u\in W^{m+3,2}$, for all $t\in[0,T]$, owing to (2) in Assumption 2 and to Theorem 4. In consequence, we also have that $(L-L_h)u\in L^2([0,T];l^{0,2})$.

We have shown that problem (16) satisfies the hypotheses of Theorem 3, therefore the following estimate holds

$$
\sup_{0\le t\le T}|u(t)-u_h(t)|^2_{l^{0,2}}+\int_0^T|u(t)-u_h(t)|^2_{l^{1,2}}\,dt
$$
$$
\le N\Big(|g-g_h|^2_{l^{0,2}}+\int_0^T|f(t)-f_h(t)|^2_{l^{0,2}}\,dt+\int_0^T|(L-L_h)u(t)|^2_{l^{0,2}}\,dt\Big).
$$

Owing again to (2) in Assumption 2 and to Theorem 4, the result follows. ■

Corollary 1. *Let the hypotheses of Theorem 6 be satisfied, and denote by u the solution of (3) in Theorem 2 and by u_h the solution of (5) in Theorem 3. If there is a constant N independent of h such that*

$$
|g-g_h|^2_{l^{0,2}}+\int_0^T|f(t)-f_h(t)|^2_{l^{0,2}}\,dt\le h^2N\Big(|g|^2_{W^{m,2}}+\int_0^T|f(t)|^2_{W^{m-1,2}}dt\Big),
$$

then

$$
\sup_{0\le t\le T}|u(t)-u_h(t)|^2_{l^{0,2}}+\int_0^T|u(t)-u_h(t)|^2_{l^{1,2}}\,dt
$$
$$
\le h^2N\Big(\int_0^T|u(t)|^2_{W^{m+3,2}}dt+|g|^2_{W^{m,2}}+\int_0^T|f(t)|^2_{W^{m-1,2}}dt\Big).
$$

P r o o f. The result follows immediately from Theorem 6. ■

5. Numerical approximation in space: Approximation results for $d = 1$

In this final section, we prove a convergence result, under weaker conditions, for the special case of one space dimension. The accuracy obtained is of order 1.

Theorem 7. *Assume that the hypotheses of Theorems 2 and 3 are satisfied. Let $d = 1$, and denote by u the solution of (3) in Theorem 2 and by u_h the solution of (5) in Theorem 3. Assume also that $u \in L^2([0,T]; W^{3,2})$. Then*

$$\sup_{0\le t\le T} |u(t) - u_h(t)|^2_{l^{0,2}} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}}\, dt$$
$$\le h^2 N \int_0^T |u(t)|^2_{W^{3,2}} dt + N\Big(|g - g_h|^2_{l^{0,2}} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}}\, dt\Big),$$

for some constant N independent of h.

Proof. From (3) and (5) we see that $u - u_h$ satisfies the problem

$$\begin{cases} L_h(u-u_h) - \dfrac{d}{dt}(u-u_h) + (L-L_h)u + (f-f_h) = 0 & \text{in } [0,T]\times Z_h \\ (u-u_h)(0,x) = (g-g_h)(x) & \text{in } Z_h. \end{cases}$$

We have that $(f - f_h) \in L^2([0,T]; l^{0,2})$ and $(g - g_h) \in l^{0,2}$. In order to use Theorem 3 and obtain an estimate for $u - u_h$, we need to prove that $(L - L_h)u \in L^2([0,T]; l^{0,2})$, that is, that

$$\begin{aligned} &\sum_{x\in Z_h} |(L-L_h)u(t)|^2 h \\ &= \sum_{x\in Z_h} \Big|a(t,x)\big(\frac{\partial^2}{\partial x^2} - \partial^-\partial^+\big)u(t,x) + b(t,x)\big(\frac{\partial}{\partial x} - \partial^+\big)u(t,x)\Big|^2 h \end{aligned} \tag{17}$$

is finite.

Note that the forward and backward discrete differences can be written

$$\partial^+ u(t,x) = h^{-1}(u(t,x+h) - u(t,x)) = \int_0^1 \frac{\partial}{\partial x} u(t,x+hq)dq \tag{18}$$

and

$$\partial^- u(t,x) = h^{-1}(u(t,x) - u(t,x-h)) = \int_0^1 \frac{\partial}{\partial x} u(t,x-hs)ds, \tag{19}$$

respectively.

Therefore, using (18) and (19), the second-order discrete difference can also be written

$$\begin{aligned}\partial^-\partial^+ u(t,x) =&\partial^- \int_0^1 \frac{\partial}{\partial x} u(t,x+hq)dq\\ =&\int_0^1 \Big(\frac{\partial}{\partial x}\int_0^1 \frac{\partial}{\partial x} u(t,x+hq-hs)dq\Big)ds\\ =&\int_0^1\int_0^1 \frac{\partial^2}{\partial x^2} u(t,x+h(q-s))dsdq.\end{aligned} \tag{20}$$

