Analyzing Tree Attachments in 2-Crossing-Critical Graphs with a V_8 Minor

by

Carter Bedsole

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2023

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The crossing number of a graph is the minimum number of pairwise edge crossings in a drawing of the graph in the plane. A graph G is k-crossing-critical if its crossing number is at least k and if every proper subgraph H of G has crossing number less than k.

It follows directly from Kuratowski's Theorem that the 1-crossing-critical graphs are precisely the subdivisions of $K_{3,3}$ and K_5. Characterizing the 2-crossing-critical graphs is an interesting open problem.

Much progress has been made in characterizing the 2-crossing-critical graphs. The only remaining unexplained such graphs are those which are 3-connected, have a V_8 minor but no V_{10} minor, and embed in the real projective plane $\mathbb{R}P^2$. This thesis seeks to extend previous attempts at classifying this particular set of graphs by examining the graphs in this category where a tree structure is attached to a subdivision of V_8.

In this paper, we analyze which of the 106 possible 3-stars can be attached to a subdivision H of V_8 in a 3-connected 2-crossing-critical graph. This analysis leads to a strong result, where we demonstrate that if a k-star is attached to a V_8 in a 2-crossing-critical graph, then $k \leq 4$. Finally, we significantly restrict the remaining trees which still need to be investigated under the same conditions.

Acknowledgements

I would like to express my gratitude to my supervisor, Bruce Richter, for taking a chance on a student with a slightly unorthodox academic background. He has been far more gracious with me than I deserved, and has been ever joyful and excited about my contributions to this problem. It has been a true delight to work with him.

I am grateful to my undergraduate advisor, Nick Willis, for encouraging me to pursue graduate studies in mathematics.

Finally, and most importantly, I am grateful to my lovely wife, KariAnna, for being willing to leave the placid tranquility of the Pacific Northwest and venture to (what seemed to us to be) an arctic tundra for me to pursue my studies. She has enjoyed this journey even more than I have, and this effervescence of hers had made it all worth it.

Dedication

This thesis is dedicated to G.K. Chesterton, who understood the risk a mathematician assumes and offered me the sole mitigation. *Ora pro nobis.*

Table of Contents

List of Figures

Chapter 1

Introduction

The *crossing number* of a graph G is the minimum number of pairwise edge crossings in a plane drawing of G. Planar graphs, by definition, have crossing number 0, and non-planar graphs have crossing number at least 1. The crossing number of a graph G is denoted $\text{cr}(G)$.

A graph G is *k-crossing-critical* if $\text{cr}(G) \geq k$ and, for every edge $e \in E(G)$, $\text{cr}(G \setminus \{e\}) < k$.

Two graphs G and H are *topologically isomorphic* if there exists some graph A such that both G and H are subdivisions of A. As remarked by Bokal, Oporowski, Richter, and Salazar in [4], if two graphs G and H are topologically isomorphic, then $\text{cr}(G) = \text{cr}(H)$ and G is k-crossing-critical if and only if H is k-crossing-critical. Practically speaking, replacing paths with internal vertices of degree 2 with edges and vice versa has no effect on the crossing number or criticality of a graph. As such, in this work, we consider graphs with no vertices of degree 2.

Under this assumption, it is a corollary of Kuratowski's Theorem that the only 1-crossing-critical graphs are $K_{3,3}$ and K_5.

Classifying the 2-crossing-critical graphs is a significantly more interesting and difficult problem. It is currently incomplete. Past attempts have made significant progress at such a classification. This thesis attempts to summarize previous partial classifications, and contribute more understanding to the remaining unclassified 2-crossing-critical graphs.

Graphs which are 2-crossing-critical were first studied by Bloom, Kennedy, and Quintas in [3], in which they discovered 21 examples. After this, Širàň demonstrated an infinite family of 3-connected n-crossing-critical graphs, for all $n \geq 3$ [12]. Kochol later discovered an infinite family of 3-connected 2-crossing-critical graphs [6].

Other notable results include Richter finding precisely eight cubic 2-crossing-critical graphs [9]. Vitray proved that there exists a single 2-crossing-critical graph G such that $\text{cr}(G) > 2$, and it is $C_3 \square C_3$, the Cartesian product of two 3-cycles [11] (it is worth noting that in earlier works, the notation $C_3 \times C_3$ is used). We note $\text{cr}(C_3 \square C_3) = 3$.

Furthermore, there are 103 graphs which minimally do not embed in the real projective plane $\mathbb{R}P^2$, as determined by Archdeacon in [1].

Most 2-crossing-critical graphs were successfully classified by Bokal, Oporowski, Richter, and Salazar in [4]. First, they were able to enumerate all of the 2-crossing-critical graphs which minimally do not embed in the real projective plane $\mathbb{R}P^2$; this is a subset of the 103 graphs found by Archdeacon in [1]. Therefore, the rest of their characterization focused on 2-crossing-critical graphs which embed in the real projective plane $\mathbb{R}P^2$.

In their partial characterization, 2-crossing-critical graphs are considered separately depending on their V_{2n} minors. The graph V_{2n} (sometimes denoted M_{2n}) is formed by adding chords to the $2n$-cycle, such that their incident vertices are at distance n on the $2n$-cycle. The V_8, comprised of an 8-cycle with four chords at distance four, is seen below.

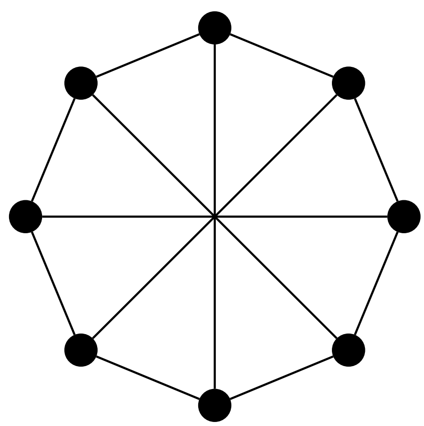

Figure 1.1: The V_8 graph.

They characterized most 2-crossing-critical graphs as follows:

1. *2-crossing-critical graphs which are not 3-connected:* These graphs were fully classified. They exist in two categories. The first contains 49 graphs which consist of two

$K_{3,3}$'s and/or K_5's which are either not connected (three examples), have a cut vertex (ten examples), or have a 2-cut (36 examples). The second category is a family of graphs which is constructed from 3-connected examples, by replacing parallel edges incident to u and v with chains of parallel edges which are incident to u and v.

2. *3-connected 2-crossing-critical graphs with no V_8 minor:* Robertson's (written with Maharry) characterization [7] of graphs without a V_8 minor is used by Bokal et al [4] to give a (fairly complicated) method for finding all the 2-crossing-critical graphs with no V_8 minor.

3. *3-connected 2-crossing-critical graphs with a V_8 minor but no V_{10} minor:* These graphs were not fully classified. Bokal, Oporowski, Richter, and Salazar demonstrated in [4] that these graphs can have at most 7 million vertices. After this, Arroyo in [5] reduced this upper bound to 4,001 vertices. Therefore, we can conclude that there are finitely many such graphs remaining to be classified.

4. *3-connected 2-crossing-critical graphs with a V_{10} minor:* These graphs were found to be completely described by an infinite family. This family can be constructed from a set of 42 *tiles*. These tiles are constructed by inserting one of 13 *pictures* (Figure 1.3) into one of two *frames* (Figure 1.2) by identifying the square of the picture with the square of the frame. To construct a 2-crossing-critical graph in this family out of tiles, a positive integer m is chosen. Then $2m + 1$ tiles are chosen and enumerated $T_0, T_1, \ldots, T_{2m}$; this is referred to as a *composition of tiles*. Tiles with odd index are flipped (top to bottom) and then tiles are attached by their endpoints. This is known as a *twisted cycling* of tiles, and twisted cyclings completely define this infinite family of graphs. See Figure 1.4 for one such example of a twisted cycling when $m = 2$.

Figure 1.2: The two *frames* used to construct all 3-connected 2-crossing-critical graphs with a V_{10} minor. (Figure 1.5 from Austin's thesis [2]).

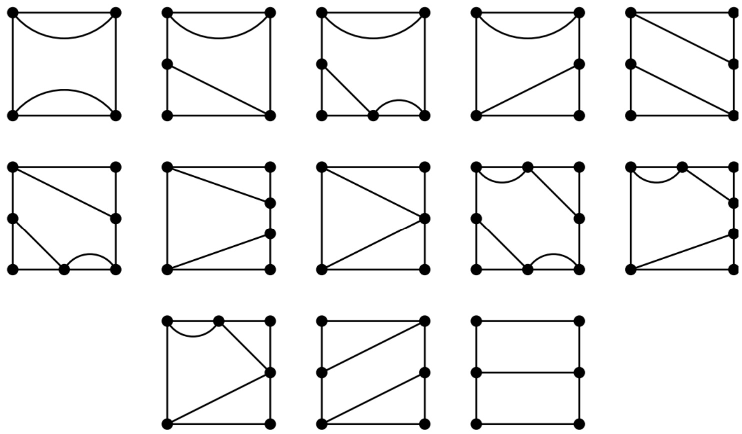

Figure 1.3: The 13 *pictures* used to construct all 3-connected 2-crossing-critical graphs with a V_{10} minor. (Figure 1.4 from Austin's thesis [2]).

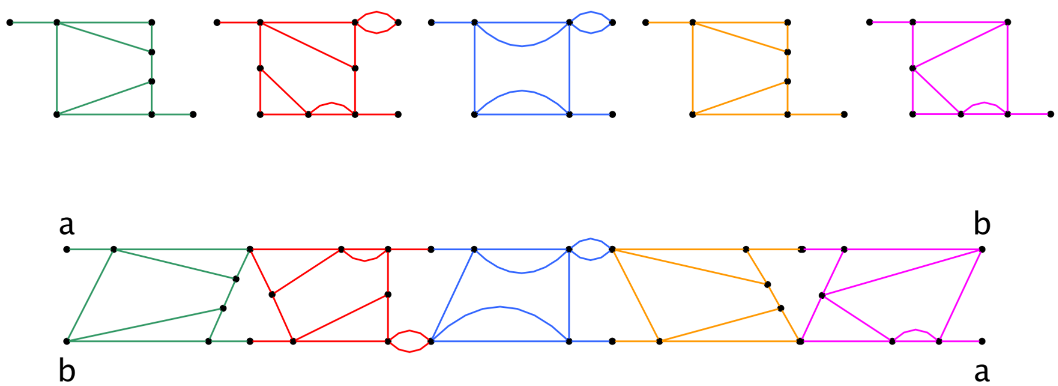

Figure 1.4: An example 3-connected 2-crossing-critical graph with a V_{10} minor formed via a *twisted cycling*. (Figure 1.0.3 from Arroyo's thesis [5]).

Further work has been done to attempt to classify the remaining finite set of 3-connected, 2-crossing-critical graphs with a V_8 minor but no V_{10} minor, which can be embedded in the real projective plane $\mathbb{R}P^2$. Urrutia Schroeder first proposed in her master's essay a method of thinking about these graphs known as *covering* [10]. Covering will be discussed later in detail. She claimed to have found 326 examples of graphs in this set, but only 214 were indeed 2-crossing-critical. Austin followed up this work by introducing the idea of *fully covering*, and was able to expand the list of these 2-crossing-critical graphs to 312 [2].

In this thesis, we seek to follow up on the work done by Urrutia Schroeder and Austin on the subject of *covering*, with the hopes to further narrow down the conditions under which a 3-connected graph with a V_8 minor but no V_{10} minor, which embeds in the real projective plane $\mathbb{R}P^2$, can be 2-crossing-critical. We first proceed to a summary of their work.

1.1 Regarding V_8's

As discussed, the graph V_8, also known as the Wagner Graph, the Möbius Ladder with eight vertices, or the M_8 graph, comprises an 8-cycle with four chords at distance four. The 8-cycle is referred to as the *rim* and the four chords are referred to as the *spokes*. The V_{10}, as one might expect, is comprised of a 10-cycle with five chords at distance five.

Since we are concerned with graphs with a V_8 minor, then the V_8's which we study may be subdivided. It is worth noting that spokes are sometimes referred to as *spoke edges* and rims are sometimes referred to as *rim edges* or *rim branches*. This does not necessarily imply that they do not contain subdivisions.

The (potentially subdivided) 4-cycles containing two rim branches and two spokes are referred to as *quads*.

The V_8 graph is embeddable in the Möbius strip and the real projective plane $\mathbb{R}P^2$. In this paper, we will represent the V_8 graph in the Möbius strip. Typically, graphs embedded in the Möbius strip are drawn as follows, where vertices or edges on one side are associated with those on the other side, to demonstrate an implied "twist" of the Möbius strip.

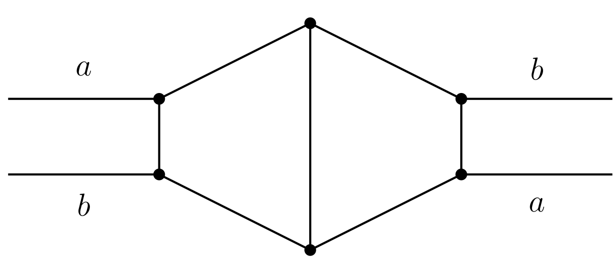

Figure 1.5: For example, a $K_{3,3}$ embedded in the Möbius strip.

Therefore, our V_8 embedded in the Möbius strip will be demonstrated as follows. Since rim edges and spokes may be subdivided, we introduce a canonical labelling of a V_8 with each edge subdivided once. (In our work, rim edges and spokes may occasionally be subdivided multiple times, in which case we make note of this and specify new notation).

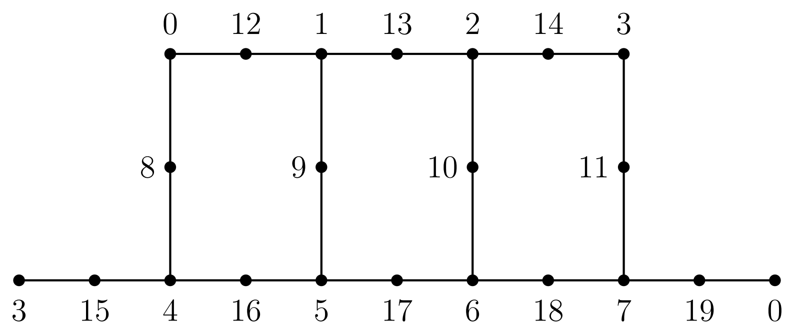

Figure 1.6: A subdivided V_8 embedded in the Möbius strip with the canonical labelling.

1.2 Crossings in a 1-Drawing of a V_8

A *k-drawing* of a graph G is a drawing in the plane which contains k pairwise edge crossings. It is important to note that in a k-drawing, pairwise edge crossings must truly be crossings; tangential intersection between edges is not permitted. Below are a couple of examples of k-drawings of graphs.

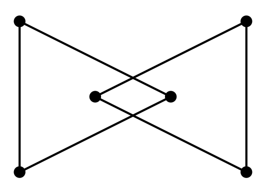

Figure 1.7: An example of a 2-drawing of a planar graph.

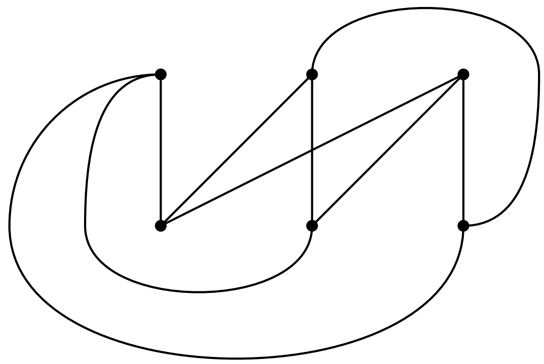

Figure 1.8: An example of a 1-drawing of a $K_{3,3}$.

Since a V_8 has crossing number 1, then its 1-drawings are of particular interest to us. In this section, we cover some helpful lemmas and definitions from Austin's work in [2] to the possible 1-drawings of both V_8 and small extensions of V_8.

Lemma 1. *Disjoint cycles do not cross in a 1-drawing.*

Proof. Suppose that disjoint cycles C_1 and C_2 cross in a 1-drawing D of a graph G. That is, suppose that there are edges $e_0 \in C_1$ and $f_0 \in C_2$ such that e and f are crossed in D. But every time C_1 crosses into C_2, it must cross out of it. Therefore, there must be a second pair of edges $e_1 \in C_1$, $f_1 \in C_2$ such that e_1 and f_1 are crossed in D, a contradiction. Therefore, C_1 and C_2 cannot cross in D. □

We note that $G \setminus A$, where A is a set of edges, is the graph formed by deleting the edges in A from G.

Lemma 2. *Let e be an edge in a graph G. If $G \setminus \{e\}$ has a $K_{3,3}$ minor, then e is not crossed in a 1-drawing.*

Proof. If e is crossed in a 1-drawing, then $G \setminus \{e\}$ is planar. By Kuratowski's Theorem, it does not have a $K_{3,3}$ minor, a contradiction. □

Let s_i through s_{i+3} denote the four spokes of a V_8 in order. Let r_i through r_{i+7} denote the eight rim branches of a V_8 in order. Indices of spokes are counted modulo 4 and indices of rim branches are counted modulo 8.

Lemma 3. *No spokes can be crossed in a 1-drawing of a* V_8*.*

Proof. The graph $V_8 \setminus \{s_j\}$ is a subdivision of a $K_{3,3}$. By the previous lemma, s_j is not crossed in a 1-drawing. □

Lemma 4. *If rim edges* r_i *and* r_j *are crossed in a 1-drawing of a* V_8*, then* $|i - j| \in \{3, 4\}$*.*

Proof. It suffices to prove that, for all $i \in [8]$, r_i crosses neither r_{i+1} nor r_{i+2}.

Two rim branches r_i and r_{i+2} are on disjoint quads. By Lemma 1, they cannot be crossed in a 1-drawing.

Now suppose two adjacent rim branches, r_i and r_{i+1}, are crossed in a 1-drawing of a V_8. Removing the spoke s_{i+1} leaves a 1-drawing of $K_{3,3}$ in which the single crossing is one edge of the $K_{3,3}$ crossing itself. Since $K_{3,3}$ has crossing number 1, there must be a crossing between two distinct edges, so such a drawing is not possible. Thus, it is not possible to have a 1-drawing of the V_8 in which adjacent rim branches are crossed.

Therefore any crossed rim branches must be at distance 3 or 4, as required. □

By the preceding lemmas, the crossing in a 1-drawing of a V_8 must be between rim edges at a rim distance of 3 or 4. Therefore, a V_8 has, up to topological isomorphism, two 1-drawings. These are demonstrated below.

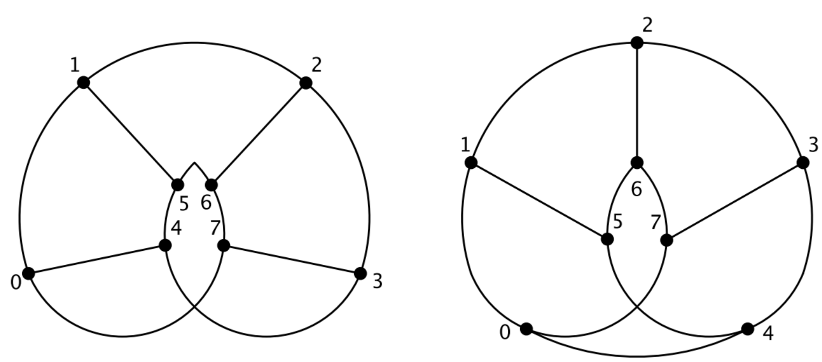

Figure 1.9: The two 1-drawings of a V_8. (Figure 2.3 in Austin's thesis [2]).

Now let G be a graph with a V_8 minor. Then a rim branch of a V_8 in G is *covered* if there does not exist a 1-drawing D of G where the rim branch is crossed. In other words, a covered rim branch is one which cannot be crossed in a 1-drawing. A V_8 in G is *fully covered* if all of its rim branches are covered. Since all 1-drawings of a V_8 involve rim edge crossings, necessarily a graph G with a fully covered V_8 has crossing number 2.

Lemma 5. *Let G be a graph with a subdivision H of V_8. If five consecutive rim branches of H in G are covered, then H in G is fully covered.*

Proof. By definition, any crossing in a 1-drawing G must be between the remaining three uncovered rim branches of H. But they are not separated by a distance of three or four in the rim, and by Lemma 4 are not crossed in a 1-drawing. Therefore, this V_8 is, by definition, fully covered. □

To understand the conditions under which a V_8 can become fully covered, and therefore potentially 2-crossing-critical, following Urrutia Schroeder in [10], Austin in [2] considered the effects of adding individual edges to the (potentially subdivided) V_8. To assist in the discussion, let H be a subdivision of V_8 contained within a graph G. The *rim R in H* is the cycle in H that is the subdivision of the rim of V_8. A V_8 *vertex* is one of the eight vertices in H that is incident with spoke and distinct rim branches.

A *jump* is an edge not in H but with both endpoints in the rim of H. A *slope* is an edge joining a vertex in the rim of H with a vertex in a spoke. A *bar* joins internal vertices in two distinct spokes of H.

The *span* of a jump, slope, or bar S is a shortest section of the rim between the endpoints of S. Jumps, slopes, and bars are defined by the length of their span. An n-jump has V_8 vertices as endpoints and spans n rim branches with $n \leq 4$. An $n\frac{1}{2}$-jump has a V_8 vertex as an endpoint and another vertex in the rim as its other endpoint. Off-n-jumps span n rim branches (that is, two halves and $n-1$ whole rim branches), with endpoints on rim branches. 3-jumps are referred to as *diagonals* and $3\frac{1}{2}$-jumps are referred to as *semi-diagonals*. A 4-jump is referred to as a *spoke jump*.

Below are a couple of examples of these structures.

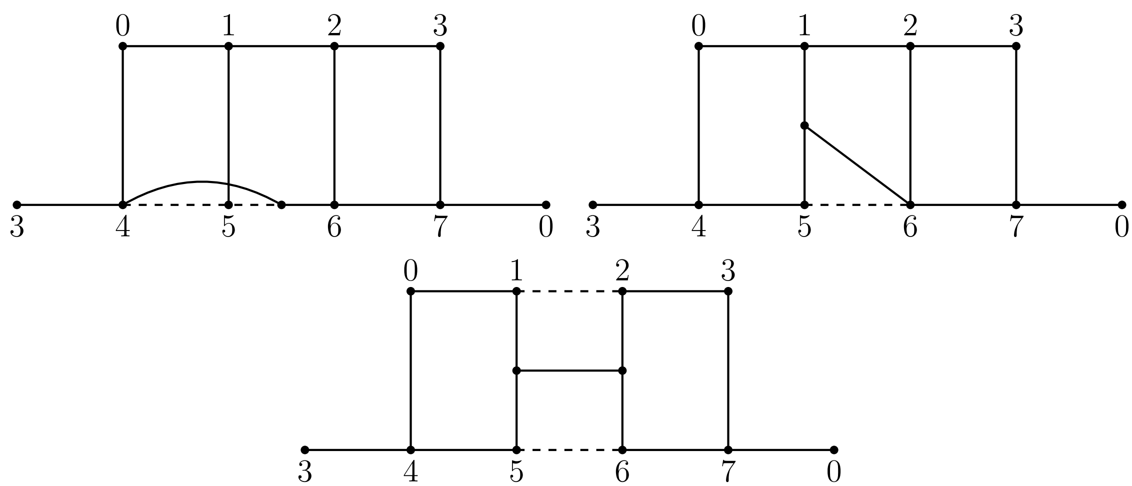

Figure 1.10: A $1\frac{1}{2}$-jump (upper left), 1-slope (upper right), and 1-bar (or bar) (lower), with their spans denoted by dashed lines.

We say that a structure S (in this context, a jump, slope, or bar) *covers* a rim branch r of a V_8 if r cannot be crossed in the graph G containing S attached to a V_8 or, consequently, any supergraph of G. The following theorem from Austin's thesis in [2] explains how each structure covers the rim branches of the V_8. This is referred to as *coverage*. The original proof of this theorem contained some errors (for instance, disjoint cycles which were not actually disjoint). For our work, we have verified the theorem with a computer. The techniques used to do so will be described in the next section.

Theorem 6 (Theorem 2.6 from Austin's work in [2])**.** *Consider a graph consisting of a V_8 with some structure S added.*

1. *If S is a slope, bar, k-jump, or off-k-jump, with $k \leq 2$, then it covers the section of rim it spans.*

2. *If S is an off-2-jump from r_i to r_{i+2}, then it also covers r_{i+5}.*

3. *If s is a $1\frac{1}{2}$-slope or 2-slope from s_i that spans r_i, then it also covers r_{i+2}, r_{i+3}, r_{i+5}, and r_{i+6}.*

4. *If S is a $2\frac{1}{2}$-jump from r_i that spans r_{i+1}, then it covers r_{i+1}, r_{i+2}, and r_{i+5}. Furthermore, the section of r_i spanned by S can only cross r_{i+3}.*

5. *If S is an off-3-jump from r_i that spans r_{i+1}, then it covers r_{i+1}, r_{i+2}, and r_{i+5}. Furthermore, the section of r_i spanned by S can only cross r_{i+3} and the section of r_{i+3} spanned by S can only cross r_i.*

6. *If S is a semi-diagonal from i to r_{i+4}, then the section of r_{i+4} spanned by S cannot cross r_{i+1}.*

Diagrams demonstrating each of these structures and the coverage they provide can be found in section A.1.

A proof for the following conjecture was also included in Austin's work in [2]. However, this proof was found to contain some errors. In particular, the proof relied on deleting rim branches from the V_8, without considering if they were subdivided and had other edges attached. If the branches in question were indeed subdivided, then the desired method can no longer be applied.

We were unable to provide a correct proof of this conjecture, which has been slightly rephrased here. However, no counterexamples are known, and it is expected that the conjecture is indeed true. Proving it would be an interesting future endeavor.

Conjecture 7 (Theorem 3.1 from Austin's work in [2])**.** *In a 3-connected 2-crossing-critical with a fully covered V_8 and no V_{10} minor, the sections of rim covered by bars, 2-bars, $\frac{1}{2}$-, 1-, $1\frac{1}{2}$-, 2-, off-$\frac{1}{2}$-, and off-1-jumps must be disjoint.*

1.3 A Note about Computing

This work, and the work that has preceded it, relies heavily on the use of computers for discovering and verifying results. We use the `nauty` C library, created and maintained by B. McKay [8], which provides the necessary structures and algorithms for working with graphs.

Contained within `nauty` is an efficient algorithm to check if a graph is planar. This algorithm is complex and its details are omitted here. We will refer to it as **isPlanar**. Also included within `nauty` are subroutines which utilize the efficient planarity checking algorithm to check if a graph has a 1-drawing with a given pairwise edge crossing, the crossing number of a graph, and if a graph is critical. These subroutines, implemented by D. Bokal, are those which are used to verify the results of this paper. We proceed to providing brief descriptions of each.

The key to the subroutine which checks if a graph has a 1-drawing where edges e and f are crossed, is to construct an auxiliary graph H. Let $e = (e_0, e_1)$ and $f = (f_0, f_1)$ in G. To construct H, remove e and f from G and add vertex r. Then add the following edges $(r, e_0), (r, e_1), (r, f_0)$, and (r, f_1). In other words, H is constructed from G by simulating a crossing of edges e and f with a vertex. We note that, if G is planar, then this simulation will yield e and f being either tangential in H or crossed. Since the examples which use this routine in this thesis are all non-planar, then we assume G is non-planar. Then we can conclude that if H is planar and G is non-planar, then G has a 1-drawing where e and f are crossed.

So, to check if a graph G has a 1-drawing with a pairwise crossing of edges e and f, one performs the following:

- Algorithm **has1Drawing**
- **Input:** graph G and edges $e = (e_0, e_1)$ and $f = (f_0, f_1)$ in G.
- **Output:** True if G has a 1-drawing where e and f are crossed; False otherwise.
- Construct auxiliary graph H from G as follows. Remove e and f from G and add vertex r. Then add the following edges $(r, e_0), (r, e_1), (r, f_0)$, and (r, f_1).
- Return True if H is planar; False otherwise.

To check if G has crossing number at most 1, we simply check if there exists a pair of edges $e, f \in E(G)$ such that e and f can be crossed in a 1-drawing of G:

- Algorithm **hasCrossingNumber1**
- **Input:** a graph G
- **Output:** True if $\text{cr}(G) \leq 1$; False otherwise
- For each pair of edges $e, f \in E(G)$
  - If has1Drawing(G, e, f), return True
- Return False

To check if G is 1-crossing-critical, we check if, for each edge $e \in E(G)$, $G \setminus \{e\}$ is planar.

- Algorithm **is1crossingCritical**
- **Input:** a graph G
- **Output:** True if G is 1-crossing-critical or planar; False otherwise
- For each edge $e \in E(G)$
  - If not isPlanar($G \setminus \{e\}$), return False
- return True

These three routines (**has1Drawing**, **hasCrossingNumber1**, and **is1crossingCritical**) can be extended for any integer n. To check if a G has an n-drawing given n pairs of edges, we simulate the n pairwise edge crossings with vertices, as in **has1Drawing**. To check if G has crossing number n, we check all possible combinations of n pairs of edges to see if G has an n-drawing with those pairs of edges. To check if a graph G is n-crossing-critical, we check if $G \setminus \{e\}$ has an $(n-1)$-drawing for all edges $e \in E(G)$.

All statements about the "crossability" of an edge or pair of edges, the crossing number of a graph, and criticality of a graph in this paper have been verified by these techniques, unless otherwise stated.

For instance, to verify Theorem 6, we used these tools to determine the coverage provided by each structure S. For a given structure S, we let G be the graph containing a subdivided V_8 with S attached. Then, using **has1Drawing**, we check, for each pair of rim edges $e, f \in E(G), e \neq f$, if G has a 1-drawing where e and f cross. If G does not have a 1-drawing where a rim edge e is crossed, then we conclude that, by definition, e is covered by S.

