Non-Adaptive Matroid Prophet Inequalities

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We consider the problem of matroid prophet inequalities. This problem has been extensively studied in case of adaptive prices, with [KW12] obtaining a tight 2-competitive mechanism for all the matroids.

However, the case non-adaptive is far from resolved, although there is a known constant-competitive mechanism for uniform and graphical matroids (see [Cha+20]).

We improve on constant-competitive mechanism from [Cha+20] for graphical matroids, present a separate mechanism for cographical matroids, and combine those to obtain constant-competitive mechanism for all regular matroids.

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Chapter 1

Introduction

Let us consider the *classical prophet inequality problem* [KS77]. A *gambler* observes a sequence of non-negative independent random variables X_1, X_2, ..., X_n, which correspond to a sequence of values for n *items*. The gambler knows the distributions of X_1, X_2, ..., X_n. The gambler is allowed to accept at most one item; and the gambler is interested in maximizing the value of the accepted item. However, the gambler cannot simply select an item of the maximum value, because the values of the n items are revealed to the gambler one by one; and each time a value of the current item is revealed the gambler has to make an irrevocable choice whether to accept the current item or not.

What stopping rule the gambler should use to maximize the expected value of the item they accept? The gambler knows only the distributions of X_1, X_2, ..., X_n while a *prophet* knows the realization of X_1, X_2, ..., X_n. Thus, in contrast to the gambler the prophet can always obtain the maximum item's value. The seminal result of Krengel and Sucheston [KS77] showed that the gambler can obtain at least a half of the expected value obtained by the prophet.

The classical prophet inequality problem led to a series of works on different variants of the problem. A natural variant of the problem is the generalization of the problem where a gambler can buy more than one item, but the set of bought items should satisfy a known feasibility constraint. Formally, let us be given a collection $\mathcal{S} \subseteq 2^{[n]}$ of item sets. Then both gambler and prophet can select any item set S from $\mathcal{S}$. So $\mathcal{S}$ defines a feasibility constraint for selecting items. In most standard examples of feasibility constraints, $\mathcal{S}$ can be defined as a collection of all item sets with cardinality at most k for some natural number k. More generally $\mathcal{S}$ can be defined as a collection of all independent sets in some matroid, in this case we speak about the *matroid prophet inequality problem*.

The result in [Sam84] showed that in the single-item setting a gambler can obtain at least half of the prophet value by using the following threshold-rule: determine a constant T as a function of known distributions and accept the first item exceeding T. This rule results in a 2-competitive mechanism, similar to the adaptive approach of [KS77]. Note, that this approximation guarantee is known to be tight. There is also another method to set a threshold presented in [KW12], which also results in a 2-competitive mechanism. This was extended by Chawla et al. in [Cha+10] and [Cha+20] to the setting of several items.

The results presented in [KW12] further extend to the *matroid prophet inequalities*, where accepted items need to form an independent set in a known matroid. It leads to a 2-competitive mechanism for every matroid, matching the single-item setting result. However, unlike the mechanism in the single-item setting, the mechanism for matroids is adaptive: the thresholds for items are computed based on the previously accepted items. By [KW12], there also exists a constant-competitive adaptive mechanism for feasibility constraints defined as an intersection of constant number of matroids. The mechanism by Kleinberg and Weinberg was further extended to a 2-competitive mechanism for polymatroids by Dütting and Kleinberg in [DK15].

Gravin and Wang [GW19] studied the bipartite matching version of this problem: in their version, the arriving items are the edges of the (known) bipartite graph. Gravin and Wang obtained a 3-competitive non-adaptive mechanism, which assigns thresholds to each vertex in the graph and an edge is accepted only if its weight is at least the sum of the thresholds associated with its endpoints.

Feldman, Svensson and Zenklusen [FSZ16] studied online item selection mechanisms called "online contention resolution schemes" (OCRS). They showed that given special properties, OCRS translate directly into a constant-competitive prophet inequality for the same problem against almighty adversary, i.e. an adversary which knows in advance realizations of all the items and the random bits generated by an algorithm. As a result, they develop a constant-competitive mechanism for prophet inequalities of the intersection of a constant number of matroids, knapsack and matching constraints. Those mechanisms are "almost" non-adaptive in a sense that they fix thresholds for all items, however their mechanisms also impose a subconstraint: an item cannot be accepted if together with previously accepted items it forms one of the fixed forbidden sets.

Finally, in a later version of their paper [FSZ21], they prove that pure non-adaptive mechanisms cannot achieve a constant-competitive approximation even against a "normal" adversary. They construct a family of gammoid matroids showing a lower bound of $\Omega(\log n / \log \log n)$ for a guarantee of non-adaptive mechanisms on gammoids with n

elements.

There have been works studying similar setups with other goals. Chawla et al. [Cha+10] studied a Bayesian item selection process in a fixed item arrival order or against an adversary in control of the order. They studied it from a perspective of the revenue maximization for the auctioneer. The performance is constant-competitive compared to the well-known Myerson mechanism [Mye81], which achieves the largest possible expected revenue among truthful mechanisms. The mechanism by Chawla et al. [Cha+10] has an advantage that it determines static thresholds together with a subconstraint so that each agent can be offered take-it-or-leave-it prices in an online fashion.

Recently, Chawla et al. [Cha+20] developed a 32-competitive non-adaptive mechanism for graphic matroids against adversary item ordering.

1.1 Our results

First, we list the known results for non-adaptive mechanism that were mentioned in the previous section.

Theorem 1 (**Uniform Rank** 1 **Matroid** [Sam84])**.** *There exists a* 2*-competitive non-adaptive mechanism for single-item setting.*

Theorem 2 (**Graphic Matroid** [Cha+20])**.** *There exists a* 32*-competitive non-adaptive mechanism for graphic matroids.*

Now let us list our results. In case of a simple graph, i.e. a graph with no parallel edges or loops, we can slightly improve the above theorem by considering essentially the same mechanism as [Cha+20] but considering a different scaling of a point from the matroid polytope. We provide this result for the sake of completeness.

Theorem 3. *There exists a* 16*-competitive non-adaptive mechanism for graphic matroids in the case of simple graphs.*

Furthermore, the mechanism [Cha+20] can be generalized to the setting of k-column sparse matroids. This result we need later to obtain Theorem 8.

Theorem 4 (**k-Column Sparse Matroids**)**.** *There exists a* $(2^{k+2}k)$*-competitive non-adaptive mechanism for k-column sparse matroids.*

Note, that Theorem 2 of [Cha+20] follows from Theorem 4, since a graphic matroid is also a 2-column sparse matroid over $\mathbb{F}_2$.

Using analogous approach to the one in [Sot13], we also develop a mechanism for cographic matroids.

Theorem 5 (**Cographic Matroids**). *There exists a* 6*-competitive non-adaptive mechanism for cographic matroids.*

The approach in [Sot13] immediately leads to the following result for γ-sparse matroids.

Theorem 6 (**γ-Sparse Matroids**). *There exists a γ-competitive non-adaptive mechanism for γ-sparse matroids.*

Combining the above results and using classic Seymour's decomposition results we obtain the following theorem.

Theorem 7 (**Regular Matroids**). *There exists a* 256*-competitive non-adaptive mechanism for regular matroids.*

Subject to the Structural Hypothesis 1 due to Geelen, Gerards and Whittle, which is stated later, we can also derive the following result.

Theorem 8. *Subject to the Structural Hypothesis 1, for every prime number p there exists a constant-competitive mechanism for every proper minor-closed class of matroids representable over $\mathbb{F}_p$.*

We also would like to observe that some of the recent results on "single sample prophet inequalities" (SSPI) lead to non-adaptive constant-competitive mechanisms. For this, the single sample required by the gambler in SSPI can be directly sampled by our gambler from the available distributions. In particular, the results in [AKW19] and [Car+21] on laminar matroids and truncated partition matroids inspired by the mechanism in [MTW16] lead to non-adaptive mechanisms for prophet inequalities. To obtain these results, it is crucial that the mechanism in [MTW16] does not involve subconstraints, i.e. each item is accepted as long as the item is not in the "observation phase", the item passes its threshold based only on the "observation phase" and the item forms an independent set with previously accepted items. In comparison, it is not clear how from the results on regular matroids in [AKW19] based on the mechanism in [DK14] one can obtain non-adaptive mechanisms.

So the following results can be directly obtained from [AKW19] and [Car+21], respectively.

Theorem 9 (**Laminar Matroid**). *There exists a* 9.6*-competitive non-adaptive mechanism for laminar matroids.*

Theorem 10 (**Truncated Partition Matroid**). *There exists an* 8*-competitive non-adaptive mechanism for truncated partition matroids.*

1.2 Comparison to known results

Our results for cographic matroids and k-column sparse matroids are obtained through modifications of the arguments in [Sot13] and [Cha+20], respectively. The results on regular matroids and minor-closed families of matroids follow the approach outlined in [HN20] for the secretary problem. As necessary building blocks we use our results for cographic and 2-column sparse matroids. Note that a biggest challenge for us is the compatibility of non-adaptive thresholds with contractions. Indeed, standard tools for deriving mechanisms for contraction minors need subconstraints, while subconstraints are not permitted in non-adaptive mechanisms. To obtain our results, we resolve this issue only in the context of matroids representable over finite fields, see arguments in Lemma 13. It would be interesting to see whether analogous results for contraction minors hold with no assumption about representability over finite fields.

1.3 Preliminaries

In this thesis, we consider the matroid prophet inequality problem, where items arrive online in adversarial order. Here, the adversary knows the distributions of all X_1, X_2, ..., X_n and knows the gambler's mechanism, but the realization of X_1, X_2, ..., X_n is not known to the adversary. Based on the available information, the adversary can decide on the order in which items and their values are observed by the gambler.

1.3.1 Prophet inequality

Def 1. *Let* M *be a matroid on the ground set* $[n] := \{1, \ldots, n\}$*, where* $[n]$ *corresponds to* n *items. Let* $\vec{X} := (X_1, \ldots, X_n)$ *be non-negative independent random variables representing the values of these* n *items.*

- *For every subset of items $S \subseteq [n]$ we define its weight as follows*

$$w(S) := \sum_{i \in S} X_i.$$

- *Let $\boldsymbol{PROPH}_M$ be the random variable corresponding to the value obtained by the prophet*

$$\boldsymbol{PROPH}_M := \max_{S \in \mathcal{I}(M)} w(S)\,,$$

where $\mathcal{I}(M)$ is a collection of independent sets for M.

- *Let $\boldsymbol{EPROPH}_M$ be the expectation of the value obtained by prophet*

$$\boldsymbol{EPROPH}_M := E[\boldsymbol{PROPH}_M]\,.$$

Def 2. *Let us be given a number $\alpha > 0$.*

- *We call a mechanism* α-competitive *(alternatively, we say that the mechanism* guarantees an α-approximation*) on the matroid M if the expected value obtained by the gambler via this mechanism is at least $\frac{1}{\alpha}\boldsymbol{EPROPH}_M$.*
- *We call a mechanism* α-competitive *(alternatively, we say that the mechanism* guarantees an α-approximation*) on the matroid class $\mathcal{M}$ if this mechanism is α-competitive for every matroid $M \in \mathcal{M}$.*

1.3.2 Non-adaptive mechanism

We say that a mechanism is *non-adaptive* if it has the following structure:

- Given the distributions of $\vec{X} = (X_1, \ldots, X_n)$, the mechanism determines the values of *thresholds* $\vec{T} = (T_1, \ldots, T_n)$, where each T_i, $i \in [n]$ is a real number or $+\infty$.
- If the value of item $i \in [n]$ is observed, the gambler accepts the item i if and only both conditions hold:
  1. the observed value of X_i is at least T_i
  2. the item i together with all previously selected items forms an independent set with respect to the matroid M.

Note, that a non-adaptive mechanism does not change thresholds during its course. So, none of the thresholds depends on the realization of $\vec{X} = (X_1, \ldots, X_n)$.

Another crucial feature of a non-adaptive mechanism is that the mechanism works only with the original matroid M. A non-adaptive mechanism does not allow us to define a new matroid M', such that a set of items is independent in M' only if it is independent in M, and modify the condition (2) based on M'.

In this work, we focus on non-adaptive mechanisms. From here and later we use the term *mechanism* to refer to non-adaptive mechanisms **exclusively**.

Remark 1. *In this work, non-adaptive mechanisms are allowed to make non-deterministic decisions. Hence, we allow a non-adaptive mechanism to construct the thresholds* $\vec{T} = (T_1, \ldots, T_n)$ *non-deterministically.*

To measure the performance of such a mechanism we use the expected total value, where the expectation is taken not only with respect to $\vec{X} = (X_1, \ldots, X_n)$ *but also with respect to* $\vec{T} = (T_1, \ldots, T_n)$.

