A NOTE ON THE NUMERICAL APPROXIMATION OF PARABOLIC EQUATIONS IN HÖLDER SPACES

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Abstract. We consider the initial-boundary value problem for a multidimensional linear parabolic PDE of second order. This problem is solvable in Hölder spaces. The solution is numerically approximated, using finite differences, and the rate of convergence of the time-space finite difference scheme is estimated. Both explicit and implicit discrete operators are given.

1. Introduction

In this article, we consider the initial-boundary value problem (with a Dirichlet boundary condition) for a multidimensional linear parabolic PDE of second order of the following type

$$Lu - u_t + f = 0 \text{ in } Q, \quad u(0,x) = g(x) \text{ for } x \in \bar{U}, \quad u = \bar{g} \text{ on } \partial_x Q,$$

where

$$L(t,x) = a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t,x)\frac{\partial}{\partial x^i} + c\,(t,x)$$

is a uniformly elliptic operator with respect to the space variables, $Q = [0,T] \times U$, with $T \in (0,\infty)$, the domain $U \subset \mathbb{R}^d$ is of class $C^{2+\delta}$, with $\delta \in (0,1)$ fixed, and $\partial_x Q := [0,T] \times \partial U$. The coefficient functions in L and the given functions f, g and $\bar{g}$ belong to Hölder spaces, as it will be explained in detail in the next sections.

We study the numerical approximation of the solution of the above initial-boundary value problem, using finite-difference methods.

The numerical methods and possible approximation results are strongly linked to the theory on the solvability of the PDEs. In this article, we make use of the theory of linear PDEs in Hölder spaces.

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In particular, we consider the approach by Krylov [3]. Using the discrete framework in Krylov [3], we show that the numerical approximation is still valid when weaker conditions are imposed over the PDE data.

Also, we construct discrete operators, using both the explicit and implicit schemes, for cases more general than those presented in Krylov [3].

We summarize the article's content. In Section 2, we go through some preliminaries on Hölder spaces, and state a main solvability result. In Section 3, the particular case where a zero boundary condition is imposed is briefly studied. We prove that the same smoothness for the solution can be obtained, under weaker conditions over the initial data, by stepping away from the origin in the time variable. In Section 4, we follow the presentation of Krylov [3] for the discretization of the PDE problem and prove an existence and uniqueness result for the solution of the discrete problem, and also a convergence result. These results are stated in Krylov [3], but proved only for an elliptic problem. In Section 5, using the same discrete framework, we prove that the numerical approximation holds under weaker conditions. In the final Section, we give discrete operators approximating the corresponding continuous operator, using both the explicit and implicit schemes. These operators are considered in Krylov [3], but for a more particular case of the PDE.

2. Preliminaries and classical results

We briefly introduce the Hölder spaces (see, e.g., Krylov [3], pp. 33-34 and 117-118).

Let U be a domain in $\mathbb{R}^d$, i.e., an open subset of $\mathbb{R}^d$. For $k = 0, 1, 2, \ldots$ we denote $C^k_{loc}(U)$ the set of all functions $u : U \to \mathbb{R}$ whose derivatives $D^\alpha u$ for $|\alpha| \leq k$ (with α a multi-index) are continuous in every bounded subset V of U. We define $|u|_{0;U} := [u]_{0;U} := \sup_U |u|$, $[u]_{k;U} := \max_{|\alpha|=k} |D^\alpha u|_{0;U}$.

Definition 1. For $k = 0, 1, 2, \ldots$, the space $C^k(U)$ is the Banach space of all real-valued functions $u \in C^k_{loc}(U)$ for which the norm $|u|_{k;U} = \sum_{j=0}^{k} [u]_{j;U}$ is finite. If $0 < \delta < 1$, we say that u is Hölder continuous with exponent δ in U if the seminorm $[u]_{\delta;U} = \sup_{x,y\in U,\, x\neq y} |u(x) - u(y)|/|x - y|^\delta$ is finite. This seminorm of u is called Hölder's constant of u of order δ.

In the above definition, the notation $|\ \ |$ stands for the Euclidean norm in $\mathbb{R}^d$.

We define $[u]_{k+\delta;U} := \max_{|\alpha|=k} [D^\alpha u]_{\delta;U}$.

Definition 2. For $0 < \delta < 1$ and $k = 0, 1, 2, \ldots$, the Hölder space $C^{k+\delta}(U)$ is the Banach space of all functions $u \in C^k(U)$ for which the norm

$$|u|_{k+\delta;U} = |u|_{k;U} + [u]_{k+\delta;U}$$

is finite.

Now denote $\mathbb{R}^{d+1} = \{(t,x) : t \in \mathbb{R},\ x \in \mathbb{R}^d\}$. In $\mathbb{R}^{d+1}$ define the parabolic distance between the points $z_1 = (t_1, x_1)$, $z_2 = (t_2, x_2)$ as

$$\rho(z_1, z_2) := |x_1 - x_2| + |t_1 - t_2|^{1/2}.$$

Fix a constant $\delta \in (0,1)$. If u is a real-valued function in $Q \subset \mathbb{R}^{d+1}$, we define

$$[u]_{\delta/2,\delta;Q} := \sup_{z_1 \neq z_2,\ z_i \in Q} |u(z_1) - u(z_2)|/\rho^{\delta}(z_1, z_2)$$

and

$$|u|_{\delta/2,\delta;Q} := |u|_{0;Q} + [u]_{\delta/2,\delta;Q}.$$

Definition 3. For $0 < \delta < 1$, $C^{\delta/2,\delta}(Q)$ is the Banach space of all functions u defined in Q for which $|u|_{\delta/2,\delta;Q} < \infty$.

We introduce the parabolic Hölder spaces.