Beyond the previously described tools, `nauty` also provides an algorithm to check if graphs are isomorphic. This algorithm is takes as input a set of graphs, and returns as output all of the non-isomorphic graphs within that set. Outlining how this algorithm works is beyond the scope of this project, so we simply note that this is the tool used to verify all statements about sets of graphs being non-isomorphic.

1.4 Attaching Stars to V_8's

We can now proceed to the main contributions of this thesis. Recall that a k-star is the complete bipartite graph $K_{\{1,k\}}$; that is, the tree with one root vertex and k leaves (when

$k > 1$). In Austin's thesis [2], following her work done with adding edges to a V_8, she outlined an unexplained class of 2-crossing-critical graphs, where a k-star is attached to a V_8. We follow her work by analyzing this class of graphs. What happens when we add edges between a single external vertex and k vertices in a subdivided V_8? How does this process make progress towards making the V_8 2-crossing-critical, without creating a V_{10} minor? This is the primary focus of this thesis.

Attaching a 1-star to a V_8 does not affect the crossing number. Attaching a 2-star ($K_{1,2}$) is topologically isomorphic to adding an edge to the V_8, which has already been discussed. Therefore, we are concerned with what happens when a 3+-star is attached to a V_8.

We first consider 3-stars. How many possibilities are there for a 3-star attached to a V_8? First of all, we do not permit the 3-star to attach to the same vertex twice, as the resulting graph would be non-critical.

Under this restriction, there are 106 non-isomorphic possible ways to attach a 3-star to a subdivided V_8, as checked by a computer. We note that, since this number is obtained by attaching a 3-star to a subdivided V_8, there are a small handful of pairs of these 106 cases which are isomorphic after smoothing all degree 2 vertices. These cases will be noted in our discussion.

We proceed by breaking these 106 non-isomorphic 3-star attachments into cases based on how the 3-star attaches to the subdivision of V_8.

We first consider the cases where the 3-star attaches to each rim or spoke of the subdivided V_8 at most once:

- 1 case where the 3-star attaches at 3 spoke edges.
- 8 cases where the 3-star attaches at 2 spoke edges.
- 6 cases where the 3-star attaches to two opposing rims (that is, two rims branches at a cyclic distance of four; in other words, two rim branches in the same quad), and therefore forms a V_{10} minor.
- 34 cases where the 3-star attaches at 1 spoke edge.
- 38 cases where the 3-star attaches to 0 spoke edges.

After this, we consider the cases where the 3-star is permitted to attach to a rim or spoke of the V_8 more than once:

- 2 cases where the 3-star attaches to the same edge 3 times.
- 7 cases where the 3-star attaches to the same spoke edge 2 times.
- 10 cases where the 3-star attaches to the same rim edge 2 times.

As explained in the previous section, with a computer it is easy to determine which rim edge crossings a given 3-star eliminates by its attachments to a V_8. However, this does not guarantee that the 3-star can be included in a 2-crossing-critical graph. Therefore, in this analysis, we seek to achieve one of two objectives for each 3-star attached to a V_8:

- Demonstrate that the 3-star can be included in a 3-connected 2-crossing-critical graph with a V_8 minor but no V_{10} minor, which embeds in the projective plane, by providing an example of one such graph. (We determine which rim edges the 3-star covers in section A.2).
- Demonstrate that the 3-star cannot be included in such a graph, typically by contradiction.

To do so, we assume that we are working with a 3-connected 2-crossing-critical graph G with a V_8 minor but no V_{10} minor, which is embeddable in the real projective plane $\mathbb{R}P^2$.

Such a graph may contain multiple V_8 minors. Therefore, we wish to consider a particular V_8. Importantly, we assume that the V_8 with which we are working is a V_8 in G with the minimum number of subdivisions. We again note that, because G is 3-connected, each vertex must have degree at least 3.

1.5 Key Results

The rest of this thesis is concerned with analyzing the effects of 3-stars on covering a subdivided V_8, and then proceeding to do the same with 4+-stars. Chapter 2 handles the cases where a 3-star attaches to a V_8 via at least two spoke edges. In fact, in Chapter 2, we were able to generalize our argument to reach a stronger conclusion about H-bridges, where H is a subdivision of V_8, which attaches to multiple spokes of H. Chapter 3 handles those cases where a 3-star attaches to precisely one spoke. Chapter 4 handles the cases where a 3-star attaches to no spokes. Chapter 5 covers those cases where a 3-star attaches to a single spoke or rim multiple times. Combining the results of Chapters 2, 3, 4, and 5 yields our primary result, Theorem 8.

Chapter 6 extends this case analysis to 4-stars, yielding our primary result regarding 4-stars, Theorem 10. Chapter 6 concludes with the strong result about 5+-stars in Theorem 12. Chapter 7 contains a discussion of future research questions, including how one might extend this discussion about k-stars attaching to a V_8 subdivision to include trees with k leaves as well.

Theorem 8. *Let G be a 3-connected 2-crossing-critical graph with a V_8 minor but no V_{10} minor, such that G embeds in the real projective plane $\mathbb{R}P^2$.*

Let G contain a 3-star T which attaches to the V_8 minor in G with the minimum number of subdivisions.

Then the connections of the 3-star T are one of the following sets of vertices, under a canonical labelling of the V_8:

- *(0, 2, 15)*
- *(0, 3, 13)*
- *(0, 5, 6)*
- *(0, 5, 17)*
- *(0, 6, 9)*
- *(0, 13, 15)*
- *(0, 13, 18)*
- *(0, 17_a, 17_b)*
- *(1, 5, 6)*
- *(1, 5, 17)*
- *(1, 17_a, 17_b)*
- *(2, 5, 17)*
- *(2, 6, 9)*
- *(3, 5, 17)*
- *(5, 6, 12)*
- *(5, 6, 19)*
- *(5, 12, 17)*
- *(5, 14, 17)*
- *(5, 17, 19)*
- *(6, 9, 12)*
- *(7, 16, 17)*
- *(8, 13, 14)*
- *(10, 14, 17)*
- *(12, 17_a, 17_b)*
- *(16, 17, 19)*

(where 17_a and 17_b represent two distinct points on the rim branch $(5,6)$).

We note that not all of the above 3-stars were demonstrated to be contained within a 2-crossing-critical graph. For a subset of the above 3-stars, it is suspected, but not yet proven, that they cannot be included in a 2-crossing-critical graph.

Conjecture 9. *Let G be a 3-connected 2-crossing-critical graph with a V_8 minor but no V_{10} minor, such that G embeds in the real projective plane $\mathbb{R}P^2$, and such that G contains a 3-star T which attaches to the V_8 minor in G with the minimum number of subdivisions. Then the connections of the 3-star T are not one of the following sets of vertices, under a canonical labelling of the V_8:*

- *(0, 2, 15)*
- *(0, 13, 18)*
- *(2, 6, 9)*
- *(5, 6, 19)*
- *(5, 17, 19)*
- *(7, 16, 17)*
- *(8, 13, 14)*

Theorem 10. *Let G be a 3-connected 2-crossing-critical graph with a V_8 minor but no V_{10} minor, such that G embeds in the real projective plane $\mathbb{R}P^2$, and such that G contains a 4-star T which attaches to the V_8 minor in G with the minimum number of subdivisions. Then the connections of the 4-star T are one of the following sets of vertices, under a canonical labelling of the V_8:*

- *(0, 1, 5, 6)*
- *(0, 1, 5, 17)*
- *(0, 2, 14, 19)*
- *(0, 4, 12, 15)*
- *(0, 12, 17_a, 17_b)*
- *(1, 2, 5, 6)*
- *(1, 12, 17_a, 17_b)*
- *(2, 5, 14, 17)*

(where 17_a and 17_b represent two distinct points on the rim branch $(5, 6)$).

We note that not all of the above 4-stars were demonstrated to be contained within a 2-crossing-critical graph. For a subset of the above 4-stars, it is suspected, but not yet proven, that they cannot be included in a 2-crossing-critical graph.

Conjecture 11. *Let G be a 3-connected 2-crossing-critical graph with a V_8 minor but no V_{10} minor, such that G embeds in the real projective plane $\mathbb{R}P^2$, and such that G contains a 4-star T which attaches to the V_8 minor in G with the minimum number of subdivisions. Then the connections of the 4-star T are not one of the following sets of vertices, under a canonical labelling of the V_8:*

- *(0, 2, 14, 19)*
- *(0, 12, 17_a, 17_b)*
- *(1, 2, 5, 6)*

Theorem 12. *Let G be a 2-crossing-critical graph with a V_8 minor but no V_{10} minor. Let T be a k-star which attaches to the V_8 minor. Then $k \leq 4$.*

1.6 Navigating This Thesis

Due to the symmetries of a V_8, a given 3-star attached to a subdivided V_8 will have many different relabellings under the canonical labelling of a V_8. Thus, our work does not organize 3-stars by their attachments under a canonical labelling, and instead opts for a more general approach. This is intended to make navigating this thesis relatively simple, while maintaining certain internal consistencies necessary if the reader wished to work with the examples in this thesis with a computer. Here, we give a brief description of the organization of the thesis so that the reader can easily find the discussion for any 3-star (which, as mentioned, will either be an example that it can be contained within a 2-crossing-critical graph, a proof that it cannot, or a brief discussion if the result is unknown). We note that the Table of Contents is also fully linked and contains the same information as below, in a more concise manner.

- Does the 3-star in question attach to a single rim or spoke edge multiple times?
- **If yes:** The 3-star is discussed in Chapter 5.
  - Does it attach three times to the same spoke edge? Then it is found in Section 5.1.
  - Does it attach three times to the same rim edge? Then it is found in Section 5.2.
  - Does it attach two times to the same spoke edge? Then it is found in Section 5.3.
  - Does it attach two times to the same rim edge? Then it is found in Section 5.4.
- **If not:** The 3-star is discussed in Chapter 2, 3, or 4.
  - How many spokes does the 3-star attach to?
  - **Two or More Spokes:** The 3-star is discussed in Chapter 2.
    * If it attaches to non-adjacent spokes, it is discussed in Section 2.2.
    * Otherwise, it attaches to adjacent spokes. If the third attachment is in the quad bounded by these spokes, it is discussed in Section 2.3.
    * Otherwise, it is discussed in Section 2.4.
  - **One or Fewer Spokes, and Opposing Rims:** If the 3-star attaches to opposing rim edges, it forms a V_{10} minor and is discussed in Section 3.1.
  - **One Spoke:** The 3-star is discussed in Chapter 3.
    * The 3-star attaches to a single spoke. Does it attach to the endpoint of an adjacent spoke? If so, it is discussed in Section 3.2.

    * Does it attach to the endpoint of the same spoke? If so, it is discussed in Section 3.3.
    * Does it attach to a rim edge incident to the endpoint of the same spoke? If so, it is discussed in Section 3.4.
    * Otherwise, it is one of a small handful of miscellaneous cases which do not have obvious similarities. These are discussed in Section 3.5 and Section 3.6.
  - **No Spokes:** The 3-star is discussed in Chapter 4.
    * Does it attach to consecutive V_8 vertices? (i.e. vertices of a non-subdivided V_8?) Then it is found in Section 4.1.
    * Does it attach to consecutive rim edges of the V_8? Then it is found in Section 4.2.
    * Does it attach to a V_8 vertex and an incident rim edge? Then it is found in Section 4.3.
    * Otherwise, it is one of a handful of miscellaneous cases which are not easily categorized. These are discussed in Section 4.4, Section 4.5, and Section 4.6, depending on their crossing number.

The linked sections above each contain multiple 3-stars. Therefore, each section has an introduction explaining how to find a given 3-star within that section. This is the final piece for navigating to a specific case within this thesis. Furthermore, for the reader's sake, all 3-stars contained in 2-crossing-critical graphs can also be found in Appendix A.2.

For example, suppose that a reader wanted to find where the following 3-star is discussed.

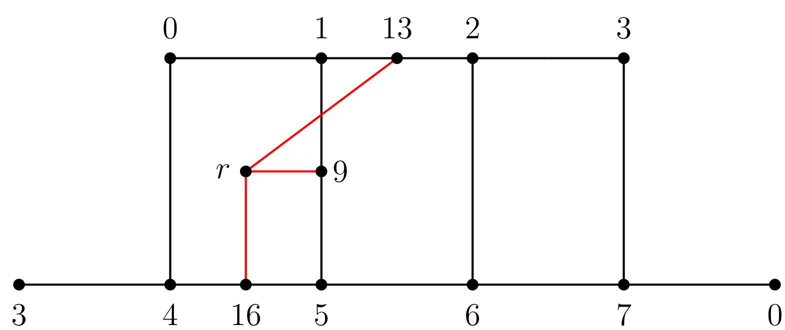

Figure 1.11: An example of a 3-star attached to a subdivided V_8.

First, the 3-star does not attach to a single rim or spoke edge multiple times. Next, it attaches to precisely one spoke, so it can be found in Chapter 3. No attachment is an end of an adjacent spoke or the end of the same spoke to which it is attached. It does attach to a rim edge incident with an end of the same spoke. Therefore, it is in Section 3.4.

Turning to Section 3.4 reveals a short introduction and the following diagram. In this section, two of the 3-star attachments have been determined, and there are eight possibilities for the third attachment. These are denoted $t_i; i \in [1, 8]$. It is easy to see that the case when $t = t_3$ (that is, vertex 14 in this canonical labelling) is isomorphic to the reader's example.

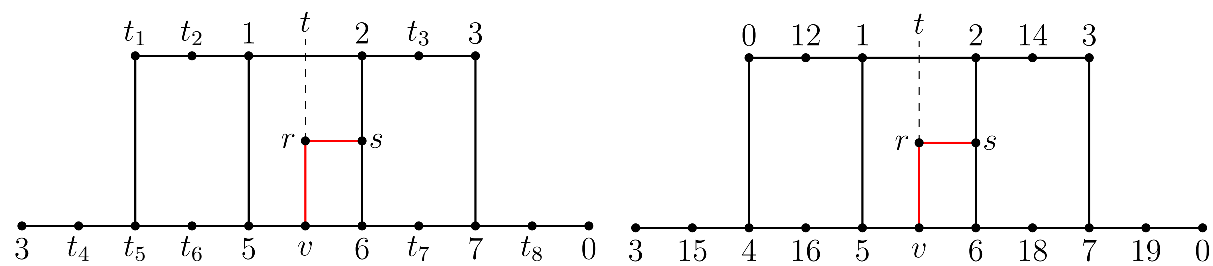

Figure 1.12: A 3-star attached to a V_8 by a spoke and an incident rim edge. Possible non-isomorphic third attachments, which have not been considered in other cases, are labeled $t_i; i \in [1, 8]$.

Indeed, turning to the case when $t = t_3$ reveals that the reader's example can be included in a 2-crossing-critical graph, and one such example is the following. Notably, the reader's example has eight relabellings under the canonical labelling, and the below example is one such relabelling.

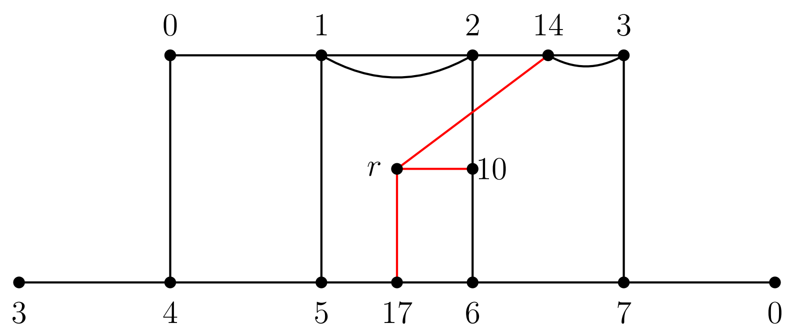

Figure 1.13: A 2-crossing-critical graph containing a V_8 with this 3-star attached when $t = t_3$.

Chapter 2

Bridges Attached to Multiple Spokes

In this chapter, we concern ourselves with proving Theorem 13. We note that the assumptions of this theorem are far more general than the assumptions of the other theorems in this paper; in particular, we need not assume that the subdivision H of V_8 in G is minimally subdivided. This chapter is the only chapter in which this assumption is not employed.

Theorem 13. *Let G be a 2-crossing-critical graph with subdivision H of V_8 and no V_{10} subdivision. If there is an H-bridge B with at least three attachments to H, two of which are in adjacent spokes, then G is isomorphic to the graph shown in Figure 2.1.*

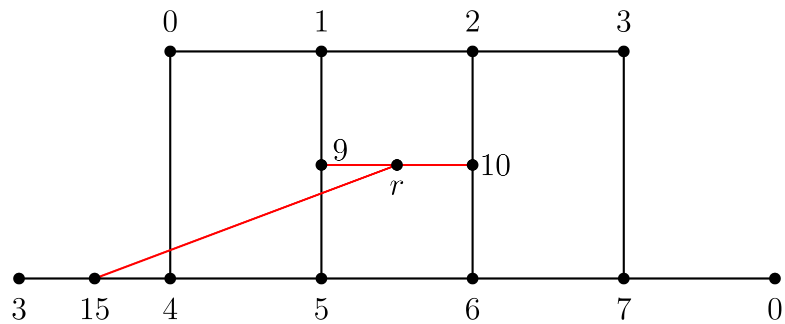

Figure 2.1: The unique 2-crossing-critical graph when a tree T attaches to two distinct spoke edges of a V_8.

In fact, the graph in Figure 2.1 does not embed in the real projective plane $\mathbb{R}P^2$. Since the stated goal of this thesis is to describe the effects of attaching k-stars to a V_8 under specific conditions including embeddability in the real projective plane, then we note the following corollary of Theorem 13.

Corollary 14. *Let G be a 3-connected 2-crossing-critical graph with a subdivision H of V_8 but no V_{10} subdivision, such that G embeds in the real projective plane. Let T be a tree which attaches to H. Then T does not attach to distinct spokes of H.*

2.1 Preliminaries

To aid with this analysis, we introduce some helpful concepts from Bokal, Oporowski, Richter, and Salazar's characterization of most 2-crossing-critical graphs [4].

The following definitions and lemma are summaries of Definition 5.1, Definition 5.2, Lemma 5.3, and Definition 5.4 in [4]. Let G be a graph and let H be a subgraph of G. Then an *H-bridge in G* is a subgraph B of G such that either B is an edge not in H, together with its ends, both of which are in H, or B is obtained from a component K of $G \setminus V(H)$ by adding to K all the edges from vertices in K to vertices in H, along with their ends in H.

For an H-bridge B in G, a vertex u of B is an *attachment of B* if $u \in V(H)$. We write $\mathrm{att}(B)$ to denote the attachments of B.

In this work, we work with the bridges of a cycle C. Let C be a cycle in graph G with distinct C-bridges B and B'. Then the *residual arcs* are the B-bridges in $B \cup C$. The C-bridges B and B' *do not overlap* if all of the attachments of B are in the same residual arc of B'; otherwise they *overlap*.

The *overlap diagram $OD(C)$* of C is an auxiliary graph with C-bridges as its vertex set and edges between two C-bridges if they overlap in C. Then C has *bipartite overlap diagram (BOD)* if its overlap diagram is bipartite. The following is a well-known result.

Lemma 15 (Lemma 5.3 in [4])**.** *Let C be a cycle in a graph G. Distinct C-bridges B and B' overlap if and only if either:*

1. *There are attachments u, v of B and u', v' of B' so that the vertices u, u', v, v' are distinct and occur in this order in C; or*

2. *$att(B) = att(B')$ and $|att(B)| = 3$.*

Now let C be a cycle in a graph G and let B be a C-bridge. Then B is a *planar C-bridge* if $C \cup B$ is planar.

Using all of the preceding definitions, we are now prepared to introduce a concept which proves to be tremendously powerful in our case analysis of 3-stars, and, more generally, H-bridges where H is a subdivision of V_8. Let G be a 2-crossing-critical graph. A cycle C in G is a *hole* if:

- C has precisely one non-planar bridge $\widehat{B}$;
- For every 1-drawing D of $C \cup \widehat{B}$, C is not crossed in D; and
- C has a bipartite overlap diagram.

We note that $C \cup \widehat{B}$ need not have any 1-drawings D for C to be a hole.

Theorem 16. *Let G be a 3-connected 2-crossing-critical graph and let C be a hole with non-planar C-bridge $\widehat{B}$. Then $\widehat{B}$ is the only C-bridge.*

Proof. By way of contradiction, suppose that there is a planar C-bridge B. Let $B^{\#}$ denote the union of C and all the C-bridges other than B. Because $B^{\#}$ is a proper subgraph of the 2-crossing-critical graph G, there is a drawing D of $B^{\#}$ in the plane with at most one crossing.

Let $\widehat{B}$ be the non-planar C-bridge. Since $C \cup \widehat{B} \subseteq B^{\#}$, and since C is not crossed in D by the definition of a hole, $D[C]$ is a simple closed curved in the plane.

Let (X, Y) be the bipartition of the overlap diagram (in G) of C such that $\widehat{B} \in X$. By moving all the planar C-bridges in $B^{\#}$, we may rearrange the drawing D into a 1-drawing drawing $\overline{D}$ such that all the C-bridges in $B^{\#}$ that are in X are on the same side of $D[C]$, while the remaining C-bridges (namely those in Y) are on the other side of $D[C]$. Now B can be added to the appropriate side, because of BOD, yielding the contradiction that $\mathrm{cr}(G) \leq 1$. □

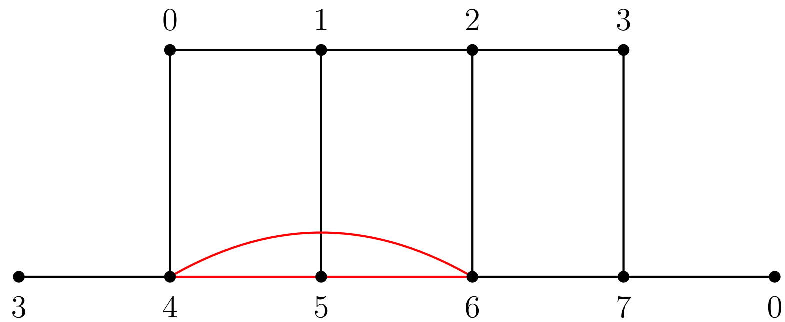

Figure 2.2: The cycle $(4, 5, 6, 4)$ is an example of a hole.

The question arises: when is a cycle C a hole? The following theorem provides a sufficient condition for a cycle C having some of the requirements of a hole. We first must introduce a definition.

The *representativity* of a graph G, embedded in a compact and connected surface Σ, is the largest integer n such that every non-contractible, simple, closed curve in Σ intersects G in at least n points. Representativity is also known as *face-width*, an idea which was used in the Graph Minors Project of Robertson and Seymour. This concept is rather technical and we need a basic understanding for only the following theorem, so we provide only a brief overview. Bokal, Oporowski, Richter, and Salazar in [4] enumerated precisely the 2-crossing-critical graphs with a V_8 minor but no V_{10} minor which embed in the real projective plane with representativity greater than 2. Furthermore, every graph embedded in the real projective plane with representativity at most one is planar. So, for our purposes, we can assume any uncharacterized 2-crossing-critical graph (i.e. those with which we concern ourselves) has representativity precisely 2.

Theorem 17. *Let G be a 2-connected graph embedded in the real projective plane with V_8-minor H with representativity equal to 2. Let C be a contractible cycle in G such that:*

- *there is a C-bridge $\hat{B}$ such that $C \cup \hat{B}$ is a non-planar graph; and*
- *for each face F of H, $C \cap \partial F$ is a path, where ∂F is the boundary of the face F.*

Then C satisfies:

- *for every other C-bridge B, $C \cup B$ is planar; and C has BOD.*

Proof. The subgraph $C \cup \widehat{B}$ of G is non-planar. Thus, its induced embedding in $\mathbb{R}P^2$ has representativity at least 2 and, since $C \cup \widehat{B}$ is 2-connected, every face is bounded by a cycle.

The representativity of G is exactly two, so there is a non-contractible curve γ in $\mathbb{R}P^2$ the meets G in exactly two points, which we may take to be vertices a and b. Add the parallel edges ab to $C \cup \widehat{B}$ to get the embedded graph H. A C-bridge B different from $\widehat{B}$ is contained in a face F of H and C meets the boundary of F in a path P_F.

It follows that:

- $C \cup B$ is planarly embedded in $\mathbb{R}P^2$; and
- B is on one side of C, either the disc side or the Möbius strip side.

Evidently, $\widehat{B}$ is on the Möbius strip side of C.

This gives the bipartition of the C-bridges. Obviously, for those C bridges on the same side of C, different faces of H yield internally-disjoint subpaths of C, so C-bridges in different faces of H and on the same side of C and if B is on the Möbius strip side of C, its attachments are in the same residual arc of $\widehat{B}$. In particular, no two C-bridges in the Möbius strip side of C overlap, nor do two C-bridges in the same face of H overlap.

□

In our work, we present cycles as holes without further justification. First, the cycles we consider will clearly have precisely one non-planar bridge. The non-planar bridge is typically a subdivision H of V_8, and the other bridges will be planar as they will be edges or trees. Second, the cycles we consider will be contractible and intersect faces of a V_8 subdivision through a path[1], so they will have BOD by Theorem 17. This is, in almost every case, plain to see as the cycles we consider are small in their intersection with a V_8 subdivision. Therefore, we omit justification for this requirement of a hole as well.

Now let C be a cycle in a graph K, where K is typically a subgraph of a purportedly 2-crossing-critical graph G. In order to verify that the cycle C fulfills the second condition in the definition of a hole, it suffices to have a computer verify that, for each edge e of C, there does not exist a 1-drawing of K where e is crossed. To do so, we can use the **has1Drawing** algorithm from before to check, for all edges $e \in E(C)$, for all edges $f \in E(K)$, if K has

[1]This is not true in one case, Figure 6.6, as the cycle in question is not contractible in all real projective planar embeddings, but we have separately verified BOD in this case.

a 1-drawing where e and f are crossed. If the answer is no for all pairs e and f, then the cycle C is not crossed in any 1-drawing of K.

Having shown that C has this property in K implies that C has this property in every supergraph of K, including G. This is the approach we have taken in this work, as we have used a computer to verify the second requirement of each hole. We make this note here and omit justification of holes throughout the paper.

A curious reader looking to check the second hole requirement by hand should refer to Theorem 6 and Appendix A as useful tools.

We can now proceed to a case analysis of bridges attached to multiples spokes of a V_8 subdivision. Throughout the remainder of this chapter, let G be a 3-connected 2-crossing-critical graph with a subdivision H of V_8 but no V_{10} subdivision, such that G contains an H-bridge B which connects to multiple spokes of H. We first consider if B is attached to non-adjacent spokes, and then proceed to analyze two cases when B is attached to adjacent spokes.

2.2 Two non-adjacent spokes

If B is attached to two non-adjacent spokes, then G contains as a proper subgraph H with a 2-bar attached. The graph with a 2-bar attached to H is 2-crossing-critical (it is one of the 103 graphs which minimally do not embed in $\mathbb{R}P^2$), so $\operatorname{cr}(G) \geq 2$ and G is non-critical, a contradiction.

2.3 Two adjacent spokes and another vertex outside of the quad

Since B does not attach to two non-adjacent spokes, then it attaches to two adjacent spokes. We now consider where a third attachment of B to H might be. Let t be this vertex. In this first case, we consider if t is outside the quad bounded by the two spokes to which B attaches. There are three non-isomorphic possibilities for t, as demonstrated in Figure 2.3.

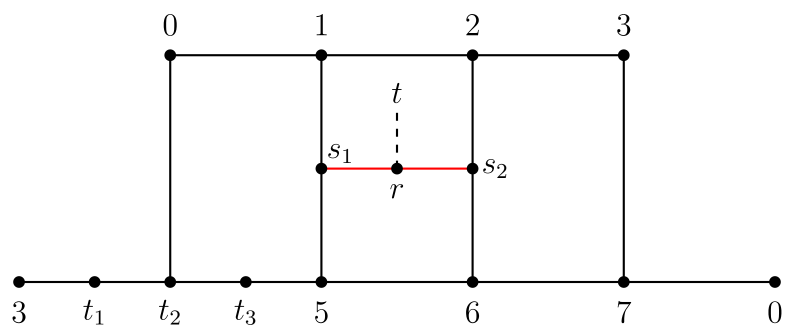

Figure 2.3: Possible non-isomorphic placements of t, denoted t_1, t_2, t_3.

2.3.1 Case $t = t_1$

When $t = t_1$, then B has a subdivision of a 3-star. As checked by a computer, B must itself be a 3-star, and the resulting graph as shown in Figure 2.4 is 2-crossing-critical.

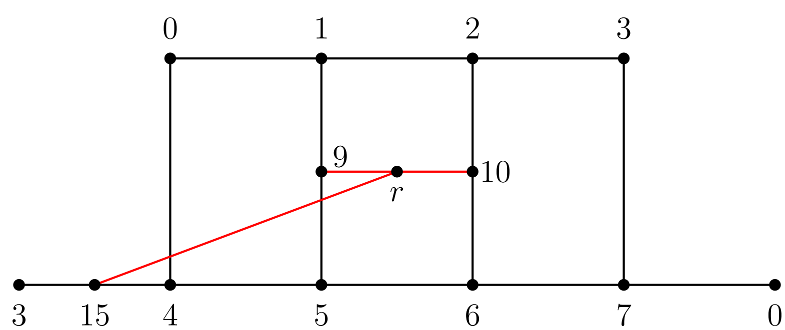

Figure 2.4: The 2-crossing-critical graph when $t = t_1$.

2.3.2 Case $t = t_2$ or $t = t_3$

When $t = t_2$ or $t = t_3$, then, as checked by a computer, G has crossing number at least 2 and is non-critical, a contradiction.