1.3.3 Matroids

We provide a review of matroids here. Experienced readers should consider skipping or skimming this section. For further results about matroids, consider consulting [Oxl06].

A *matroid* $M = (E, \mathcal{S})$ is a pair of a finite *ground set* E and a collection $\mathcal{S} \subseteq 2^E$ of *independent* sets. The collection $\mathcal{S} \subseteq 2^E$ of subsets of E satisfies the following conditions:

(i) Empty set is an independent set, so $\varnothing \in \mathcal{S}$.

(ii) The collection $\mathcal{S}$ is closed with respect to taking subsets, so for all $A \subseteq B \subseteq E$ if B is in $\mathcal{S}$ then A is in $\mathcal{S}$.

(iii) The collection $\mathcal{S}$ satisfies so called *augmenation property*. In other words, for all $A, B \subseteq E$ such that $A, B \in \mathcal{S}$ and $|A| > |B|$, there exists $c \in A \setminus B$ such that $B \cup \{c\} \in \mathcal{S}$.

A subset of E is called *dependent* if it is not in $\mathcal{S}$. The inclusion-maximal independent sets are called *bases* and the inclusion-minimal dependent sets are called *circuits*. For every two bases, their cardinalities are equal: for every bases A and B of M we have $|A| = |B|$. A rank function for the matroid M is a function $r_M : 2^E \to \mathbb{N}$ such that for every $A \subseteq E$

the value $r_M(A)$ equals the cardinality of an inclusion-maximal independent subset of A. In the cases when the choice of the matroid is clear from the context, we write r instead of r_M.

Given a matroid M, we can define the *dual matroid* M^* over the same ground set E. A set A is independent for matroid M^* if and only if $E \setminus A$ contains a basis of M. An element $c \in E$ is called a *loop* in M if $r_M(c) = 0$. An element $c \in E$ is called a *free element* in M if $r_{M^*}(c) = 0$. To put it another way, an element c is free, if and only if for every set A, which is independent in M, $A \cup \{c\}$ is also independent in M. We say that elements c and $d \in E$ are *parallel* in matroid M, denoted by $c \parallel d$, if $r_M(c) = r_M(d) = r_M(\{c, d\}) = 1$. One can show that "being parallel" defines an equivalence relation on the non-loop elements of M. A matroid is called *simple* if it has no loops and no parallel elements.

Let $M = (E, \mathcal{S})$ be a matroid and $A \subseteq E$. The *contraction* of M by A, denoted as M/A, is a matroid over ground set $E \setminus A$ with the following independent sets

$$\{S \subseteq E \setminus A \,:\, S \cup A' \in \mathcal{S}\}\,,$$

where A' is an inclusion-maximal independent subset of A.

The restriction of M to A, denoted as $M\,|_A$ or $M \setminus \overline{A}$, is a matroid over the ground set A where a set $S \subseteq A$ is independent in $M\,|_A$ if and only if it is independent in M.

A matroid M' is called a *simple version* of M if M' is obtained from M by deleting all loops and contracting every parallel class of elements into a single element.

For matroids M, N, we say that N is a *minor* of $M = (E, \mathcal{S})$ if N is isomorphic to $M/A \backslash B$ for some disjoint sets $A, B \subseteq E$. A matroid class $\mathcal{M}$ is called *minor-closed* if for any $M \in \mathcal{M}$ every minor of M is also in $\mathcal{M}$.

Let us now list some of the classical examples of matroids, which were extensively studied in the context of various mathematical fields.

- A *uniform* matroid $M = (E, \mathcal{S})$ of rank k is matroid where

  $$\mathcal{S} := \{A \subseteq E \,:\, |A| \leq k\}\,.$$

  When $|E| = n$, we denote the uniform matroid of rank k as $U_{k,n}$.

- A *graphic* matroid over graph $G = (V, E)$ is a matroid $M = (E, \mathcal{S})$, where

  $$\mathcal{S} := \{A \subseteq E \,:\, A \text{ is acyclic}\}\,.$$

  The graphic matroid over graph G is denoted as $M(G)$.

- A *cographic* matroid over graph $G = (V, E)$ is a dual matroid $M = (E, \mathcal{S})$ to the graphic matroid over the same graph G. In this case we have

$$\mathcal{S} := \{A \subseteq E \,:\, (V, E \setminus A) \text{ has the same number of components as } (V, E)\}\,.$$

- A *vector* matroid $M = (E, \mathcal{S})$ is a matroid such that there is a vector space V and a map $\phi : E \to V$ satisfying

$$\mathcal{S} := \{A \subseteq E \,:\, \text{multiset } \phi(A) \text{ is linearly independent}\}\,.$$

  Given a field $\mathbb{F}$, we say that M is *representable* over field $\mathbb{F}$ if M is isomorphic to the vector matroid where V is a vector space over field $\mathbb{F}$.

  A matroid is called *regular* if it is representable over every field. A matroid is called *binary* if it is representable over $\mathbb{F}_2$.

- A *k-column sparse* matroid $M = (E, \mathcal{S})$ is a matroid such that there is a field $\mathbb{F}$ and dimension m and a map $\phi : E \to \mathbb{F}^m$ such that

$$\mathcal{S} := \{A \subseteq E \,:\, \text{multiset } \phi(A) \text{ is linearly independent over } \mathbb{F}\}\,;$$

  and moreover $\phi(c) \in \mathbb{F}^m$ has at most k nonzero coordinates for every $c \in E$.

- A *γ-sparse* matroid $M = (E, \mathcal{S})$ is a matroid such that the inequality $|S| \leqslant \gamma r_M(S)$ holds for every $S \subseteq E$.

- A *laminar* matroid $M = (E, \mathcal{S})$ is a matroid such that there exists a laminar family $\mathcal{F}$ over the ground set E and there are numbers $c_F \in \mathbb{N}$, $F \in \mathcal{F}$ such that

$$\mathcal{S} := \{A \subseteq E \,:\, |A \cap F| \leq c_F \text{ for every } F \in \mathcal{F}\}\,.$$

  Moreover, if $\mathcal{F} = \{E, E_1, \ldots, E_k\}$, where $E_1, \ldots, E_k$ form a partition of the ground set E, then M is called a *truncated partition matroid.* Recall, that a family $\mathcal{F}$ is called *laminar* if for every $A, B \in \mathcal{F}$ we have $A \subseteq B$ or $B \subseteq A$ or $A \cap B = \varnothing$.

Given a matroid $M = (E, \mathcal{S})$ we can define the corresponding polytope $P_M \subseteq \mathbb{R}^E$ as the convex hull of points corresponding to the characteristic vectors of independent sets. The polytope P_M is known to admit the following outer description [Sch03b].

$$\begin{aligned} P_M = \{x \in \mathbb{R}^E \,:\, & x \geq 0 \text{ and} \\ & x(S) \leq r_M(S) \text{ for every } S \subseteq E\}\,, \end{aligned}$$

where $x(S)$ stands for $\sum_{c \in S} x_c$.

For a matroid $M = (E, \mathcal{S})$ and a set $A \subseteq E$ we can define the *closure* of A as the following set

$$\mathrm{cl}_M(A) := \{c \in E \mid r_M(A \cup \{c\}) = r_M(A)\} .$$

For a matroid $M = (E, \mathcal{S})$, we call the following function $\sqcap_M \colon E \times E \to \mathbb{Z}$ a *local connectivity* function

$$\sqcap_M(X, Y) = r(X) + r(Y) - r(X \cup Y) .$$

The following function $\lambda_M \colon E \to \mathbb{Z}_{\geqslant 0}$ is called a *connectivity* function

$$\lambda_M(X) := \sqcap_M(X, E \setminus X) = r(X) + r(E \setminus X) - r(E) .$$

Informally, connectivity functions measure dependence with respect to the matroid between parts of the ground set. To illustrate it, let us consider the connectivity function for vector matroids. Suppose $M = (E, \mathcal{S})$ is a vector matroid defined by a vector space V and a map $\phi : E \to V$. Then we have

$$\begin{aligned}\lambda_M(S) =& r(S) + r(E \setminus S) - r(E) = \\ & \dim(\operatorname{span} \phi(S)) + \dim(\operatorname{span} \phi(E \setminus S)) - \dim(\phi(E)) = \\ & \dim\left((\operatorname{span} \phi(S)) \cap (\operatorname{span} \phi(E \setminus S))\right) .\end{aligned}$$

1.3.4 Ex-ante relaxation to the matroid polytope

The goal of ex-ante relaxation [FSZ16] or [Cha+20] is to reduce the original problem to the problem where item values are distributed as independent Bernoulli random variables. Note, that both problems are using the same matroid.

In the original problem item values $\vec{X} = (X_1, \ldots, X_n)$ are independent random variables with known distributions. For $i \in [n]$ let F_i be the cumulative distribution function of X_i. The reduction of the original problem to a new problem is done using a point p in the matroid polytope P_M. Let us first show that there is a point $p \in P_M$ with properties that prove to be desirable later following the argumentation in [Cha+20].

Lemma 1. *Given a matroid M over the ground set $[n]$ and random variables $\vec{X} = (X_1, \ldots, X_n)$, there exists $p \in P_M$ such that*

$$\boldsymbol{EPROPH}_M \leqslant \sum_{i=1}^{n} p_i t_i ,$$

where $t_i := E[X_i \mid X_i \geqslant F_i^{-1}(1-p_i)]$ *for every* $i \in [n]$[1].

Proof. Let I_{opt} be a random variable indicating an optimal independent set in M with respect to $\vec{X} = (X_1, \ldots, X_n)$. In case when for some realization of $\vec{X} = (X_1, \ldots, X_n)$ there are several optimal independent sets, I_{opt} can be selected as any of these sets. For $i \in [n]$, let p_i be the probability that element i is in I_{opt}. Note that $p = (p_1, \ldots, p_m)$ is a convex combination of independent sets of M, and so lies in P_M.

Due to $\mathbf{EPROPH}_M = E[\sum_{i \in I_{opt}} X_i]$, it remains to show that

$$E[\sum_{i \in I_{opt}} X_i] \leqslant \sum_{i=1}^{n} p_i t_i \,.$$

We have

$$E[\sum_{i \in I_{opt}} X_i] = \sum_{i=1}^{n} P[i \in I_{opt}] E[X_i \mid i \in I_{opt}] = \sum_{i=1}^{n} p_i E[X_i \mid i \in I_{opt}] \,.$$

For every $i \in [n]$ we have that t_i and $E[X_i \mid i \in I_{opt}]$ are expectations of the same random variable X_i but conditioned on the event $X_i \geqslant F_i^{-1}(1-p_i)$ and on the event $i \in I_{opt}$, respectively. Note, that the probability of both these events equals p_i. However, the expectation of X_i conditioned on $X_i \geqslant F_i^{-1}(1-p_i)$ is the "largest" conditional expectation of X_i on an event of probability p_i. Thus, we have $p_i E[X_i \mid i \in I_{opt}] \leqslant p_i t_i$ for every $i \in [n]$ and so we get the desired inequality

$$\sum_{i=1}^{n} p_i E[X_i \mid i \in I_{opt}] \leqslant \sum_{i=1}^{n} p_i t_i \,.$$

□

Let us show how one can use the point $p = (p_1, \ldots, p_n)$ guaranteed by Lemma 1 to reduce the original problem. Let us define independent Bernoulli random variables $\vec{X}' = (X_1', \ldots, X_n')$ as follows, for each $i \in [n]$

$$X_i' = \begin{cases} t_i & \text{with probability } \; p_i \\ 0 & \text{with probability } \; 1 - p_i \,, \end{cases}$$

[1]Here, we assume that for every $i \in [n]$ the event $X_i = F_i^{-1}(1-p_i)$ happens with the zero probability, which is true for all continuous distributions. In case of discrete distributions one needs to introduce appropriate tie-breaking.

where $t_i := E[X_i \mid X_i \geqslant F_i^{-1}(1-p_i)]$.

Let us assume that we have a non-adaptive mechanism for the original matroid M and item values $\vec{X}' = (X_1', \ldots, X_n')$, which sets nonnegative thresholds $\vec{T}' = (T_1', \ldots, T_n')$. By definition of $\vec{X}' = (X_1', \ldots, X_n')$, for every $i \in [n]$ the exact value of T_i' is not relevant per se, but it is crucial whether $t_i \geq T_i'$ or $t_i < T_i'$. If for some $i \in [n]$ we have $T_i' > t_i$ then this item i is "inactive" and so is never selected by the gambler working with M and $\vec{X}' = (X_1', \ldots, X_n')$.