Definition 4. For $0 < \delta < 1$, the parabolic Hölder space $C^{1+\delta/2,2+\delta}(Q)$ is the Banach space of all real-valued functions $u(z)$ defined in Q for which both

(1) $[u]_{1+\delta/2,2+\delta;Q} := [u_t]_{\delta/2,\delta;Q} + \sum_{i,j=1}^{d} [u_{x^i x^j}]_{\delta/2,\delta;Q}$

(2) $|u|_{1+\delta/2,2+\delta;Q} := |u|_{0;Q} + |u_x|_{0;Q} + |u_t|_{0;Q} + \sum_{i,j=1}^{d} |u_{x^i x^j}|_{0;Q} + [u]_{1+\delta/2,2+\delta;Q}$

are finite.

Next, we will set an initial-boundary value problem for a second-order parabolic PDE, and state a classical result on its solvability in Hölder spaces.

We define the elliptic operator of order m.

Definition 5. Let $m \geq 1$ be an integer and $a^{\alpha}(x)$ be some real-valued functions in $\mathbb{R}^d$, given for any multi-index α with $|\alpha| \leq m$. The operator $L = \sum_{|\alpha| \leq m} a^{\alpha}(x) D^{\alpha}$ is called m^{th} order (uniformly) elliptic if there exists a constant $\lambda > 0$ (the constant of ellipticity), such that $\sum_{|\alpha| \leq m} a^{\alpha}(x)\xi^{\alpha} \geq \lambda |\xi|^m$ $\forall x, \xi \in \mathbb{R}^d$.

We consider the second-order operator (in the non-divergence form)[1]

$$L(t,x) = a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t,x)\frac{\partial}{\partial x^i} + c\,(t,x), \tag{2.1}$$

with real coefficients. We assume that, for some $\lambda > 0$ and for each $t > 0$, the operator satisfies $a^{ij}(t,x)\xi^i\xi^j \geq \lambda|\xi|^2$, for all $x, \xi \in \mathbb{R}^d$, so that L is uniformly elliptic with respect to the space variables, with constant of ellipticity λ. Then, for each t, the symmetric matrix $(a^{ij}(x,t))$ is positive definite for any

[1] The operator L is written assuming the usual summation convention. In the sequel, this convention is used whenever it makes the writing simpler.

$x \in \mathbb{R}^d$. We also assume that there exists a constant K such that $|a|_{\delta/2,\delta} \le K$, $|b|_{\delta/2,\delta} \le K$, $|c|_{\delta/2,\delta} \le K$, where $\delta \in (0,1)$ is fixed.

As, for any constant μ, the function $v(t,x) = u(t,x)e^{-\mu t}$ satisfies $Lv - \mu v - v_t + fe^{-\mu t} = 0$ if and only if u satisfies $Lu - u_t + f = 0$, we set $c \le 0$ without loss of generality.

Let $U \subset \mathbb{R}^d$ be a bounded domain. We give a preliminary definition concerning the straightening of the boundary of U (see Krylov [3], p. 78). Denote $B_R(x_0) \subset \mathbb{R}^d$ the open ball in $\mathbb{R}^d$ with center x_0 and radius R. For any $U \subset \mathbb{R}^d$, denote ∂U the boundary of U. Denote also

$$\mathbb{R}^d_+ = \{(x', x^d) : x' = (x^1, \ldots, x^{d-1}) \in \mathbb{R}^{d-1},\ x^d > 0\}.$$

Definition 6. Let $r > 0$ and U be a bounded domain in $\mathbb{R}^d$. We write $U \in C^r$ (or $\partial U \in C^r$) and say that the domain U is of class C^r if there are numbers $\rho_0, K_0 > 0$ such that for any point $x_0 \in \partial U$ there exists a one-to-one mapping ψ of $B_{\rho_0}(x_0)$ onto a domain $D \subset \mathbb{R}^d$ such that

(1) $D_+ := \psi(B_{\rho_0}(x_0) \cap U) \subset \mathbb{R}^d_+$ and $\psi(x_0) = 0$;
(2) $\psi(B_{\rho_0}(x_0) \cap \partial U) = D \cap \{y \in \mathbb{R}^d : y^d = 0\}$;
(3) $[\psi]_{s;B_{\rho_0}(x_0)} + [\psi^{-1}]_{s;D} \le K_0$ for any $s \in [0,r]$;
(4) $|\psi^{-1}(y_1) - \psi^{-1}(y_2)| \le K_0\, |y_1 - y_2|$ for any $y_i \in D$.

Consider the initial-boundary problem, with Dirichlet boundary conditions

$$Lu - u_t + f = 0 \ \text{ in } \ Q, \quad u(0,x) = g(x) \ \text{ for } \ x \in \bar{U}, \quad u = \bar{g} \ \text{ on } \ \partial_x Q, \tag{2.2}$$

where $Q = [0,T] \times U$, with $T \in (0,\infty)$, the domain $U \subset \mathbb{R}^d$ is of class $C^{2+\delta}$, $\partial_x Q := [0,T] \times \partial U$ and f, g and $\bar{g}$ are given functions.

Notation. We denote $\partial_t Q := \{0\} \times \bar{U}$ and $\partial Q := \partial_x Q \cup \partial_t Q$.

We make some assumptions.

Assumption 1. (Consistency conditions)

(1) $\bar{g}(0,x) = g(x)$ for $x \in \partial U$;
(2) $L(0,x)g(x) - \bar{g}_t(0,x) + f(0,x) = 0$ for $x \in \partial U$.

The following result states the solvability of the problem in Hölder spaces (see Krylov [3], p. 153). Denote $\mathbb{R}^{d+1}_+ = \{(t,x) : t \ge 0,\ x \in \mathbb{R}^d\}$.

Theorem 1. *Let $f \in C^{\delta/2,\delta}(\mathbb{R}^{d+1}_+)$, $g \in C^{2+\delta}(\mathbb{R}^d)$, $\bar{g} \in C^{1+\delta/2,2+\delta}(Q)$, with $Q = [0,\infty) \times U$. Let (1)–(2) in Assumption 1 be satisfied. Then there exists a unique function $u \in C^{1+\delta/2,2+\delta}(Q)$ satisfying (2.2). Moreover*

$$|u|_{1+\delta/2,2+\delta;Q} \le N\big(|f|_{\delta/2,\delta;\mathbb{R}^{d+1}_+} + |g|_{2+\delta;\mathbb{R}^d} + |\bar{g}|_{1+\delta/2,2+\delta;Q}\big),$$

where N is a constant depending on d, λ, δ, K, ρ_0, K_0 and the diameter of U.