2.4 Two adjacent spokes and another vertex within the quad

So now we consider if B is attached to two adjacent spokes, and B has a third attachment to H contained within the quad bounded by those spokes. Let t be the vertex on H which is this third attachment of B.

2.4.1 t is on a rim edge of the quad

If t is on a rim edge of the quad, then G has a V_{10} minor, a contradiction.

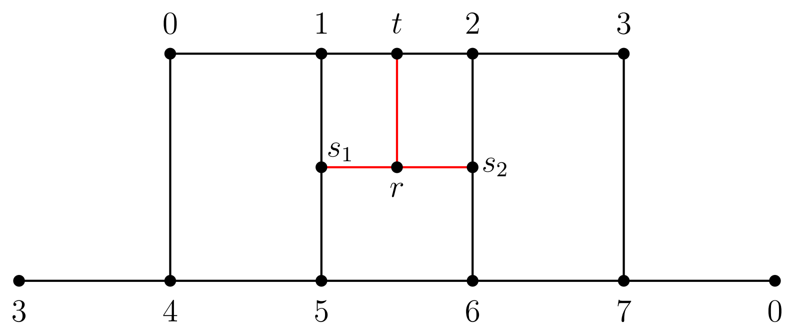

Figure 2.5: A 3-star attached to a V_8 by adjacent spokes and a rim edge.

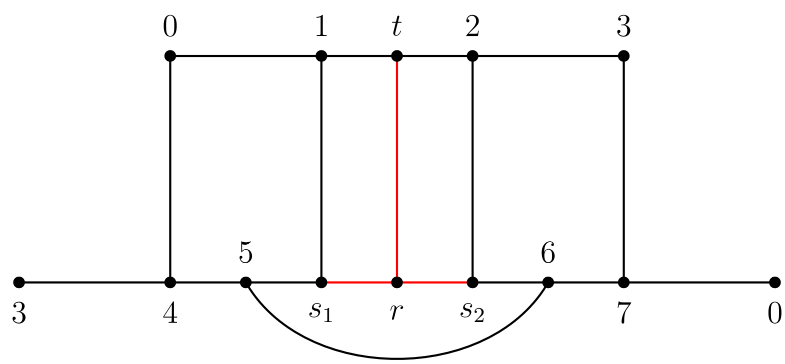

Figure 2.6: Transformation of Figure 2.5 demonstrating a V_{10} minor.

2.4.2 t is not on a rim edge of the quad

Now suppose t is one of the four original V_8 vertices contained within the quad. Without loss of generality, let $t = 1$ as in Figure 2.7. If (r, t) is an edge, then the cycle $(r, s_1, t, 2, s_2, r)$ is a hole with planar bridge (r, t), a contradiction with Theorem 16.

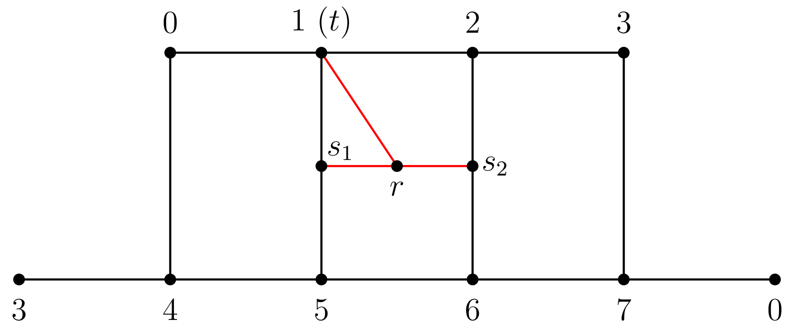

Figure 2.7: A 3-star attached to a V_8 by adjacent spokes and the endpoint of one of them.

Then (r, t) is not an edge. Let T be the 3-tree subdivision of B with attachments to H at s_1, s_2, and t. Let i be an internal vertex of the path (r, t) with degree at least 3. Then there exists an $H \cup T$-avoiding path from i to a vertex a in $H \cup T$. Then, as checked by a computer, if a is not contained within the (potentially subdivided) cycle $(r, s_1, t, 2, s_2, r)$, the resulting graph has crossing number at least 2 and is non-critical. So a is contained within the cycle $(r, s_1, t, 2, s_2, r)$. Then the cycle $(r, s_1, t, 2, s_2, r)$ is a hole with a planar bridge, a contradiction with Theorem 16. Therefore, a contradiction arises in all cases.

2.5 Conclusion

In conclusion, every case of an H-bridge B with at least three attachments to H, two of which are in adjacent spokes, has been considered. The only case which is 2-crossing-critical is demonstrated in Figure 2.4; in every other case, a contradiction was demonstrated. This completes the proof of Theorem 13.

Chapter 3

3-Star Case Analysis: One Spoke Attachment

We now continue with the 3-star case analysis. Recall that Corollary 14 in Chapter 2 explained all 3-stars which attach to at least two spokes of a V_8 subdivision. In this chapter, we consider 3-stars which attach to precisely one spoke of a V_8 subdivision. As discussed previously, henceforth we assume that we are working with a V_8 subdivision in our graph G which has the fewest vertices.

Theorem 18. *Let G be a 3-connected 2-crossing-critical graph with a V_8 minor but no V_{10} minor, such that G embeds in the real projective plane $\mathbb{R}P^2$, and such that G contains a 3-star T which attaches to the V_8 subdivision H in G with the minimum number of subdivisions. Suppose that T attaches to H at precisely one spoke edge. Then T attaches to H at one of the following sets of vertices, under the canonical labelling of a V_8 subdivision and up to symmetry:*

- *(0, 6, 9)*
- *(2, 6, 9)*
- *(6, 9, 12)*
- *(8, 13, 14)*
- *(10, 14, 17)*

Further, some of the above 3-stars were not demonstrated to be included in a 2-crossing-critical graph. It is suspected, but not yet proven, that these cannot be included in a 2-crossing-critical graph.

Conjecture 19. *Let G be a 3-connected 2-crossing-critical graph with a V_8 minor but no V_{10} minor, such that G embeds in the real projective plane $\mathbb{R}P^2$, and such that G contains* *a 3-star T which attaches to the V_8 minor in G with the minimum number of subdivisions. Then the connections of the 3-star T are not one of the following sets of vertices, under a canonical labelling of the V_8:*

- *(2, 6, 9)*
- *(8, 13, 14)*

There are 34 non-isomorphic cases to be considered in this section, as verified by a computer.

3.1 Opposing rims

In this section, we consider 3-stars attached to opposing rims of a V_8 subdivision. This section can be explained by a useful, more general Lemma.

Lemma 20. *Let G be a graph with a V_8 minor but no V_{10} minor. Let H be a V_8 subdivision in G. Let B be an H-bridge. Then B does not attach to opposing rim edges of H.*

Proof. Suppose towards a contradiction that an H-bridge B attaches to opposing rim edges of H. Then the path in B between opposing rim edges of H forms a fifth spoke, and thus G contains a V_{10} minor, a contradiction. □

3.2 One spoke and an endpoint of an adjacent spoke

Now let a 3-star with root r be attached to one spoke of the V_8 at s, the endpoint of an adjacent spoke at v, and at a third non-spoke attachment at vertex t. The vertex t can be at any point on the rim of the subdivided V_8, and this case contains 15 non-isomorphic possibilities depending on the placement of t.

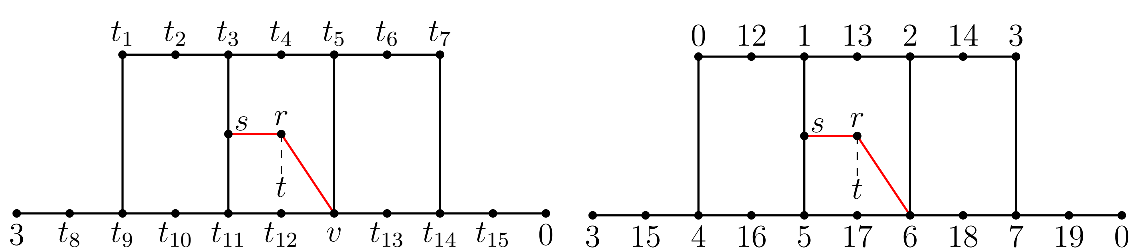

Figure 3.1: A 3-star attached to one spoke and opposing V_8 vertex. Possibilities for a third attachment t are denoted $t_i; i \in [1 : 15]$.

We note that, throughout this discussion, labelling of t on the left and the canonical labelling of the V_8 will both be referenced as appropriate. We include the canonical labelling here again for the sake of the reader.

3.2.1 Case $t = t_1 = 0$

If t is at t_1, then a V_8 with this 3-star attached can be contained in a 2-crossing-critical graph. One such example is Figure 3.2.

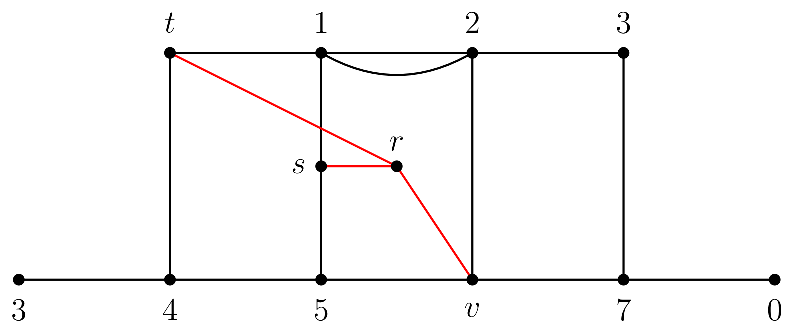

Figure 3.2: A fully-covered 2-crossing-critical graph containing the 3-star when $t = t_1$.

3.2.2 Case $t = t_2 = 12$

If t is at t_2, then a V_8 with this 3-star attached can be contained in a 2-crossing-critical graph. One such example is Figure 3.3.

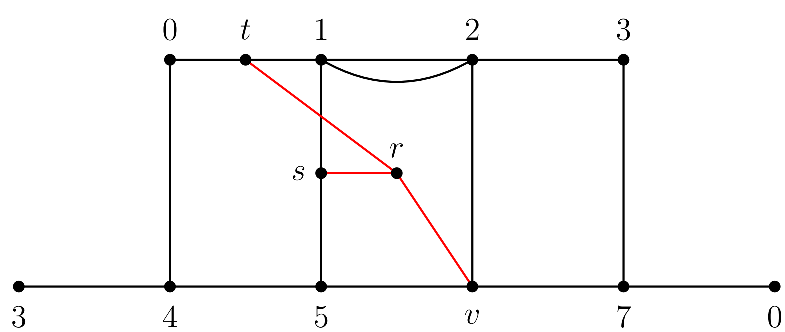

Figure 3.3: A fully-covered 2-crossing-critical graph containing the 3-star when $t = t_2$.

3.2.3 Case $t = t_3 = 1$

Now suppose that $t = t_3$. If $(s, 1)$ is an edge, then delete it. Redefine the spoke $(1, s, 5)$ to go through $(1, r, s, 5)$. Then $(s, 1)$ can be drawn parallel to the non-crossed path $(1, r, s)$ in $G \setminus \{(s, 1)\}$, resulting in a 1-drawing of G, a contradiction.

So $(s, 1)$ is a path. By the minimality of V_8 subdivisions, $(s, 1)$ contains precisely one subdivision (this can be seen by again redefining the spoke $(1, s, 5)$ to be $(1, r, s, 5)$). Let i be the internal vertex of the $(s, 1)$ path. Then there is an H-bridge which attaches to i and some other vertex a, which is somewhere in H, the V_8 subdivision.

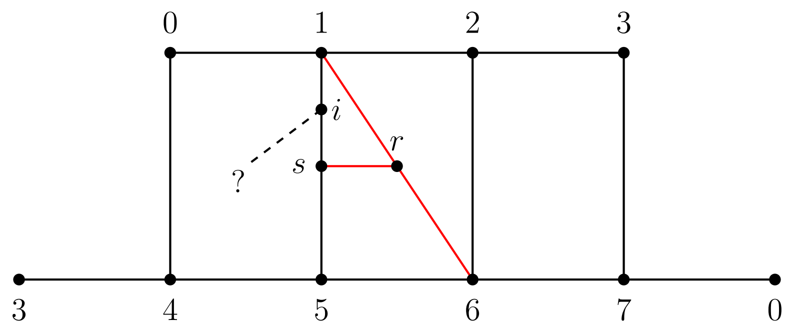

Figure 3.4: The 3-star when $t = t_3$, if $(s, 1)$ contains an internal vertex i.

If a is on a spoke, then we have a 3-star with root i which attaches to two distinct spokes. This has already been fully explained by Theorem 13.

If a is in the $7, 0, 1, 2, 3$ path, then let the cycle $C = (s, i, a, P, 1, r, s)$, with P being the path from a through $7, 0, 1, 2, 3$ to 1. Then C is a hole with planar bridge $(i, 1)$, a contradiction with Theorem 16.

If a is in the $3, 4, 5, 6, 7$ path, then let the cycle $C = (i, a, P, 5, s, r, 1, i)$, with P being the path from a through $3, 4, 5, 6, 7$ to 5. Then C is a hole with planar bridge (s, i), a contradiction with Theorem 16.

Since these cases cover all possible placements of a, then a contradiction arises in all cases and we conclude that $t \neq t_3$.

3.2.4 Case $t = t_4 = 13$

If t is at t_4, then the resulting graph contains a V_{10} minor, a contradiction.

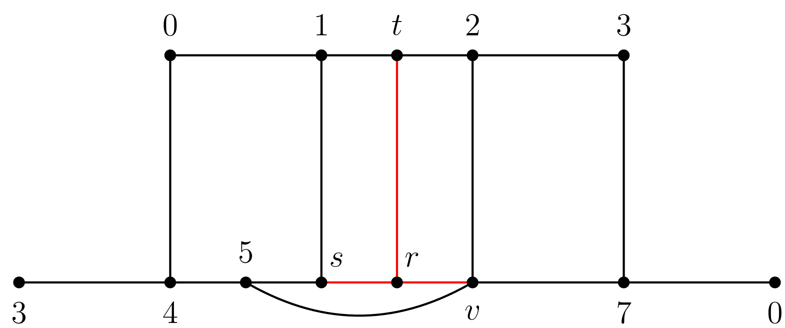

Figure 3.5: Transformation of Figure 3.1 when t is at t_4, demonstrating a V_{10} minor.

3.2.5 Case $t = t_5 = 2$

If t is at t_5 as in Figure 3.6, then it is suspected, but not yet proven, that the resulting graph cannot be the subgraph of a 2-crossing-critical graph. This is the first of a small handful of interesting cases which are beyond the scope of this project. See section 7.1 for further discussion.

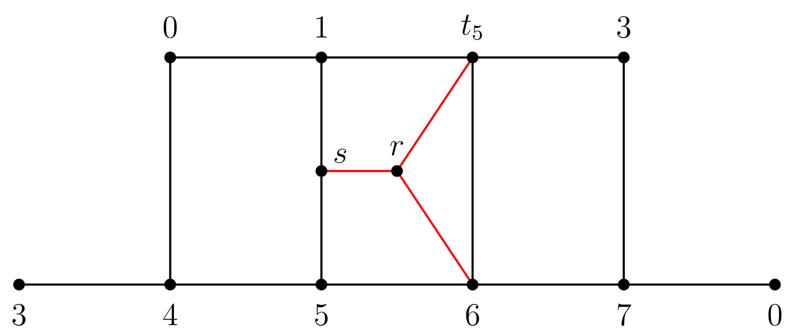

Figure 3.6: Figure 3.1 when t is placed at t_5.

3.2.6 Case $t = t_6 = 14$ or $t = t_7 = 3$

If t is at t_6 or t_7, then the resulting graph has crossing number 2 and is not critical (as checked by a computer), a contradiction.

3.2.7 Case $t = t_8 = 15$

If t is at t_8, then the additional transformation below demonstrates that the resulting graph contains an V_8 subdivision H' and an H'-bridge connecting to two spokes. This case has already been explained by Theorem 13 and is therefore not considered here.

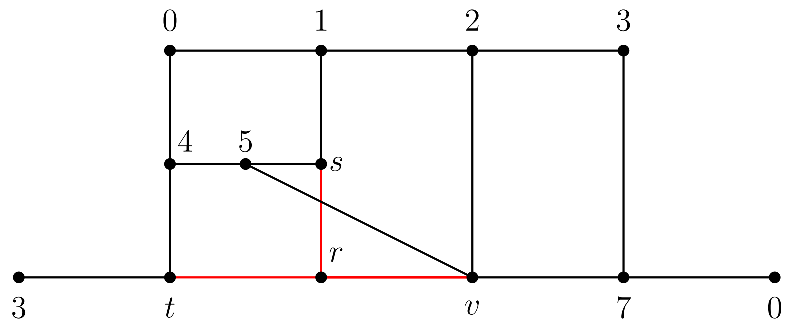

Figure 3.7: Transformation of Figure 3.1 when t is at t_8.

3.2.8 Case $t = t_9 = 4$ or $t = t_{10} = 16$

If t is at t_9 or t_{10}, then the resulting graph has crossing number at least 2 and is non-critical (as checked by a computer), a contradiction.

3.2.9 Case $t = t_{11} = 5$ or $t = t_{12} = 17$

If t is at t_{11} or t_{12}, then the cycle (t, s, r, v, t) is a hole with a planar bridge $(r, 5)$, a contradiction with Theorem 16. Therefore, this case cannot be 2-crossing-critical.

3.2.10 Case $t = t_{13} = 18$ or $t = t_{14} = 7$

If t is at t_{13} or t_{14}, then the cycle $(5, s, r, t, v, 5)$ is a hole with planar bridge (r, v), a contradiction with Theorem 16. Therefore, this case cannot be 2-crossing-critical.

3.2.11 Case $t = t_{15} = 19$

Now suppose that t is placed at t_{15}. If the path $(6, 7)$ is an edge, then the cycle $(r, 6, 2, 3, 7, t, r)$ is a hole with planar bridge $(6, 7)$, a contradiction with Theorem 16.

So $(6, 7)$ contains at least one subdivision. But under the transformation in Figure 3.8, the resulting graph has a V_8 subdivision H' with an H'-bridge attached to two consecutive spokes. This case has already been fully explained by Theorem 13, and is therefore not considered here.

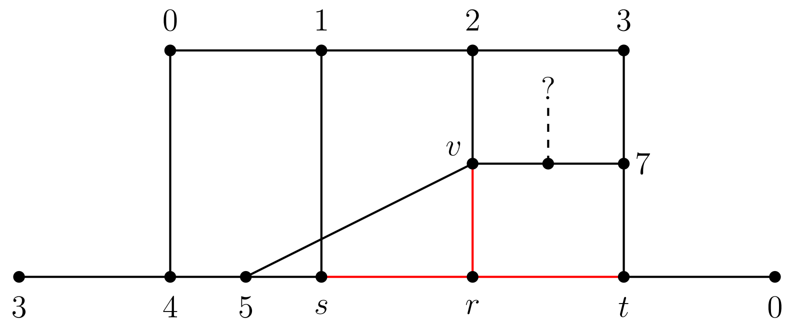

Figure 3.8: Transformation of Figure 3.1 when t is placed at t_{15}, under the assumption that $(6, 7)$ contains at least one subdivision.

Therefore, in all cases when t is at t_{15}, the resulting graph has already been explained or yields a contradiction.

3.2.12 Conclusion

To conclude this section's analysis, if a 3-star T is attached to a subdivision H of a V_8 at one spoke, and the endpoint of an adjacent spoke, there are fifteen possibilities. If this occurs in a 3-connected 2-crossing-critical graph G, then T is attached to H at one of the following two sets of vertices, under the canonical labelling of a subdivided V_8 and up to symmetry:

- (0, 6, 9)
- (6, 9, 12)
- (2, 6, 9)

The first two have been shown to be contained in 2-crossing-critical examples, and it is suspected, but not yet proven, that third is not contained in a 2-crossing-critical example.

3.3 One spoke and an endpoint of the same spoke

Let the 3-star with root r be attached to a spoke of the V_8 at s, and the endpoint of that spoke at v, and a third non-spoke attachment at vertex t. Then there are six possible non-isomorphic placements for t. The vertex t cannot be at 1, 3, 5, or 7, as these were considered in the previous case. This leaves 12 remaining options, but only six are non-isomorphic as s and v are symmetric with respect to the quads $[1, 2, 6, 5]$ and $[3, 2, 6, 7]$.

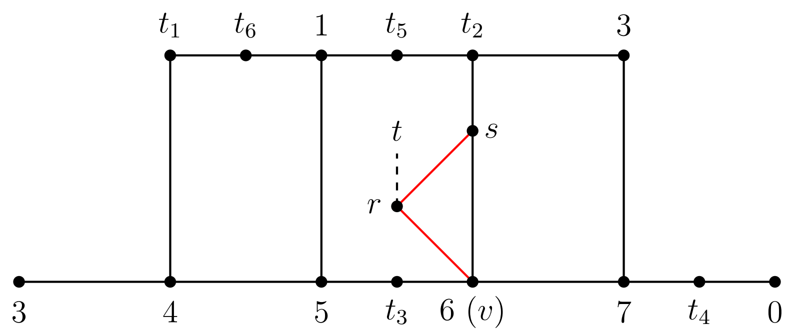

Figure 3.9: Possible non-isomorphic placements of t are denoted $t_1, t_2, t_3, t_4, t_5, t_6$.

3.3.1 Case $t = t_1 = 0$

If t is placed at t_1, then the cycle $(6, s, r, 0, 7, 6)$ is a hole with a planar bridge $(r, 6)$, a contradiction with Theorem 16.

3.3.2 Case $t = t_2 = 2$

If t is placed at t_2, then the cycle $(6, r, 2, s, 6)$ is a hole with a planar bridge (r, s), a contradiction with Theorem 16.

3.3.3 Case $t = t_3 = 17$

If t is placed at t_3, then the cycle $(6, t, r, s, 6)$ is a hole with planar bridge $(r, 6)$, a contradiction with Theorem 16.

3.3.4 Case $t = t_4 = 19$

If t is placed at t_4, then the cycle $(6, 7, t, r, s, 6)$ is a hole with planar bridge $(r, 6)$, a contradiction with Theorem 16.

3.3.5 Case $t = t_5 = 13$

Now suppose that t is placed at t_5. By the minimality of V_8 subdivisions, the replacement spoke $(t, r, 6)$ shows that $(s, 6)$ is not subdivided. So $(s, 6)$ is an edge. Redefining the spoke as $(2, s, r, 6)$, and deleting $(s, 6)$ gives a 1-drawing of the resulting graph. But then $(s, 6)$ can be added back alongside the uncrossed path $(s, r, 6)$, yielding a 1-drawing of the original graph, a contradiction.

3.3.6 Case $t = t_6 = 12$

Now suppose t is placed at t_6. First, we suppose that $(s, 6)$ is an edge. Now redefine the spoke $(2, s, 6)$ to instead be $(2, s, r, 6)$. Delete this edge to form G'. Since the spoke $(2, s, r, 6)$ is not crossed in any 1-drawing D' of G', then the edge $(s, 6)$ can be added back alongside the spoke in D', resulting in a 1-drawing of G, a contradiction. Therefore, this case cannot be 2-crossing-critical and is discarded.

Thus, $(s, 6)$ is not an edge. By the minimality of V_8 subdivisions, $(s, 6)$ can have at most one subdivision. To see this, redefine the spoke $(2, s, 6)$ to be $(2, s, r, v)$. If $(s, 6)$ had more than one subdivision, then this new spoke would be a part of a V_8 with fewer subdivisions than the original one, a contradiction.

So let k be the single internal vertex on the path $(s, 6)$. Then there is an H-avoiding path from k to another vertex i on H. As checked by a computer, except for the vertices 14,11,18,17,2,3, and 7, all other cases yield a non-critical graph with crossing number at least two. The vertices 14,11,18,17,2,3, and 7, are denoted $i_j; j \in [7]$ below.

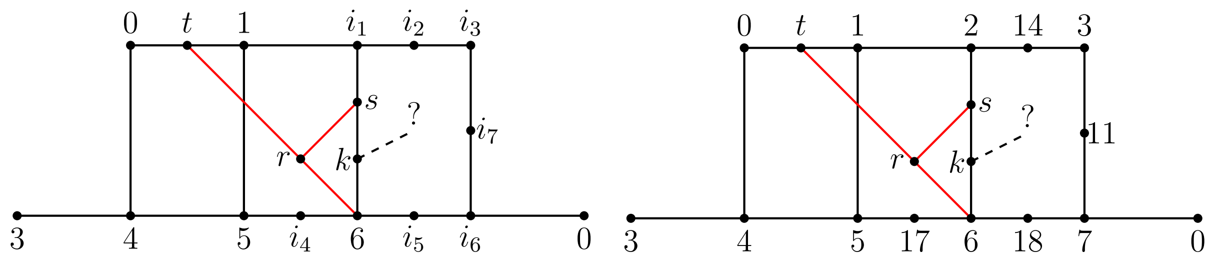

Figure 3.10: Possible non-isomorphic attachments of a second attachment of an H-bridge attached at i, when $t = t_6$, are denoted $i_j; j \in [7]$. A partial canonical labelling of a subdivided V_8 is included for the reader's sake.

If $i = i_1 = 2$ or $i = i_2 = 14$ or $i = i_3 = 3$, then the cycle $(k, [2, 3], s, r, 6, k)$, where $[2, 3]$ represents the path from i through the rim branch $(2, 3)$ to 2, is a hole with planar bridge (k, s), a contradiction with Theorem 16

If $i = i_4 = 17$, then the cycle $(k, 17, 6, r, s, k)$ is a hole with planar bridge $(k, 6)$, a contradiction with Theorem 16.

If $i = i_5 = 18$ or $i = i_6 = 7$, then the cycle $(k, [6, 7], 6, r, s, k)$, where $[6, 7]$ represents the path from i through the rim branch $(6, 7)$ to 7, is a hole with planar bridge $(k, 6)$, a contradiction with Theorem 16.

If $i = i_7 = 11$, then the cycle $(k, 11, 7, 6, r, s, k)$ is a cycle with planar bridge $(k, 6)$, a contradiction with Theorem 16.

Therefore, in all cases of an H-bridge with k as an endpoint, a contradiction arises.

Therefore, a contradiction arises when $(s, 6)$ is an edge and when it is not an edge, and we conclude that $t \neq t_6$.

3.3.7 Conclusion

Therefore, in a 3-connected 2-crossing-critical graph G with a subdivision H of a V_8, such that H is the minimally subdivided V_8 subdivision in G, a 3-star T cannot attach to a spoke of H and the endpoint of that same spoke.

3.4 One spoke, and a rim edge incident to the endpoint of the spoke

In this case, let the 3-star connect to a spoke of the V_8 at s, and a rim edge incident to an endpoint of this spoke at v. Let the third connection of the 3-star be connected to the V_8 at t. Connections at vertices 1,3,5, and 7 were considered in section 3.2, and connections at vertices 2 and 6 were considered in section 3.3. Furthermore, the connection to the $(1,2)$ rim edge forms a V_{10} minor as covered in section 3.1. As such, there are 8 non-isomorphic possibilities for t.

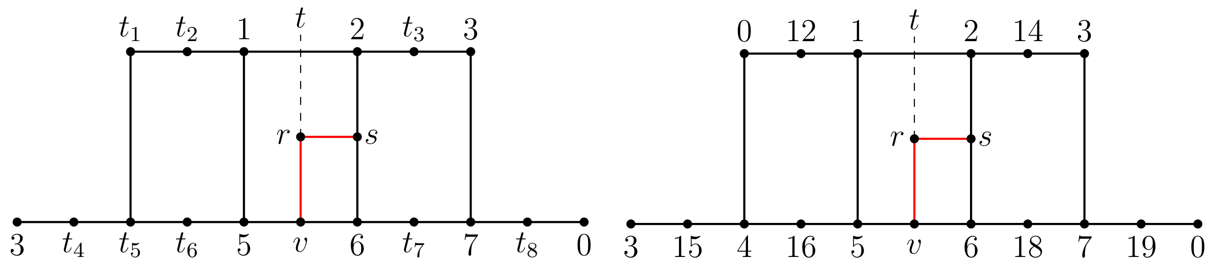

Figure 3.11: A 3-star attached to a V_8 by a spoke and an incident rim edge. Possible non-isomorphic third attachments, which have not been considered in other cases, are labeled $t_i; i \in [1,8]$.

3.4.1 Case $t = t_1 = 0$

If t is placed at t_1, then the resulting graph has crossing number 2 and is non-critical (as checked by a computer), a contradiction.

3.4.2 Case $t = t_2 = 12$

If t is placed at t_2, then the resulting graph has a V_{10} minor under the transformation in Figure 3.12, a contradiction.

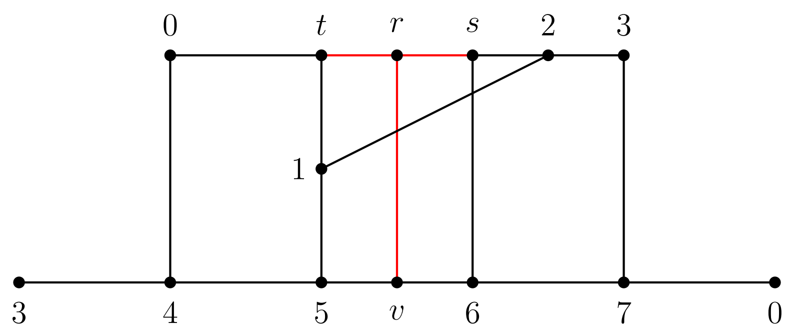

Figure 3.12: Transformation of Figure 3.11 demonstrating a V_{10} minor when t is placed at t_2.

3.4.3 Case $t = t_3 = 14$

If t is placed at t_3, then a V_8 with this 3-star attached can be contained in a 2-crossing-critical graph. One such example is Figure 3.13.