The key is to construct a non-adaptive mechanism for the original matroid M and item values $\vec{X}' = (X_1', \ldots, X_n')$ with positive thresholds $\vec{T}' = (T_1', \ldots, T_n')$ such that for each item $i \in [n]$ the probability that i is selected by the gambler is at least αp_i. Now we can use such a non-adaptive mechanism for the original matroid M and item values $\vec{X}' = (X_1', \ldots, X_n')$ to construct a non-adaptive α-competitive mechanism for the same matroid M and random variables $\vec{X} = (X_1, \ldots, X_n)$. Let us define the thresholds $\vec{T} = (T_1, \ldots, T_n)$ as follows, for every $i \in [n]$

$$T_i := \begin{cases} +\infty & \text{if } t_i < T_i' \\ F_i^{-1}(1-p_i) & \text{otherwise}\,. \end{cases}$$

To see that the thresholds $\vec{T} = (T_1, \ldots, T_n)$ lead to an α-competitive mechanism for M and $\vec{X} = (X_1, \ldots, X_n)$, let us couple random variables X_i' with random variables X_i as follows

$$X_i' := \begin{cases} t_i & \text{if } X_i \geq F_i^{-1}(1-p_i) \\ 0 & \text{otherwise}\,. \end{cases}$$

Note that $\vec{X}' = (X_1', \ldots, X_n')$ are independent Bernoulli random variables, where for each $i \in [n]$ the variable X_i' equals t_i with probability p_i and equals 0 with probability $1 - p_i$. When $\vec{X}'$ are coupled with $\vec{X}$ this way, X_i and X_i' have the same expected value when conditioned on X_i' being t_i. The mechanism with thresholds $\vec{T}$ selects an item $i \in [n]$ when run for $\vec{X}$ only if the mechanism with thresholds $\vec{T}'$ selects the item i when run for $\vec{X}'$. Moreover, for both of these algorithms, conditionally on the event that the item i is selected the expected value of i equals t_i. Now, α-competitiveness guarantee of the thresholds $\vec{T}$ for M and $\vec{X}$ follows from Lemma 1.

Chapter 2

2-competitive adaptive mechanism

Here, we recap the 2-competitive adaptive mechanism for matroid prophet inequalities problem presented by Kleinberg and Weinberg in [KW12].

In their approach, n items are presented one by one to the mechanism in adversarial order — the adversary can adaptively select item arrival order based on information of what's happened so far, but not based on the unknown realizations of item valuations. Let $w : [n] \to \mathbb{R}_+$ be the assignment of weights, as selected from random distribution. Let $w' : [n] \to \mathbb{R}_+$ be another sample from the same distribution, selected independently from w and all other random variables.

Let A be a variable of the item set, accepted by the mechanism. Naturally, A depends on random variables of item weights, as well as decisions made by mechanism and adversary. Let B be the maximum-weight basis of matroid M in weights w'.

Given A, let us define $C = C(A)$ and $R = R(A)$ as follows. The matroid exchange axiom ensures that there is at least one way to select a subset R of B and combine it with A so that $A \cup R$ is a basis, and R is disjoint from A. We select R so that $w'(R)$ is maximum, given constraints above. Let $C = B \setminus R$.

Lemma 2. *R is a maximum-weight basis in $M/_A$ with respect to weights w'.*

Proof. Observe that by definition R is a maximum-weight basis in $M/_A$ among those contained in B. So if R is not a maximum basis in $M/_A$, then such a maximum is attained on a set not contained in B.

Let the ground set of M be $\{e_1, \ldots, e_k\}$, where elements are sorted in decreasing order, i.e. $w'(e_i) \geqslant w'(e_{i+1})$, and the equal elements are ordered so that B is obtained by greedy

Rado-Edmonds algorithm by going over elements e_i in that order and by accepting each element e_i it is possible to accept.

Let the ground set of $M/_A$ be $\{c_1, \ldots, c_t\}$, where elements $c_1, c_2, \ldots, c_t$ form a subsequence of $e_1, \ldots, e_k$.

Let R' be the maximum-weight basis of $M/_A$ obtained by Rado-Edmonds using the order $c_1, c_2, \ldots, c_t$.

We claim that $R' \subseteq B$. Observe that Rado-Edmonds select c_i into R' if and only if $c_i \notin \mathrm{cl}_{M/_A}\{c_1, \ldots, c_{i-1}\}$, which happens if and only if $c_i \notin \mathrm{cl}_M(A \cup \{c_1, \ldots, c_{i-1}\})$. Thus if c_i is in R', the greedy algorithm also puts c_i in B.

Observe that R' is a feasible choice when selecting R, and R is a feasible choice when selecting R', thus $w'(R) = w'(R')$, and R is a maximum-weight basis in $M/_A$. □

Now consider a class of deterministic threshold-based algorithms: an item i is accepted if $X_i \geqslant T_i$, where threshold T_i can depend on the previously seen elements and their valuations. Since the mechanism can't know the arrival order in advance, this implies that the adaptive mechanism might need to recompute the thresholds before each further item arrives. If the mechanism is unable or unwilling to accept an item, it can set T_i to be infinity. This way threshold T_i is a function of a of sequence elements revealed previously.

From here and further, for analysis purposes, let us use T_i as the value of **the threshold at the moment item i arrives**.

Def 3. *For a constant $\alpha > 1$, the deterministic threshold-based algorithm is α-balanced, if for every possible arrival order of the items, and for every set V disjoint from A so that $A \cup V$ is independent, we have that*

$$\sum_{i \in A} T_i \geqslant (1/\alpha) E[w'(C)],$$

$$\sum_{i \in V} T_i \leqslant (1 - 1/\alpha) E[w'(R)].$$

Informally, the first property guarantees that the mechanism obtains a good fraction of the part of the optimum, and the second property guarantees that mechanism doesn't waste much of the other part of optimum.

Lemma 3. *If a deterministic threshold-based algorithm is α-balanced, it's α-competitive.*

Proof. By definition, $OPT = E[w'(B)] = E[w'(C)] + E[w'(R)]$.

To prove the lemma, we use two inequalities:

$$E\left[\sum_{i \in A} T_i\right] \geqslant (1/\alpha) E[w'(C)], \tag{2.1}$$

$$E\left[\sum_{i \in A} (w(i) - T_i)^+\right] \geqslant (1/\alpha) E[w'(R)]. \tag{2.2}$$

Where c^+ denotes $\max(c, 0)$. Observe that for each item we have $w(i) \leqslant T_i + (w(i) - T_i)^+$. Moreover, for each $i \in A$ we have an equality, since those items are accepted by the mechanism, and thus exceed the threshold.

Assuming we have 2.1, 2.2, we prove the lemma in one line:

$$E[w(A)] = E\left[\sum_{i \in A} T_i\right] + E\left[\sum_{i \in A} (w(i) - T_i)^+\right] \geqslant \frac{E[w'(C)] + E[w'(R)]}{\alpha} = \frac{OPT}{\alpha}$$

The inequality 2.1 is immediate by definition of α-balanced mechanism, and now the proof of the inequality 2.2 follows. We split it into claims.

Claim 1. $E\left[\sum_{i \in A} (w(i) - T_i)^+\right] \geqslant E\left[\sum_{i \in R} (w'(i) - T_i)^+\right]$.

Proof. First, note that since mechanism accepts any item satisfying $w(i) \geqslant T_i$, we have

$$E\left[\sum_{i \in A} (w(i) - T_i)^+\right] = E\left[\sum_{i \in [n]} (w(i) - T_i)^+\right].$$

Since $w(i)$ and $w'(i)$ have identical distribution, we have

$$E\left[\sum_{i \in [n]} (w(i) - T_i)^+\right] = E\left[\sum_{i \in [n]} (w'(i) - T_i)^+\right].$$

Finally we note, that since $R \subseteq [n]$, we have

$$E\left[\sum_{i \in [n]} (w'(i) - T_i)^+\right] \geqslant E\left[\sum_{i \in R} (w'(i) - T_i)^+\right].$$

Combining all the inequalities, we obtain the goal. $\square$

Claim 2. $E\left[\sum_{i \in R}(w'(i) - T_i)^+\right] \geqslant (1/\alpha)E[w'(R)]$.

Proof. Recall we have $w'(i) \leqslant T_i + (w'(i) - T_i)^+$, $\forall i \in [n]$. Summing up, we have:

$$E\left[\sum_{i \in R} w'(i)\right] \leqslant E\left[\sum_{i \in R} T_i\right] + E\left[\sum_{i \in R}(w'(i) - T_i)^+\right]$$

Using definition of mechanism being α-balanced, we have:

$$E\left[\sum_{i \in R} T_i\right] + E\left[\sum_{i \in R}(w'(i) - T_i)^+\right] \leqslant (1 - 1/\alpha)E[w'(R)] + E\left[\sum_{i \in R}(w'(i) - T_i)^+\right]$$

Let us combine the above:

$$E\left[\sum_{i \in R} w'(i)\right] \leqslant (1 - 1/\alpha)E[w'(R)] + E\left[\sum_{i \in R}(w'(i) - T_i)^+\right]$$

Regrouping summands, we obtain

$$E\left[\sum_{i \in R}(w'(i) - T_i)^+\right] \geqslant (1/\alpha)E[w'(R)]$$

□

The claims combined give the inequality we needed to show. □

Now we can discuss the mechanism itself. We will show that for every matroid there exist 2-balanced thresholds for it.

Let A_i be the set of items the mechanism accepted after i steps. This way, $A = A_n$.

Suppose we are at the point where mechanism adjusts thresholds after the first i items have already been processed (thus, it knows A_i). Naturally, if item j no longer can be accepted (i.e. $A_i \cup \{j\}$ is dependent set), $T_j = \infty$. Otherwise, let us set

$$T_j = 0.5E[w'(R(A_i)) - w'(R(A_i \cup \{j\}))] = 0.5E[w'(C(A_i \cup \{j\})) - w'(C(A_i))],$$

where the second equality comes from the following considerations:

$$w'(R(A_i)) + w'(C(A_i)) = w'(B) = w'(R(A_i \cup \{j\})) + w'(C(A_i \cup \{j\}))$$

The idea here is that $E[w'(R(A_i))]$ is equal to the expected value the prophet can obtain if they are forced to start with A_i being taken.

This way $E[w'(R(A_i)) - w'(R(A_i \cup \{j\}))]$ equals the expected loss in prophet's utility if they commit to take j, given A_i. The mechanism then accepts the item if it's weight is at least half of the expected prophet loss.

The main downside of Kleinberg and Weinberg's mechanism is that not only mechanism is forced to recompute the thresholds after every subsequent item, but also that those computations might not be tractable depending on the distributions of the items.

Theorem 11. *The above-described threshold mechanism is* 2*-balanced (and consequently* 2*-competitive).*

To prove the theorem, we need to prove two properties from the definition of being 2-balanced (Def 3). The two lemmas below prove them separately. Let x_i be the i-th arrived item.

Lemma 4.

$$\sum_{j \in A} T_j \geqslant (1/\alpha) E[w'(C)]$$

Proof. Inserting the definition of thresholds, we have that sum of theholds is equal to the following.

$$\sum_{x_i \in A} T_{x_i} = 0.5 \sum_{x_i \in A} E[w'(C(A_{i-1} \cup \{x_i\})) - w'(C(A_{i-1})]$$

However, this sum is a sum of telescopic series and most terms cancel out:

$$\sum_{x_i \in A} T_{x_i} = 0.5 E[w'(C(A))] - 0.5 E[w'(C(\varnothing))] \geqslant 0.5 E[w'(C(A))]$$

□

Lemma 5. *Suppose V is disjoint from A, and $A \cup V$ is independent. Then*

$$\sum_{x_i \in V} T_{x_i} \leqslant (1 - 1/\alpha) E[w'(R)]$$

Before we can prove the lemma, we need to state necessary proposition.

Proposition 1. *Suppose that M is a matroid and X and Y are two bases.*

Then there is a bijection $\varphi : X \to Y$, so that $(Y \setminus \{\varphi(x)\}) \cup \{x\}$ is an independent set for each $x \in X$.

If we remove the requirement of φ being bijection, this statement is known as strong basis exchange property. However, it's possible to argue that φ can be made a bijection. See Corollary 39.12a to Theorem 39.12 in [Sch03a], section 39.3.

Lemma 6. *For any independent subset I, function $f(S) = w'(R(S))$ is a submodular function on subsets of I.*

Proof. To prove submodularity, it's sufficient to show the property of diminishing returns (for all S, x, y, where $x \neq y$, and neither of x or y is in S):

$$f(S) - f(S \cup \{x\}) \leqslant f(S \cup \{y\}) - f(S \cup \{x, y\}) \tag{2.3}$$

Consider matroid $M/_S$ instead. Then the property above becomes

$$f(\varnothing) - f(\{x\}) \leqslant f(\{y\}) - f(\{x, y\})$$

Consider a sequence of elements in ground set of $M/_S$, ordered largest to small by weight w'. Due to Rado-Edmonds algorithm, the basis of $M/_S$ can be determined by going over the elements in $M/_S$ in that order, and greedily selecting each element it's possible to add due to matroid constraints.