3. Further results under weaker conditions

Consider now the particular case of the initial-boundary value problem (2.2) where $\bar{g} \equiv 0$, under weaker smoothness imposed over the initial data g.

We state a main result on the existence and uniqueness of the solution of (2.2) (proved, e.g., in Ladyženskaja et al. [4], pp. 412-413, for interior and exterior domains).

Theorem 2. *Let $f \in C^{\delta/2,\delta}(Q)$, $g \in C(\bar{U})$, with $Q = [0,T] \times U$, $T \in (0,\infty)$. Assume that (1) in Assumption 1 is satisfied. Then problem (2.2) with $\bar{g} \equiv 0$ has a unique solution $u(t,x)$ in Q. Moreover*

$$u(t,x) = \int_0^t d\tau \int_U G(t,\tau,x,y) f(\tau,y) dy + \int_U G(t,0,x,y) g(y) dy,$$

where G is the Green's function for problem (2.2).

The following estimates for the Green's function and its derivatives hold (see, e.g., Ladyženskaja et al. [4], pp. 412-414).

Proposition 1. *Let G be the Green's function considered in Theorem 2. The following inequalities hold:*

(1) $|D_t^\alpha D_x^\beta G(t,\tau,x,y)| \leq K(t-\tau)^{-(d+2|\alpha|+|\beta|)/2} \exp(-M|x-y|^2/(t-\tau))$, *where K, M constants, $2|\alpha| + |\beta| \leq 2$ and $\tau < t$;*

(2) $|D_t^\alpha D_x^\beta G(t,\tau,x,y) - D_{t'}^\alpha D_x^\beta G(t',\tau,x,y)|$
$\leq K(t-t')^{(\delta-2|\alpha|-|\beta|+2)/2}(t'-\tau)^{-(\delta+d+2)/2} \exp(-M|x-y|^2/(t-\tau))$, *where K, M constants, $2|\alpha| + |\beta| = 1, 2$ and $\tau < t' < t$;*

(3) $|D_t^\alpha D_x^\beta G(t,\tau,x,y) - D_t^\alpha D_{x'}^\beta G(t,\tau,x',y)|$
$\leq K|x-x'|^\delta (t-\tau)^{-(\delta+d+2)/2} \exp(-M|x''-y|^2/(t-\tau))$, *where K, M constants, $2|\alpha| + |\beta| = 2$, $\tau < t$ and x'' is the one of the points x and x' which is closest to y.*

It can be easily shown from estimate (1) in Proposition 1 that the solution u of problem (2.2) in Theorem 2 belongs to $C^{1,2}(Q)$.

The smoothness of the solution u can be improved stepping away from the time origin in problem (2.2). We prove that, in this case, we obtain a $C^{1+\delta/2,2+\delta}$ solution.

Theorem 3. *Assume that the hypotheses of Theorem 2 are satisfied and denote by u the corresponding solution of problem (2.2) with $\bar{g} \equiv 0$. Let the set $Q_\varepsilon = [\varepsilon, T] \times U$, where ε is a positive constant. Then $u \in C^{1+\delta/2,2+\delta}(Q_\varepsilon)$.*

Proof. Denote

$$u_1(t,x) = \int_0^t d\tau \int_U G(t,\tau,x,y) f(\tau,y) dy \quad \text{and} \quad u_2(t,x) = \int_U G(t,0,x,y) g(y) dy,$$

so that $u(t,x) = u_1(t,x) + u_2(t,x)$.

We note that $u_1(t,x)$ solves the problem

$$Lu - u_t + f = 0 \text{ in } Q, \quad u(0,x) = 0 \text{ for } x \in \bar{U}, \quad u = 0 \text{ on } \partial_x Q,$$

and that $u_2(t,x)$ solves the problem

$$Lu - u_t = 0 \text{ in } Q, \quad u(0,x) = g(x) \text{ for } x \in \bar{U}, \quad u = 0 \text{ on } \partial_x Q.$$

From Theorem 1, we obtain immediately that $u_1 \in C^{1+\delta/2,2+\delta}(Q)$. Thus $u_1 \in C^{1+\delta/2,2+\delta}(Q_\varepsilon)$. It remains to prove that $u_2 \in C^{1+\delta/2,2+\delta}(Q_\varepsilon)$.

From estimate (2) in Proposition 1, with $|\alpha| = 1$, $|\beta| = 0$ and $0 < \varepsilon < t' < t$,

$$\begin{aligned} |D_t G(t,0,x,y) - D_{t'} G(t',0,x,y)| &\le K(t-t')^{\frac{\delta}{2}} \, t'^{-\frac{d+2+\delta}{2}} \exp\Big(-M\frac{|x-y|^2}{t}\Big) \\ &\le N_\varepsilon (t-t')^{\frac{\delta}{2}}, \end{aligned} \tag{3.1}$$

with N_ε a constant independent of t, t' and x.

From estimate (3) in Proposition 1, with $|\alpha| = 0$, $|\beta| = 2$ and $0 < \varepsilon < t$,

$$\begin{aligned} |D_x^\beta G(t,0,x,y) - D_{x'}^\beta G(t,0,x',y)| &\le K|x-x'|^\delta t^{-\frac{d+2+\delta}{2}} \exp\Big(-M\frac{|x''-y|^2}{t}\Big) \\ &\le N_\varepsilon |x-x'|^\delta, \end{aligned} \tag{3.2}$$

with N_ε a constant independent of t, x and x'.

From estimate (1) in Proposition 1, with $0 < \varepsilon < t$ and $|\alpha|$, $|\beta|$ taking the appropriate values,

$$|G(t,0,x,y)| \le N_\varepsilon, \; |D_t G(t,0,x,y)| \le N_\varepsilon, \; |D_x^\beta G(t,0,x,y)| \le N_\varepsilon, \; |\beta| = 1, 2, \tag{3.3}$$

with N_ε a constant independent of t and x.