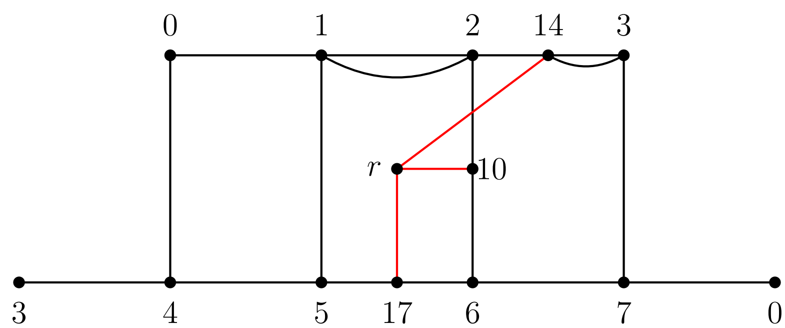

Figure 3.13: A 2-crossing-critical graph containing a V_8 with this 3-star attached when $t = t_3$.

3.4.4 Case $t = t_4 = 15$

Now let t be placed at t_4. If the path $(4, 5)$ is an edge, then the cycle $(t, 4, 0, 1, 5, v, r)$ is a hole with planar bridge $(4, 5)$, a contradiction with Theorem 16.

Therefore, the path $(4, 5)$ contains at least one subdivision. But under the transformation in Figure 3.14, the resulting graph has a subdivision H' of a V_8 with an H'-bridge attached to two consecutive spokes. This case has already been fully explained by Theorem 13, and is therefore not considered here.

Therefore, all cases when $t = t_4$ lead to a contradiction or have already been explained.

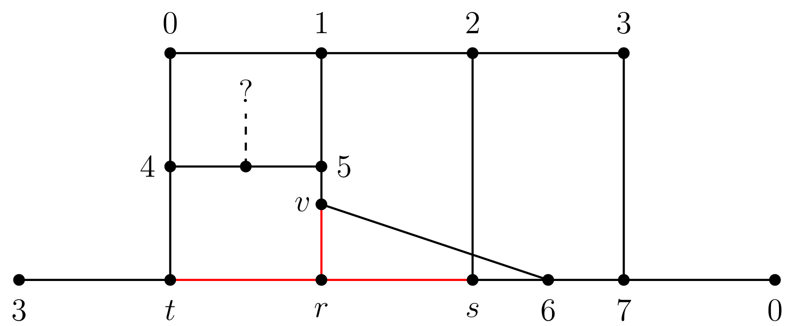

Figure 3.14: Further transformation of Figure 3.11 when t is placed at t_4.

3.4.5 Case $t = t_5 = 4$ or $t = t_6 = 16$

If t is at t_5 or t_6, then the cycle $(t, 5, v, 6, s, r, t)$ is a hole with planar bridge (r, v), a contradiction with Theorem 16.

3.4.6 Case $t = t_7 = 18$

If t is at t_7, then the resulting graph has crossing number at least 2 and is non-critical (as checked by a computer), a contradiction.

3.4.7 Case $t = t_8 = 19$

If t is placed at t_8, then this case has already been considered as it is equivalent to a 3-star attached to a V_8 at two spokes, as demonstrated under the transformation in Figure 3.15.

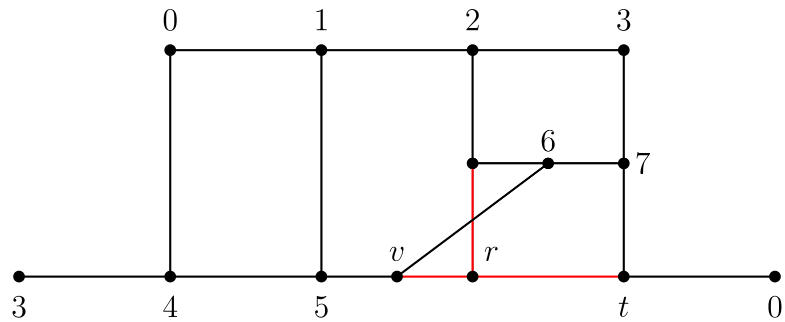

Figure 3.15: Further transformation of Figure 3.11 when t is at t_8, demonstrating that this case has already been considered (and discarded) in the Two Spoke case.

3.4.8 Conclusion

To conclude this section's analysis, if a 3-star T is attached to a subdivision H of a V_8 at one spoke, and a rim edge incident to the endpoint of that spoke, then there were eight cases which had not been previously considered. If this scenario occurs in a 3-connected 2-crossing-critical graph G, then T is attached to H at the vertices (10, 14, 17), under the canonical labelling of a V_8 subdivision and up to symmetry.

3.5 One spoke, such that a 2-bar is formed

At this point, as checked by a computer, all cases of a 3-star attaching to a V_8 at precisely one spoke have been considered, except for five. Three of the remaining cases have a similar structure and are considered here.

In this case, let the 3-star connect to a spoke of the V_8 at s. Let the remaining connections between the 3-star and the V_8 be such that the 3-star contains a two-bar, following some transformation as depicted in Figure 3.16. Then the resulting graph contains a 2-crossing-critical subgraph, but is not 2-crossing critical itself. This can be seen as the resulting graph contains a 2-bar, and a V_8 with a 2-bar is 2-crossing-critical. However, the resulting graph also has an extra edge; by the labelling in Figure 3.16, this is the edge $(1, 5)$.

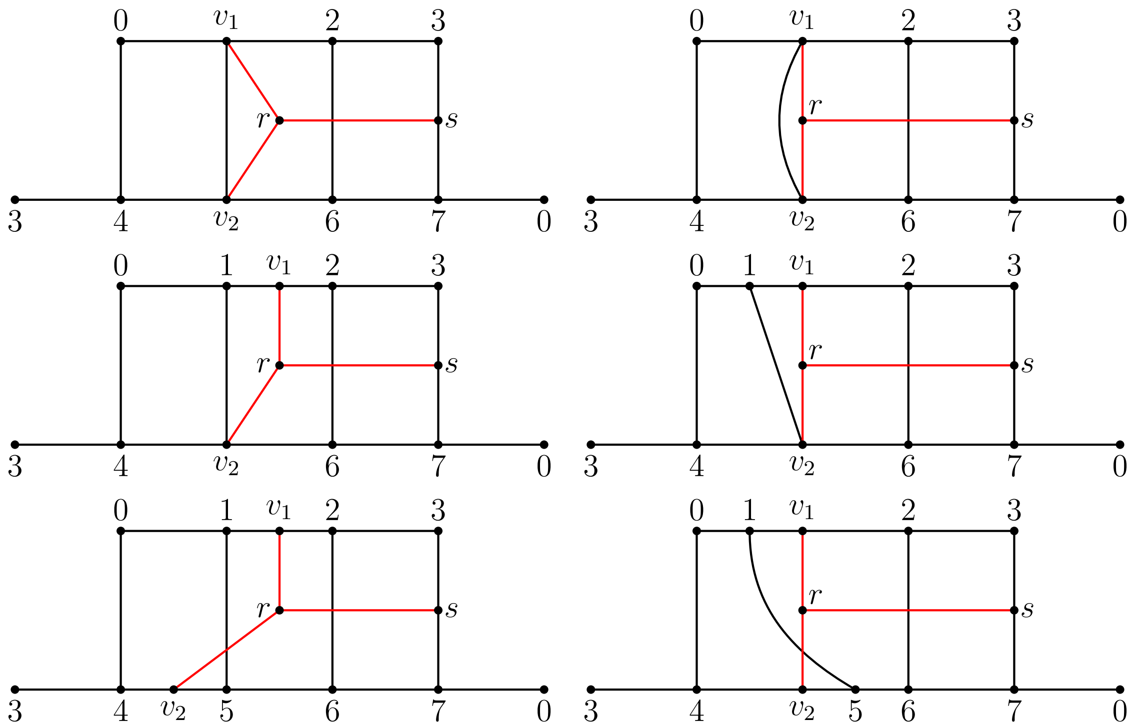

Figure 3.16: All possible non-isomorphic 3-star attachments in this case (left); and their transformations (right), demonstrating that the resulting graph contains a 2-bar and thus has crossing number at least 2, but also contains an extra edge and is therefore not critical.

3.6 One spoke, remaining cases

There remain two non-isomorphic cases of a 3-star attaching to a V_8 at precisely one spoke edge; all other cases have been considered as verified by a computer. These two do not fall neatly into any category and we consider them individually here.

3.6.1 Case 1

In the first special case, let the 3-star attach to the V_8 as in Figure 3.17. Then the cycle $(s, r, 0, v, 1, 2, s)$ is a hole with a planar bridge (r, v), a contradiction with Theorem 16.

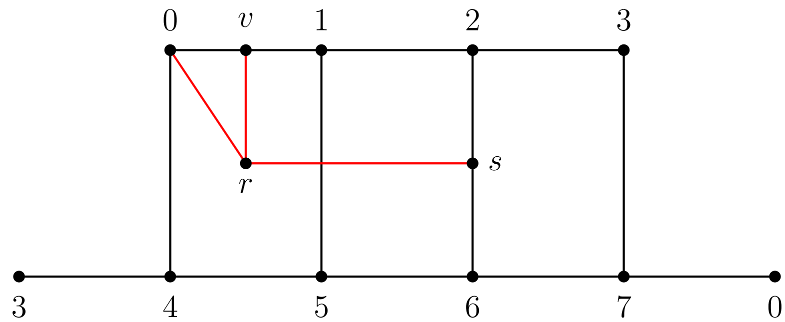

Figure 3.17: The first of two special cases of a 3-star attaching to a V_8 at precisely one spoke.

3.6.2 Case 2

In the second of the two special cases, let the 3-star attach to the V_8 as in Figure 3.18. It is suspected, but not yet proven, that this case cannot be included in a 2-crossing-critical graph. See section 7.1 for further discussion.

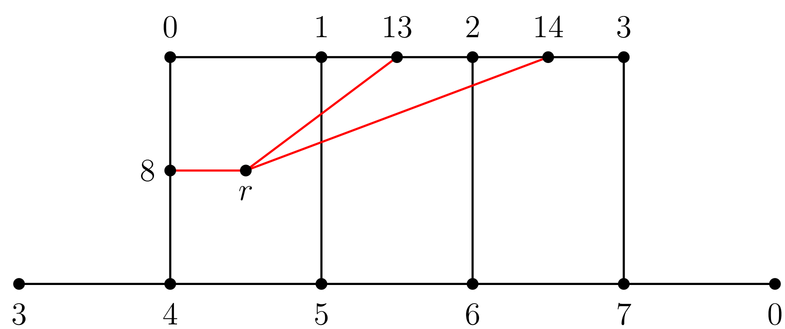

Figure 3.18: The second of two special cases of a 3-star attaching to a V_8 at precisely one spoke.

3.7 Conclusion

Thus concludes the analysis of the 34 non-isomorphic cases where a 3-star T attaches to a subdivision H of V_8, such that T attaches at precisely one spoke edge. As we have seen, if a 3-star T is attached to a subdivision H of a V_8 at one spoke in a 3-connected 2-crossing-critical graph G, then T is attached to H at one of the following two sets of vertices, under the canonical labelling of a subdivided V_8 and up to symmetry:

- (0, 6, 9)
- (2, 6, 9)
- (8, 13, 14)
- (6, 9, 12)
- (10, 14, 17)

Furthermore, it is suspected, but not yet proven, that the 3-stars attached at vertices (2, 6, 9) or (8, 13, 14) cannot be contained in 2-crossing-critical examples.

Chapter 4

3-Star Case Analysis: No Spoke Attachments

We now continue our 3-star case analysis. We consider 3-stars with no attachment in any spoke of a V_8 subdivision. The majority of 2-crossing-critical examples come from this case.

Theorem 21. *Let G be a 3-connected 2-crossing-critical graph with a V_8 minor but no V_{10} minor, such that G embeds in the real projective plane $\mathbb{R}P^2$, and such that G contains a 3-star T which attaches to the V_8 subdivision H in G with the minimum number of subdivisions. Suppose that T attaches to H at no spoke edge. Then T attaches to H at one of the following sets of vertices, under the canonical labelling of a V_8 subdivision and up to symmetry:*

- *(0, 2, 15)*
- *(0, 3, 13)*
- *(0, 5, 6)*
- *(0, 5, 17)*
- *(0, 13, 15)*
- *(0, 13, 18)*
- *(1, 5, 6)*
- *(1, 5, 17)*
- *(2, 5, 17)*
- *(3, 5, 17)*
- *(5, 6, 12)*
- *(5, 6, 19)*
- *(5, 12, 17)*
- *(5, 14, 17)*
- *(5, 17, 19)*
- *(7, 16, 17)*
- *(16, 17, 19)*

Further, some of the above 3-stars were not demonstrated to be included in a 2-crossing-critical graph. It is suspected, but not yet proven, that these cannot be included in a

2-crossing-critical graph.

Conjecture 22. *Let G be a 3-connected 2-crossing-critical graph with a V_8 minor but no V_{10} minor, such that G embeds in the real projective plane $\mathbb{R}P^2$, and such that G contains a 3-star T which attaches to the V_8 minor in G with the minimum number of subdivisions. Then the connections of the 3-star T are not one of the following sets of vertices, under a canonical labelling of the V_8:*

- *(0, 2, 15)*
- *(0, 13, 18)*
- *(5, 6, 19)*
- *(5, 17, 19)*
- *(7, 16, 17)*

There are 38 non-isomorphic cases to be considered in this section, as verified by a computer.

4.1 No spokes, consecutive vertices of the V_8

In this case, let the 3-star connect to consecutive vertices of the V_8, 5 and 6 under the canonical labelling. Let the third connection of the 3-star be connected to the V_8 at vertex t. Since t cannot attach to a spoke edge as these cases have already been considered, then there are 8 non-isomorphic possibilities for t. We note that this number is 8 rather than 15 due to the symmetry between the quads $[0, 1, 5, 4]$ and $[3, 2, 6, 7]$.

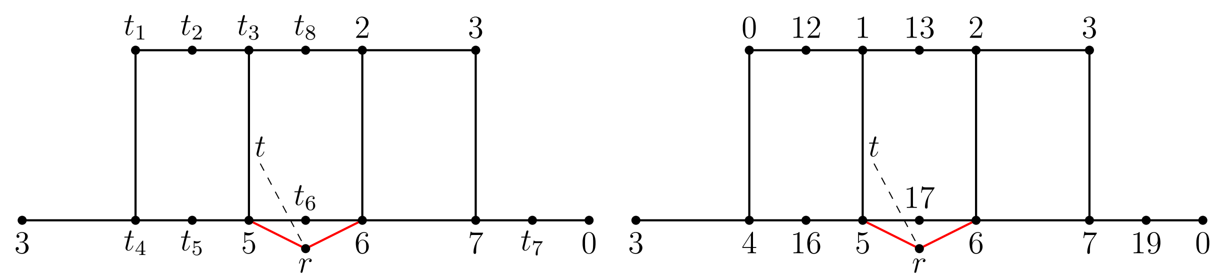

Figure 4.1: A 3-star attached to two adjacent vertices of the V_8. Possible non-isomorphic attachment are denoted $t_i; i \in [1, 8]$.

4.1.1 Case $t = t_1 = 0$ or $t = t_3 = 1$

If t is placed at t_1 or t_3, then a V_8 with this 3-star attached can be contained in a 2-crossing-critical graph. One such example containing both the 3-star when $t = t_1$ and the 3-star when $t = t_3$ is Figure 4.2.

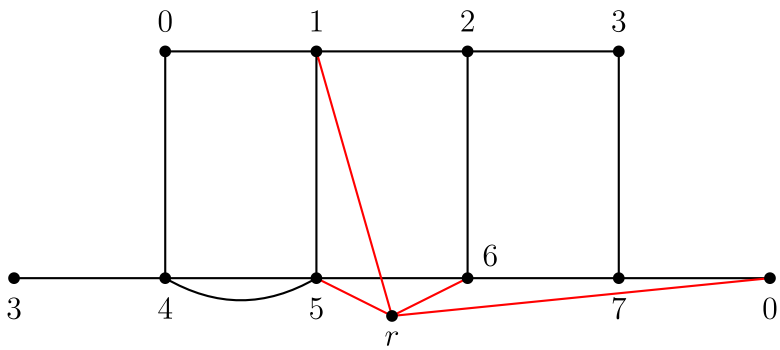

Figure 4.2: A 2-crossing-critical graph containing a V_8 with the 3-star when $t = t_1$ and the 3-star when $t = t_3$ attached.

4.1.2 Case $t = t_2 = 12$

If t is placed at t_2, then a V_8 with this 3-star attached can be contained in a 2-crossing-critical graph. One such example is Figure 4.3.

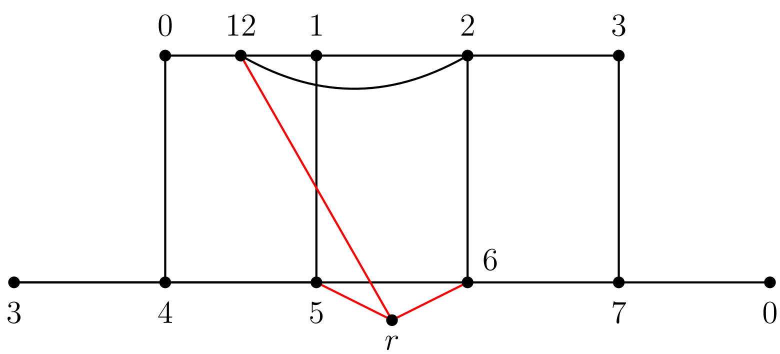

Figure 4.3: A 2-crossing-critical graph containing a V_8 with the 3-star when $t = t_2$ attached.

4.1.3 Case $t = t_4 = 4$ or $t = t_5 = 16$

If t is placed at t_4 or t_5, then the cycle $(t, 5, 6, r, t)$ is a hole with planar bridge $(r, 5)$, a contradiction with Theorem 16.

4.1.4 Case $t = t_6 = 17$

If t is at t_6, then the cycle $(5, t, 6, r, 5)$ is a hole with planar bridge (r, t), a contradiction with Theorem 16.

4.1.5 Case $t = t_7 = 19$

If t is placed at t_7, then it is suspected, but not yet proven, that the resulting graph cannot be included in a 2-crossing-critical graph. See section 7.1 for further discussion.

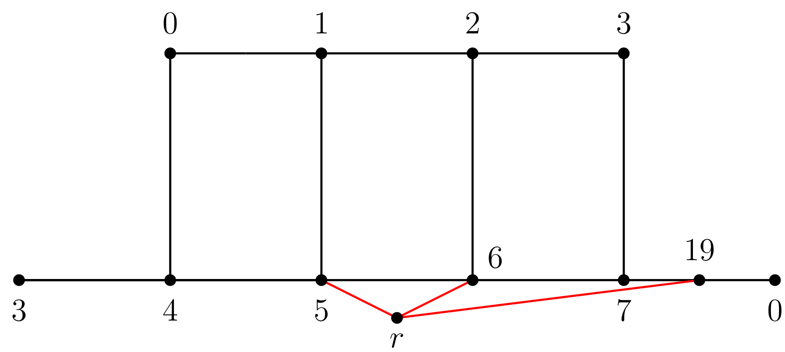

Figure 4.4: Figure 4.1 when $t = t_7$.

4.1.6 Case $t = t_8 = 13$

If $t = t_8$, then a V_{10} minor is formed, a contradiction, as seen in Figure 4.5.

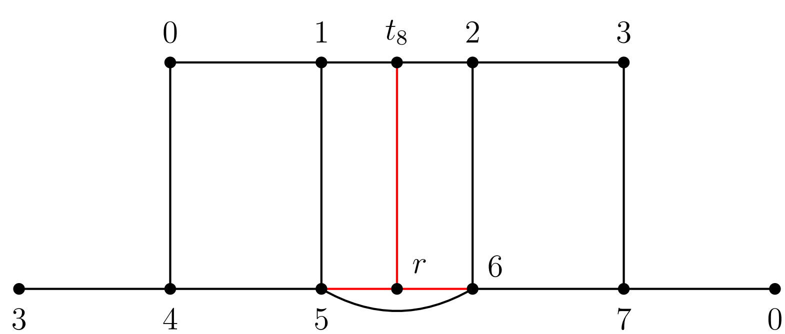

Figure 4.5: Transformation of Figure 4.1 when $t = t_8$ demonstrating a V_{10} minor.

4.1.7 Conclusion

To conclude this section's analysis, when a 3-star T attaches to a subdivision H of V_8 in a 2-crossing-critical graph G, such that T attaches to H at two consecutive V_8 vertices and no spoke edges, then T attaches to H at one of the following sets of vertices, under the canonical labelling of H and up to symmetry: (0,5,6); (1,5,6); (5,6,12); or (5,6,19). Furthermore, it is suspected, but not yet proven, that T cannot attach to (5,6,19).

4.2 No spokes, consecutive rim edges of the V_8

In this case, let the 3-star connect to consecutive rim edges of the V_8 at vertices v_1 and v_2. Let the third connection of the 3-star be connected to the V_8 at vertex t. Since t cannot be a spoke edge, or an opposing rim edge to v_1 or v_2, then there are 7 possibilities for t. We note that it is 7 as opposed to 13 due to the symmetry of the quads $[0, 1, 5, 4]$ and $[2, 1, 5, 6]$.

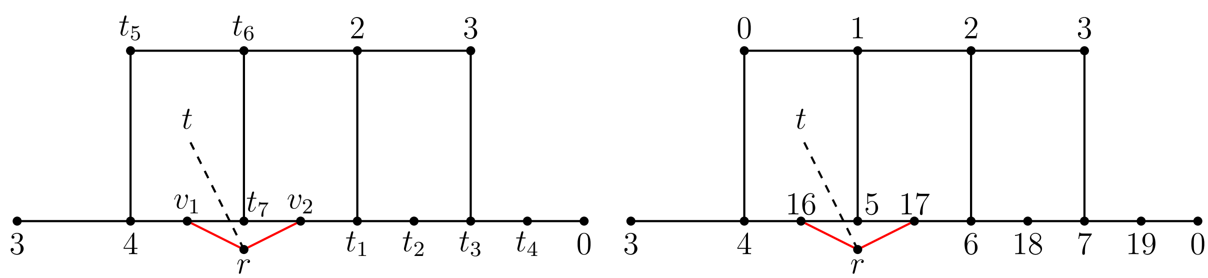

Figure 4.6: A 3-star attached to two adjacent rim edges.

4.2.1 Case $t = t_1 = 6$

If t is placed at t_1, then the cycle $(v_1, r, t, v_2, 5, v_1)$ is a hole with planar bridge (r, v_2), a contradiction with Theorem 16.

4.2.2 Case $t = t_2 = 18$

If t is placed at t_2, then the cycle $(v_1, r, t, 6, v_2, 5, v_1)$ is a hole with planar bridge (r, v_2), a contradiction with Theorem 16.

4.2.3 Case $t = t_3 = 7$

If t is placed at t_3, then it is suspected, but not yet proven, that the resulting graph cannot be included in a 2-crossing-critical graph. See section 7.1 for further discussion.

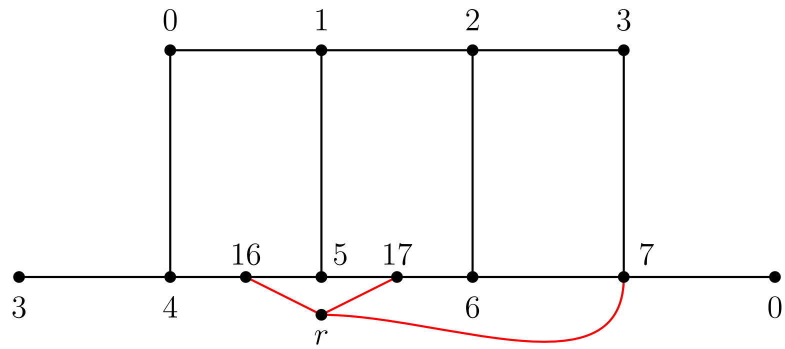

Figure 4.7: Figure 4.6 when $t = t_3 = 7$.

4.2.4 Case $t = t_4 = 19$

When t is placed at t_4, then a V_8 with this 3-star attached can be contained in a 2-crossing-critical graph. One such example is Figure 4.8.

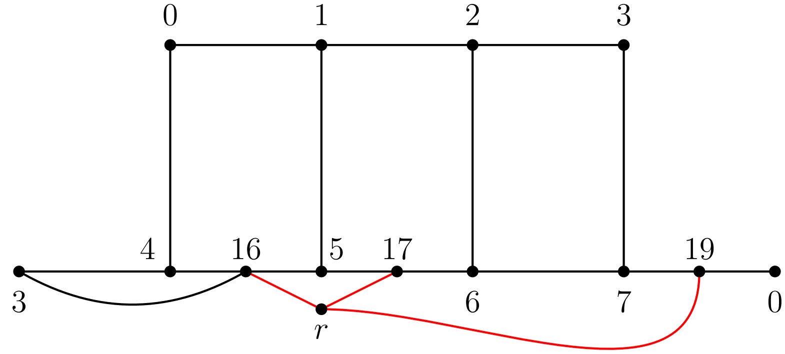

Figure 4.8: A 2-crossing-critical graph containing a V_8 with the 3-star when $t = t_4$ attached.

4.2.5 Case $t = t_5 = 0$ or $t = t_6 = 1$

If t is at t_5 or t_6, then the resulting graph has crossing number at least 2 and is not critical (as checked by a computer), a contradiction.

4.2.6 Case $t = t_7 = 5$

If t is at t_7, then the cycle (v_1, t, v_2, r, v_1) is a hole with planar bridge (r, t), a contradiction with Theorem 16.

4.2.7 Conclusion

To conclude this section's analysis, when a 3-star T attaches to a subdivision H of V_8 in a 2-crossing-critical graph G, such that T attaches to H at two rim branches and no spoke edges, then T attaches to H at one of the following sets of vertices, under the canonical labelling of H and up to symmetry: (7,16,17) or (16,17,19). Only the latter is known to be contained in a 2-crossing-critical example.

4.3 No spokes, a consecutive vertex and rim edge

In this case, let the 3-star connect to a vertex and an incident rim edge of the V_8 at vertices 5 and 17. Let the third connection of the 3-star be connected to the V_8 at vertex t. The vertex t cannot be 4 or 6, nor at the rim edges $(1, 2)$, $(4, 5)$, $(6, 7)$, as these were considered in the previous cases. Therefore, there are 9 non-isomorphic possibilities for t.

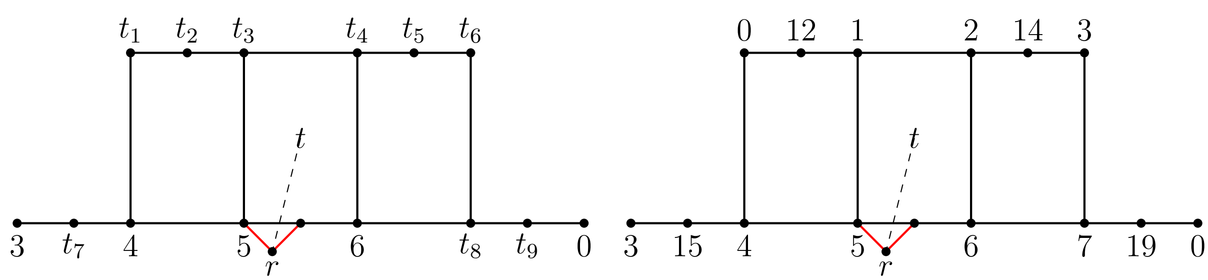

Figure 4.9: A 3-star attached to a rim edge and incident vertex. Possible third attachments are denoted $t_i; i \in [1, 9]$.

4.3.1 Case $t = t_1 = 0$ or $t = t_3 = 1$

If t is at t_1 or t is at t_3, then a V_8 with one of these 3-stars attached can be contained in a 2-crossing-critical graph. One such example containing both 3-stars in question is Figure 4.10.

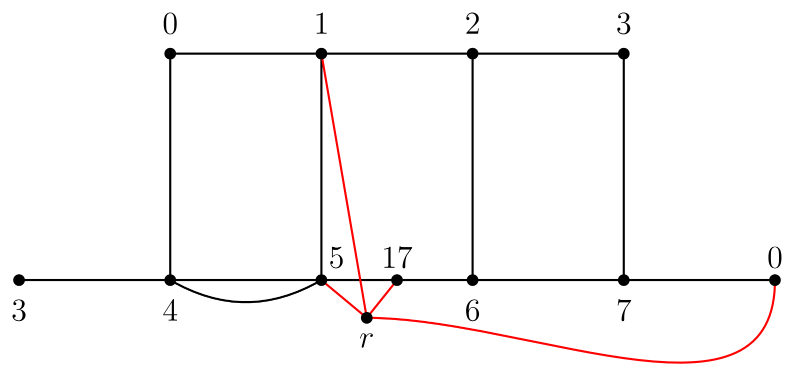

Figure 4.10: A 2-crossing-critical graph with a V_8 subdivision and the 3-stars when $t = t_1$ and $t = t_3$.

4.3.2 Case $t = t_2 = 12$

If t is at t_2, then a V_8 with this 3-star attached can be contained in a 2-crossing-critical graph. One such example is Figure 4.11.

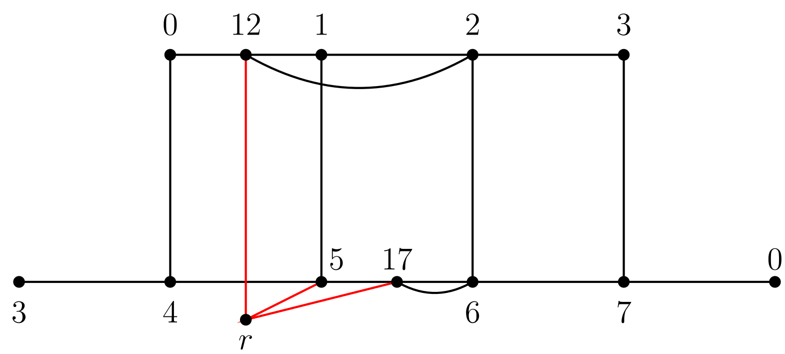

Figure 4.11: A fully covered 2-crossing-critical graph containing Figure A.29 as a subgraph.