Similarly, set $R_{M/_S}(T)$, which is a max-weight basis in $M/_{S\cup T}$, can be determined by the same greedy procedure, if we start this procedure with a set, which is a basis of T.

Let us define B as the max-weight basis in $M/_S$, i.e. $B = R_{M/_S}(\varnothing)$, and let $B = \{b_1, b_2, \ldots, b_k\}$, where elements b_i are ordered in the same order they are considered by the greedy algorithm.

Then when we consider $f(T)$, which is weight of $R_{M/_S}(T)$, it is obtained by going over $b_1, b_2, \ldots, b_k$, and taking elements if they are independent when combined with already taken elements and basis of T. Note that we don't need to consider going over elements which are in $M/_S$, but not in B, since those were not taken by greedy algorithm into $R_{M/_S}(\varnothing)$, so they wouldn't be taken into $R_{M/_S}(T)$ as well.

Let B_x, B_y, B_{xy} be the sets $R_{M/_S}(T)$, with T being $\{x\}$, $\{y\}$ and $\{x, y\}$ correspondingly. We need to show $w'(B) - w'(B_x) \leqslant w'(B_y) - w'(B_{xy})$.

We considered function f only on subsets of independent set I, thus all S, $S \cup \{x\}$, $S \cup \{y\}$, $S \cup \{x, y\}$, are independent. Then x is not a loop in neither $M/_S$, nor $M_{S\cup\{y\}}$, and y is not a loop in M_S. Since $S \cup \{x\}$ is an independent set, B_x being the w'-maximum basis of $M/_{S\cup\{x\}}$, satisfies $B_x = B \setminus \{b_i\}$, for some b_i, where i is smallest index so that set $\{b_1, b_2, \dots, b_i, x\}$ either has duplicate elements (that is, $b_i = x$), or is a dependent set in $M/_S$. For simplicity, below we refer to both of those situations as multiset being dependent.

By the same argument, $B_y = B \setminus \{b_j\}$ for some b_j determined likewise.

Without loss of generality assume $j \leqslant i$ (inequality 2.3 we need to show is invariant when swapping x and y, and thus swapping i and j). Then $B_{xy} = B \setminus \{b_j, b_{i'}\}$ for some $b_{i'}$ — since $j \leqslant i$, the element j gets removed in both B_y, B_{xy}, but other removed element $b_{i'}$ might not be the same as b_i.

Claim 3. *$i' \leqslant i$.*

Proof. Suppose for the sake of contradiction $i' > i$. Then by definition of b_i, set $\{b_1, b_2, \dots, b_i, x\}$ is dependent in $M/_S$.

However, by definition of the $b_{i'}$, set $\{b_1, b_2, \dots, b_i, x, y\} \setminus \{b_j\}$ is independent in $M/_S$.

But then $\{b_1, b_2, \dots, b_i, x\}$ is also independent: consider set $\{b_1, b_2, \dots, b_i, x, y\}$. By above, it contains at most one circuit, and it also contains the same circuit as the set $\{b_1, b_2, \dots, b_j, y\}$ contains, and thus removing y from $\{b_1, b_2, \dots, b_i, x, y\}$ makes it independent set.

This way $\{b_1, b_2, \dots, b_i, x\}$ has to be simultaneously dependent and independent, which is a contradiction. □

Consequently, $w'(b_{i'}) \geqslant w'(b_i)$.

This way,

$$f(\varnothing) - f(\{x\}) = w'(b_i) \leqslant w'(b_{i'}) = f(\{y\}) - f(\{x, y\})$$

□

Now let us prove lemma 5.

Proof. Applying the definition of thresholds, we actually need to prove the following:

$$0.5 \sum_{x_i \in V} E[w'(R(A_{i-1}) - w'(R(A_{i-1} \cup \{x_i\}))] \leqslant 0.5E[w'(R)]$$

We see that 0.5 cancels out. We proceed to prove that the inequalities holds not merely in expectation, but for every weight assignment:

$$\sum_{x_i \in V} w'(R(A_{i-1})) - w'(R(A_{i-1} \cup \{x_i\})) \leqslant w'(R)$$

Since $w'(R(S))$ is submodular on subsets of $A \cup V$, using Lemma 6, we have

$$\sum_{x_i \in V} w'(R(A_{i-1}) - w'(R(A_{i-1} \cup \{x_i\})) \leqslant \sum_{x_i \in V} w'(R(A)) - w'(R(A \cup \{x_i\}))$$

By definition, $R(A)$ is a basis in $M/_A$. Without loss of generality assume that V is also a basis in $M/_A$: recall that V is independent in $M/_A$, and in case it is not a basis, we can arbitrarily extend V to be a basis in $M/_A$, since this can only increase the left hand side of the goal inequality.

By Proposition 1, we have bijection $\varphi \colon V \to R(A)$, so that $(R(A) \setminus \varphi(x)) \cup \{x\}$ is independent in $M/_A$, or in other words, $(R(A) \setminus \varphi(x))$ is independent in $M/_{A \cup \{x\}}$. However, $R(A \cup \{x\})$ is the max-weight basis in $M/_{A \cup \{x\}}$. Thus we have

$$w'(R(A)) - w'(\varphi(x)) \leqslant w'(R(A \cup \{x\})).$$

Rewriting, we have

$$w'(R(A)) - w'(R(A \cup \{x\})) \leqslant w'(\varphi(x)).$$

This way, we can bound

$$\sum_{x_i \in V} w'(R(A_{i-1})) - w'(R(A_{i-1} \cup \{x_i\})) \leqslant \sum_{x_i \in V} w'(\varphi(x)) = w'(R),$$

which completes the proof. □

Chapter 3

Graphic and k-column sparse matroids

First, we construct a 16-competitive non-adaptive mechanism for graphic matroids without parallel edges. Our construction is done through the ex-ante relaxation to the matroid polytope, following the works in [FSZ16] or [Cha+20]. Later, we present a constant-competitive non-adaptive mechanism for k-column sparse matroids whenever k is constant.

3.1 Graphic matroids

Now we are ready to provide a 16-competitive non-adaptive mechanism for graphic matroid. The provided mechanism is essentially the one constructed in [Cha+20] but with saving a factor of 2 in the guarantee, which is achieved by rescaling the point from the matroid polytope by 2 and not by 4.

Let us be given a simple graph $G = (V, E)$ and let us consider the corresponding graphic matroid M over the ground set E. Recall that a subset of E is independent with respect to M if and only if it is acyclic in G. Let us also assume that the graph G has n edges and so $E = \{e_1, e_2, \ldots, e_n\}$.

Lemma 7. *Let $p = (p_1, \ldots, p_n)$ be a point in the polytope P_M. Thus we assume that for every $i \in [n]$ the coordinate p_i of p corresponds to the edge e_i. Then there exists an orientation of edges $E = \{e_1, e_2, \ldots, e_n\}$ in the graph $G = (V, E)$ such that for every vertex $v \in V$ we have $\sum_{i\in[n]:e_i\in\delta^-(v)} p_i \leq 2$.*

Proof. Observe that the average degree of a vertex in a forest on $|V|$ vertices is at most $(2|V| - 2)/|V| = 2 - 1/|V| \leqslant 2$.

Let us use this fact to prove the desired statement by induction on the number of vertices in the graph G.

If the graph G has at most two vertices then the orientation is trivial. Otherwise, since p is a convex combination of points corresponding to forests in G, we have that the average of the value $\sum_{i\in[n]:e_i\in\delta(v)} p_i$ over all vertices $v \in V$ is at most 2. Thus there exists a vertex $v \in V$ such that we have $\sum_{i\in[n]:e_i\in\delta(v)} p_i \le 2$. We orient all edges incident to v as edges in $\delta^-(v)$, so these edges are incoming with respect to v. Then we remove the vertex v and all edges incident to it and orient the remaining edges according to the orientation guaranteed by the inductive hypothesis. □

Now we present an algorithm for graphic matroids of simple graphs.

Algorithm 1 A non-adaptive 16-competitive mechanisms for graphic matroids of a simple graph

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1: Let p be a point in the polytope P_M so that the statement of Lemma 1 is satisfied.
2: Let the edges of the original graph $G = (V, E)$ be oriented so that the statement of Lemma 7 is satisfied.
3: For every edge $e_i \in E$, $i \in [n]$, mark the edge e_i as "discarded" independently at random with probability 1/2.
4: Select a cut $S \subseteq V$ uniformly at random, mark all edges not in $[S; \overline{S}]$ as "discarded". Here, $[S; \overline{S}]$ stands for the set of edges which are oriented such that their tail is in S and their head is in $\overline{S}$.
5: Set thresholds $\vec{T} = (T_1, \ldots, T_n)$ as follows, for each $i \in [n]$

$$T_i := \begin{cases} +\infty & \text{if } e_i \text{ is "discarded"} \\ F_i^{-1}(1 - p_i) & \text{otherwise}\,. \end{cases}$$

Lemma 8. *For every $i \in [n]$, we have*

$$P[e_i \textit{ is selected} \mid X_i \ge T_i \textit{ and } e_i \textit{ is not "discarded"}] \ge 1/2\,.$$

Proof. Let us assume that the vertex v is the head of the oriented edge e_i. Let us also assume that e_i is not marked as "discarded" and $X_i \ge T_i$.

Since the edge e_i is not "discarded", the edge e_i is in the selected set $[S; \overline{S}]$. Hence, every not "discarded" edge incident to v has the vertex v as its head.

Thus, as long as no other edge with the head at the vertex v is selected by the gambler, the gambler has to select e_i. We claim, that with probability at least 1/2 no other edge with the head at v was selected by the gambler.

Let I be the event indicating that "the gambler selected an edge e_j, $j \neq i$ such that v is the head of e_j", in other words "there is $j \in [n]$, $j \neq i$ such that v is the head of e_j and $X_j \geq T_j$ and e_j is not "discarded"". Let J indicate the event that "e_i is not marked as "discarded" after the selection of the cut", in other words, "the head of e_i is in $\overline{S}$ and the tail of e_i is in S".

Let us show

$$P[I \mid J] \leq 1/2\,.$$

By the union bound, we have

$$P[I \mid J] \leqslant \sum_{j\in[n]\setminus\{i\}:e_j\in\delta^-(v)} P[X_j \geq T_j \text{ and } e_j \text{ is not "discarded"} \mid J]$$

Note that for each edge $e_j \in \delta^-(v)$ we have $P[X_j \geq T_j | J] = p_j$ and we also have $P[e_j \text{ is not "discarded"} | J] = 1/4$. Note that any edge is not "discarded" in Step 3 of Algorithm 1 with probability 1/2, and not "discarded" in Step 4 of Algorithm 1 with probability 1/4. However, since the probabilities are with respect to the edge $e_j \in \delta^-(v)$ and are counted conditioned on the event J, the conditioned probability of not being "discarded" in Step 4 of Algorithm 1 is 1/2. Moreover, even conditioned on J the events "$X_j \geq T_j$" and "e_j is not "discarded"" are independent events. Thus we have

$$\sum_{j\in[n]\setminus\{i\}:e_j\in\delta^-(v)} P[X_j \geq T_j \text{ and } e_j \text{ is not "discarded"} \mid J] \leq$$

$$\sum_{j\in[n]\setminus\{i\}:e_j\in\delta^-(v)} p_j/4 \leqslant 1/2\,,$$

where the last inequality follows from the orientation. □

We are ready to prove Theorem 3 by showing that Algorithm 1 is a 16-competitive for graphic matroids without parallel edges.

Proof of Theorem 3. By Lemma 8 for every $i \in [n]$ the probability of edge e_i being accepted conditional on $X_i \geq T_i$ and being not "discarded" is at least 1/2.

Overall, the probability of edge e_i being accepted is at least $\frac{1}{16}p_i$. Thus mechanism guarantees at least $\sum_{i=1}^{n} \frac{1}{16}p_i t_i$ of the expected total value. By Lemma 1, we have $\sum_{i\in[n]} \frac{1}{16}p_i t_i \geqslant \frac{1}{16}\mathbf{EPROPH}_M$, finishing the proof. □

3.2 k-column sparse matroids

There are known constant-competitive mechanisms for k-column sparse matroids in the context of the secretary problem [Sot13]. However they do not immediately lead to a non-adaptive mechanism of constant competitiveness guarantee. The reason for that are not the updated thresholds but implicit changes to the considered matroid.

Here, we present a constant competitive mechanism for k-column sparse matroid class for each constant k. Note, graphic matroids form a subclass of 2-column sparse matroids . Because of their significance, 2-column sparse matroids are also known in literature as *represented frame matroids*. Later, we use 2-column sparse matroids to prove results in Section 5.4.