As g is a bounded function in $\bar{U}$, from (3.1) and (3.2) we obtain $[u_2]_{1+\delta/2,2+\delta;Q_\varepsilon} < \infty$ and, from (3.3)

$$|u_2|_{0;Q_\varepsilon} + |D_x u_2|_{0;Q_\varepsilon} + |D_t u_2|_{0;Q_\varepsilon} + \sum_{i,j=1}^{d} |D_{x^j} D_{x^i} u_2|_{0;Q_\varepsilon} < \infty.$$

Thus $u_2 \in C^{1+\delta/2,2+\delta}(Q_\varepsilon)$, and the result is proved. □

4. Numerical Approximation

We want to discretize problem (2.2). For the discretization, we use the setting in Krylov [3], p. 155, where the time and the space steps are connected.

Take a number $T \in (0,\infty)$ and denote $Q = [0,T] \times U$, with U a bounded domain in $\mathbb{R}^d$. Let $l(h)$ be a function on $(0,1]$ such that $l(h) > 0$ and $l(h) \to 0$ as $h \downarrow 0$. For $h \in (0,1]$ define the $(l(h),h)-$grid on $\mathbb{R}^{d+1}_+$

$$Z_h^{d+1} = \{(t,x) : t = l(h)k, \; x = h\sum_{i=1}^{d} e_i n_i, \;\; k = 0,1,2,\ldots, \; n_i = 0, \pm 1, \pm 2, \ldots\}.$$

Here, (e_i), $i = 1, \ldots, d$ denotes the standard basis of $\mathbb{R}^d$. Let $Q(h) = Q \cap Z_h^{d+1}$ and $Q^0(h) = \{(t,x) \in Q(h) : \operatorname{dist}(x, \partial U) \geq h \text{ and } t \geq l(h)\}$. Denote $\partial' Q(h) = Q(h) \setminus Q^0(h) = \partial'_x Q(h) \cup \partial'_t Q(h)$, with

$$\partial'_x Q(h) = \{(t,x) \in Q(h) : \operatorname{dist}(x, \partial U) < h\}, \quad \partial'_t Q(h) = \{(t,x) \in Q(h) : t < l(h)\}.$$

For any $h \in (0,1]$, $z \in Q^0(h)$, $z_1 \in Q(h)$ denote

$$\mathcal{L}_h u(z) = \sum_{z_1 \in Q(h)} p_h(z, z_1) u(z_1), \tag{4.1}$$

where $p_h(z, z_1)$ are some given numbers.

Consider the following problem, discrete version of problem (2.2)

$$\mathcal{L}_h u(z) + f(z) = 0 \ \ \forall z \in Q^0(h), \quad u(z) = \bar{g}(z) \ \ \forall z \in \partial' Q(h), \tag{4.2}$$

We make assumptions on the behaviour of the discrete operator $\mathcal{L}_h$.

Assumption 2. (Maximum principle). If u is a function defined on $Q(h)$ and for a point $z_0 \in Q^0(h)$ we have $u(z_0) = \max_{Q(h)} u(z) > 0$, then $\mathcal{L}_h u(z_0) \leq 0$.

Assumption 3. The operators $\mathcal{L}_h$ approximate $L - \partial/\partial t$. More precisely, for any $u \in C^{1+\delta/2, 2+\delta}(Q)$ and any $z \in Q^0(h)$ we have

$$|Lu(z) - u_t(z) - \mathcal{L}_h u(z)| \leq K h^\delta |u|_{1+\delta/2, 2+\delta; Q},$$

with K a constant.

Note that Assumption 3 regards the consistency of the discretization.

We next study the stability of the numerical approximation. Under Assumptions 2 and 3, we prove an existence and uniqueness result for the discretized problem and give estimates for the solution (this result is stated in Krylov [3], p. 154, but only proved for an elliptic version).

Theorem 4. *Let Assumptions 2 and 3 be satisfied. Then there is a constant $h_0 > 0$ depending only on κ, K, δ, d and the diameter of U such that for $h \in (0, h_0]$ and for any bounded functions f, $\bar{g}$, the system of linear equations (4.2) has a unique solution $u_h(z)$, $z \in Q(h)$. In addition*

$$\begin{aligned} \max_{Q(h)} (u_h(z))_+ &\leq N \max_{Q^0(h)} f_+(z) + \max_{\partial' Q(h)} \bar{g}_+(z), \\ \max_{Q(h)} (u_h(z))_- &\leq N \max_{Q^0(h)} f_-(z) + \max_{\partial' Q(h)} \bar{g}_-(z), \\ \max_{Q(h)} |u_h(z)| &\leq N \max_{Q^0(h)} |f(z)| + \max_{\partial' Q(h)} |\bar{g}(z)|, \end{aligned}$$

where the constant N depends only on λ, K, d and the diameter of U.

For the proof of the above theorem, we need the following lemma (see Krylov [3], p. 77):

Lemma 1. *For any $R > 0$ there exists a function $v_0 \in C^\infty(\bar{B}_R)$, with $B_R \subset \mathbb{R}^d$, such that $Lv_0 \leq -1$ in B_R. Moreover, $0 < v_0 \leq N_0 = N_0(\lambda, K, R, d)$ in B_R and $v_0 = 0$ on ∂B_R.*

We now prove Theorem 4.[2]

Proof. (Theorem 4) Let n be the number of points in $Q(h)$. Then the linear system (4.2) is a system of n equations about n variables $u_h(z), z \in Q(h)$. Therefore, to prove the first assertion we only need to prove uniqueness of the trivial solution for $f \equiv \bar{g} \equiv 0$. This uniqueness follows at once from the second assertion.