4.3.3 Case $t = t_4 = 2$

If t is at t_4, then a V_8 with this 3-star attached can be contained in a 2-crossing-critical graph. Indeed, Figure 4.12 is one such example.

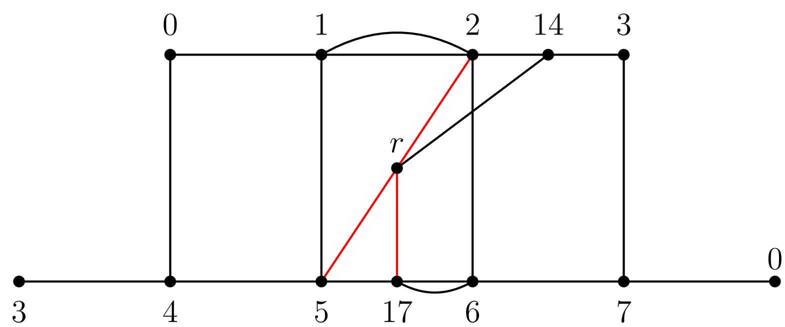

Figure 4.12: A 2-crossing-critical graph containing a V_8 with the 3-star when $t = t_4$ attached.

4.3.4 Case $t = t_5 = 14$

If t is at t_5, then a V_8 with this 3-star attached can be contained in a 2-crossing-critical graph. One such example is Figure 4.13.

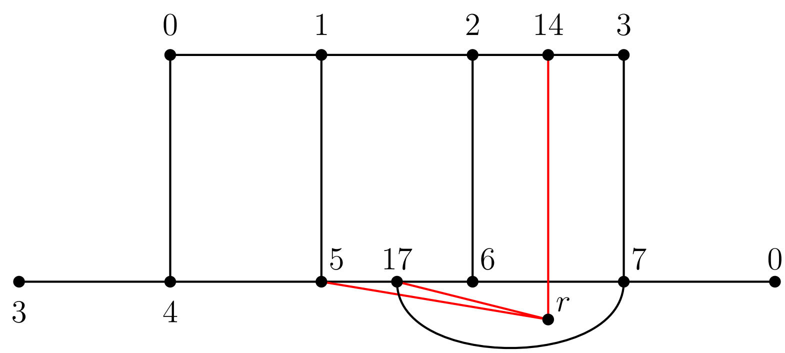

Figure 4.13: A fully-covered 2-crossing-critical graph containing a V_8 with the 3-star when $t = t_5$ attached.

4.3.5 Case $t = t_6 = 3$

If t is at t_6, then a V_8 with this 3-star attached can be contained in a 2-crossing-critical graph. Indeed, Figure 4.14 is one such example.

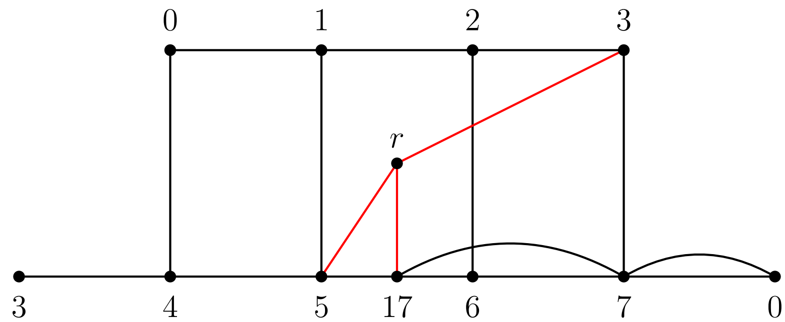

Figure 4.14: A fully-covered 2-crossing-critical graph containing a V_8 with the 3-star when $t = t_6$ attached.

4.3.6 Case $t = t_7 = 15$

If t is at t_7, then the cycle $(t, 4, 5, v, r, t)$ is a hole with planar bridge $(r, 5)$, a contradiction with Theorem 16.

4.3.7 Case $t = t_8 = 7$

If t is at t_8, then the cycle $(5, r, t, 6, 17, 5)$ is a hole with planar bridge $(r, 17)$, a contradiction with Theorem 16.

4.3.8 Case $t = t_9 = 19$

If t is at t_9, then it is suspected, but not yet proven, that the resulting graph cannot be the subgraph of a 2-crossing-critical graph. See section 7.1 for further discussion.

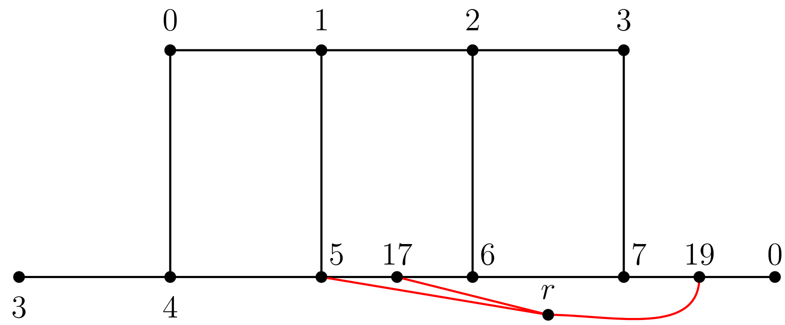

Figure 4.15: Figure 4.9 when $t = t_9 = 19$.

4.3.9 Conclusion

To conclude this section's analysis, when a 3-star T attaches to a subdivision H of V_8 in a 2-crossing-critical graph G, such that T attaches to H at a rim edge and incident V_8 vertex, then T attaches to H at one of the following sets of vertices, under the canonical labelling of H and up to symmetry: (0,5,17); (1,5,17); (5,12,17); (2,5,17); (5,14,17); (3,5,17); or (5,17,19). Furthermore, it is suspected, but not yet proven, that T cannot attach to (5,17,19) in a 2-crossing-critical example.

4.4 No spokes, and a V_{10} minor is formed

There remain two additional cases of a 3-star attachment to a V_8 at no spoke edges, such that a V_{10} minor is formed. These two cases, and the transformations demonstrating a V_{10} minor, are shown below.

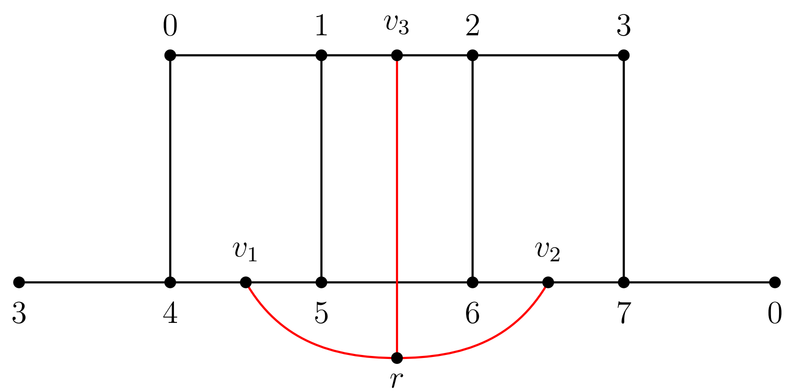

Figure 4.16: A 3-star connected to a V_8, at no spokes, such that a V_{10} minor is formed.

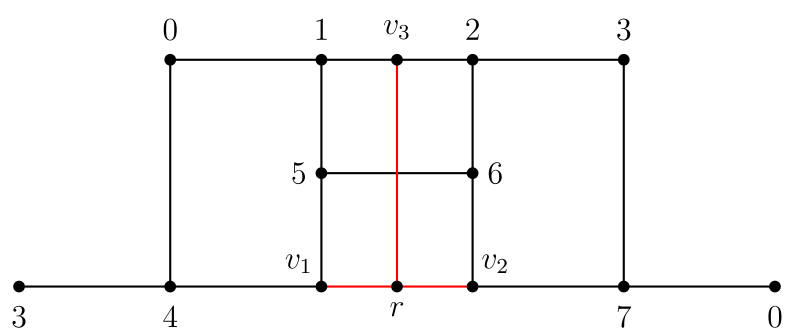

Figure 4.17: Transformation of Figure 4.16 demonstrating a V_{10} minor.

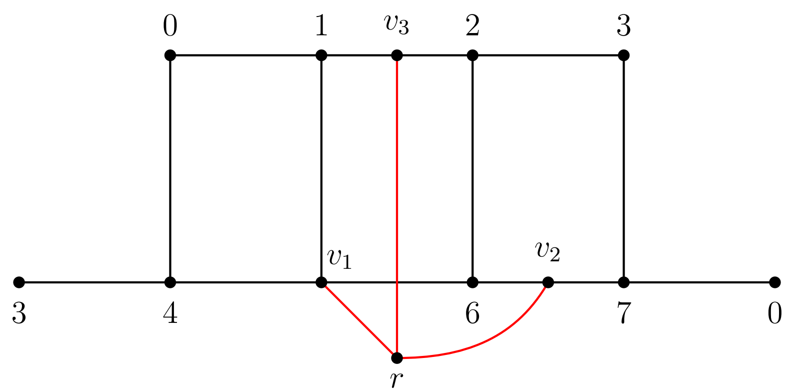

Figure 4.18: A second example of a 3-star connected to a V_8, at no spokes, such that a V_{10} is formed.

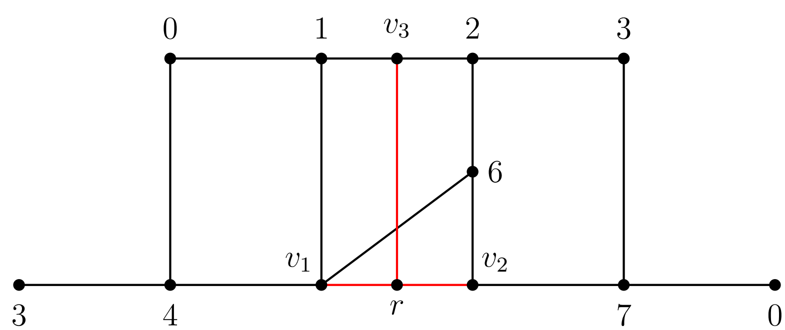

Figure 4.19: Transformation of Figure 4.18 demonstrating a V_{10} minor.

4.5 No spokes, remaining cases which have crossing number at least 2 and are non-critical

As checked by a computer, there are twelve remaining cases of attaching a 3-star to a V_8, such that it attaches to no spoke edges. Six of these have crossing number at least 2 and are non-critical, as checked by a computer. These cases do not fall neatly into any other category, but are represented below in Figure 4.20.

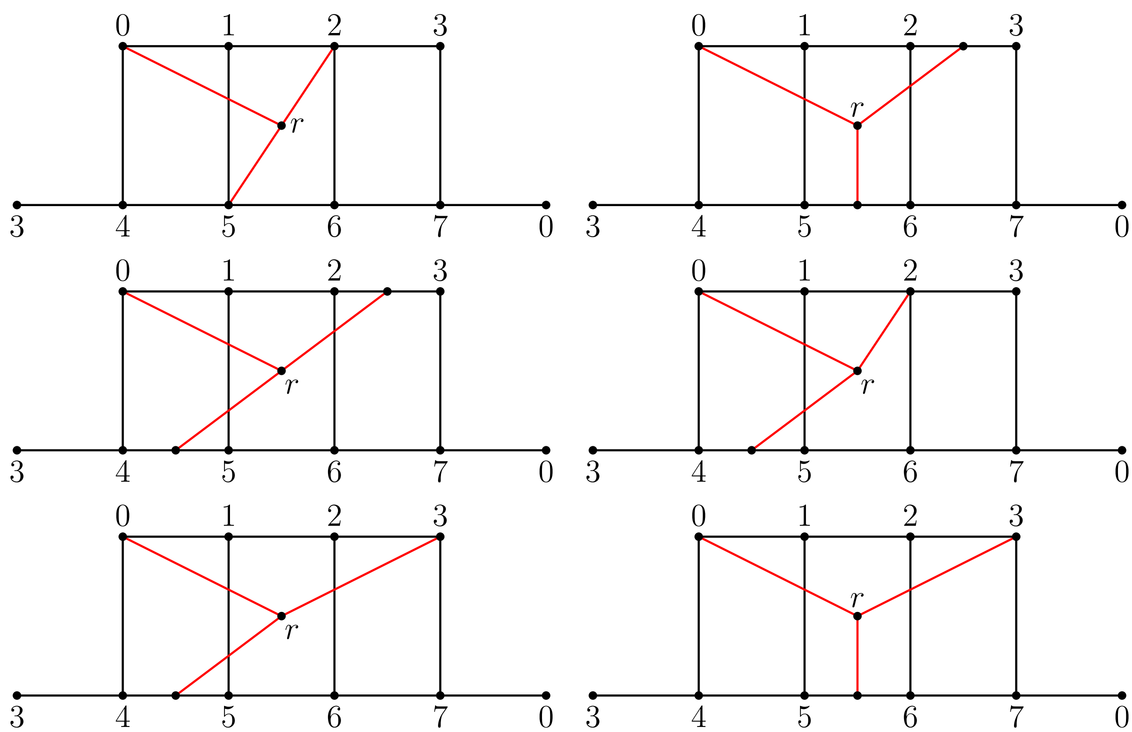

Figure 4.20: Six non-isomorphic 3-star attachments to a subdivision H of V_8, which have crossing number at least 2 and are non-critical.

4.6 No spokes, remaining cases

As checked by a computer, there remain six non-isomorphic 3-star attachments to a V_8 which do not fit neatly into any of the previous cases. All other 3-star attachments to the V_8, such that the 3-star does not attach to any spoke edges, have already been considered. We handle these six individually here.

4.6.1 Cases 1 and 2

In this case, suppose that the 3-star is connected to the subdivision H of V_8 as in Figure 4.21 or as in Figure 4.22. The arguments for these two cases are the same.

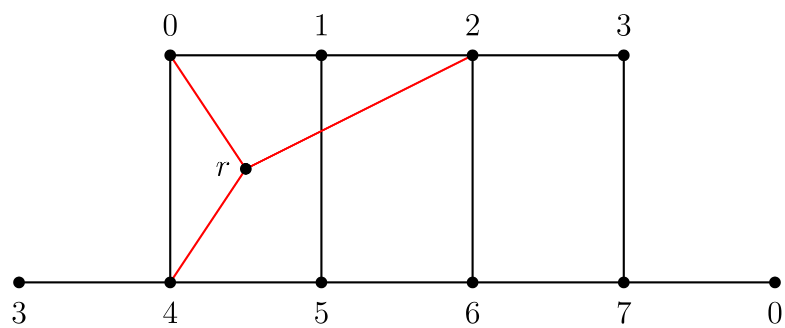

Figure 4.21: The first of six special cases where a 3-star is connected to a V_8 at no spoke edges.

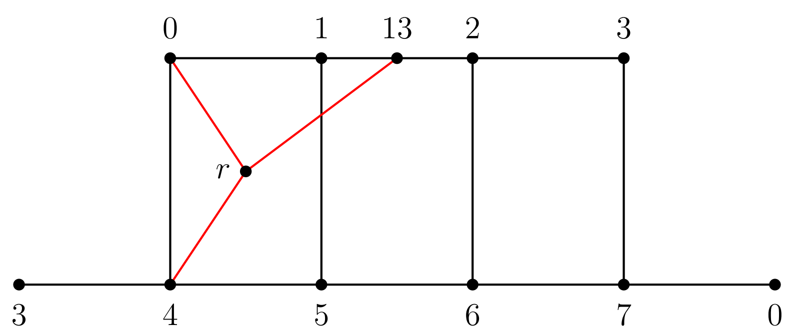

Figure 4.22: The second of six special cases where a 3-star is connected to a V_8 at no spoke edges.

If the path $(0, 4)$ contains no internal vertices, then it may be deleted to yield a 1-drawing where the path $(0, r, 4)$ is not crossed (this can be seen by replacing the $(0, 4)$ spoke with $(0, r, 4)$). Then $(0, 4)$ can be added parallel to $(0, r, 4)$, yielding a 1-drawing of the original graph G, a contradiction.

So $(0, 4)$ contains an internal vertex. By minimality, by the same argument as before (that is, replacing $(0, 4)$ with $(0, r, 4)$), the $(0, 4)$ path has precisely one internal vertex. So let i be this vertex.

Let H' be the V_8 subdivision formed by replacing $(0, i, 4)$ with $(0, r, 4)$. Then i is part of an H'-bridge B', which is attached to H' via at least one other vertex j. As checked by a computer, if j is not in either $[4, 5]$ or $[0, 7]$, then the resulting graph has crossing number at least 2 and is non-critical, a contradiction. So j is in either $[4, 5]$ or $[0, 7]$.

If j is in $[4,5]$, then the cycle $(0,r,4,j,i,0)$ is a hole with a planar bridge $(i,4)$, a contradiction with Theorem 16. If j is on $[0,7]$, then the cycle $(4,r,0,j,i,4)$ is a hole with a planar bridge $(i,0)$, a contradiction with Theorem 16. Since a contradiction arises in all cases, then we conclude that this case cannot lead to a 2-crossing-critical graph.

4.6.2 Case 3

In this case, let the 3-star attach to the V_8 as in Figure A.34. Indeed, this graph is a 2-crossing-critical graph, and thus this 3-star can be included in a 2-crossing-critical graph.

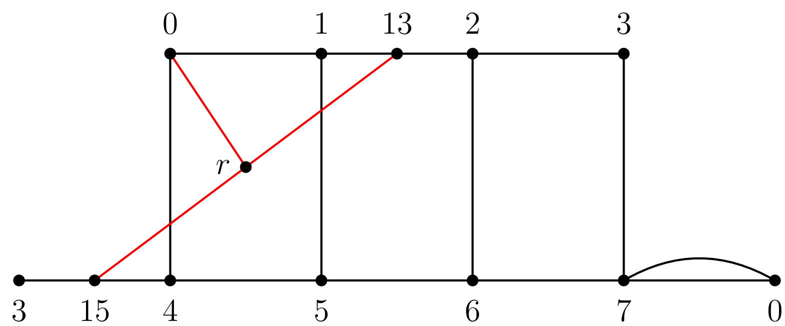

Figure 4.23: A 2-crossing-critical graph containing the third of six special cases where a 3-star is connected to a V_8 at no spoke edges.

4.6.3 Case 4

In this case, let the 3-star attach to the V_8 as in Figure 4.24. It is suspected, but not yet proven, that this graph cannot be the subgraph of a 2-crossing-critical graph.

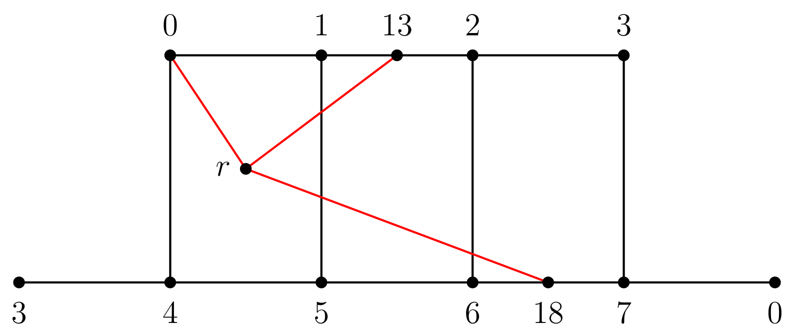

Figure 4.24: The fourth of six special cases where a 3-star is connected to a V_8 at no spoke edges.

4.6.4 Case 5

In this case, let the 3-star attach to the V_8 as in Figure 4.25. It is suspected, but not yet proven, that this graph cannot be the subgraph of a 2-crossing-critical graph.

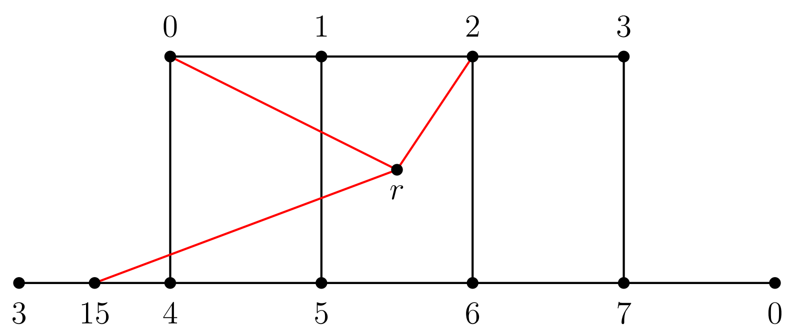

Figure 4.25: The fifth of six special cases where a 3-star is connected to a V_8 at no spoke edges.

4.6.5 Case 6

In this case, let the 3-star attach to the V_8 as in Figure A.35. Indeed, this graph is a 2-crossing-critical graph.

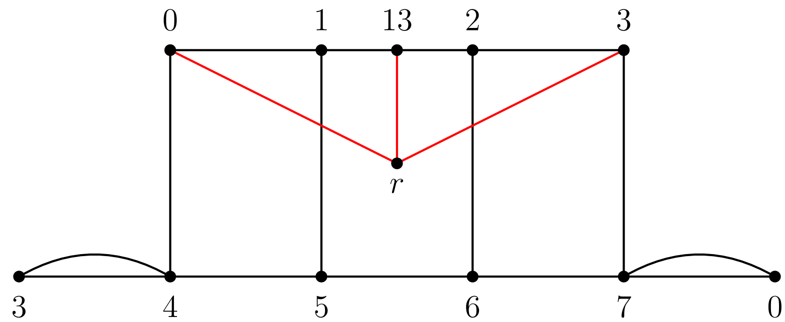

Figure 4.26: A 2-crossing-critical graph containing the sixth of six special cases where a 3-star is connected to a V_8 at no spoke edges.

4.6.6 Conclusion

In conclusion of this section's analysis, six special cases were considered. Let T be a 3-star attached to a subdivision H of V_8 in a 3-connected 2-crossing-critical graph G. Under the canonical labelling and up to symmetry, T cannot be attached at (0,2,4) or (0,4,13). The 3-star T could be attached at (0,13,15) or (0,3,13). It is suspected, but not yet proven, that T cannot be attached at (0,13,18) or (0,2,15).

4.7 Conclusion

In conclusion of this chapter, we examined 38 non-isomorphic 3-star attachments to a subdivided V_8, such that the 3-star attaches to the subdivided V_8 at no spoke edges. We concluded that 12 such cases can certainly be included in a 2-crossing-critical graph; it is suspected that 5 other cases cannot be included in a 2-crossing-critical graph; and the remaining 21 cases were proven not to be contained in a 2-crossing-critical graph.

Chapter 5

3-Star Analysis: Multiple Attachments to a Spoke or Rim

We now proceed to consider what happens when a 3-star is permitted to attach multiple times to a single edge of a V_8. As mentioned in the introduction, there are 19 cases to be considered in this chapter:

- 2 cases where the 3-star attaches to the same edge 3 times.
- 7 cases where the 3-star attaches to the same spoke edge 2 times.
- 10 cases where the 3-star attaches to the same rim edge 2 times.

The primary result of this chapter is as follows.

Theorem 23. *Let G be a 3-connected 2-crossing-critical graph with a V_8 minor but no V_{10} minor, such that G embeds in the real projective plane $\mathbb{R}P^2$, and such that G contains a 3-star T which attaches to subdivision H of V_8 in G such that H has the minimum number of subdivisions. Suppose that T attaches to the same spoke or rim of H at multiple distinct points. Then the connections of the 3-star T are one of the following sets of vertices, under the canonical labelling of the V_8 and up to symmetry:*

- *(0, 17_a, 17_b)*
- *(1, 17_a, 17_b)*
- *(12, 17_a, 17_b)*

(where 17_a and 17_b represent two distinct points on the rim branch $(5, 6)$).

5.1 Three attachments to a spoke edge

If a 3-star attaches to the same spoke edge of a V_8 at three points as in Figure 5.1, then the resulting graph contains a cycle (r, s_1, s_2, s_3, r) with planar bridge (r, s_2), a contradiction with Theorem 16.

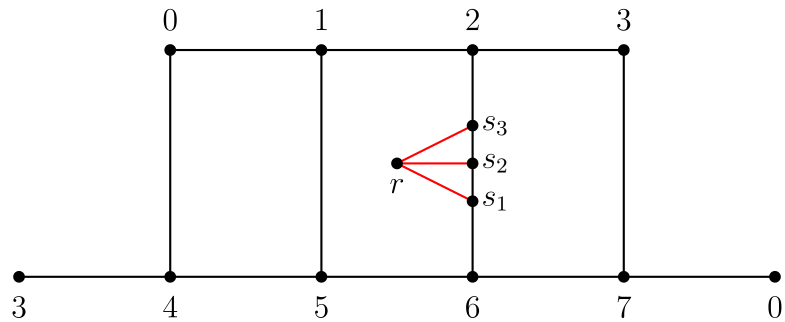

Figure 5.1: A 3-star attached to a V_8 spoke at three distinct points.

5.2 Three attachments to a rim edge

If a 3-star attaches to the same rim edge of a V_8 at three points as in Figure 5.2, then the resulting graph contains a cycle (r, v_1, v_2, v_3, r) with planar bridge (r, v_2), a contradiction with Theorem 16.

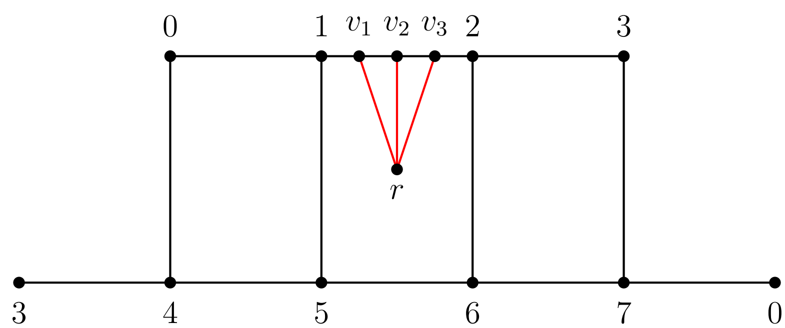

Figure 5.2: A 3-star attached to a V_8 rim branch at three distinct points.

5.3 Two attachments to a spoke edge

In this case, let the 3-star attach to a V_8 twice on the same spoke edge, and at another attachment. This scenario contains 7 non-isomorphic cases, due to the symmetry between the quads $[1, 2, 6, 5]$ and $[3, 2, 6, 7]$, as well as the vertical symmetry along the spoke $(2, s_2, s_1, 6)$.

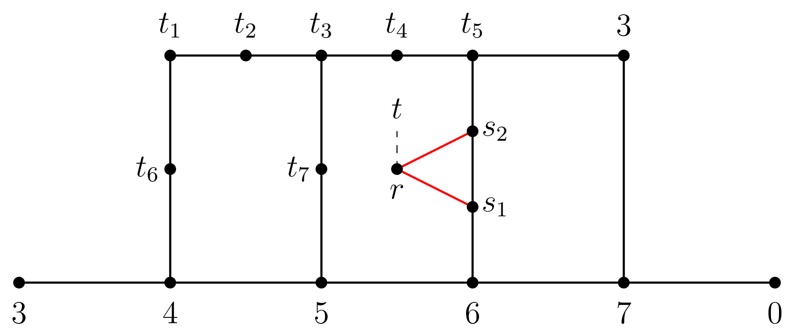

Figure 5.3: Possible 3-star attachments to a V_8, where the 3-star attaches to one spoke edge twice. Possible non-isomorphic third attachment locations are denoted $t_i; i \in [1, 5]$.

5.3.1 Case $t = t_1$ or $t = t_2$

If t is at t_1 or t_2, then the cycle $(s_1, t, 1, 2, s_2, s_1)$ is a hole with a planar bridge (r, s_2), a contradiction with Theorem 16. Therefore, the graphs in these cases cannot be 2-crossing-critical.

5.3.2 Case $t = t_3$ or $t = t_4$

If t is at t_3 or t_4, then the cycle $(s_1, t, 2, s_2, s_1)$ is a hole with a planar bridge (r, s_2), a contradiction with Theorem 16. Therefore, the graphs in these cases cannot be 2-crossing-critical.

5.3.3 Case $t = t_5$

If t is at t_5, then the cycle (s_1, r, t, s_2, s_1) is a hole with planar bridge (r, s_2), a contradiction with Theorem 16. Therefore, this case cannot be 2-crossing-critical.

5.3.4 Case $t = t_6$ or $t = t_7$

If t is at t_6 or t_7, then the 3-star is an H-bridge between two spokes of a V_8 subdivision H. These cases have already been eliminated by Theorem 13.

5.4 Two attachments to a rim edge

In this case, let the 3-star attach to a V_8 twice on the same rim edge, and at another attachment. This scenario contains 10 non-isomorphic cases due to the symmetry between quads $[0, 1, 5, 4]$ and $[3, 2, 6, 7]$.

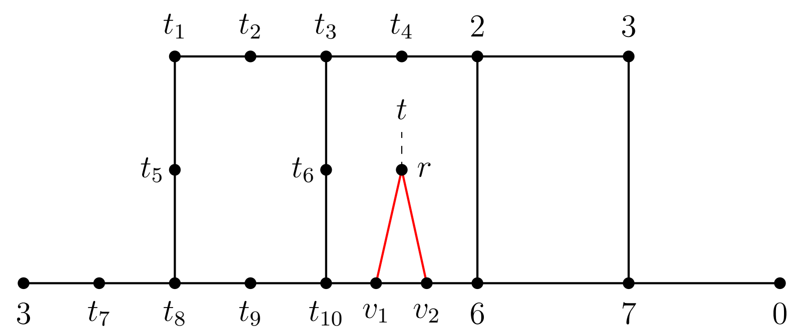

Figure 5.4: Possible 3-star attachments to a V_8, where the 3-star attaches to one rim edge twice. Possible non-isomorphic third attachment locations are denoted $t_i; i \in [1, 10]$.