Suppose M is a k-column sparse matroid over field $\mathbb{F}$. In this section, we prove that there exists a $(2^{k+2}k)$-competitive mechanism for M.

Suppose a k-sparse representation of $M = (E, \mathcal{S})$ is defined by a map $\phi : E \to \mathbb{F}^d$. Note, if for some element $t \in E$ the vector $\phi(t)$ is a zero vector then c is a loop and therefore can be removed from consideration.

Now we consider an undirected hyper-multigraph G with vertex set $[d]$. Each matroid element $t \in E$ induces a hyperedge e_t in this graph between non-zero coordinates of $\phi(t)$. Formally, the hyperedge e_t is defined as follows $e_t := \{i \in [d] \,:\, \phi(t)_i \neq 0\}$. We say that a vertex $i \in [d]$ of the hyper-multigraph G is incident to every edge e of G such that $i \in e$. For a vertex $i \in [d]$ we denote the collection of incident hyperedges by $\delta(i)$. The degree of a vertex i in the hyper-multigraph G equals $|\delta(i)|$.

Claim 4. *Suppose I is an independent set of the matroid M. Then the average degree of a vertex is at most k when one considers the hyper-multigraph with vertices $[d]$ and hyperedges $\{e_t \,:\, t \in I\}$.*

Proof. Observe that $|I| \leqslant d$ because having more than d vectors in d-dimensional vector space $\mathbb{F}^d$ leads to a a linear dependency.

Since M is k-column sparse, we have that every edge in $\{e_t \,:\, t \in I\}$ is incident to at most k vertices in $[d]$. Hence, the total degree is at most kd and thus the average degree of a vertex is at most k. □

Now we consider *orientations* of the graph G. An orientation of the graph G is a function φ which maps every edge e_t into one vertex of G incident to e_t. We call $\varphi(e_t)$ to be the head of the edge e_t, and all other vertices, if any, to be tails. For every vertex $i \in [d]$ we denote the set of incoming edges by $\delta^-(i)$, formally $\delta^-(i) = \{e_t \,:\, \varphi(e_t) = i,\, t \in E\}$.

Lemma 9. *Let p be a point in the polytope P_M. We assume that for every $t \in E$, the coordinate p_t of p corresponds to the element t. Then there exists an orientation φ of hyperedges in the hyper-mulrigraph G such that for every vertex $i \in [d]$ we have $\sum_{t \in E: e_t \in \delta^-(i)} p_t \leqslant k$.*

The proof of Lemma 9 is analogous to the proof of Lemma 7. Now let us describe an algorithm for k-column sparse matroids.

Algorithm 2 A non-adaptive $2^{k+2}k$-competitive mechanisms for k-column sparse matroids

1: Let p be a point in the polytope P_M so that the statement of Lemma 1 is satisfied.
2: Let the edges of the hyper-multigraph G be oriented so that the statement of Lemma 9 is satisfied.
3: For every edge $e_i \in E$, $i \in [n]$, mark the edge e_i as "discarded" independently at random with probability $1 - \frac{1}{2k}$.
4: Select a cut $S \subseteq [d]$ uniformly at random, mark all edges not in $[S; \overline{S}]$ as "discarded". Here, $[S; \overline{S}]$ stands for the set of edges which are oriented such that all their tails are in S and their head is in $\overline{S}$. In particular, for $t \in E$ we say that e_t lies in a cut $[S; \overline{S}]$ with respect to the orientation φ if $\varphi(e_t) \in \overline{S}$ and for every $i \in e_t \setminus \{\varphi(e_t)\}$ we have $i \in S$.
5: Set thresholds $\{T_t : t \in E\}$ as follows, for each $t \in E$

$$T_t := \begin{cases} +\infty & \text{if } t \text{ is "discarded"} \\ F_t^{-1}(1 - p_t) & \text{otherwise}\,. \end{cases}$$

Lemma 10. *For every $t \in E$ we have*

$$P[t \text{ is selected} \mid X_t \geq T_t \text{ and } t \text{ is not "discarded"}] \geq 1/2\,.$$

Proof. Note that item $t \in E$ is accepted whenever $X_t \geq T_t$ and no other item was selected from non-discarded edges in $\delta^-(\varphi(t))$. By the union bound, for every event J we can upper bound the probability that

$$P[\text{there } j \in E \setminus \{t\} \text{ such that } j \text{ is selected and } e_j \in \delta^-(\varphi(t)) \mid J] \leqslant \sum_{j \in E \setminus \{t\}: e_j \in \delta^-(\varphi(t))} P[e_j \text{ is not "discarded" and } X_j \geq T_j \mid J]\,.$$

Let J indicate the event that "e_t is not marked as "discarded" after the selection of the cut". Then for each $j \in E \setminus \{t\}$ we have $P[e_j \text{ is not "discarded" and } X_j \geq T_j \mid J] \leq \frac{1}{2k} p_j$. By Lemma 9, we have $\sum_{j \in E: e_j \in \delta^-(\varphi(t))} p_j \leqslant k$, leading to the desired inequality. □

Note that the proof of Lemma 10 is analogous to the proof of Lemma 8. We are ready to prove Theorem by showing that the Algorithm 2 is a $2^{k+2}k$-competitive for k-column sparse matroids.

Proof of Theorem 4. For every item $t \in E$ we have $P[X_t \geq T_t] = p_t$ and $P[t \text{ is not "discarded"}] \geq \frac{1}{2^{k+1}k}$. By Lemma 10, we have that with probability at least $1/2$ the item t is selected when it is not "discarded" and $X_t \geq T_t$. Thus the expected total value of Algorithm 2 is at least $\sum_{j \in E} \frac{1}{2^{k+2}k} p_j t_j$ which is at least $\frac{1}{2^{k+2}k}\mathbf{EPROPH}_M$ by Lemma 1.

$\square$

Chapter 4

Cographic and gamma-sparse matroids

4.1 Cographic matroids

Let us revisit a mechanism of Soto [Sot13] for the cographic matroid secretary problem which is based on the following corollary of Edmond's matroid partitioning theorem [Edm65]. This mechanism leads to a non-adaptive mechanism for cographic matroids.

Proposition 2. *Let $G = (V, E)$ be a three edge-connected graph. Then there exist spanning trees H_1, H_2, H_3 in G such that the union of their complements contains all the edges E, i.e. $E = (E \setminus H_1) \cup (E \setminus H_2) \cup (E \setminus H_3)$.*

Algorithm 3 A non-adaptive 3-competitive mechanisms for cographic matroids in the case of three edge-connectivity

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1: Let H_1, H_2 and H_3 be the spanning trees as in Proposition 11.

2: Uniformly at random select a spanning tree H^* from H_1, H_2 and H_3. Set thresholds $\{T_e : e \in E\}$ as follows, for each $e \in E$

$$T_e := \begin{cases} +\infty & \text{if } e \text{ is not in } H^* \\ 0 & \text{otherwise}\,. \end{cases}$$

Lemma 11. *Let $G = (V, E)$ be a three edge-connected graph and let M be the cographic matroid over G. Then Algorithm 3 is a 3-competitive non-adaptive mechanism for the matroid M.*

Proof. The expected total value of the mechanism provided by Algorithm 3 equals $E[\sum_{e\in E\setminus H^*} X_e]$ which can be estimated as follows

$$E[\sum_{e\in E\setminus H^*} X_e] = \frac{1}{3}E[\sum_{i\in[3]}\sum_{e\in E\setminus H_i} X_e] \geq \frac{1}{3}E[\sum_{e\in E} X_e] \geq \frac{1}{3}\mathbf{EPROPH}_M\,.$$

$\square$

Algorithm 4 A non-adaptive 6-competitive mechanisms for cographic matroids

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1. Delete all loops of M to obtain a matroid M'. Remove all bridges from $G = (V, E)$ and obtain a graph $G' = (V', E')$.
2. Let $C_1, \ldots, C_k$ be equivalence classes of M' with respect to the relation of being parallel. Construct the matroid M'' from M' by contracting all but one edge in each class C_1, C_2, ..., C_k. Note, that the ground set of M'' has k elements and matroid M'' is the cographic matroid over a graph G'', where each connected component of G'' is three edge-connected. Abusing the notation we refer to the elements of the ground set of M'' as C_1, C_2, ..., C_k.
3. Let H_1, H_2 and H_3 be forests in G'' such that the restriction of H_1, H_2 and H_3 to each connected component of G'' satisfies Proposition 11 for the respective connected component.
4. Uniformly at random select a forest H^* from H_1, H_2 and H_3.
5. For each $i \in [k]$ select thresholds $\overline{T}_e$, $e \in C_i$ according to Theorem 1 when the gambler is allowed to accept only one item of C_i and the distributions of X_e, $e \in C_i$ are the same as original distributions of values for $e \in C_i$.
6. Set thresholds $\{T_e : e \in E\}$ as follows, for each $e \in E$

$$T_e := \begin{cases} \overline{T}_e & \text{if } e \in C_i \text{ and } C_i \in H^* \text{ for some } i \in [k] \\ +\infty & \text{otherwise}\,. \end{cases}$$

The next theorem provides a proof for Theorem 5.

Theorem 12. *Let $G = (V, E)$ be a graph and let M be the cographic matroid over G. Then Algorithm 4 is a 6-competitive non-adaptive mechanism for the matroid M.*

Proof. We can assume that G does not have bridges, because every such bridge is a loop in M. Thus these edges can be selected neither by the gambler nor by the prophet. So we can assume $G = G'$ and $M = M'$.

In the case when each connected component of G is three edge-connected, then Algorithm 4 runs Algorithm 3 for each component to obtain a 3-competitive non-adaptive mechanism.

Otherwise, there is one or more pairs of edges e,e' such that $\{e, e'\}$ corresponds to a cut in G. In this case, the edges e,e' correspond to parallel elements of the cographic matroid M.

Algorithm 4 considers the partition of E into classes of parallel elements C_1, C_2, ..., C_k. Let us construct the matroid M'' from M by contracting all but one edge in each class C_1, C_2, ..., C_k. Note, that the ground set of M'' has k elements. Abusing the notation we refer to these elements of the ground set as C_1, C_2, ..., C_k. The matroid M'' is isomorphic to the cographic matroid over a graph G'', where each connected component of G'' is three edge-connected. Following Lemma 11, Algorithm 4 constructs forests H_1, H_2, H_3 for the graph G''.

So Algorithm 4 leads us to a 6-competitive mechanism. Indeed, the prophet with M and with the original distributions of X_e, $e \in E$ performs exactly as the prophet with M'' and with the corresponding distributions of $X''_i := \max_{e\in C_i} X_e$, $i \in [k]$. By selecting forests in Algorithm 4 the gambler acheives in expectation $E[\sum_{i\in[k]} X''_i]/3$ when all classes C_1, C_2, ..., C_k are singletons. However, for classes that are not singletons we need to take into account another 2 approximation factor with respect to the prophet, who can achieve the expected value $E[X''_i]$ for each $i \in [k]$, while the gambler is guaranteed in expectation to achieve only $E[X''_i]/2$ for each $i \in [k]$. $\square$

4.2 Gamma-sparse matroids

Let us also revisit a mechanism of Soto [Sot13] for γ-sparse matroids to verify that it directly leads to a non-adaptive mechanism.

Theorem 13. *Let $M = (E, \mathcal{S})$ be a γ-sparse matroid. There exists a γ-competitive non-adaptive mechanism for M.*

Proof. First observe that the point $x := \mathbb{1}/\gamma$ lies in the matroid polytope P_M. Indeed, it is non-negative and for every set $S \subseteq E(M)$ we have $x(S) = |S|/\gamma \leqslant r_M(S)$.

Then x can be expressed as a convex combination of indicator variables corresponding to the independent sets of M. In other words, we have $x = \sum_{S\in\mathcal{S}} \alpha_S \mathbb{1}_S$ for some $\alpha \geqslant 0$, $\sum_{S\in\mathcal{S}} \alpha_S = 1$, where $\mathbb{1}_S$ refers to the characteristic vector of S.

Now sample an independent set S in matroid M randomly with probability α_S. Let the gambler select all items in S and let the gambler leave all the items not in S unselected.

If X_e is the random variable corresponding to the weight of element $e \in E(M)$, then this mechanism results in a total expected value as follows

$$\sum_{S \in \mathcal{S}} \alpha_S \sum_{e \in S} E[X_e] = \sum_{e \in E} (1/\gamma) E[X_e] = E[\sum_{e \in E} X_e]/\gamma \geqslant \mathbf{EPROPH}/\gamma\,,$$

finishing the proof. □

Observe that Proposition 2 implies that for a three edge-connected graph G, the co-graphic matroid of G is 3-sparse. Thus Lemma 11 is a corollary of Theorem 13.

Similarly, for a planar graph G the graphic matroid is 3-sparse, leading us to the following corollary.