To prove the second assertion, it suffices only to prove the first estimate. In fact, if

$$\max_{Q(h)}(u_h(z))_+ \le N \max_{Q^0(h)} f_+(z) + \max_{\partial' Q(h)} \bar{g}_+(z),$$

then

$$\begin{aligned}\max_{Q(h)}(u_h(z))_- = \max_{Q(h)}((-u_h(z))_+ &\le N \max_{Q^0(h)}(-f)_+(z) + \max_{\partial' Q(h)}(-\bar{g})_+(z)\\ &= N \max_{Q^0(h)} f_-(z) + \max_{\partial' Q(h)} \bar{g}_-(z).\end{aligned}$$

Note that if u_h solves (4.2) then $-u_h$ is a solution of the system obtained from (4.2) taking $-f$ and $-\bar{g}$ instead of f and $\bar{g}$, respectively.

Also

$$\max_{Q(h)}|u_h(z)| = \max_{Q(h)}\big((u_h(z))_+ + (u_h(z))_-\big) = \max_{Q(h)}(u_h(z))_+ + \max_{Q(h)}(u_h(z))_-,$$

and the third estimate follows.

In the proof of the first estimate we assume without loss of generality that $0 \in \bar{Q}$. We take the function v_0 from Lemma 1 with R defined as the diameter of U. Define $v_0^*(t,x) := v_0(x)$ for all $(t,x) \in \bar{Q}$. Note that $(L - \partial/\partial t)v_0^* \le -1$ in Q, so that, by Assumption 3, we can choose h_0 to have $\mathcal{L}_h v_0^* \le -1/2$, for any $h \in (0, h_0]$ and for any $z \in Q^0(h)$. In fact

$$\begin{aligned}&|Lv_0^*(z) - \frac{\partial}{\partial t}v_0^*(z) - \mathcal{L}_h v_0^*(z)| \le Kh^\delta |v_0^*|_{1+\delta/2,2+\delta;Q}\\ &\Longrightarrow \mathcal{L}_h v_0^*(z) \le Kh^\delta |v_0^*|_{1+\delta/2,2+\delta;Q} + Lv_0^*(z) - \frac{\partial}{\partial t}v_0^*(z),\end{aligned}$$

and, as $(L - \partial/\partial t)v_0^* \le -1$ in Q, then $\mathcal{L}_h v_0^*(z) \le Kh^\delta |v_0^*|_{1+\delta/2,2+\delta;Q} - 1$. If we take $h \le ((2K|v_0^*|_{1+\delta/2,2+\delta;Q})^{-1})^{1/\delta}$, then

$$\mathcal{L}_h v_0^*(z) \le -\frac{1}{2}, \ \forall z \in Q^0(h).$$

Now, we take a solution u_h of (4.2) and consider $w = u_h - 2(F + \varepsilon)v_0^* - \bar{G}$ where $F = \max_{Q^0(h)} f_+$, $\bar{G} = \max_{\partial' Q(h)} \bar{g}_+$ and ε is a positive constant.

If we prove that for any ε we have $w \le 0$ in $Q(h)$, then the first estimate will obviously follow. In fact if $w \le 0$ in $Q(h)$ then

$$u_h \le 2(\max_{Q^0(h)} f_+ + \varepsilon)v_0^* + \max_{\partial' Q(h)} \bar{g}_+$$

[2] The proof we give is an adaptation of the proof given in Krylov [3] for an elliptic version of the result.

and

$$\max_{Q(h)}(u_h)_+ = \max_{Q(h)} u_h = \sup_{Q(h)} u_h \le 2v_0^* \max_{Q^0(h)} f_+ + \max_{\partial' Q(h)} \bar{g}_+.$$

By Lemma 1, $v_0^* \le N_0 = N_0(\lambda, K, R, d)$ in Q (with $R =$ diameter of U) and we obtain

$$\max_{Q(h)}(u_h)_+ \le 2N_0 \max_{Q^0(h)} f_+ + \max_{\partial' Q(h)} \bar{g}_+.$$

Assume that $w > 0$ at some points and define z_0 as a point in $Q(h)$, where w takes its maximum value $w(z_0) > 0$. Since $u_h = \bar{g}$ and $v_0^* \ge 0$ on $\partial' Q(h)$,

$$w = \bar{g} - \max_{\partial' Q(h)} \bar{g}_+ - 2v_0^*(\max_{Q^0(h)} f_+ + \varepsilon) \le 0, \quad \text{on} \ \ \partial' Q(h),$$

so that $z_0 \in Q^0(h)$.

By Assumption 2 we obtain $\mathcal{L}_h \bar{G} \le 0$ and $\mathcal{L}_h w(z_0) \le 0$. Note that if $\bar{G} = \max_{\partial' Q(h)} \bar{g}_+ = 0$ then $\mathcal{L}_h \bar{G} = 0 \le 0$ trivially.

Then

$$\begin{aligned} 0 &\ge \mathcal{L}_h w(z_0) = \mathcal{L}_h u_h(z_0) - 2(F+\varepsilon)\mathcal{L}_h v_0^*(z_0) - \mathcal{L}_h \bar{G}(z_0) \\ &= -f(z_0) - 2(F+\varepsilon)\mathcal{L}_h v_0^*(z_0) - \mathcal{L}_h \bar{G}(v_0^*) \\ &\ge -f(z_0) + F + \varepsilon \ge \varepsilon > 0. \end{aligned}$$

We obtained a contradiction and the proposition is proved. □

Furthermore, we prove that the solution of the discrete problem (4.2) converges to the solution of the continuous problem (2.2), and determine the rate of convergence (this result is also stated in Krylov [3], p. 155, but only proved for the elliptic case).

Theorem 5. *Let $f \in C^{\delta/2,\delta}(\mathbb{R}^{d+1}_+)$, $\bar{g} \in C^{1+\delta/2,2+\delta}(\mathbb{R}^{d+1}_+)$. Take $g(x) = \bar{g}(0,x)$ in Theorem 1, and assume that its hypotheses are satisfied. Let $u \in C^{1+\delta/2,2+\delta}(Q)$ be the solution of (2.2). Take a number $h \in (0, h_0]$ and denote by u_h the corresponding solution of (4.2). Then*

$$|u - u_h|_{0,Q(h)} \le Nh^\delta \left(|f|_{\delta/2,\delta;\mathbb{R}^{d+1}_+} + |\bar{g}|_{1+\delta/2,2+\delta;\mathbb{R}^{d+1}_+} \right),$$

where the constant N depends only on d, K, δ, λ, ρ_0, K_0 and the diameter of U.