Using (18), we obtain

$$\begin{aligned}(\frac{\partial}{\partial x} - \partial^+)u(t,x) =& \int_0^1 (\frac{\partial}{\partial x}u(t,x) - \frac{\partial}{\partial x} u(t,x+hq))dq\\ =&h\int_0^1\int_0^1 q\frac{\partial^2}{\partial x^2} u(t,x+hqs)dsdq\end{aligned} \tag{21}$$

and, using (20),

$$\begin{aligned}&(\frac{\partial^2}{\partial x^2} - \partial^-\partial^+)u(t,x)\\ &= \int_0^1\int_0^1 (\frac{\partial^2}{\partial x^2}(t,x) - \frac{\partial^2}{\partial x^2} u(t,x+h(q-s)))dsdq\\ &= h\int_0^1\int_0^1\int_0^1 (q-s)\frac{\partial^3}{\partial x^3}\, u(t,x+hv(q-s))\,dvdsdq.\end{aligned} \tag{22}$$

For (21) we have, using Jensen's inequality,

$$\begin{aligned}\big|(\frac{\partial}{\partial x} - \partial^+)u(t,x)\big|^2 \le& h^2\int_0^1\int_0^1 q^2\big|\frac{\partial^2}{\partial x^2} u(t,x+hqs)\big|^2 dsdq\\ =&h\int_0^1\int_0^{hq} q\big|\frac{\partial^2}{\partial x^2} u(t,x+v)\big|^2 dvdq\\ \le&h\int_0^1 qdq\int_0^h \big|\frac{\partial^2}{\partial x^2} u(t,x+v)\big|^2 dv\\ \le&\frac{h}{2}\int_0^h \big|\frac{\partial^2}{\partial x^2} u(t,x+v)\big|^2 dv = \frac{h}{2}\int_x^{x+h}\big|\frac{\partial^2}{\partial z^2} u(t,z)\big|^2 dz.\end{aligned} \tag{23}$$

Finally, from (23), summing up over Z_h^d

$$\sum_{x\in Z_h^d} \big|\big(\frac{\partial}{\partial x} - \partial^+\big)u(t,x)\big|^2 h \le h^2 N |u(t)|^2_{W^{2,2}}, \tag{24}$$

with N a constant independent of h.

Now for (22), also using Jensen's inequality,

$$\begin{aligned}
&\big|\big(\frac{\partial^2}{\partial x^2} - \partial^-\partial^+\big)u(t,x)\big|^2 \\
&\le h^2 \int_0^1\int_0^1\int_0^1 \big|q-s\big|^2 \big|\frac{\partial^3}{\partial x^3}u(t,x+hv(q-s))\big|^2 dvdsdq \\
&= h^2 \int_0^1\int_0^1\int_0^{h(q-s)} \frac{q-s}{h}\big|\frac{\partial^3}{\partial x^3}\,u(t,x+w)\big|^2 dwdsdq \\
&\le h \int_0^1\int_0^1 \big|q-s\big| dsdq \int_0^h \big|\frac{\partial^3}{\partial x^3}\,u(t,x+w)\big|^2 dw \\
&\le h \int_0^h \big|\frac{\partial^3}{\partial x^3}\,u(t,x+w)\big|^2 dw \le h\int_x^{x+h} \big|\frac{\partial^3}{\partial z^3}\,u(t,z)\big|^2 dz.
\end{aligned} \tag{25}$$

And, by summing up over Z_h^d, from (25) we obtain the estimate

$$\sum_{x\in Z_h^d} \big|\big(\frac{\partial^2}{\partial x^2} - \partial^-\partial^+\big)u(t,x)\big|^2 h \le h^2 N |u(t)|^2_{W^{3,2}} \tag{26}$$

with N a constant independent of h.

From (24) and (26), and owing to (2) in Assumption 2, we proved that (17) is finite. Thus $(L-L_h)u \in L^2([0,T]; l^{0,2})$. We now use Theorem 3 and obtain the estimate

$$\begin{aligned}
&\sup_{0\le t\le T} |u(t)-u_h(t)|^2_{l^{0,2}} + \int_0^T |u(t)-u_h(t)|^2_{l^{1,2}}\,dt \\
&\le N\big(|g-g_h|^2_{l^{0,2}} + \int_0^T |f(t)-f_h(t)|^2_{l^{0,2}}\,dt + \int_0^T |(L-L_h)u(t)|^2_{l^{0,2}}\,dt\big).
\end{aligned}$$

Owing again to (24) and (26), and to (2) in Assumption 2, the result is proved. ■

Corollary 2. *Assume that the hypotheses of Theorem 7 are satisfied, and denote by u the solution of (3) in Theorem 2 and by u_h the solution of (5) in Theorem 3. If there is a constant N independent of h such that*

$$|g-g_h|^2_{l^{0,2}} + \int_0^T |f(t)-f_h(t)|^2_{l^{0,2}}\,dt \le h^2 N\Big(|g|^2_{W^{m,2}} + \int_0^T |f(t)|^2_{W^{m-1,2}} dt\Big),$$

then

$$\sup_{0\le t\le T} |u(t)-u_h(t)|^2_{l^{0,2}} + \int_0^T |u(t)-u_h(t)|^2_{l^{1,2}}\,dt$$
$$\le h^2 N(\int_0^T |u(t)|^2_{W^{3,2}}dt + |g|^2_{W^{m,2}} + \int_0^T |f(t)|^2_{W^{m-1,2}}dt).$$

Proof. The result is an immediate consequence of Theorem 7. ∎

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