Corollary 1. *Let G is a planar graph and let M be the corresponding graphic matroid. There is a 3-competitive non-adaptive mechanism for M.*

Chapter 5

Representable matroids

Many results in the theory of matroids make use of minors coming from restrictions and contractions. To get access to the toolbox provided by matroid theory, we need to understand how prophet inequality guarantees change when we consider minors.

5.1 Preliminaries

Lemma 12. *Let M be a matroid and let matroid N be a restriction of the matroid M. If there exists an α-competitive non-adaptive mechanism on M, then there is an α-competitive non-adaptive mechanism for N.*

Proof. To obtain a mechanism for the matroid N, we can impose thresholds $+\infty$ for the items that were removed from the ground set to obtain the restriction N from the matroid M. The remaining items are assigned the same thresholds in both mechanisms. $\square$

A similar result for contractions is harder to obtain in the case of non-adaptive mechanisms. Indeed, a straightforward approach would require us to impose the thresholds $+\infty$ for the contracted items, while using the given mechanism on the remaining items. Unfortunately, this would also require us to "change" the underlying matroid, in other words a gambler might be forced to reject an item even though its value is over the assigned threshold and its addition to the currently selected items keeps the selected set independent with respect to M.

Because of this difficulty, in this work we provide a matching result for contractions only for matroids representable over a finite field. This result is sufficient for the purpose of this work.

Lemma 13. *Let $M = (E, \mathcal{S})$ be a matroid representable over the field $\mathbb{F}_p$ for some p. Let $T \subseteq E$ be a subset of the ground set such that $\lambda_M(T) \leqslant k$ for some k.*

Then there exists $S \subseteq T$ so that every set that is independent in $M\mid_S$ is also independent in $M/_{\overline{T}}$ and

$$\boldsymbol{EPROPH}_{M|_S} \geqslant \frac{1}{p^{k+1}} \boldsymbol{EPROPH}_{M/_{\overline{T}}} .$$

Recall that $\overline{T}$ stands for the complement of T with respect to the ground set E.

Proof. Consider the representation of the matroid M over $\mathbb{F}_p$. Let $\phi : E \to \mathbb{F}_p^m$ be the map describing the representation of M. Thus, for every $S \subseteq E$ we have that the set $\phi(S) = \{\phi(e) \in \mathbb{F}_p^m : e \in S\}$ is independent over the field $\mathbb{F}_p$ if and only if S is an independent set for the matroid M.

Since $\lambda_M(T) \leqslant k$ holds, by definition of λ_M we have

$$r_M(T) + r_M(\overline{T}) - r_M(E) \leqslant k .$$

We have $r_M(R) = \dim \operatorname{span}(\phi(R))$ for every $R \subseteq E$. Thus, we have

$$\dim \operatorname{span} \phi(E) = \dim \operatorname{span} \phi(T) + \dim \operatorname{span} \phi(\overline{T}) - \dim \big((\operatorname{span} \phi(T)) \cap (\operatorname{span} \phi(\overline{T})) \big) .$$

and so

$$\dim \big((\operatorname{span} \phi(T)) \cap (\operatorname{span} \phi(\overline{T})) \big) \leqslant k .$$

Since we are working over the field $\mathbb{F}_p$, the linear space $L := (\operatorname{span} \phi(T)) \cap (\operatorname{span} \phi(\overline{T}))$ has at most p^k vectors. Let C be the orthogonal complement of the linear space L in the space $\operatorname{span} \phi(T)$. Thus, we can represent $\operatorname{span} \phi(T)$ as $L \oplus C$. For every vector $v \in \operatorname{span} \phi(T)$ we denote v orthogonal projection to L and C by $v\mid_L$ and $v\mid_C$, respectively.

For each vector $a \in L$, define the set $T_a := \{t \in T : \phi(t)\mid_L = a,\ \phi(t) \neq a\}$. Note that by definition for every $a \in L$ we have $T_a \cap L = \varnothing$. Now let us select a uniformly at random from L.

Claim 5. $E_a[\boldsymbol{EPROPH}_{M|_{T_a}}] \geq \frac{1}{p^k} \boldsymbol{EPROPH}_{M/_{\overline{T}}}$.

Proof. To prove the desired inequality, we prove the corresponding inequality for any realization of item values. From now on we consider the realization of item values fixed and thus we prove the following inequality

$$E_a[\mathbf{PROPH}_{M|_{T_a}}] \geq \frac{1}{p^k}\mathbf{PROPH}_{M/_{\overline{T}}}$$

Let us consider the set I_{opt} on which the prophet achieves $\mathbf{PROPH}_{M/_{\overline{T}}}$. Note that the set I_{opt} does not contain any item e such that $\phi(e)$ is in L, because every such an item e is a loop in $M/_{\overline{T}}$. Thus, the set I_{opt} can be partitioned into sets $I_{opt,a}$, $a \in L$ where $I_{opt,a}$ is a subset of T_a.

The set I_{opt} is independent in $M/_{\overline{T}}$ and so I_{opt} is also independent in M. Hence the sets $I_{opt,a}$, $a \in L$ are also independent in M. Thus for every $a \in L$, $\mathbf{PROPH}_{M|_{T_a}} \geqslant w(I_{opt,a})$. Then we have

$$E_a[\mathbf{PROPH}_{M|_{T_a}}] \geqslant \frac{\sum_{a \in L} w(I_{opt,a})}{|L|} = \frac{1}{|L|}w(I_{opt}) \geqslant \frac{1}{p^k}\mathbf{PROPH}_{M/_{\overline{T}}},$$

finishing the proof of the claim.

□

Let us now select $a^* \in L$ such that $\mathbf{EPROPH}_{M|_{T_a}}$ is maximized. By the previous claim, we have

$$\mathbf{PROPH}_{M|_{T_{a^*}}} \geqslant \frac{1}{p^k}\mathbf{PROPH}_{M/_{\overline{T}}}.$$

Now for every $c \in C$ define set $H_c := \{t \in T_{a^*} : (\phi(t) \mid_C) \cdot c = 1\}$. Now let us select c uniformly at random from C.

Claim 6. $E_c[\boldsymbol{EPROPH}_{M|_{H_c}}] \geq \frac{1}{p}\boldsymbol{EPROPH}_{M|_{\overline{T}_{a^*}}}$.

Proof. To prove the desired inequality, we prove the corresponding inequality for any realization of item values. From now on we consider the realization of item values fixed and thus we prove the following inequality

$$E_c[\mathbf{PROPH}_{M|_{H_c}}] \geq \frac{1}{p}\mathbf{PROPH}_{M|_{\overline{T}_{a^*}}}.$$

Let I_{opt} be the set corresponding to $\mathbf{PROPH}_{M|_{T_{a^*}}}$. Thus, we have that for every $e \in I_{opt}$, $\phi(e)$ is not in L and hence $\phi(e) \mid_C$ is not the zero vector. Due to $P_c[c \cdot t = 1] = 1/p$, for every $t \in T_{a^*}$, we have

$$E_c[w(I_{opt} \cap H_c)] = \sum_{t \in I_{opt}} P_c[c \cdot t = 1] w(t) = \frac{1}{p} \sum_{t \in I_{opt}} w(t) = \frac{1}{p} w(I_{opt}) = \mathbf{PROPH}_{M|_{T_{a^*}}} .$$

Finally, since I_{opt} is independent in M so is $I_{opt} \cap H_c$. Thus, we have

$$E_c[\mathbf{PROPH}_{M|_{H_c}}] \geq \frac{1}{p} \mathbf{PROPH}_{M|_{\overline{T}_{a^*}}} ,$$

finishing the proof of the claim. □

Now let us select c^* so that $\mathbf{EPROPH}_{M|_{H_c}}$ is maximized and let $S^* := H_{c^*}$. Then we have $\mathbf{EPROPH}(M \mid_{S^*}) \geqslant \frac{1}{p^{k+1}} \mathbf{EPROPH}(M/_{\overline{T}})$.

Finally, we need to show that every set independent in $M \mid_{S^*}$ is an independent set in $M/_{\overline{T}}$. Suppose the contrary, i.e. there exists a set that is independent in $M \mid_{S^*}$ but is not an independent set in $M/_{\overline{T}}$. Then $\operatorname{span} S^*$ has a non-trivial intersection with $\operatorname{span} \overline{T}$, suppose $x \in (\operatorname{span} \phi(S^*)) \cup (\operatorname{span} \phi(\overline{T}))$. Let us show that x is a zero vector. Since $x \in \operatorname{span} S^*$, we have $x = \sum_{s \in S^*} \alpha_s \phi(s)$ for some $\alpha_s \in \mathbb{F}_p$, $s \in S^*$.

Let us consider the projections of x on C and L. Since $x \in \operatorname{span} \phi(\overline{T})$ we have that x lies in L and so $x \mid_C$ is the zero vector. Thus $x \mid_C = \sum_{s \in S^*} \alpha_s (\phi(s) \mid_C)$ is the zero vector.

Note that by definition, $\phi(s) \mid_L = a^*$ and $c^* \cdot (\phi(s) \mid_C) = 1$ hold for every $s \in S^*$. Thus over the field $\mathbb{F}_p$ we have

$$\sum_{s \in S^*} \alpha_s = \sum_{s \in S^*} \alpha_s (c^* \cdot (\phi(s) \mid_C)) = c^* \cdot \left(\sum_{s \in S^*} \alpha_s (\phi(s) \mid_C) \right) = c^* \cdot (x \mid_C) = 0 .$$

Now let us consider $x \mid_L$. We have

$$x \mid_L = \sum_{s \in S^*} \alpha_s (\phi(s) \mid_L) = \left(\sum_{s \in S^*} \alpha_s \right) a^* ,$$

where the last expression equals the zero vector since $\sum_{s \in S^*} \alpha_s = 0$. Thus we have a vector $x \in L \oplus C$ such that both projections $x \mid_L$ and $x \mid_C$ are the zero vector. Hence, the vector x is the zero vector, finishing the proof.

□

5.2 Tree Decompositions

Similarly to the approach [HN20] for the matroid secretary problem, we extensively use the tree decomposition of matroids. A tree decomposition of bounded thickness allows us to construct non-adaptive mechanisms with good approximation ratios. Before proceeding with these constructions, let us introduce tree decompositions.

A *tree decomposition* of a matroid $M = (E, \mathcal{S})$ is a pair $(T, \mathcal{X})$ where T is a tree and $\mathcal{X} = \{X_v \subseteq E : v \in V(T)\}$, where sets in $\mathcal{X}$ form a partition of E. Here, we refer to the vertex and edge sets of the tree T as $V(T)$ and $E(T)$, respectively.

Given an edge $e = \{v_1, v_2\} \in E(T)$ of the tree T, let T_1 and T_2 be two connected components of $T - e$, in other word the removal of the edge e from T leads to two connected components T_1 and T_2. The thickness of the edge $e = (v_1, v_2)$ is denoted as $\lambda(e)$ and is defined as follows

$$\lambda(e) := \lambda_M(\cup_{v \in V(T_1)} X_v)\,.$$

The *thickness of the tree decomposition* is the maximum thickness of the edge e in $E(T)$.

Let $\mathcal{M}$ be a family of matroids, M be a matroid and $(T, \mathcal{X})$ be a tree decomposition of M. We say that tree decomposition $(T, \mathcal{X})$ is *$\mathcal{M}$-tree decomposition* if $M\mid_{cl_M(X_v)} \in \mathcal{M}$ holds for every $v \in V(T)$. Let $t_k(\mathcal{M})$ be a set of matroids which have $\mathcal{M}$-tree decomposition of thickness at most k.

Theorem 14. *Let $\mathcal{M}_{\alpha,p}$ be the family of matroids which admit α-competitive non-adaptive mechanisms and are representable over the finite field $\mathbb{F}_p$. Then for every natural number k and every matroid M in $t_k(\mathcal{M}_{\alpha,p})$, the matroid M has an (αp^{k+1})-competitive non-adaptive mechanism.*

Proof. For a natural number m, let $t_{k,m}(\mathcal{M}_{\alpha,p})$ be the set of matroids which have an $\mathcal{M}_{\alpha,p}$-tree decomposition $(T, \mathcal{X})$ of thickness at most k satisfying $|V(T)| = m$.

Let us prove the statement of the lemma by induction on m. The base case follows from the definition of the family $\mathcal{M}_{\alpha,p}$ and the fact that $\mathcal{M}_{\alpha,p} = t_{k,1}(\mathcal{M}_{\alpha,p})$.

Let us now show how to do the inductive step. Let us assume $m \geq 2$ and consider a matroid $M = (E, \mathcal{S})$ in $t_{k,m}(\mathcal{M}_{\alpha,p})$ with its $\mathcal{M}_{\alpha,p}$-tree decomposition $(T, \mathcal{X})$ of thickness at most k and with $|V(T)| = m$. Let ℓ be a leaf of the tree T and let u be the neighbour of the vertex ℓ in the tree T.