Proof. For $z \in Q^0(h)$

$$\begin{aligned} |\mathcal{L}_h(u_h - u)(z)| &= |-f(z) - \mathcal{L}_h u(z)| = |Lu(z) - u_t(z) - \mathcal{L}_h u(z)| \\ &\le Kh^\delta |u|_{1+\delta/2,2+\delta;\mathbb{R}^{d+1}_+} \le Nh^\delta \left(|f|_{\delta/2,2;\mathbb{R}^{d+1}_+} + |\bar{g}|_{1+\delta/2,2+\delta;\mathbb{R}^{d+1}_+} \right), \end{aligned}$$

owing to Assumption 3, and to Theorem 1.

Now, notice that $u_h - u$ satisfies the problem

$$\begin{cases} \mathcal{L}_h(u_h - u)(z) = -f(z) - \mathcal{L}_h u(z) & \forall z \in Q^0(h) \\ (u_h - u)(z) = 0 & \forall z \in \partial' Q(h) \cap \partial Q \\ (u_h - u)(z) = (\bar{g} - u)(z) & \forall z \in \partial' Q(h) \setminus \partial Q\,. \end{cases}$$

Therefore, owing to Theorem 4, the desired estimate is obtained.

If $z \in \partial' Q(h)$, then the distance from z to ∂Q is less than h, so that there is a $y \in \partial Q$ satisfying $\rho(z, y) \le h$. Notice that $\partial'_t Q(h) \subset \partial_t Q$ so that if $z \in \partial'_t Q(h)$ then $\rho(z, \partial Q) = 0$ and the inequality is satisfied trivially.

We obtain, using the mean-value theorem and Theorem 1,

$$\begin{aligned} &|(u_h - u)(z)| = |\bar{g}(z) - u(z)| = |\bar{g}(z) - u(z) + u(y) - \bar{g}(y)| \\ &\le |\bar{g}(z) - \bar{g}(y)| + |u(z) - u(y)| \\ &\le h\Big(\sup_{w \in [z,y]} |\nabla \bar{g}(w)| + \sup_{w \in [z,y]} |\nabla u(w)|\Big) \\ &\le h\Big(\sup_{w \in [z,y]} \Big(\sum_{i=1}^{d} |\bar{g}_{x^i}(w)| + |\bar{g}_t(w)|\Big) + \sup_{w \in [z,y]} \Big(\sum_{i=1}^{d} |u_{x^i}(w)| + |u_t(w)|\Big)\Big) \\ &\le h\big(|\bar{g}|_{1+\delta/2, 2+\delta; \mathbb{R}^{d+1}_+} + |u|_{1+\delta/2, 2+\delta; \mathbb{R}^{d+1}_+}\big) \\ &\le h\big(|\bar{g}|_{1+\delta/2, 2+\delta; \mathbb{R}^{d+1}_+} + N\big(|f|_{\delta/2, \delta; \mathbb{R}^{d+1}_+} + |\bar{g}|_{1+\delta/2, 2+\delta; \mathbb{R}^{d+1}_+}\big)\big), \end{aligned}$$

and the result is proved. □

5. Approximation under weaker conditions

In Section 3, we considered the case where weaker smoothness was imposed over the initial data g, for the case of a zero Dirichlet boundary. Under the discrete framework we set in Section 4, Theorem 4 still holds in this case, and for the same reasons.

For the convergence, we state a new proposition. Let $Q_\varepsilon = [\varepsilon, T] \times U$, with $\varepsilon > 0$ a constant, and $Q_\varepsilon(h) = Q(h) \cap Q_\varepsilon$.

Theorem 6. *Let $f \in C^{\delta/2,\delta}(Q)$, $g \in C(\bar{U})$, with $Q = [0, T] \times U$, for $T \in (0, \infty)$. Define*

$$\bar{g}(t, x) = \begin{cases} 0, & x \in \partial U \\ g(x), & \textit{otherwise}\,. \end{cases}$$

Assume that the hypotheses in Theorem 2 are satisfied. Let u be the solution of (2.2), and denote by u_ε its restriction to Q_ε. Take a number $h \in (0, h_0]$, let u_h be the solution of (4.2), and denote by $u_{h\varepsilon}$ its restriction to $Q_\varepsilon(h)$. Then

$$|u_\varepsilon - u_{h\varepsilon}|_{0, Q_\varepsilon(h)} \le N h^\delta \left(|f|_{\delta/2, \delta; \bar{Q}_\varepsilon} + |\bar{g}|_{1+\delta/2, 2+\delta; \bar{Q}_\varepsilon}\right),$$

where the constant N depends only on d, K, δ, λ, ρ_0, K_0, ε and the diameter of U.

Proof. The proof is the same as for Theorem 5 taking, when needed, Q_ε and $Q_\varepsilon(h)$ in place of Q and $Q(h)$, respectively. □

6. Two examples

From what we showed in the previous section, in order to obtain an approximation for the solution of the continuous problem (2.2), with a known rate of convergence, it suffices to consider a discrete operator with the form of operator $\mathcal{L}_h$ in (4.1), and satisfying Assumptions 2 and 3.

We will now construct particular operators, using both the explicit and implicit schemes.