5.4.1 Case $t = t_1$

If t is at t_1, then the 3-star can be included in a 2-crossing-critical graph. Figure 5.5 is an example.

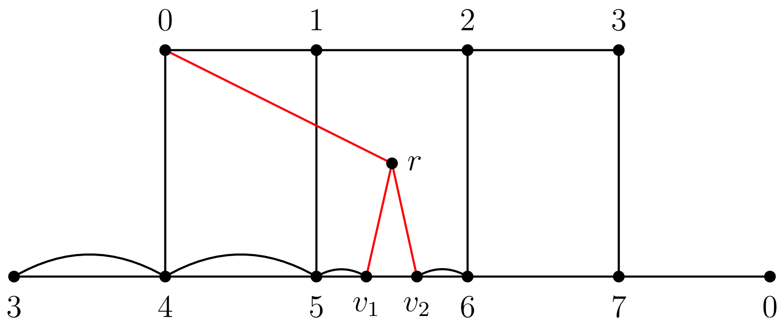

Figure 5.5: A 2-crossing-critical graph containing the 3-star when $t = t_1$.

5.4.2 Case $t = t_2$

If t is at t_2, then the 3-star can be included in a 2-crossing-critical graph. Figure 5.6 is an example.

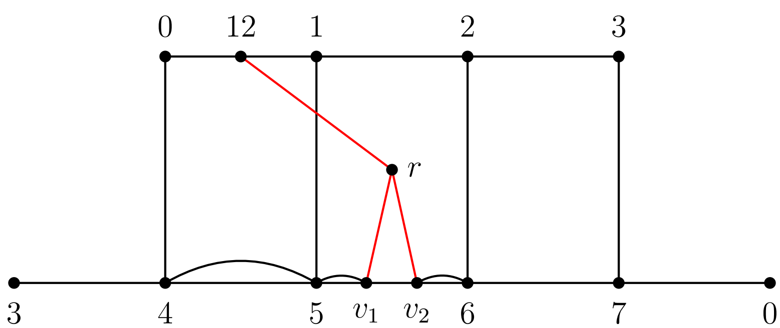

Figure 5.6: A 2-crossing-critical graph containing the 3-star when $t = t_2$.

5.4.3 Case $t = t_3$

If t is at t_3, then the 3-star can be included in a 2-crossing-critical graph. Figure 5.7 is an example.

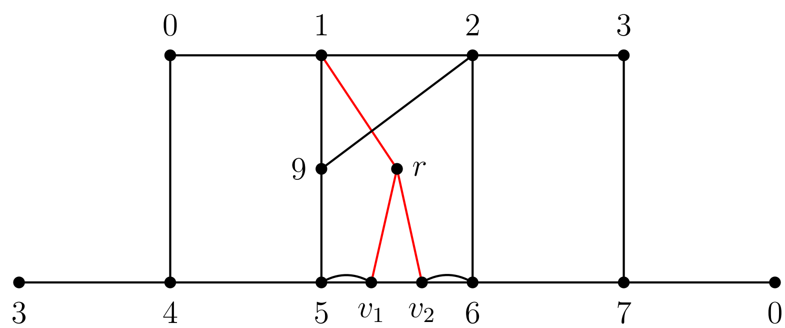

Figure 5.7: A 2-crossing-critical graph containing the 3-star when $t = t_2$.

5.4.4 Case $t = t_4$

If t is at t_4, then the resulting graph contains a 3-star attached to two opposing rim edges of H, a contradiction with Lemma 19.

5.4.5 Case $t = t_5$

If t is at t_5, then the cycle $(t, v_2, v_1, 5, 4, t)$ is a hole with planar bridge (r, v_1), a contradiction with Theorem 16.

5.4.6 Case $t = t_6$

If t is at t_6, then the cycle $(t, r, v_2, v_1, 5, t)$ is a hole with planar bridge (r, v_1), a contradiction with Theorem 16.

5.4.7 Case $t = t_7$, $t = t_8$, $t = t_9$, or $t = t_{10}$

If t is at t_7, t_8, t_9, or t_{10}, then the cycle (r, t, v_1, v_2, r) is a hole with planar bridge (r, v_1), a contradiction with Theorem 16.

5.5 Conclusion

To conclude this chapter's analysis, of the 19 possible 3-star attachments to a subdivided V_8, such that the 3-star attaches to one rim or spoke edge twice, there are only three which can be included in a 3-connected 2-crossing-critical graph G. These are, under the canonical labelling of the V_8 and up to symmetry:

- $(0,\ 17_a,\ 17_b)$
- $(12,\ 17_a,\ 17_b)$
- $(1,\ 17_a,\ 17_b)$

(where 17_a and 17_b represent two distinct points on the rim branch $(5, 6)$). All of these cases were found to be contained in 2-crossing-critical examples.

Chapter 6

4+-Star Case Analysis

We now proceed to consider the possible ways that a 4+-star could be attached to a fully-covered, 2-crossing-critical V_8. The two main goals of this chapter are to prove Theorem 10 and Theorem 12, restated below. In this chapter, as with before, let G be a 3-connected 2-crossing-critical graph which embeds in the real projective plan $\mathbb{R}P^2$, such that G contains a V_8 subdivision H but no V_{10} subdivision. We again emphasize here that we assume that H is a V_8 subdivision in G with the minimum number of vertices. Let T be a 4+-star which is an H-bridge in G.

Theorem 10. *Let G be a 3-connected 2-crossing-critical graph with a V_8 minor but no V_{10} minor, such that G embeds in the real projective plane $\mathbb{R}P^2$, and such that G contains a 4-star T which attaches to the V_8 minor in G with the minimum number of subdivisions. Then the connections of the 4-star T are one of the following sets of vertices, under a canonical labelling of the V_8:*

- *(0, 1, 5, 6)*
- *(0, 1, 5, 17)*
- *(0, 2, 14, 19)*
- *(0, 4, 12, 15)*
- *(0, 12, 17_a, 17_b)*
- *(1, 2, 5, 6)*
- *(1, 12, 17_a, 17_b)*
- *(2, 5, 14, 17)*

(where 17_a and 17_b represent two distinct points on the rim branch $(5, 6)$).

Conjecture 11. *Let G be a 3-connected 2-crossing-critical graph with a V_8 minor but no V_{10} minor, such that G embeds in the real projective plane $\mathbb{R}P^2$, and such that G contains a 4-star T which attaches to the V_8 minor in G with the minimum number of subdivisions. Then the connections of the 4-star T are not one of the following sets of vertices, under a canonical labelling of the V_8:*

- *(0, 2, 14, 19)*
- *(0, 12,* 17_a*,* 17_b*)*
- *(1, 2, 5, 6)*

Theorem 12. *Let G be a 2-crossing-critical graph with a V_8 minor but no V_{10} minor. Let T be a k-star which attaches to the V_8 minor. Then $k \leq 4$.*

In Chapters 2, 3, 4, and 5, we were able to demonstrate that some 3-stars cannot be attached to a fully-covered V_8 in a 2-crossing-critical graph. We will refer to these as the *eliminated* 3-stars. The other 3-stars will be referred to as the *viable* 3-stars.

A 4-star has four 3-star subgraphs. Clearly, none of the 3-star subgraphs of T can be in the eliminated 3-stars, as we have already demonstrated that the eliminated 3-stars cannot be included in a graph G under our assumptions.

Therefore, we consider only those possibilities for a 4-star H-bridge T, for which all four 3-star subgraphs of T are included in the viable 3-stars. Using a computer to check all possible 3-star subgraphs of all possible 4-stars, we determine that there are 11 such 4-stars attached to a V_8 which fulfill this requirement.

This computer check was completed with a brief Python script. We first stored the set of 3-stars as a set of lists of 3 vertices (v_0, v_1, v_2), where each v_i is an integer in [20] so that v_i refers to a vertex in the canonical labelling of a subdivided V_8. Then we designed a routine which takes in a list of vertices $l = (v_0, v_1, v_2, v_3)$ representing the vertex attachments of a 4-star under the canonical labelling of a V_8. This routine iterated over each 3-subset $l_3 \subset l$, and checked if l_3 could be relabelled to any of the viable 3-stars, by checking all possible relabellings of the subdivided V_8. If l_3 is not a viable 3-star or cannot be relabelled as such, the 4-star is discarded. Checking all 4-stars in this manner yielded the list of 11 viable non-isomorphic 4-stars.

Three of these 4-stars we have already encountered, and we have already demonstrated that these can be contained within a 2-crossing-critical graph. These can be found in Figure 4.2, Figure 4.10, and Figure 4.12. We proceed to analyzing the remaining eight cases to complete the proof of Theorem 10.

6.1 4-Star 1

When we have a 4-star attached to a V_8 as in Figure 6.1, then it is suspected, but not yet proven, that the resulting graph cannot be contained within a 2-crossing-critical graph.

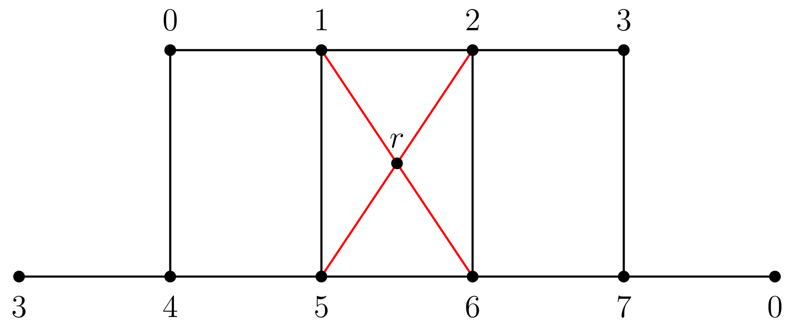

Figure 6.1: A 4-star in Layout 4. Neither (1,2) nor (5,6) can be crossed in a 1-drawing.

6.2 4-Star 2

With the 4-star in Figure 6.2, the cycle $(r, 1, 2, 3, 15, r)$ is a hole with a planar bridge $(r, 3)$, contradicting Theorem 16.

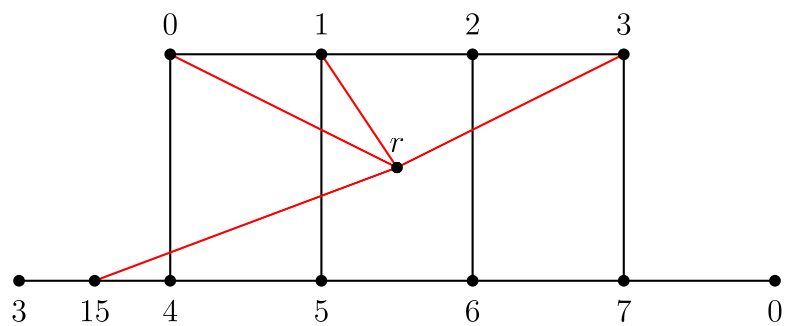

Figure 6.2: 4-Star 2.

6.3 4-Star 3

When we have a 4-star attached to a V_8 as in Figure 6.3, then it is suspected, but not yet proven, that the resulting graph cannot be contained within a 2-crossing-critical graph.

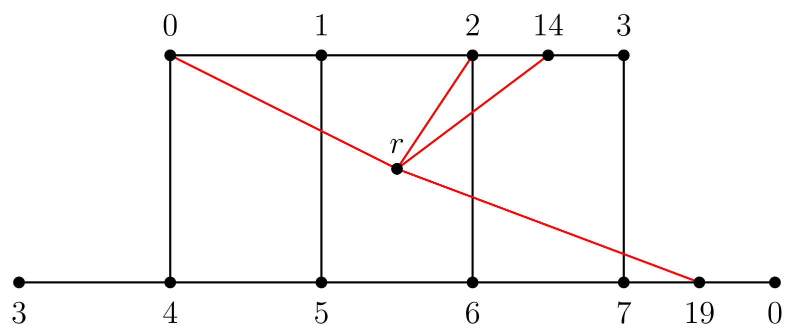

Figure 6.3: 4-Star 3.

6.4 4-Star 4

This 4-star can be included in a 2-crossing-critical graph, as seen in Figure 6.4.

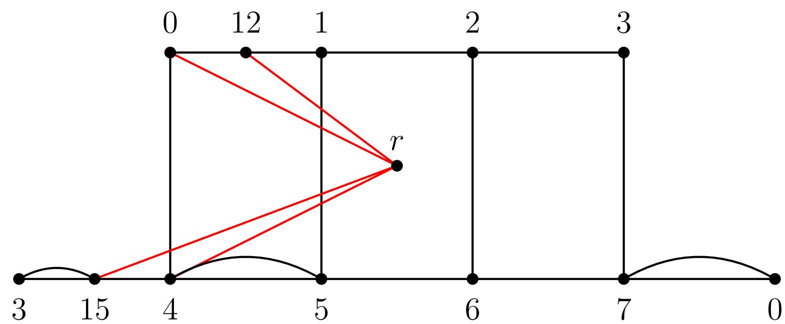

Figure 6.4: 4-Star 4.

6.5 4-Star 5

Now suppose that we have a 4-star as in Figure 6.5. If the path $(1, 2)$ is an edge, then the cycle $(r, 12, 1, 5, 6, 2, 14, r)$ is a hole with a planar bridge $(1, 2)$, contradicting Theorem 16.

If $(1, 2)$ is not an edge, then any H-bridge connecting to $(1, 2)$ must have all of its attachments to H in the cycle $(r, 12, 1, 5, 6, 2, 14, r)$, as a computer check demonstrates that all other options yield a graph G with crossing number at least two such that G is non-critical. But then the cycle $(r, 12, 1, 5, 6, 2, 14, r)$ is still a hole with a planar bridge $(1, 2)$, contradicting Theorem 16.

Therefore, a contradiction arises in all cases.

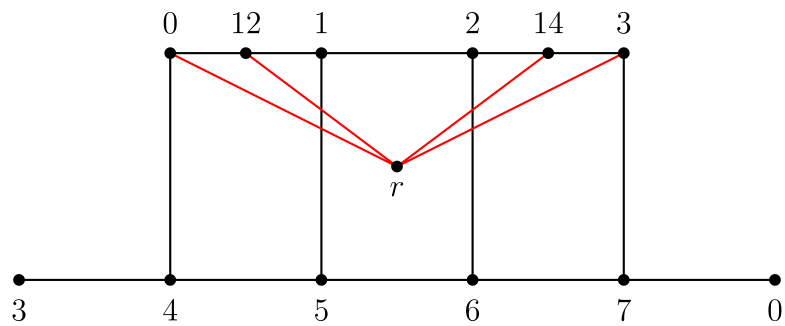

Figure 6.5: 4-Star 5.

6.6 4-Star 6

With the 4-star in Figure 6.6, the cycle $(0, r, v_1, v_2, 6, 7, 0)$ is a hole with a planar bridge (r, v_2), contradicting Theorem 16.

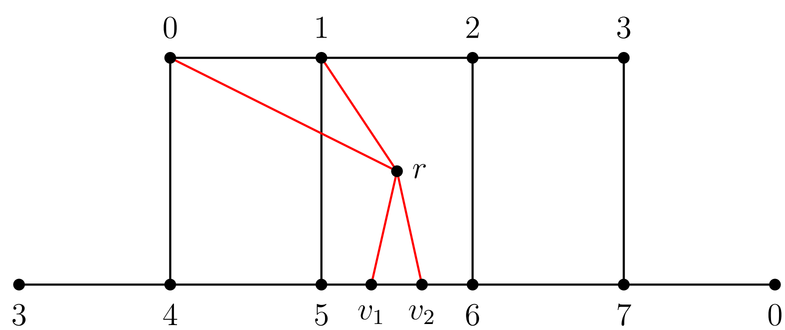

Figure 6.6: 4-Star 6.

6.7 4-Star 7

When we have a 4-star attached to a V_8 as in Figure 6.7, then it is suspected, but not yet proven, that the resulting graph cannot be contained within a 2-crossing-critical graph.

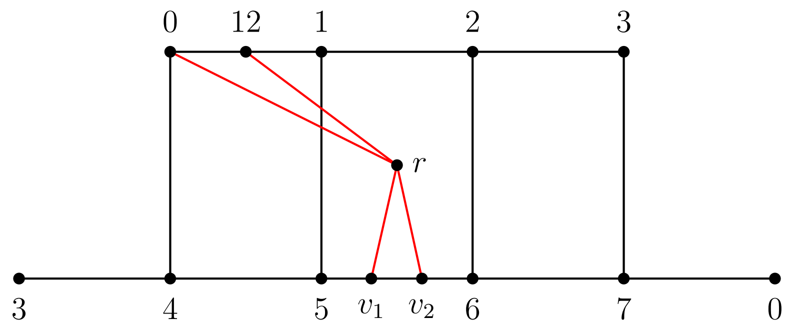

Figure 6.7: 4-Star 7.

6.8 4-Star 8

This 4-star can be included in a 2-crossing-critical graph, as seen in Figure 6.8.

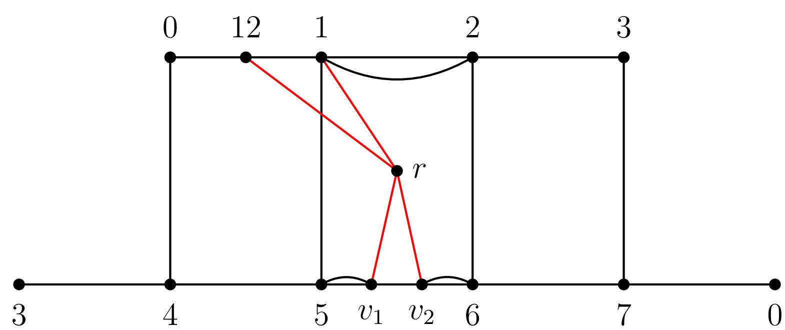

Figure 6.8: 4-Star 8.

6.9 5-Star Case Analysis

A 5-star has 10 3-star subgraphs. As with the 4-star analysis, to have a 2-crossing-critical graph with a V_8 minor and a 5-star attached to it, then none of the 5-star's 3-star subgraphs can be in the eliminated 3-stars. Therefore, we consider only those 5-star for which all 10 3-star subgraphs are included in the viable 3-stars.

Indeed, as checked by a computer in the same manner as before, we determined that there are no such 5-stars. By the same reasoning, there are no such 5+-stars. Therefore, the proof of Theorem 12 is complete.

Chapter 7

Future Work

Following the analysis of star attachments to a V_8, we have a significantly stronger understanding of the conditions under which a 3-connected 2-crossing-critical graph containing a V_8 minor but no V_{10} minor can occur. Still, such graphs are far from being fully characterized. In this section, we present some remaining questions which may need to be answered to complete the characterization.

7.1 Remaining 3- and 4-Stars

The analysis in this paper left seven 3-stars and three 4-stars unexplained, as enumerated in Corollary 9 and Corollary 11. These will likely require additional tools and techniques. Here, we present the outline of a proof technique that may be sufficient in explaining some or all of these remaining cases. We use Figure 7.1 as an example to demonstrate this technique.

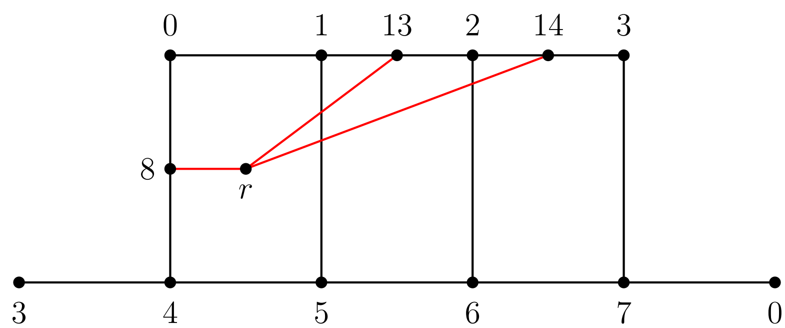

Figure 7.1: The second of two special cases of a 3-star attaching to a V_8 at precisely one spoke.

In this example, the only permitted crossing in a 1-drawing is between rim branches $(0, 7)$ and $(4, 5)$. By Theorem 6, covering $(0, 7)$ with additional structure(s) yields a graph with crossing number at least 2. However, doing so and then removing the edge $(r, 13)$ still yields a graph with crossing number at least 2, by Theorem 6, provided that removing $(r, 13)$ does not affect the coverage of $(0, 7)$.

The same is true symmetrically for $(4, 5)$. Covering $(4, 5)$ with additional structure(s) yields a graph with crossing number at least 2, but removing $(r, 14)$ still yields a graph with crossing number at least 2, provided that removing $(r, 14)$ does not affect the coverage of $(4, 5)$.

If we can demonstrate that either $(0, 7)$ or $(4, 5)$ is covered (i.e. a second crossing is caused by some other effect) and that removing $(r, 13)$ or $(r, 14)$ does not affect the covering of $(0, 7)$ or $(4, 5)$, respectively, then the argument is complete and we have shown that this 3-star cannot be included in a 2-crossing-critical example.

Let *large bridges* refer to H-bridges in which every H-avoiding path is a large jump (i.e. spanning more than two rim edges). Demonstrating that large bridges cover neither $(0, 7)$ nor $(4, 5)$, nor force as second crossing between themselves, accomplishes the two objectives in the preceding paragraph, as small structures are well-understood by Austin's work in [2].

To do so, we examine all possible large bridges which may attach to $(0, 7)$ or $(4, 5)$. We need not investigate any other large bridges, as any structure embedded in the projective plane in one of the five faces bounded by a quad or incident to r has the same embedding in a 1-drawing. However, large bridges outside of these faces may have a different embedding in a 1-drawing from the projective plane. This could lead to overlap, and therefore a second crossing. In this argument, we seek to rule out this possibility, as well as the possibility that large bridges could cover $(0, 7)$ or $(4, 5)$.

Let e and f be the sections of $(0, 7)$ and $(4, 5)$, respectively, which can be crossed in a 1-drawing. There are three possibilities for the locations of e and f in this embedding; here, we have shown the one where e and f are both on the top of the drawing. The other two are e and f both on the left (symmetrically the right), and e and f both on the bottom.

In Figure 7.2, we examine two possible large bridges. Here, they are represented by I and J, but they may have more than two H-attachments provide that the requirements for a large bridge are met.

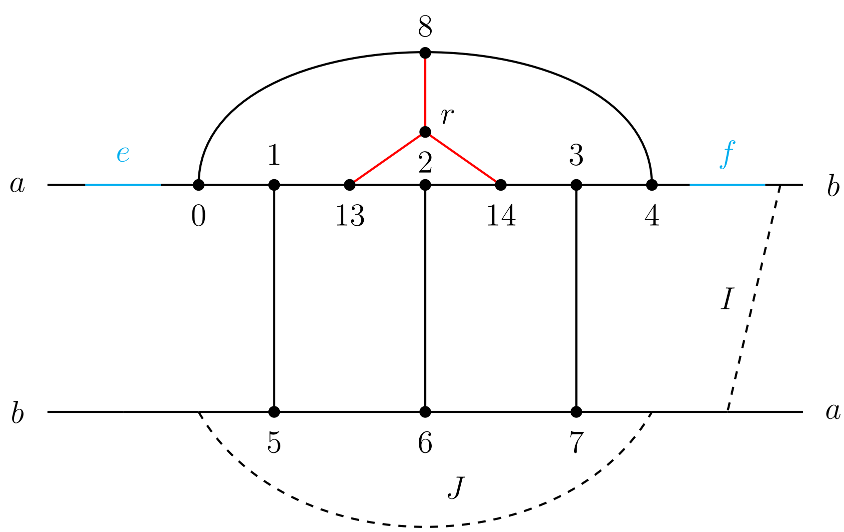

Figure 7.2: The only embedding of Figure 7.1 in the real projective plane $\mathbb{R}P^2$ with representativity 2.

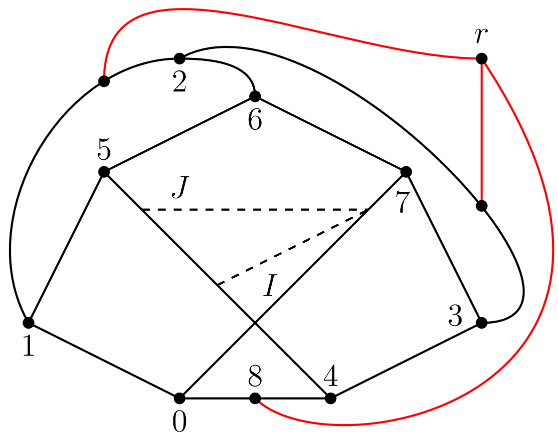

Figure 7.3: The 1-drawing of Figure 7.1.

If J and I do not overlap in Figure 7.3, then there is no problem as e and f remain uncovered, and therefore the crossing number remains at most 1. If J and I overlap in Figure 7.3, then their attachments to either $(0,7)$ or $(4,5)$ must be equal or interlace. In either case, a hole with a planar bridge can be demonstrated. Therefore, we conclude that I and J cover neither e nor f, and that I and J can be embedded in a 1-drawing.

Performing this analysis for all possible combinations of large bridges in the three possible layouts of e and f can demonstrate that there are no large bridges attaching to $(0,7)$ or $(4,5)$, and that the resulting graph has crossing number 1. Therefore, we can conclude that $(0,7)$ or $(4,5)$ is covered by a set of small structures, and the rest of the argument becomes valid.

As mentioned, it is believed that this approach could eliminate the remaining ten 3- and 4-stars. However, the amount of effort required to examine all possible large bridges in every possible planar embedding and 1-drawing of the remaining cases forced this approach outside the scope of this paper.

7.2 4+-Trees

In the previous chapter, we considered 4+-stars attaching to V_8. However, unlike in the case of a 3-tree, where a 3-star is the only tree with three leaves, there are multiple possible

4+-trees which could attach to a V_8. In the case of a 4-tree, the two tree arrangements are a 4-star and a perfect binary tree with four leaves.

In contrast to the 4+-star analysis, we can no longer eliminate any 4+-tree which contains an eliminated 3-star subgraph. The 3-star elimination arguments which use Theorem 16 do not work if the planar bridge in question, typically an edge of the 3-star, is permitted to have attachments to the rest of the V_8. As such, the 4+-trees which remain to be considered are precisely the following:

1. 3-stars which were eliminated by a contradiction with Theorem 16, where the edge which formed the planar bridge in question is permitted to be subdivided; and
2. 4+-trees which are attached at the same points as the 4-stars in chapter 6.

It is hypothesized that the 4+-trees in the first category cannot be included in a 2-crossing-critical graph with a fully covered V_8.

Conjecture 24. *Let G be a 3-connected 2-crossing-critical graph with a subdivision H of V_8 but no V_{10} subdivision. Let T be a 4+-tree which is an H-bridge in G. Then G does not contain as a subgraph any eliminated 3-star T' attached to the V_8 subdivision H.*

This conjecture may be slightly generalized as the following conjecture.

Conjecture 25. *Let G be a 3-connected 2-crossing-critical graph with a subdivision H of V_8 but no V_{10} subdivision, such that G is embeddable in the real projective plane $\mathbb{R}P^2$. Let T be a 4+-tree which is an H-bridge in G. Then contracting the internal vertices of T to a single vertex to yield T', where T' a 4+-star with the same attachments to H as T, yields a 3-connected 2-crossing-critical graph which is embeddable in the real projective plane $\mathbb{R}P^2$.*

Furthermore, Bokal, Oporowski, Richter, and Salazar demonstrated that any H-bridge with three attachments is a 3-star. Following this reasoning, the previous conjecture raises one more question.

Conjecture 26. *Let G be a 3-connected 2-crossing-critical graph with a subdivision H of V_8 but no V_{10} subdivision, such that G is embeddable in the real projective plane $\mathbb{R}P^2$. If B is an H-bridge, then B has at most four H-attachments, and B is a tree.*

Addressing these three conjectures is beyond the scope of this work. But, we are able to give some consideration to the 4-trees in the second category above, to attempt to fully explain those H-bridges which would not explained by the preceding three conjectures.

Unlike in the case of a 3-tree, where a 3-star is the only tree with three leaves, there are two possible 4-trees which could attach to a V_8: a 4-star and a complete binary tree with four leaves (sometimes referred to as a perfect binary tree). The complete binary trees with the same attachments to the V_8 as in chapter 6 are the trees with which we concern ourselves here.

In the complete binary tree with four leaves, there are three possible layouts of the tree with respect to the attachments to the V_8. Let a, b, c, d be the four connections of the tree to the V_8, in that order around the rim. The binary tree which pairs (a, c) and (b, d) cannot be embedded in the real projective plane $\mathbb{R}P^2$ when attached to a V_8 in the manner described, unless this occurs within a quad of the V_8. Therefore, we only consider this possibility in Layout 4. We do consider the other two layouts, which pair (a, d) and (b, c), and which pair (a, b) and (c, d).

7.2.1 Layout 1

Suppose that a complete binary tree with this layout is attached to a V_8, with leaf pairing (1,0) and (5,6), as in Figure 7.4. Then the cycle $(0, r_0, r_1, 5, 1, 0)$ is a hole with planar bridge $(r_0, 1)$, a contradiction with Theorem 16. Therefore, this case cannot be contained within a 2-crossing-critical graph.