Observe that the tree $(V(T) \setminus \{\ell\}, E(T) \setminus \{\ell u\})$ together with the subfamily $\{X_w : w \in V(T) \setminus \{\ell\}\}$ defines an $\mathcal{M}_{\alpha,p}$-tree decomposition of the matroid $M \setminus X_\ell$. Thus the matroid $M \setminus X_\ell$ is in $M \in t_{k,m-1}(\mathcal{M}_{\alpha,p})$. Hence, by the inductive hypothesis there are thresholds T'_e, $e \in E \setminus X_\ell$ guaranteeing αp^{k+1}-competitiveness of the gambler in comparison to the prophet on the matroid $M \setminus X_\ell$.

Claim 7. *There are thresholds T''_e, $e \in X_\ell$ leading to an $(\alpha \cdot p^{k+1})$-competitive non-adaptive mechanism for matroid $M\mid_{X_\ell}$, such that the gambler always selects a set that is independent in $M/_{\overline{X_\ell}}$.*

Proof. By Lemma 13 there exists a set $S \subseteq X_l$ such that every set independent in $M\mid_S$ is also independent in the matroid $M/\overline{X_\ell}$ and

$$\mathbf{EPROPH}_{M|_S} \geqslant \frac{1}{p^{k+1}} \mathbf{EPROPH}_{M/_{\overline{X_\ell}}}.$$

By definition of $\mathcal{M}_{\alpha,p}$ and the appearance of X_ℓ in the tree decomposition, we have that $M\mid_{X_\ell}$ is in the family $\mathcal{M}_{\alpha,p}$. By Lemma 12, since S is a subset of X_ℓ the matroid $M\mid_S$ is also in the family $\mathcal{M}_{\alpha,p}$. Thus, there are thresholds T''_e, $e \in S$ that lead to an α-competitive non-adaptive mechanism on $M\mid_S$. The thresholds T''_e, $e \in X_\ell \setminus S$ can be defined as $+\infty$, finishing the proof of the claim. □

Now we can define thresholds T_e, $e \in E$ for all elements of the matroid M as follows

$$T_e := \begin{cases} T'_e & \text{if } e \notin X_\ell \\ T''_e & \text{otherwise}. \end{cases}$$

Let us now demonstrate that such thresholds T_e, $e \in E$ lead to an (αp^{k+1})-competitive non-adaptive mechanism for M.

First, by the above claim the selected items from X_ℓ always form an independent set in $M/\overline{X_\ell}$ when used with the thresholds T_e, $e \in X_\ell$ on the matroid $M\mid_{X_\ell}$. Thus the definition of the thresholds guarantees that in expectation the value of selected items from X_ℓ is at least $\mathbf{EPROPH}_{M|_{X_\ell}}/(\alpha p^{k+1})$; and in expectation the value of selected items from $E \setminus X_\ell$ is at least $\mathbf{EPROPH}_{M\setminus X_\ell}/(\alpha p^{k+1})$.

To finish the proof, note that we have

$$\mathbf{PROPH}_{M|_{X_\ell}} + \mathbf{PROPH}_{M\setminus X_\ell} \geq \mathbf{PROPH}_M$$

and so

$$\mathbf{EPROPH}_{M|_{X_\ell}} + \mathbf{EPROPH}_{M\setminus X_\ell} \geq \mathbf{EPROPH}_M.$$

□

5.3 Regular matroids

In this section, we prove Theorem 7. Before we proceed to the proof, let us define key notions related to regular matroids.

A subset of the matroid's ground set is called a *circuit*, if it is an inclusion-minimal dependent set. A *cycle* is a subset of the ground set which can be partitioned into a disjoint union of circuits.

Let $M_1 = (E_1, \mathcal{S}_1)$, $M_2 = (E_2, \mathcal{S}_2)$ be two binary matroids. Then the matroid sum $M_1 \triangle M_2$ has the ground set $E_1 \triangle E_2$ and the cycles of $M_1 \triangle M_2$ are all sets of the form $C_1 \triangle C_2$, where C_1 is a cycle for M_1 and C_2 is a cycle for M_2.

Def 4. *Consider two binary matroids $M_1 = (E_1, \mathcal{S}_1)$, $M_2 = (E_2, \mathcal{S}_2)$ and $M = M_1 \triangle M_2$.*

1. *If $|E_1 \cap E_2| = 0$, and $E_1 \neq \varnothing$, $E_2 \neq \varnothing$, M is called a* 1-sum *of M_1 and M_2.*
2. *If $|E_1 \cap E_2| = 1$, $|E_1| \geq 3$, $|E_2| \geq 3$ and $E_1 \cap E_2$ is not a loop of M_1 or M_2 or their dual matroids, M is called a* 2-sum *of M_1 and M_2.*
3. *If $|E_1 \cap E_2| = 3$, $|E_1| \geq 7$, $|E_2| \geq 7$ and $E_1 \cap E_2$ is a circuit in both M_1 and M_2, and $E_1 \cap E_2$ does not contain a circuit in their dual matroids, then M is called a* 3-sum *of M_1 and M_2.*

Proof of Theorem 7. By Seymour's regular matroid decomposition theorem [Sey80], every regular matroid M can be obtained from graphic, cographic or a special matroid R_{10} through a sequence of 1-sums, 2-sums or 3-sums.

This gives a tree decomposition $(T, \mathcal{X})$ of thickness at most 2 so that each $M\ |_{X_v}$, $v \in V(T)$ is either a graphic, cographic or a special matroid R_{10}.

By performing parallel extensions of the elements to be deleted before each 2-sum and 3-sum, we construct a matroid M', so that M is a restriction of M' and M' has a tree decomposition $(T, \mathcal{X}')$ so that each $M'\ |_{\mathrm{cl}_{M'}(X'_v)}$, $v \in V(T)$ is either graphic, cographic or a parallel extension of R_{10}.

By Theorem 2, every graphic matroid has a 32-competitive non-adaptive mechanism. By Theorem 5, every cographic matroid has a 6-competitive non-adaptive mechanism. Since matroid R_{10} has ground set of size 10, by Theorem 1 every parallel extension of R_{10} has a 20-competitive non-adaptive mechanism.

Note that by definition every regular matroid is representable over finite field $\mathbb{F}_2$. Thus, by Theorem 14 with $p = 2$, $k = 2$ and $\alpha = 32$ there is a 256-competitive non-adaptive mechanism for matroid M'. Since M is a restriction of M', by Lemma 12, there is a 256-competitive non-adaptive mechanism for M, finishing the proof. $\square$

5.4 Minor-closed representable matroid families

In this section we show that every minor-clossed subclass of matroids representable over $\mathbb{F}_p$ has a constant-competitive non-adaptive mechanism, where the constant is a function only of p. The proof of this fact is analogous to the proof in [HN20].

Theorem 15 (Theorem 4.3 in [Gee11]). *Given natural numbers $q \geqslant 2$ and $n \geqslant 1$, let $M = (E, \mathcal{S})$ be a matroid with no $U_{2,q+2}$ or $M(K_n)$ minors. Then we have $|E| \leq q^{q^{3n}} r_M(E)$.*

Corollary 2. *Given natural numbers $q \geqslant 2$ and $n \geqslant 1$, let $M = (E, \mathcal{S})$ be a matroid with no $U_{2,q+2}$ or $M(K_n)$ minors. Then there exists a $q^{q^{3n}}$-competitive non-adaptive mechanism for M.*

Proof. If M has no $U_{2,q+2}$ or $M(K_n)$ minors, then every restriction of M also has no $U_{2,q+2}$ or $M(K_n)$ minors. Thus for every $X \subseteq E$ we have $|X| \leqslant q^{q^{3n}} r_M(X)$. So, M is a $q^{q^{3n}}$-sparse matroid and by Theorem 13 there exists a $q^{q^{3n}}$-competitive non-adaptive mechanism for M. $\square$

Projections and lifts

Let M be a matroid and x be an element of the ground set, which is a not a loop and not a free element of the matroid M. Then M/x is called a *projection* of $M \setminus x$; $M \setminus x$ is called a *lift* of M/x. Note that here and later we write M/x and $M \setminus x$ instead of $M/\{x\}$ and $M \setminus \{x\}$, repsectively.

Let M and N be two matroids with the same ground set. We say that the distance between M and N is t, denoted by $\text{dist}(M, N) = t$ if t is the smallest integer such that there exists a sequence of matroids $P_0, P_1, \ldots, P_t$ where $P_0 = M$ and $P_t = N$ and for every $i \in [t]$ the matroid P_i is either a lift or a projection of P_{i-1}.

Lemma 14. *Let N be a lift of the matroid M. If there is an α-competitive non-adaptive mechanism for M then there exists a $(2\alpha + 2)$-competitive non-adaptive mechanism for N.*

Proof. Since N is a lift of M, there exists a matroid $L = (E, \mathcal{S})$ and an element x of its ground set, such that $M = L/x$, $N = L\backslash x$. Here, x is not a loop and not a free element of L.

Let P be the set of elements in L that are parallel to x, in other words $P := \{x' \in E : x' \parallel x\}$. Note that $N\mid_{P\backslash\{x\}}$ is a uniform matroid of rank 1. Note also that elements in $P \setminus \{x\}$ are loops in M and so $\mathbf{EPROPH}_M = \mathbf{EPROPH}_{M\backslash P}$.

Let T'_e, $e \in E \setminus \{x\}$ be the thresholds imposed by an α-competitive non-adaptive mechanism for the matroid M. Let T''_e, $e \in P$ be the thresholds guaranteeing 2-competitive non-adaptive mechanism as in Theorem 1 for the uniform matroid of rank 1 on the ground set $P\backslash\{x\}$; and let T''_e, $e \in E\backslash(P\cup\{x\})$ be $+\infty$. We select one of these two sets of thresholds for the matroid N as described below. The constructed mechanism for the matroid N selects one of those two sets at random, where first set of thresholds T'_e, $e \in E \setminus \{x\}$ is selected with probability $\gamma := \alpha/(\alpha+1)$ and the second set T''_e, $e \in E\setminus\{x\}$ with probability $1 - \gamma = 1/(\alpha + 1)$.

Next part is dedicated to the analysis of how thresholds T'_e, $e \in E \setminus \{x\}$ perform on the matroid N. Note, that these thresholds are coming from a mechanism for the matroid M, while they are used for the matroid N with probability γ. We show that the total expected value achieved by thresholds T'_e, $e \in E \setminus \{x\}$ on N is at least the total expected value achieved by these thresholds on M. For this we can assume that for every realization of item values, the orders of items in matroid N and M are the same. To see that this assumption is valid, we can assume that the order for N is chosen in an adversarial way and is used also as the items order for M.

Claim 8. *Let us assume that the items order for M and N is the same for a given realization of item values. Let us also assume that for every item $e \in E \setminus \{x\}$ the threshold T'_e is used. Then the gambler with matroid N selects all items that the gambler with matroid M selects.*

Proof. We fix the item values realization and items order. Let e_1, e_2,..., e_k be the items with their values being at least their threshold and with the corresponding order.

Now we need to show that if the gambler with matroid N selects items greedily from e_1, e_2,..., e_k starting from e_1, then the set of selected items is a superset of the items greedily selected by the gambler with matroid M. If both gamblers end up selecting exactly the same set of items, then proof of the claim is complete. Otherwise consider the first index $i \in [k]$ such that the item e_i is selected by exactly one of the two gamblers. Since $N = L\backslash x$ and $M = L/x$ we have that it is only possible if e_i is selected by the gambler with the matroid N and rejected by the gambler with the matroid M.

Now we claim that every subsequent item, in other words an item in $e_{i+1}, \ldots, e_k$, is either selected by both gamblers or rejected by both gamblers. Suppose the contrary and consider the first item e_j, $i+1 \leq j \leq k$ that is selected by one gambler and rejected by another gambler. Let $S := \{e_1, e_2, \ldots, e_{j-1}\}$ and let T be the set of items selected by the gambler with M from the set S. Thus the gambler with N selected $T \cup \{e_i\}$ from the set S. So $T \cup \{e_i\}$ is a basis of $(L \setminus x) \mid_S$ and T is a basis of $(L/x) \mid_S$. Thus, both $T \cup \{e_i\}$ and $T \cup \{x\}$ are bases of $L \mid_S$. If only one of the two gamblers accepts the item s_j then the matroid $L \mid_{S \cup \{s_j\}}$ has two bases of different cardinality, attaining a contradiction and finishing the proof. □

Thus we have that the thresholds T'_e, $e \in E \setminus \{x\}$ guarantee at least $\mathbf{EPROPH}_M$ as the expected total value of the gambler with N. To prove that the constructed mechanism is $1/(2\alpha+2)$-competitive it is enough to show the following claim. Note that in our construction we used α-competitive non-adaptive mechanism for the matroid M and 2-competitive non-adaptive mechanism for the uniform matroid of rank 1 on $P \setminus \{x\}$.