In Krylov [3], pp. 155-156, discrete operators are considered for the particular case where $L = \sum_{i=1}^d a^{ii}(t,x)D_i^2$.[3] We will construct discrete operators for the more general case

$$L = \sum_{i,j=1}^d a^{ij}(t,x)D_iD_j + \sum_{i=1}^d b^i(t,x)D_i + c(t,x),$$

where the coefficients $a^{ij}(t,x)$ satisfy $\sum_{j=1}^d a^{ij}(t,x) \geq 0$, for $i = 1,2,\ldots,d$,and $a^{ij}(t,x) < 0$, for $i \neq j$, $i,j = 1,2,\ldots,d$. Note that there is a large class of positive definite matrices $(a^{ij}(t,x))$ satisfying the preceding conditions. The matrix defined by $a^{ii}(t,x) = d$ for $i = 1,2,\ldots,d$ and $a^{ij}(t,x) = -1$ for $i \neq j$, $i,j = 1,2,\ldots,d$, with eigenvalues 1 and $d+1$ with multiplicity 1 and $d-1$, respectively, is an example.

In the sequel, $\sum_i$, $\sum_j$, $\sum_{i,j}$ denote de summation over $i = 1,\ldots,d$, and $j = 1,\ldots,d$.

First, we consider the explicit scheme. For $(t,x) \in Q^0(h)$, we define the operator

$$\begin{aligned}\mathcal{L}_h u(t,x) :=& -\varepsilon^{-1}h^{-2}\big(u(t,x) - u(t-\varepsilon h^2,x)\big)\\ &+ \sum_{i,j} a^{ij}(t-\varepsilon h^2,x)2^{-1}h^{-2}\Big(u(t-\varepsilon h^2,x+he_i)\\ &+ u(t-\varepsilon h^2,x-he_i) - u(t-\varepsilon h^2,x+h(e_i-e_j))\\ &- 2u(t-\varepsilon h^2,x) - u(t-\varepsilon h^2,x-h(e_i-e_j))\\ &+ u(t-\varepsilon h^2,x-he_j) + u(t-\varepsilon h^2,x+he_j)\Big)\\ &+ \sum_i |b^i(t-\varepsilon h^2,x)|h^{-1}\Big(u(t-\varepsilon h^2,x+he_i\,\mathrm{sign}\,b^i(t-\varepsilon h^2,x))\\ &- u(t-\varepsilon h^2,x)\Big) + c(t,x)u(t,x).\end{aligned} \tag{6.1}$$

[3] In Krylov [3], pp. 86-87, discrete schemes for the operator $L = \sum_{i=1}^d a^{ii}(x)D_i^2 + \sum_{i=1}^d b^i(x)D_i$ are also introduced, in connection to an elliptic problem).

Theorem 7. *Assume the coefficients* $a^{ij}(t,x)$ *are such that* $\sum_j a^{ij}(t,x) \geq 0$, *for* $i = 1,2,\dots,d$ *and* $a^{ij}(t,x) < 0$ *for* $i \neq j$, $i,j = 1,2,\dots,d$. *Let* $l(h) = \varepsilon h^2$, *where* $\varepsilon^{-1} \geq \sup_z(2\sum_{i\leq j} a^{ij}(z) + \sum_i |b^i(z)|)$. *Then the discrete operator* $\mathcal{L}_h$ *defined by (6.1) satisfies Assumptions 2 and 3.*

Proof. To check Assumption 2, let $z_0 = (t_0, x_0) \in Q_h^0$ and

$$u(t_0, x_0) = M = \max_{Q(h)} u(z) > 0.$$

Denote $t_0^{'} = t_0 - \varepsilon h^2$. We then have

$$\begin{aligned} h^2\mathcal{L}_h u(t_0,x_0) = &- M\varepsilon^{-1} + u(t_0^{'},x_0)\Big(\varepsilon^{-1} - 2\sum_{i\leq j} a^{ij}(t_0^{'},x_0) - h\sum_i |b^i(t_0^{'},x_0)|\Big) \\ &+ \frac{1}{2}\sum_i\sum_j a^{ij}(t_0^{'},x_0)\Big(u(t_0^{'},x_0+he_i) + u(t_0^{'},x_0-he_i)\Big) \\ &+ \frac{1}{2}\sum_j\sum_i a^{ij}(t_0^{'},x_0)\Big(u(t_0^{'},x_0+he_j) + u(t_0^{'},x_0-he_j)\Big) \\ &- \sum_{i<j} a^{ij}(t_0^{'},x_0)\Big(u(t_0^{'},x_0+h(e_i-e_j)) + u(t_0^{'},x_0-h(e_i-e_j))\Big) \\ &+ h\sum_i |b^i(t_0^{'},x_0)| u\big(t_0^{'},x_0+he_i \operatorname{sign} b^i(t_0^{'},x_0)\big) + h^2 M c(t_0,x_0) \end{aligned}$$

Owing to the hypotheses over the matrix $(a^{ij}(t,x))$ and also over ε, we obtain

$$\begin{aligned} h^2\mathcal{L}_h u(t_0,x_0) \leq &- M\varepsilon^{-1} + M\Big(\varepsilon^{-1} - 2\sum_{i\leq j} a^{ij}(t_0^{'},x_0) - h\sum_i |b^i(t_0^{'},x_0)|\Big) \\ &+ 2M\sum_i\sum_j a^{ij}(t_0^{'},x_0) - 2M\sum_{i<j} a^{ij}(t_0^{'},x_0) \\ &+ hM\sum_i |b^i(t_0^{'},x_0)| \\ = &\, 0, \end{aligned}$$

and Assumption 2 is satisfied.