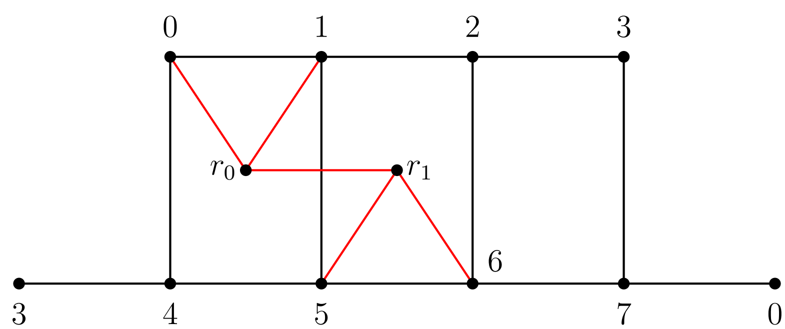

Figure 7.4: A complete binary tree in Layout 1.

Now suppose that a complete binary tree with this layout is attached to a V_8, with leaf pairing $(0, 6)$ and $(1, 5)$, as in Figure 7.5.

If $(1, 5)$ is an edge, then delete it. The resulting graph has a 1-drawing D. But then, in D, the path $(1, r_1, 5)$ is not crossed, as this can be defined to replace the $(1, 5)$ spoke in a V_8. So $(1, 5)$ can be drawn alongside this path, and we have a 1-drawing of the original graph, a contradiction.

Then $(1, 5)$ has at least one subdivision. And it has at most one subdivision, by the minimality of subdivisions of the V_8. (To see this, redefine the $(1, 5)$ spoke to be $(1, r_1, 5)$ as before). Let i be the internal vertex of $(1, 5)$. Then, besides 1 and 5 (which would yield a non-critical edge), an H-bridge attaching to i could only be adjacent to r_1 or 13, as checked by a computer (the other possibilities have crossing number at least 2 and are non-critical). Let B' be this H-bridge.

If B' is adjacent to r_1, then the cycle $(1, 5, r_1, 1)$ is a hole with planar bridge (i, r_1), a contradiction with Theorem 16.

If B' is adjacent to 13, then the cycle $(r_0, r_1, 1, 0, 7, 6, r_0)$ is a hole with planar bridge $(r_0, 0)$, a contradiction with Theorem 16.

Since a contradiction arises in all cases, then this case cannot be contained within a 2-crossing-critical graph.

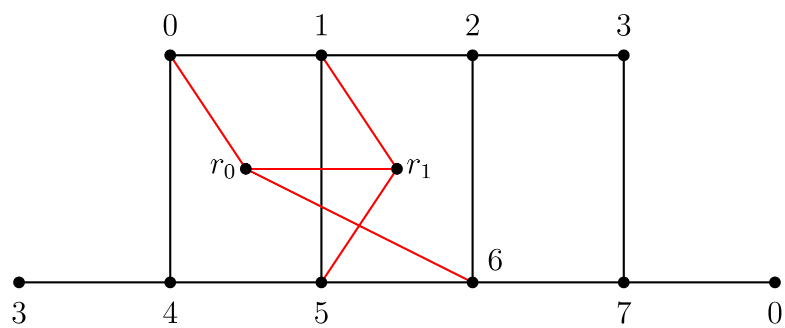

Figure 7.5: Another complete binary tree in Layout 1.

7.2.2 Layout 2

In this layout, if the 4-tree is a complete binary tree with leaf pairing (1,0) and (5,17), then we can produce the following 2-crossing-critical graph. It is worth noting that contracting the edge (r_0, r_1) yields the 2-crossing-critical graph seen in Figure 4.2.

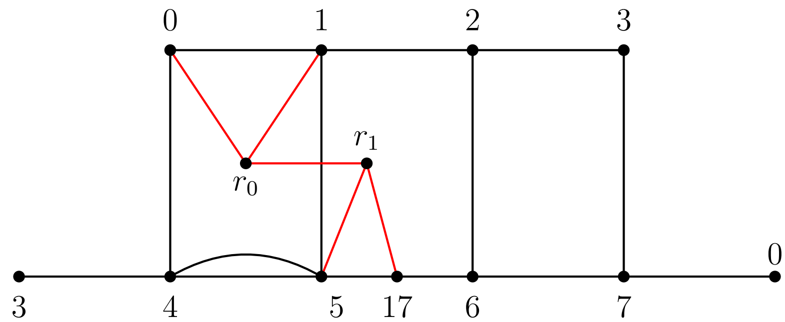

Figure 7.6: A 2-crossing-critical graph with a fully covered V_8 and a 4-tree with Layout 2.

It is suspected, but not yet proven, that the other possible complete binary tree in this layout, with leaf pairing (0,17) and (1,5), cannot be included in a 2-crossing-critical graph.

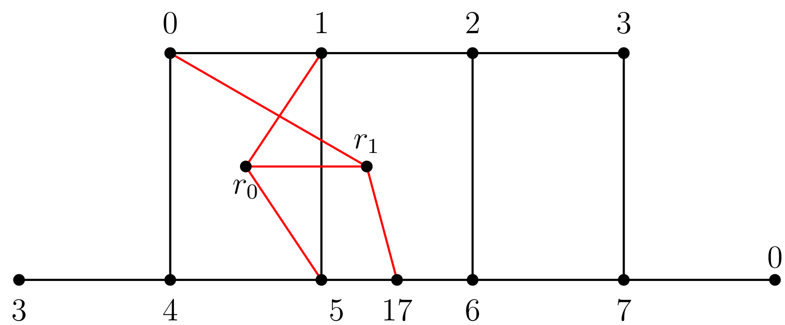

Figure 7.7: Another complete binary tree with four leaves in Layout 2.

7.2.3 Layout 3

In this layout, if the 4-tree is a complete binary tree with leaf pairing (2,14) and (5,17), then we can produce a 2-crossing-critical graph. It is worth noting that contracting the edge (r_0, r_1) yields the 2-crossing-critical graph seen in Figure 4.12.

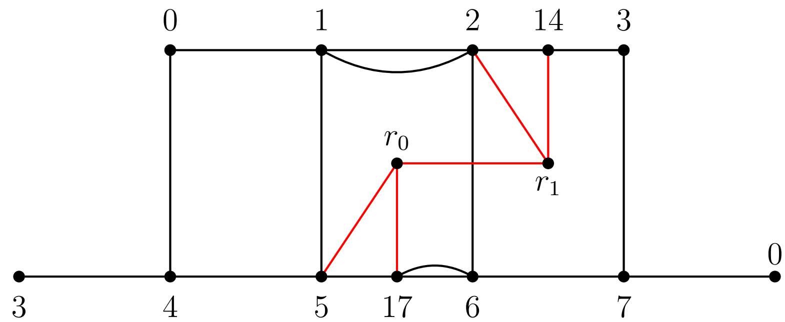

Figure 7.8: A 2-crossing-critical graph with a fully covered V_8 and a 4-tree with Layout 3.

It is suspected, but not yet proven, that the other possible complete binary tree in this layout, with leaf pairing (2,17) and (5,14) cannot be included in a 2-crossing-critical graph.

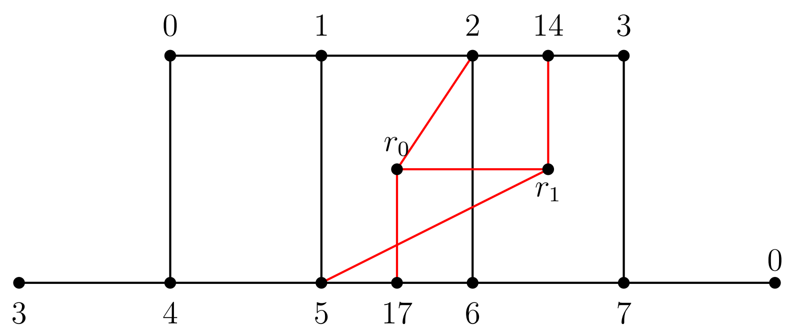

Figure 7.9: Another complete binary tree with four leaves in Layout 3.

7.2.4 Layout 4

In Layout 4, it is possible to embed all three complete binary trees attached to the V_8 in the real projective plane $\mathbb{R}P^2$, so all three cases are considered here.

When we have a complete binary tree as in Figure 7.10, then the graph cannot be 2-crossing-critical. The cycle $(1, 5, r_0, 6, 2, r_1, 1)$ is a hole with planar bridge (r_0, r_1), a contradiction with Theorem 16. Therefore, this case cannot be contained within a 2-crossing-critical graph.

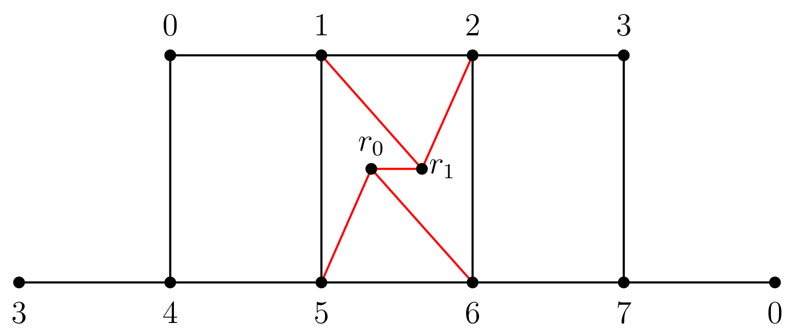

Figure 7.10: A complete binary 4-tree in Layout 4.

It is suspected, but not yet proven, that the other possible complete binary tree in this layout, as seen below, cannot be included in a 2-crossing-critical graph.

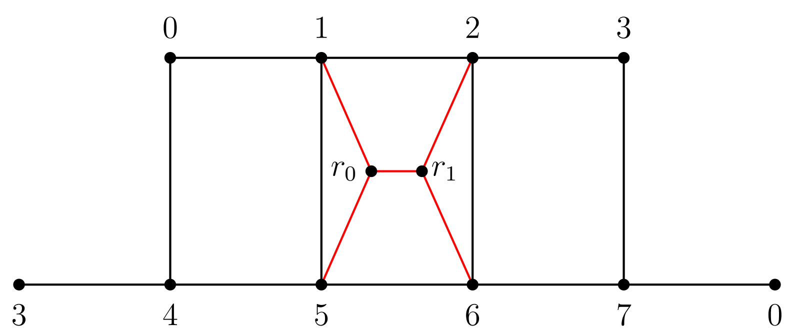

Figure 7.11: Another complete binary 4-tree in Layout 4.

When we have a complete binary tree as in Figure 7.12, then the resulting graph has crossing number at least 2 and is non-critical, as checked by a computer. Therefore, this case cannot be contained within a 2-crossing-critical graph.

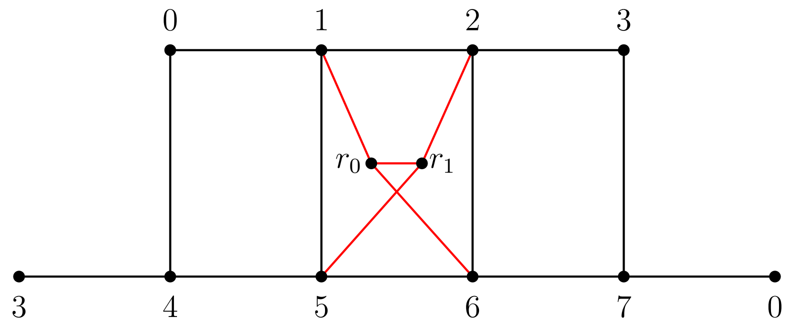

Figure 7.12: The third complete binary 4-tree in Layout 4.

7.2.5 Layout 6

With the complete binary tree in Layout 6 with leaf pairing $(0, 1)$ and $(3, 15)$, the cycle $(r_1, r_0, 1, 2, 3, 15, r_1)$ is a hole with a planar bridge $(r_1, 3)$, contradicting Theorem 16. Therefore, this case cannot be contained within a 2-crossing-critical graph.

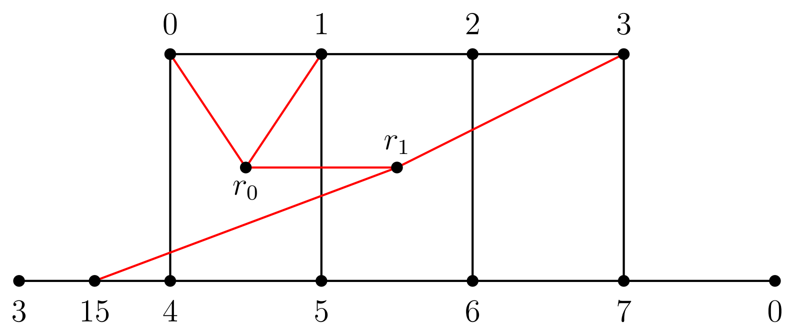

Figure 7.13: A complete binary tree in Layout 6.

With the complete binary tree in Layout 6 with leaf pairing $(0, 15)$ and $(1, 3)$, the cycle $(3, 2, 1, r_1, r_0, 15, 3)$ is a hole with a planar bridge $(r_1, 3)$, contradicting Theorem 16. Therefore, this case cannot be contained within a 2-crossing-critical graph.

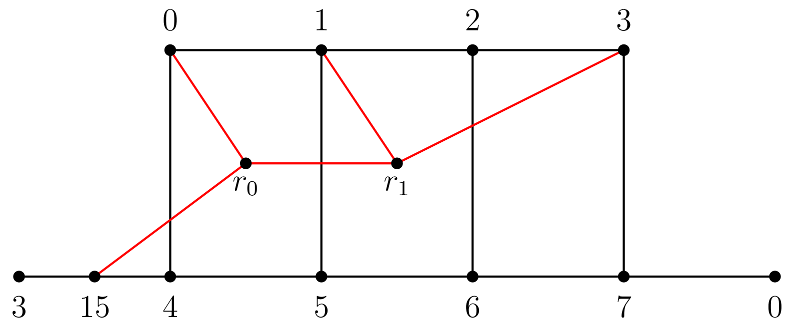

Figure 7.14: Another complete binary tree in Layout 6.

7.2.6 Layout 7

Now let the complete binary tree in Layout 7, with leaf pairings $(0, 19)$ and $(2, 14)$, attach to a V_8. It is suspected, but not yet proven, that this case cannot lead to a 2-crossing-critical graph.

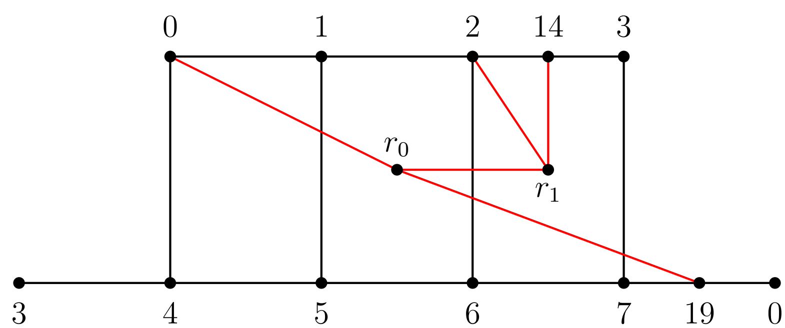

Figure 7.15: A complete binary tree in Layout 7.

Now let the complete binary tree in Layout 7, with leaf pairings $(0, 19)$ and $(2, 14)$, attach to a V_8. It is suspected, but not yet proven, that this case cannot lead to a 2-crossing-critical graph.

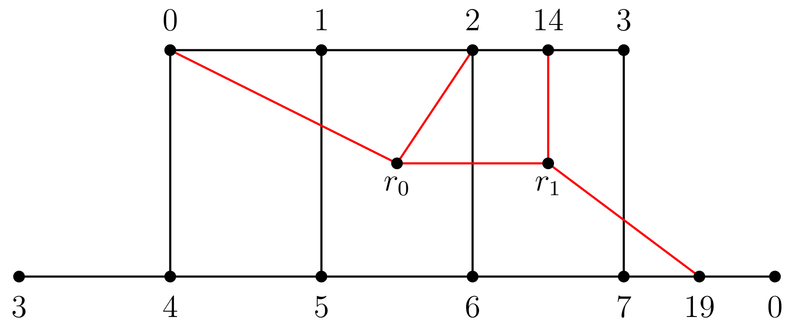

Figure 7.16: Another complete binary tree in Layout 7.

7.2.7 Layout 8

In this layout, if the 4-tree is a complete binary tree with leaf pairings $(0, 12)$ and $(4, 15)$, then we can produce a 2-crossing-critical graph. It is worth noting that contracting the edge (r_0, r_1) yields the 2-crossing-critical graph seen in Figure 6.4.

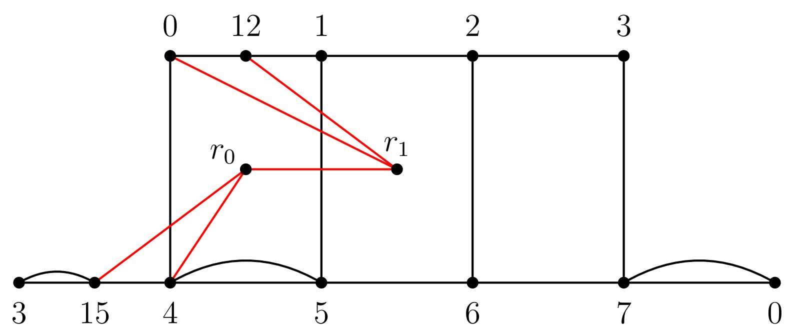

Figure 7.17: A fully covered, 2-crossing-critical graph containing the 4-tree in Layout 8 with leaf pairing (0,12) and (15,4).

Now let the complete binary tree in Layout 8 with leaf pairings (0,4) and (12,15) attach to a V_8. It is suspected, but not yet proven, that this case cannot yield a 2-crossing-critical graph.

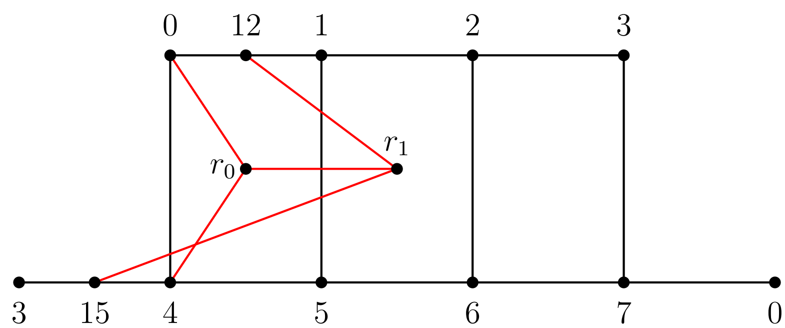

Figure 7.18: A 4-tree in Layout 8 with leaf pairings (0,4) and (12,15).

7.2.8 Layouts 5, 9, 10, and 11

It is unknown if the complete binary trees in these layouts (as in Figure 6.6, Figure 6.7, Figure 6.8, and Figure 6.5) can be included in 2-crossing-critical graphs.

7.3 Structures with Multiple V_8 Embeddings

The 3-jump (diagonal) and $3\frac{1}{2}$-jump (semi-diagonal) can be drawn inside of a V_8 quad or outside of the V_8. When permitted to be drawn inside of a V_8 quad in a 1-drawing, these structures provide very little coverage. By itself, the diagonal eliminates no crossings from the V_8 and the semi-diagonal only eliminates one. However, if either one is prevented from being drawn inside of a V_8 quad in a 1-drawing by some other structure, then the structure provides significantly more coverage. Understanding the possible combinations of structures which take advantage of this fact and can be included in a 2-crossing-critical graph is an open question.

For example, the 3-star in Figure 4.10 can be transformed into a V_8 with a semi-diagonal, $\frac{1}{2}$-jump, and $2\frac{1}{2}$-jump, as seen below.

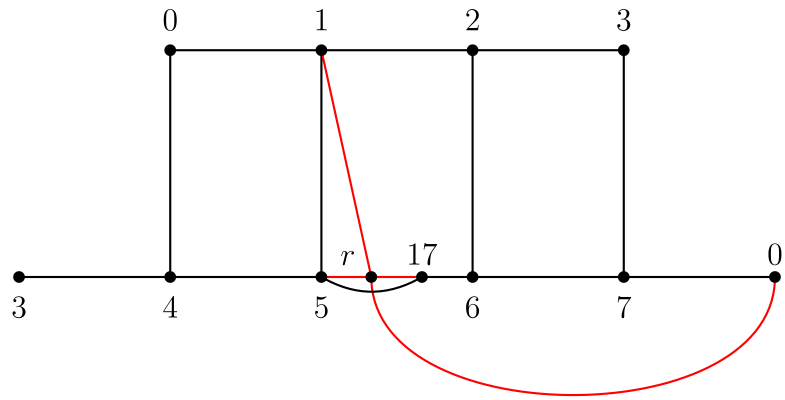

Figure 7.19: A transformation of Figure 4.10.

In this case, the $\frac{1}{2}$-jump forces the semi-diagonal to be drawn outside of the $(1, 2, 6, 5)$ quad in a 1-drawing. But this, combined with the $2\frac{1}{2}$-jump $(r, 0)$, eliminates all possible 1-drawings of the V_8, by fully covering 5 consecutive rim branches.

Another simpler example occurs when two opposing diagonals are placed in a quad, as seen in Figure 7.20. Recall that, by themselves, diagonals do not eliminate any rim edge crossings of a V_8. However, two opposing diagonals cover the edges $(3, 4)$ and $(0, 7)$. This can be seen by considering which diagonal of the two is drawn outside of the quad in a 1-drawing. If $(1, 6)$ is outside of the quad in a 1-drawing, then $(6, 7)$, $(0, 7)$, and $(0, 1)$ cannot be crossed. As a consequence, $(3, 4)$ cannot be crossed as well, since all of its crossing pairs have been eliminated. If $(2, 5)$ is drawn outside of the quad in a 1-drawing, then $(2, 3)$, $(3, 4)$, and $(4, 5)$ cannot be crossed. As a consequence, $(0, 7)$ cannot be crossed as well.

In either scenario, neither $(3, 4)$ nor $(0, 7)$ can be crossed. Therefore, we conclude that this combination of structures covers these edges. (This fact has been verified by a computer).

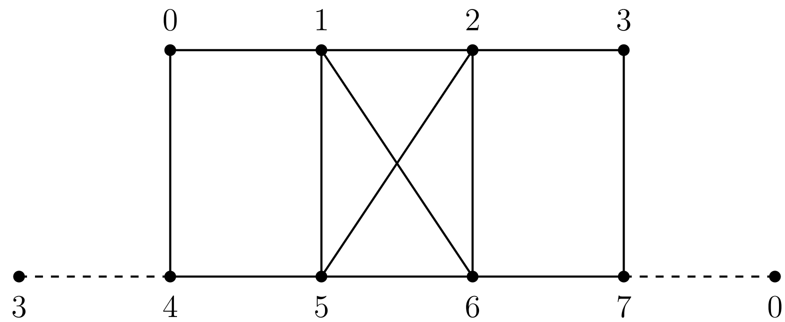

Figure 7.20: A V_8 with two opposing diagonals inside of a quad. Covered edges are denoted with dashed lines.

Explaining the possible ways that combinations of structures can yield greater coverage than the union of the coverage of the individual structures is an important step towards classifying 3-connected 2-crossing-critical graphs with a V_8 minor but no V_{10} minor.

7.4 2-crossing-critical Graphs with a V_8 Minor, without a Fully Covered V_8

Perhaps the most important question towards a full characterization still remains: are graphs with a fully covered V_8 subdivision the only 3-connected 2-crossing-critical graphs with a V_8 minor but no V_{10} minor which can be embedded in the real projective plane $\mathbb{R}P^2$? It is hypothesized that this is the case. Resolving this conjecture remains, at the moment, the biggest hurdle towards fully characterizing 2-crossing-critical graphs with a V_8 minor but no V_{10} minor.

Conjecture 27. *If G is a 3-connected 2-crossing-critical graph with a V_8 subdivision but no V_{10} subdivision, such that G can be embedded in the real projective plane $\mathbb{R}P^2$, then G contains a fully covered V_8 subdivision.*

7.5 Miscellaneous

Finally, we conclude with some miscellaneous open questions. These may or may not be directly helpful in fully classifying 2-crossing-critical graphs with a V_8 minor but no V_{10} minor, but are interesting nonetheless.

Let T be a tree attached to a fully covered V_8 in a 2-crossing-critical graph G. The tree T will have some number of 2-star subgraphs, which are topologically isomorphic to attaching an edge to the V_8. It is a direct corollary of Theorem 6 that the 1-drawing crossings eliminated by T are at least those 1-drawing crossings which are eliminated by all of the 2-star subgraphs of T. It is an interesting question to determine if these are all of the 1-drawing crossings which are eliminated by T.

Conjecture 28. *Let G be a 2-crossing-critical graph with a V_8 minor but no V_{10} minor, such that the V_8 is fully covered. Then the coverage provided by a tree attached to the V_8 is precisely the union of the coverage provided by its 2-star subgraphs.*

Finally, finding a correct proof or counterexample of the following conjecture (as discussed in Chapter 1) is another interesting open question.

Conjecture 29 (Theorem 3.1 from Austin's work in [2])**.** *In a 3-connected 2-crossing-critical with a fully covered V_8 and no V_{10} minor, the sections of rim covered by bars, 2-bars, $\frac{1}{2}$-, 1-, $1\frac{1}{2}$-, 2-, off-$\frac{1}{2}$-, and off-1-jumps must be disjoint.*

Chapter 8

Conclusion

In conclusion, significantly further progress has been made towards understanding the 3-connected 2-crossing-critical graphs with a V_8 minor but no V_{10} minor, which embed in the real projective plane $\mathbb{R}P^2$. The remaining possibilities to be considered have been narrowed, and we have achieved a deeper understanding of the ways in which preventing rim edge crossings (i.e. coverage) can lead to 2-crossing-critical graphs.

However, there still remain open questions. Many of these are still in the area of determining which structures can be used to fully cover a V_8. There remain a handful of trickier cases of trees which need to be dealt with in order to fully explain how trees can fully cover a V_8. Combinations of structures, both trees and edges, also need to be examined. Additionally, it is still an open question if fully covered V_8's can lead to a full characterization of this class of graphs, or if some additional analysis will be required.

Still, investigating these questions appears to be, at this moment, the best way to finish characterizing all 2-crossing-critical graphs.

References

[1] D. Archdeacon. A Kuratowski theorem for the projective plane. *J. Graph Theory*, 5:243–246, 1981.

[2] E. Austin. 2-crossing critical graphs with a V_8 minor. MMath Thesis, University of Waterloo, 2012.

[3] G. S. Bloom, J. W. Kennedy, and L. V. Quintas. On crossing numbers and linguistic structures. *Lecture Notes in Math*, 1018:14–22, 1983.

[4] D. Bokal, B. Oporowski, R. B. Richter, and G. Salazar. Characterizing 3-connected, 2-crossing-critical graphs. unpublished.

[5] A. Arroyo Guevara. On 2-crossing critical graphs with a V_8 minor. MMath Thesis, University of Waterloo, 2014.

[6] M. Kochol. Construction of crossing-critical graphs. *Discrete Math*, 66:311–313, 1987.

[7] J. Maharry and N. Robertson. The structure of graphs not topologically containing the Wagner graph. *J. Combin. Theory*, Ser. B 121:398–420, 2016.

[8] Brendan D. McKay and Adolfo Piperno. Practical graph isomorphism, ii, 2013. 10.48550/ARXIV.1301.1493, `https://arxiv.org/abs/1301.1493`.

[9] B. Richter. Cubic graphs with crossing number 2. *J. Graph Theory*, 12.3:363–374, 1988.

[10] I. Urrutia Schroeder. Finding 3-connected 2-crossing-critical graphs with V_8 minors and no V_{10} minors. MMath Essay, University of Waterloo, 2011.

[11] R. P. Vitray. Graphs containing graphs of crossing number 2. Presentation at AMS Summer Conference, Ohio State University, August 1990.

[12] J. Širàň. Infinite families of crossing-critical graphs with a given crossing number. *Discrete Math*, 48.1:129–132, 1984.

APPENDICES

Appendix A

Coverage Provided by Structures Attached to a V_8

When adding structure to a V_8 in order to increase the crossing number, it is helpful to know the rim edge crossings which a given structure eliminates. For each 3- and 4-star which we determined can be contained in a 2-crossing-critical graph, the coverage provided is contained below. Dashed rim edges represent rim edges which cannot be crossed in a 1-drawing of the graph.

It is important to note that the coverage given here is a minimum for each structure. Some structures may also eliminate a single crossing from a given rim edge, without eliminating all three crossings (recall that, in a 1-drawing of a V_8 with no structure added, a given rim edge has three possible crossings with other rim edges).

Furthermore, some structures may eliminate a small number of crossings when embedded in one face, but a great many crossings when embedded in another face.

A.1 Edges

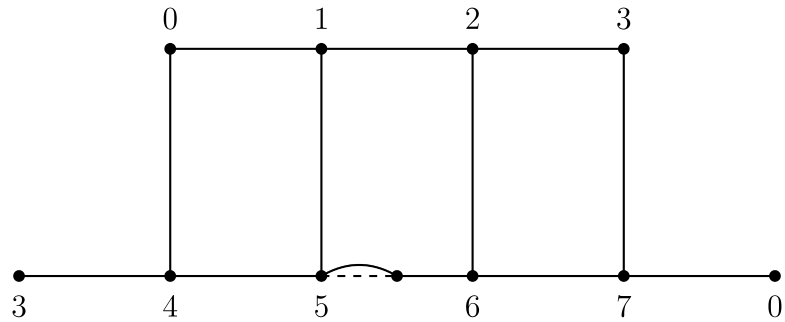

Figure A.1: Coverage provided by a $\frac{1}{2}$-jump.