Claim 9. $\gamma \frac{1}{\alpha} \boldsymbol{EPROPH}_M + (1-\gamma)\frac{1}{2}\boldsymbol{EPROPH}_{P\setminus\{x\}} \geqslant \frac{1}{2\alpha+2}\boldsymbol{EPROPH}_N$

Proof. Let us consider the inclusion-maximal set I_{opt} on which the prophet achieves $\mathbf{PROPH}_N$. Let C_{opt} be a random variable corresponding to the unique circuit of $I_{opt} \cup \{x\}$ in L. Recall that x is not a free element of L so such a circuit exists and is unique and contains x.

First consider the events when $|C_{opt}| \geqslant 3$. Note that by definition of a circuit, for every $y \in C_{opt} \setminus \{x\}$ the set $(I_{opt} \cup \{x\}) \setminus \{y\}$ is independent in L. Hence, for every $y \in C_{opt} \setminus \{x\}$ the set $I_{opt} \setminus \{y\}$ is independent in M. So we have that conditioned on $|C_{opt}| \geqslant 3$ we have $\mathbf{PROPH}_M \geq w(I_{opt} \setminus \{y\})$ for every $y \in C_{opt} \setminus \{x\}$. Let y_{opt} be the random variable representing the element in $C_{opt} \setminus \{x\}$ of smallest value. Then conditioned on $|C_{opt}| \geqslant 3$, we have $w(C_{opt} \setminus \{y_{opt}, x\}) \geq w(C \setminus \{x\})/2$. Thus, conditioned on $|C_{opt}| \geqslant 3$ we have

$$\begin{aligned}\mathbf{PROPH}_M &\geqslant w(I_{opt} \setminus \{y_{opt}\}) = w(I_{opt} \setminus C_{opt}) + w(C_{opt} \setminus \{y_{opt}\}) \\ &\geq w(I_{opt} \setminus C_{opt}) + \frac{1}{2}w(C_{opt} \setminus \{x\}) \geq \frac{1}{2}w(I_{opt}) = \frac{1}{2}\mathbf{PROPH}_N\,.\end{aligned}$$

Second consider the event that $|C_{opt}| < 3$. Since x is not a loop of L by definition, we have $|C_{opt}| = 2$ and so $C_{opt} = \{x, x_{opt}\}$ for some random variable element $x_{opt} \in P \setminus \{x\}$. For the event $|C_{opt}| \geq 3$ let us define the random variable element x_{opt} to be an arbitrary element in $C_{opt} \setminus \{x\}$. Thus, if $|C_{opt}| < 3$ we have $\mathbf{PROPH}_{P\setminus\{x\}} \geq w(x_{opt})$. Now let us define $J_{opt} := I_{opt} \setminus \{x_{opt}\}$ and note that J_{opt} is independent in the matroid M. Moreover, since I_{opt} is the set on which the prophet achieves $\mathbf{PROPH}_N$, we have that conditioned on $|C_{opt}| < 3$ the prophet achieves $\mathbf{PROPH}_M$ on the set J_{opt}.

Combining everything together we have

$$\begin{aligned}
&\gamma\frac{1}{\alpha}\mathbf{EPROPH}_M + (1-\gamma)\frac{1}{2}\mathbf{EPROPH}_{P\setminus\{x\}} = \\
&\frac{1}{\alpha+1}\mathbf{EPROPH}_M + \frac{1}{2\alpha+2}\mathbf{EPROPH}_{P\setminus\{x\}} \geq \\
&\quad E\left[\frac{w(x_{opt})}{2\alpha+2} + \frac{\mathbf{PROPH}_M}{\alpha+1}\,\middle|\, |C_{opt}| < 3\right] P\left[|C_{opt}| < 3\right] \\
&\quad + E\left[\frac{\mathbf{PROPH}_M}{\alpha+1}\,\middle|\, |C_{opt}| \geqslant 3\right] P\left[|C_{opt}| \geqslant 3\right] = \\
&\quad E\left[\frac{w(x_{opt})}{2\alpha+2} + \frac{w(I_{opt}\setminus\{x_{opt}\})}{\alpha+1}\,\middle|\, |C_{opt}| < 3\right] P\left[|C_{opt}| < 3\right] \\
&\quad + E\left[\frac{\mathbf{PROPH}_M}{\alpha+1}\,\middle|\, |C_{opt}| \geqslant 3\right] P\left[|C_{opt}| \geqslant 3\right] \geq \\
&\quad E\left[\frac{\mathbf{PROPH}_N}{2\alpha+2}\,\middle|\, |C_{opt}| < 3\right] P\left[|C_{opt}| < 3\right] \\
&\quad + E\left[\frac{\mathbf{PROPH}_M}{\alpha+1}\,\middle|\, |C_{opt}| \geqslant 3\right] P\left[|C_{opt}| \geqslant 3\right] \geq \frac{1}{2\alpha+2}\mathbf{EPROPH}_N\,.
\end{aligned}$$

□

□

Lemma 15. *Let N be a matroid obtained from a matroid M by a sequence of t projections. Let L be the set of loops in the matroid N. Let there exist an α-competitive non-adaptive mechanism for the matroid M. Then there exists a non-adaptive mechanism for $N \setminus L$ such that the expected total value of this mechanism is at least $\frac{1}{\alpha\cdot 3^t}\boldsymbol{EPROPH}_{M\setminus L}$.*

In the context of Lemma 15, every set that is independent for the matroid $N \setminus L$ is also independent for the matroid $M \setminus L$. Hence, we have $\mathbf{EPROPH}_{M\setminus L} \geq \mathbf{EPROPH}_{N\setminus L}$. Thus in case $t = 1$, Lemma 15 leads us to the following corollary.

Corollary 3. *Let N be a projection of the matroid M. If there is an α-competitive non-adaptive mechanism for M then there exists a 3α-competitive non-adaptive mechanism for N.*

Proof of Lemma 15. Let us prove the statement by induction. Of course, in case $t = 0$ we have $M = N$ and the statement is trivially true.

Let us now assume that t is at least 1. Let N' be a matroid such that N' is obtained from the matroid M by a sequence of $t-1$ projections and N is a projection of N'. Since N is a projection of N' there is a matroid $P = (E, \mathcal{S})$ and $x \in E$ such that $P \setminus x = N'$ and $P/x = N$. Let L' be the set of loops in the matroid N'.

By induction hypothesis, there exist thresholds T'_e, $e \in E \setminus (L' \cup \{x\})$ such that the gambler with the matroid $N' \setminus L'$ achieves at least $\frac{1}{\alpha \cdot 3^{t-1}} \mathbf{EPROPH}_{M \setminus L'}$ as the expected total value. Let us assume that to compute thresholds T'_e, $e \in E \setminus (L' \cup \{x\})$ the values of items in L were set to be 0 while the distribution of values for other items remain the same. Since $L' \subseteq L$, analogously to Lemma 12 we can define thresholds

$$T''_e := \begin{cases} +\infty & \text{if } e \in L \\ T'_e & \text{otherwise} \end{cases}$$

such that the gambler with the matroid $N' \setminus L$ achieves at least $\frac{1}{\alpha \cdot 3^{t-1}} \mathbf{EPROPH}_{M \setminus L}$ as the expected total value. Let T'''_e, $e \in E \setminus (L \cup \{x\})$ be the thresholds guaranteeing 2-competitive non-adaptive mechanism as in Theorem 1 for the uniform matroid of rank 1 on the ground set $E \setminus (L \cup \{x\})$.

The constructed mechanism for the matroid $N \setminus L$ selects one of two threshohold sets at random, where first set of thresholds T''_e, $e \in E \setminus (L \cup \{x\})$ is selected with probability 1/3 and the thresholds T'''_e, $e \in E \setminus (L \cup \{x\})$ with probability 2/3. Note that the thresholds T''_e, $e \in E \setminus (L \cup \{x\})$ were designed for the matroid $N' \setminus L$ but are used for the matroid $N \setminus L$; hence less items might be selected than when it is used for $N' \setminus L$. Also note, that the thresholds T'''_e, $e \in E \setminus (L \cup \{x\})$ are used for $N \setminus L$ but were designed for the uniform matroid of rank 1.

For the analysis, let I_{alg} be the random variable indicating the items set selected by the gambler with matroid $N' \setminus L$ when the thresholds T''_e, $e \in E \setminus (L \cup \{x\})$ are used. Analogously to a claim in the proof of Lemma 14, we can assume that when the thresholds T''_e, $e \in E \setminus (L \cup \{x\})$ are used the gambler with $N \setminus L$ select all items in I_{alg} with an exception for possibly one item. Let x_{opt} be the random variable indicating the element of maximum value in $E \setminus (L \cup \{x\})$.

To finish the proof it is enough to show the following inequality

$$\frac{1}{3} E[w(I_{alg}) - w(x_{opt})] + \frac{2}{3}\frac{1}{2} E[w(x_{opt})] \geq \frac{1}{\alpha \cdot 3^t} \mathbf{EPROPH}_{M \setminus L}\,.$$

To obtain this inequality we can do estimations as follows

$$\frac{1}{3}E[w(I_{alg}) - w(x_{opt})] + \frac{2}{3}\frac{1}{2}E[w(x_{opt})] = \frac{1}{3}E[w(I_{alg})] \geq \frac{1}{3}\frac{1}{\alpha \cdot 3^{t-1}}\mathbf{EPROPH}_{M \setminus L}\,.$$

□

Now let us combine Corollary 3 and Lemma 14.

Lemma 16. *Let M and N be matroids such that $\operatorname{dist}(M, N) \leq t$. If there exists an α-competitive non-adaptive mechanism for the matroid M with $\alpha \geq 2$ then there exists a $3^t\alpha$-competitive non-adaptive mechanism for the matroid N.*

Proof. Note that for $\alpha \geq 2$ we have $3\alpha \geq 2\alpha + 2$. Since N can be obtained from M by a sequence of t projection and lift steps, we can use Corollary 3 or Lemma 14 for each of these steps to obtain the desired competitiveness ratio. □

Minor-closed families theorem

Lemma 17 (Lemma 6 in [HN20]). *Let p and n be integers such that $p \leqslant n-2$ and p is prime. The matroid $U_{2,n}$ is not representable over the field $\mathbb{F}_p$.*

The following Structural Hypothesis is due to Geelen, Gerards and Whittle. The proof of this Structural Hypothesis has not appeared in print.

Hypothesis 1. *Let p be a prime number and $\mathcal{M}$ is a proper minor-closed class of matroids representable over $\mathbb{F}_p$.*

Then there exist k, n, t such that every $M \in \mathcal{M}$ is a restriction of an $\mathbb{F}_p$-representable matroid M' having a full tree-decomposition $(T, \mathcal{X})$ of thickness at most k so that for every $v \in V(T)$ if $M' |_{\operatorname{cl}_{M'}(X_v)}$ has a $M(K_n)$ minor, then there exists a 2-column sparse matroid N with $\operatorname{dist}(M' |_{\operatorname{cl}_{M'}(X_v)}, N) \leqslant t$.

Proof of Theorem 8. Let k, n, t are as stated in the Structural Hypothesis 1 on $\mathcal{M}$.

Let $\mathcal{M}_1$ be the set of matroids on distance t or less from some 2-column sparse matroid and are representable over $\mathbb{F}_p$. By Theorem 4 all 2-column sparse matroids have a 32-competitive non-adaptive mechanism. By Lemma 16 there exists a $(3^t \cdot 32)$-competitive mechanism for matroids in $\mathcal{M}_1$.

Let $\mathcal{M}_2$ be the set of matroids without $M(K_n)$ minor and are representable over $\mathbb{F}_p$. By Lemma 17 all matroids in $\mathcal{M}_2$ do not have $U_{2,p+2}$ as a minor. Then by Corollary 2, we have that there is a $p^{p^{3n}}$-competitive non-adaptive mechanism for every matroid in $\mathcal{M}_2$.

By the Structural Hypothesis 1 we have that every $M \in \mathcal{M}$ is a restriction of some M' with a full tree-decomposition $(T, \mathcal{X})$ of thickness at most k so that for every $v \in V(T)$ $M' |_{\mathrm{cl}_{M'}(X_v)} \in \mathcal{M}_1 \cup \mathcal{M}_2$.

Thus by Theorem 14, matroid M' has a $\gamma := (\max(3^t \cdot 32, p^{p^{3n}}) \cdot p^{k+1})$-competitive non-adaptive mechanism. By Lemma 12 the matroid M has also a γ-competitive non-adaptive mechanism.

$\square$

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