We now check Assumption 3. Denote $t' = t - \varepsilon h^2$. The expression we want to estimate, $|Lu(t,x) - u_t(t,x) - \mathcal{L}_h u(t,x)|$, writes

$$
\begin{aligned}
&\big|Lu(t,x) - u_t(t,x) - \mathcal{L}_h u(t,x)\big| \\
&= \Big|\sum_{i,j} a^{ij}(t,x) u_{x^i x^j}(t,x) \\
&\quad + \sum_i b^i(t,x) u_{x^i}(t,x) + c(t,x)u(t,x) - u_t(t,x) + \varepsilon^{-1}h^{-2}(u(t,x) - u(t',x)) \\
&\quad - \sum_{i,j} a^{ij}(t',x) 2^{-1} h^{-2}\Big(u(t',x+he_i) + u(t',x-he_i) - u(t',x+h(e_i-e_j)) \\
&\qquad - 2u(t',x) - u(t',x-h(e_i-e_j)) + u(t',x+he_j) + u(t',x-he_j)\Big) \\
&\quad - \sum_i |b^i(t',x)| h^{-1}\big(u(t',x+he_i \operatorname{sign} b^i(t',x)) - u(t',x)\big) - c(t,x)u(t,x)\Big|.
\end{aligned}
$$

Rearranging and manipulating the terms, we obtain

$$
\begin{aligned}
&\big|Lu(t,x) - u_t(t,x) - \mathcal{L}_h u(t,x)\big| \\
&\le \Big|\sum_{i,j} a^{ij}(t,x) u_{x^i x^j}(t,x) \\
&\quad - \sum_{i,j} \big(a^{ij}(t,x) + a^{ij}(t',x) - a^{ij}(t,x)\big) 2^{-1} h^{-2}\Big(\big(u(t',x+he_i) - u(t',x)\big) \\
&\qquad - \big(u(t',x) - u(t',x-he_i)\big) + \big(u(t',x+he_j) - u(t',x-h(e_i-e_j))\big) \\
&\qquad - \big(u(t',x+h(e_i-e_j)) - u(t',x-he_j)\big)\Big)\Big| \\
&\quad + \Big|\sum_i b^i(t,x) u_{x^i}(t,x) - \sum_i |b^i(t',x)| h^{-1}\big(u(t',x+he_i \operatorname{sign} b^i(t',x)) - u(t',x)\big)\Big| \\
&\quad + \Big|u_t(t,x) - \varepsilon^{-1}h^{-2}\big(u(t,x) - u(t',x)\big)\Big|.
\end{aligned}
$$

Using the mean-value theorem repeatedly, owing the hypotheses over the smoothness of the coefficients the result is easily obtained. □

The operator we have constructed furnishes an explicit scheme for the approximation. It allows the computation of $u_h(t,x)$ on $Q(h)$, starting from $u(0,x)$ (which is given), and then finding $u_h(\varepsilon h^2, x), u_h(2\varepsilon h^2, x)$, and so on, recursively from the formula

$$\begin{aligned}u_h(t,x) =& c'\varepsilon h^2 f(t,x) - c'u(t',x)\\ &- 2^{-1}c'\varepsilon\sum_{i,j} a^{ij}(t',x)\Big(u(t',x+he_i)+u(t',x-he_i)\\ &\qquad - u\big(t',x+h(e_i-e_j)\big) - 2u(t',x)\\ &\qquad - u\big(t',x-h(e_i-e_j)\big)+u(t',x-he_j)+u(t',x+he_j)\Big)\\ &- c'\varepsilon h\sum_i |b^i(t',x)|\Big(u\big(t',x+he_i \operatorname{sign} b^i(t',x)\big) - u(t',x)\Big),\end{aligned}$$

where $t' = t - \varepsilon h^2$ and $c' = (\varepsilon h^2 c(t,x) - 1)^{-1}$.

We note that the restrictions over ε corresponding to the cases where $(a^{ij}(t,x))$ is diagonal or where there are no first-order partial derivatives (as in Krylov [3]) can be obtained immediately from the more general condition presented in Theorem 7.

We consider now the implicit scheme. For the same particular case of the continuous operator L, we define, for $(t,x) \in Q^0(h)$, the discrete operator

$$\begin{aligned}\mathcal{L}_h u(t,x) :=& -\varepsilon^{-1}h^{-2}\big(u(t,x) - u(t-\varepsilon h^2,x)\big)\\ &+ \sum_{i,j} a^{ij}(t,x)2^{-1}h^{-2}\Big(u(t,x+he_i) + u(t,x-he_i)\\ &\qquad - u\big(t,x+h(e_i-e_j)\big) - 2u(t,x) - u\big(t,x-h(e_i-e_j)\big)\\ &\qquad + u(t,x-he_j) + u(t,x+he_j)\Big)\\ &+ \sum_i |b^i(t,x)|h^{-1}\Big(u\big(t,x+he_i \operatorname{sign} b^i(t,x)\big) - u(t,x)\Big)\\ &+ c(t,x)u(t,x).\end{aligned} \tag{6.2}$$

Theorem 8. *Assume that the coefficients $a^{ij}(t,x)$ satisfy the hypotheses in Theorem 7. Then the discrete operator defined by (6.2) satisfies Assumptions 2 and 3.*

Proof. The proof is the same as for Theorem 7. The operator $\mathcal{L}_h$ satisfies Assumption 3 for the same reasons and Assumption 2 with no restrictions over ε. □

The method of computation of $u_h(t,x)$ on $Q(h)$ is implicit: in order to find $u_h((k+1)\varepsilon h^2,x)$ from $u_h(k\varepsilon h^2,x)$ a linear system of n equations about n variables (with n the number of points in $Q(h)$) has to be solved.

References

[1] A. Friedman, *Partial Differential Equations of Parabolic Type*, Dover (2008).

[2] F. F. Gonçalves, *Numerical Approximation of Partial Differential Equations Arising in Financial Option Pricing.* Ph.D. thesis, The University of Edinburgh, UK (2007).

[3] N. V. Krylov, *Lectures on Elliptic and Parabolic Equations in Hölder Spaces*, Amer. Math. Soc., Providence, RI (1996).

[4] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural′ceva, *Linear and Quasi-Linear Equations of Parabolic Type*, Transl. Math. Monogr., Vol. 23, Amer. Math. Soc., Providence, RI (1968).

[5] K. W. Morton and D. F. Mayers, *Numerical Solutions of Partial Differential Equations*, 2nd Ed., Cambridge University Press, Cambridge, UK (2005).

[6] V. Thomée, Finite difference methods for linear parabolic equations, *Finite Difference Methods (Part 1), Solution of Equations in R^n (Part 1)*, P. G. Ciarlet and J. L. Lions, Handbook of Numerical Analysis, Vol. I, North-Holland (1990), 3-196.

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