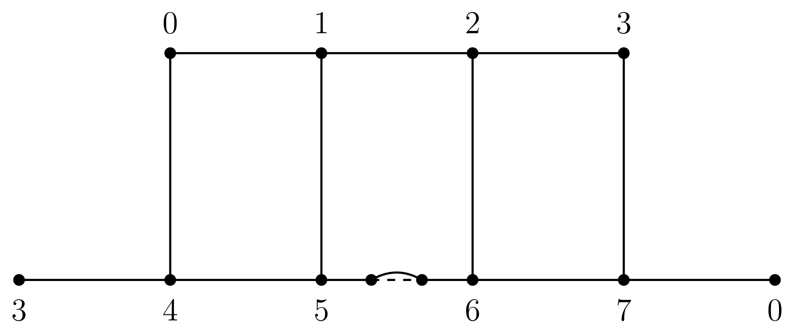

Figure A.2: Coverage provided by an off-$\frac{1}{2}$-jump.

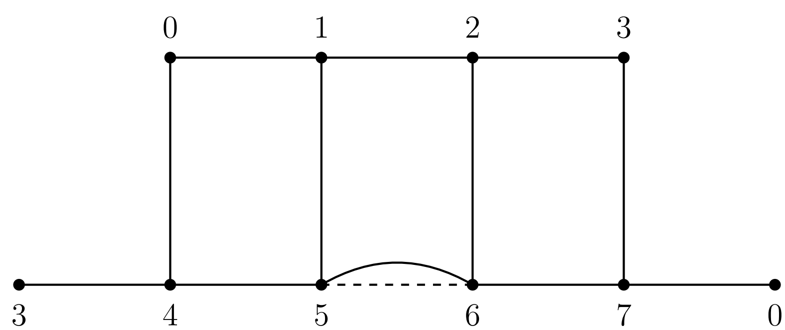

Figure A.3: Coverage provided by a 1-jump.

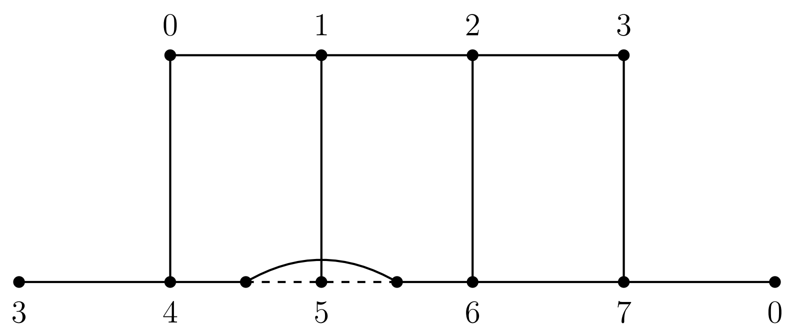

Figure A.4: Coverage provided by a off-1-jump.

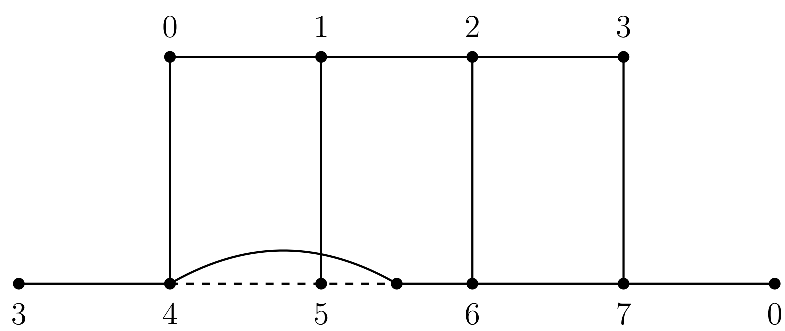

Figure A.5: Coverage provided by a 1-$\frac{1}{2}$-jump.

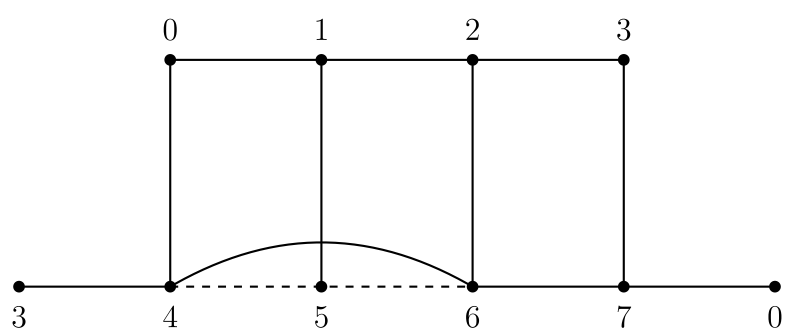

Figure A.6: Coverage provided by a 2-jump.

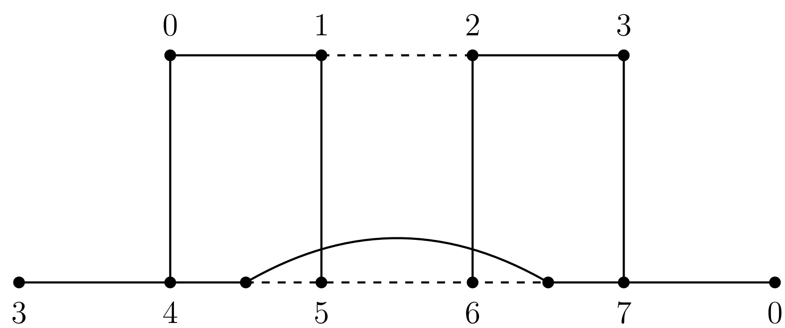

Figure A.7: Coverage provided by an off-2-jump.

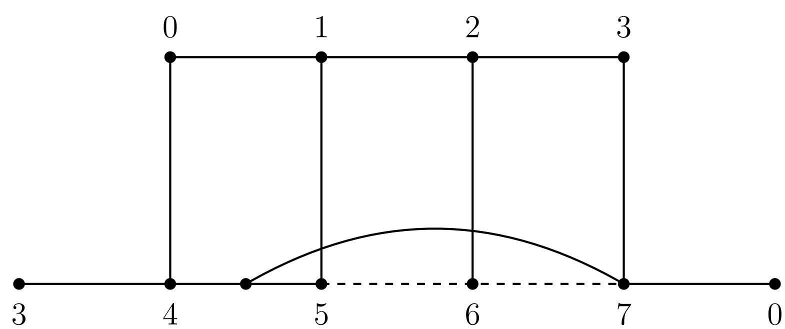

Figure A.8: Coverage provided by a $2\frac{1}{2}$-jump.

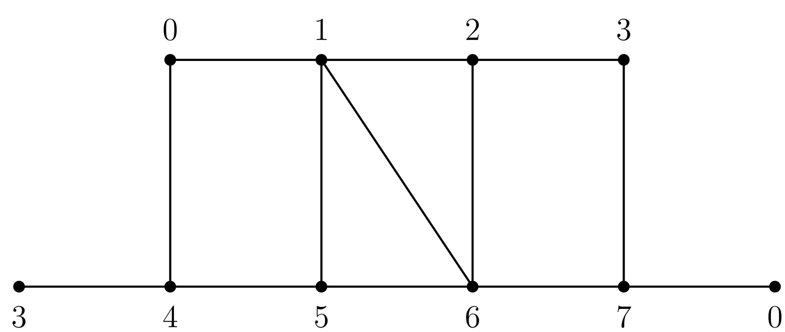

Figure A.9: Coverage provided by a 3-jump (diagonal).

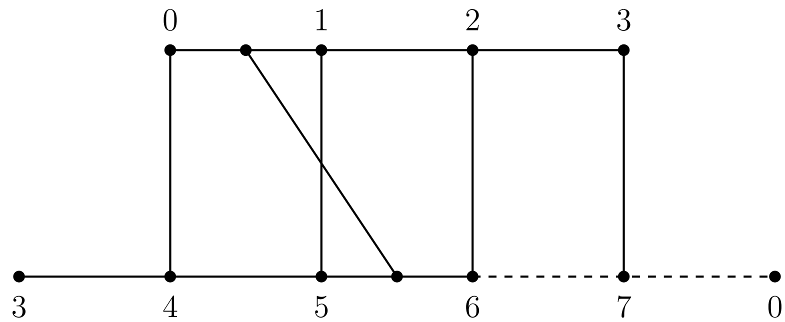

Figure A.10: Coverage provided by an off-3-jump.

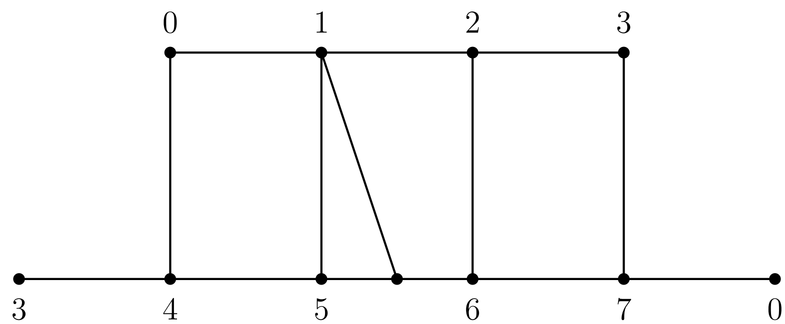

Figure A.11: Coverage provided by a $3\frac{1}{2}$-jump (semi-diagonal).

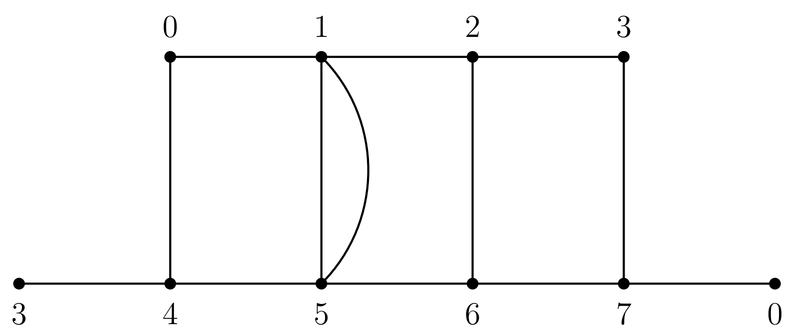

Figure A.12: Coverage provided by a spoke jump.

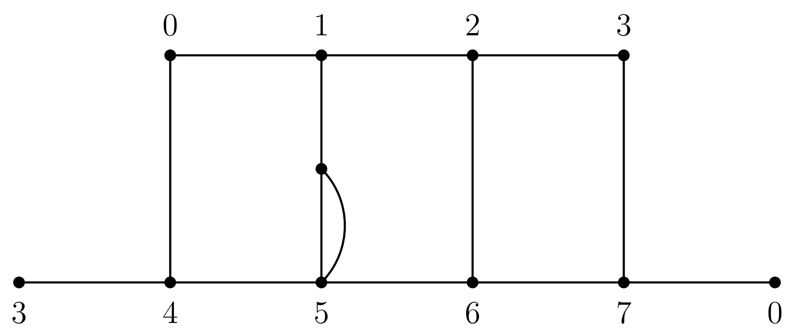

Figure A.13: Coverage provided by a $\frac{1}{2}$-spoke jump.

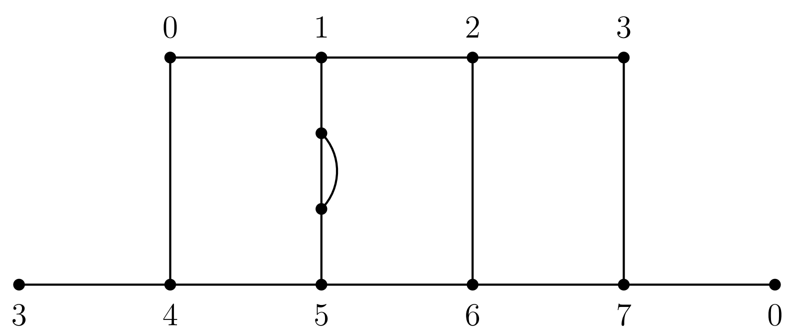

Figure A.14: Coverage provided by an off-$\frac{1}{2}$-spoke jump.

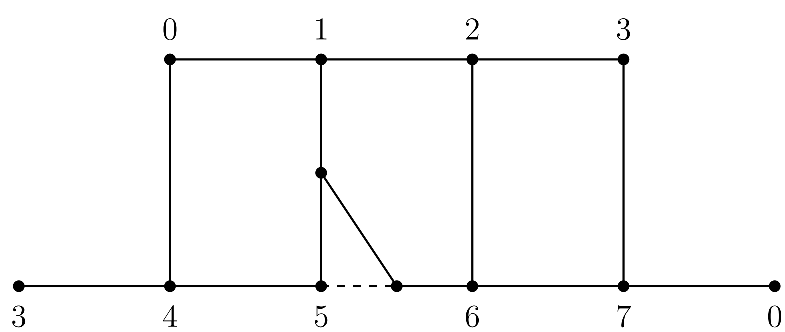

Figure A.15: Coverage provided by a $\frac{1}{2}$-slope.

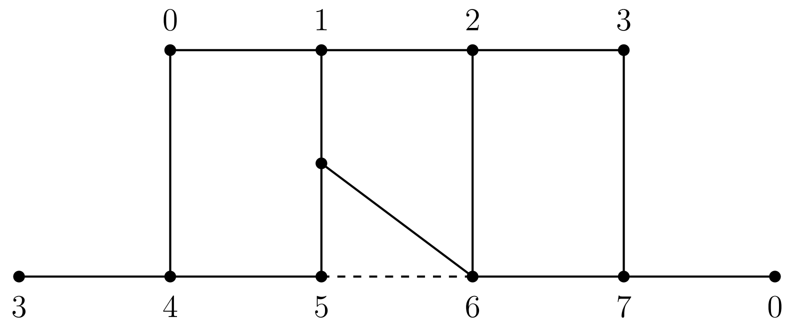

Figure A.16: Coverage provided by a 1-slope.

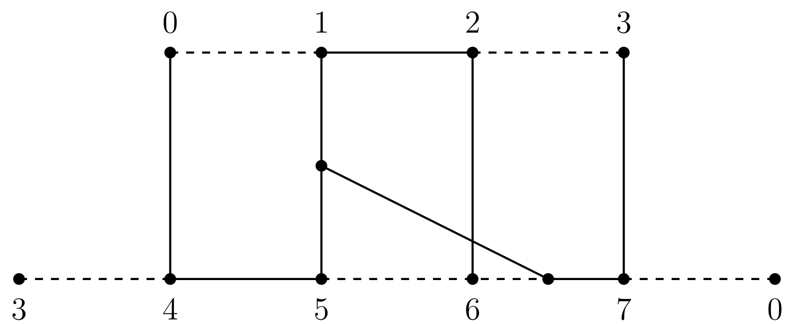

Figure A.17: Coverage provided by a $1\frac{1}{2}$-slope.

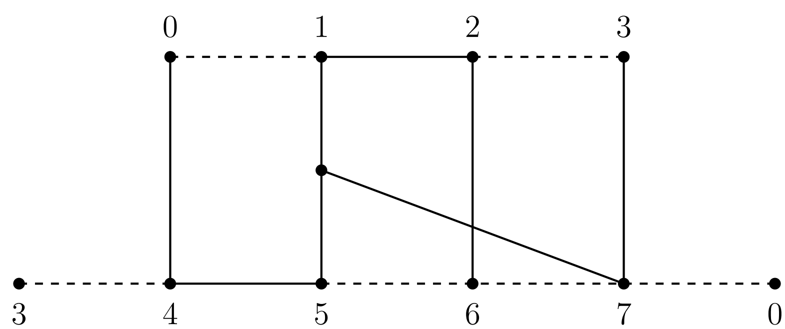

Figure A.18: Coverage provided by a 2-slope.

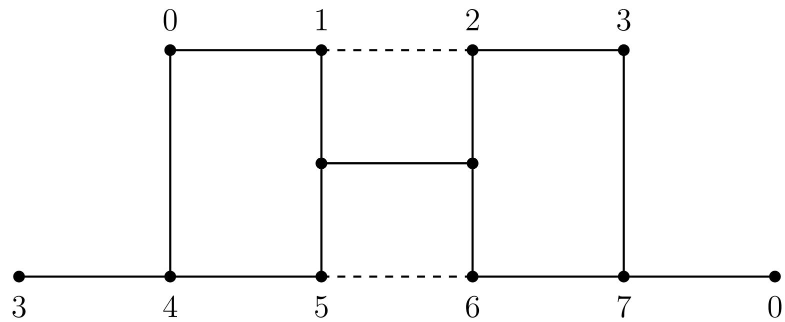

Figure A.19: Coverage provided by a 1-bar.

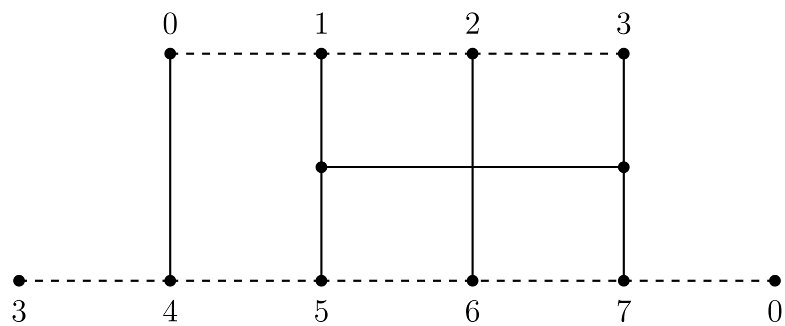

Figure A.20: Coverage provided by a 2-bar. This graph is 2-crossing-critical.

A.2 3-Stars

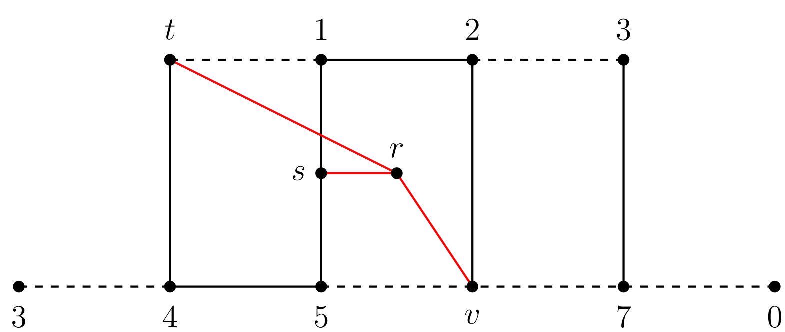

Figure A.21: Coverage provided by the 3-star in Figure 3.2.

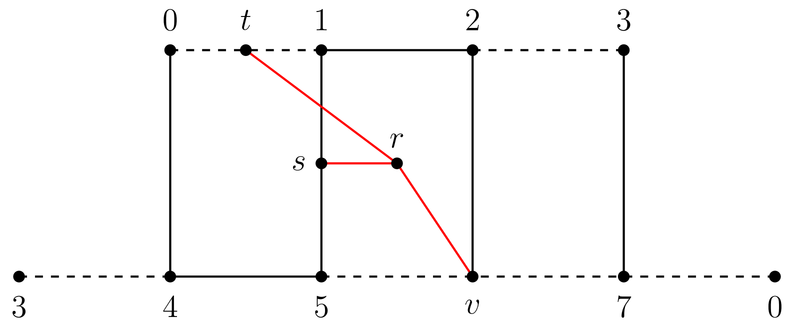

Figure A.22: Coverage provided by the 3-star in Figure 3.3.

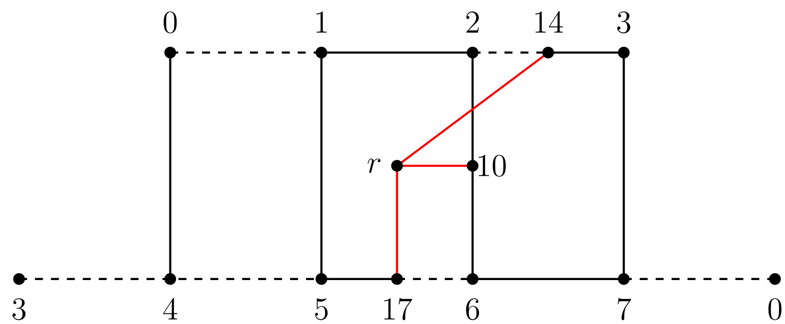

Figure A.23: Coverage provided by the 3-star in Figure 3.13.

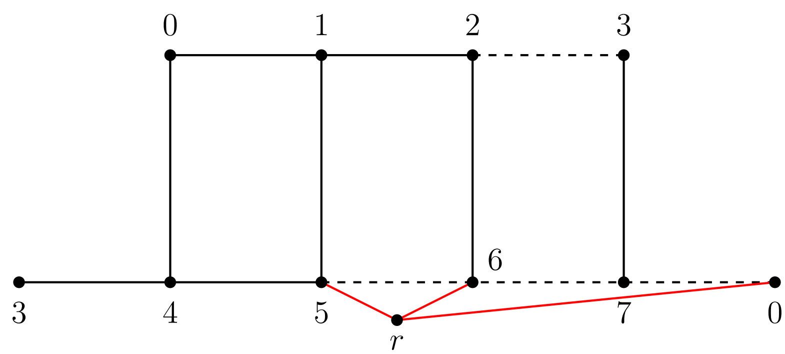

Figure A.24: Coverage provided by the 3-star in Figure 4.2.

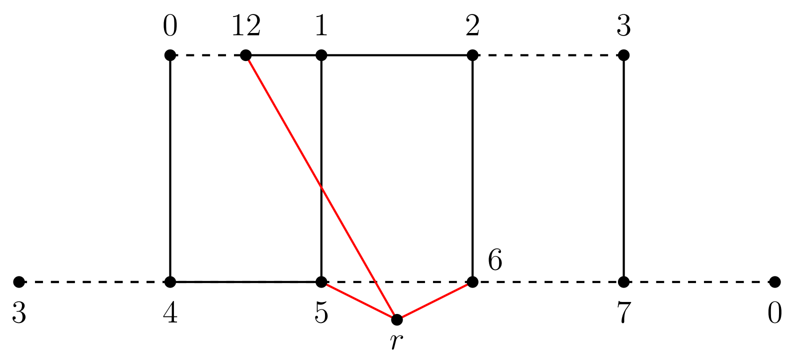

Figure A.25: Coverage provided by the 3-star when $t = t_1$ in Figure 4.3.

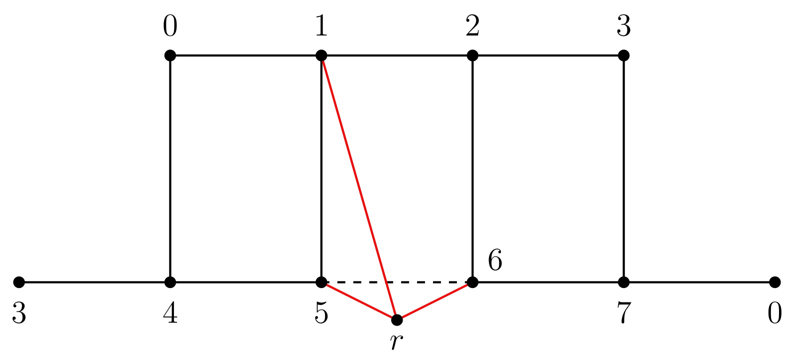

Figure A.26: Coverage provided by the 3-star when $t = t_3$ in Figure 4.3.

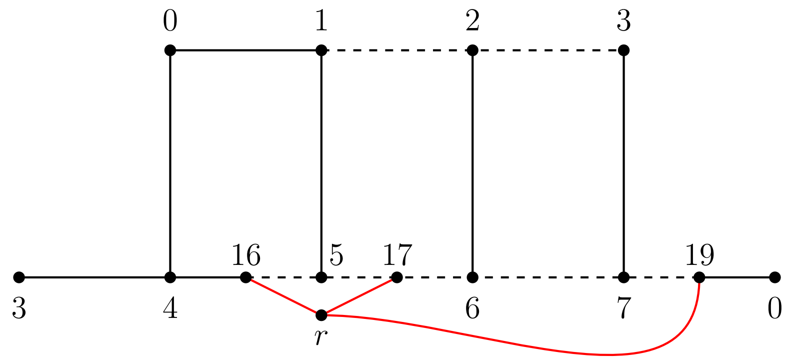

Figure A.27: Coverage provided by the 3-star in Figure 4.8.

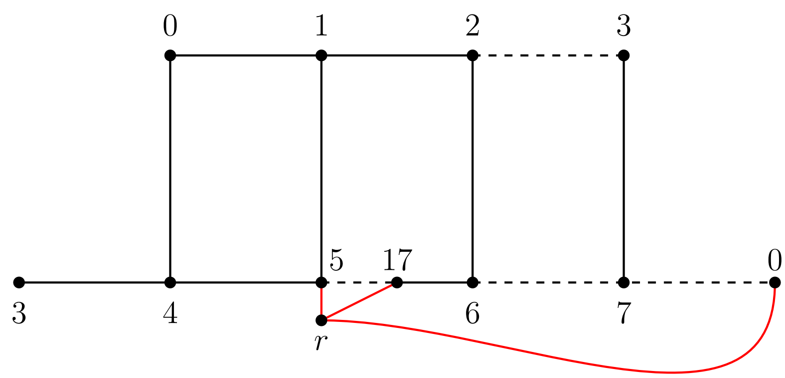

Figure A.28: Coverage provided by the 3-star in Figure 4.10, when $t = t_1$.

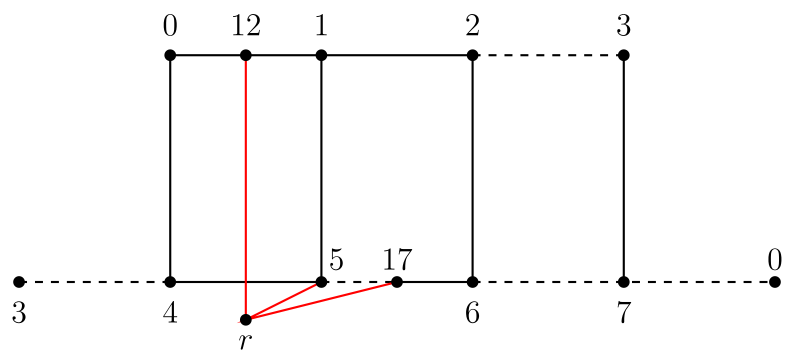

Figure A.29: Coverage provided by the 3-star in Figure 4.11.

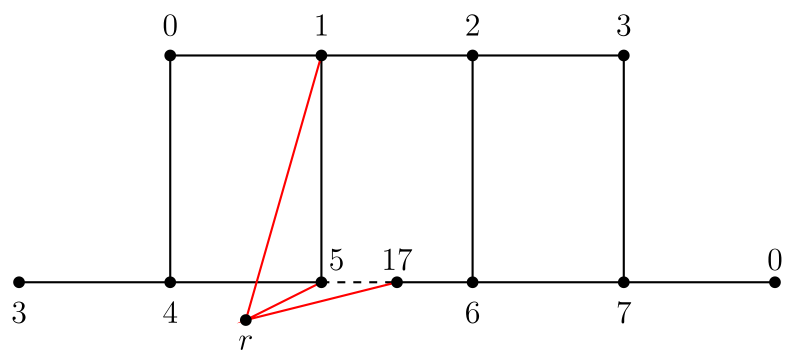

Figure A.30: Coverage provided by the 3-star in Figure 4.10 when $t = t_3$.

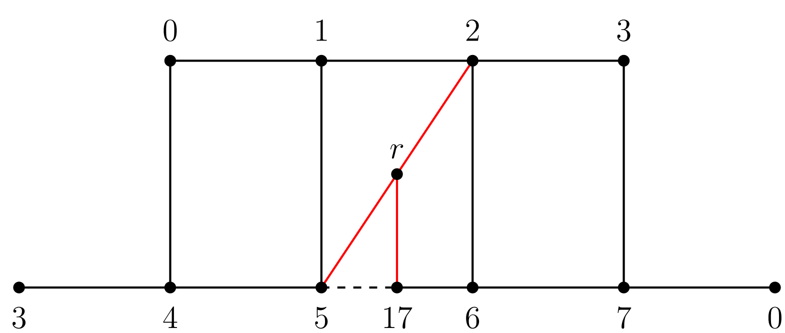

Figure A.31: Coverage provided by the 3-star in Figure 4.12 when $t = t_4$.

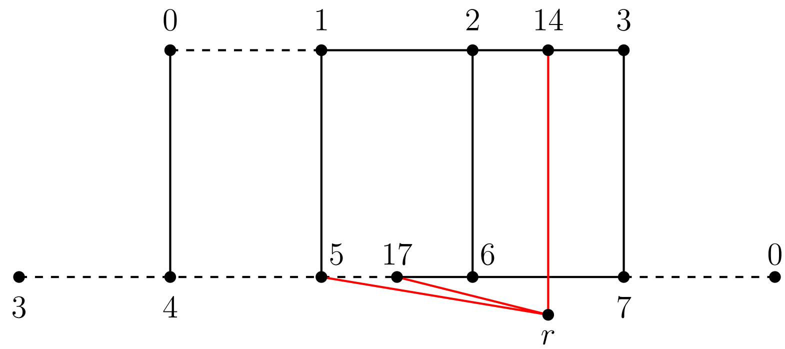

Figure A.32: Coverage provided by the 3-star in Figure 4.13 when $t = t_5$.

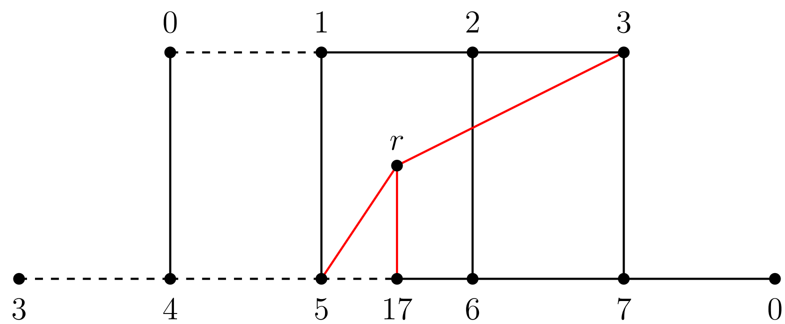

Figure A.33: Coverage provided by the 3-star in Figure 4.14, when $t = t_6$.

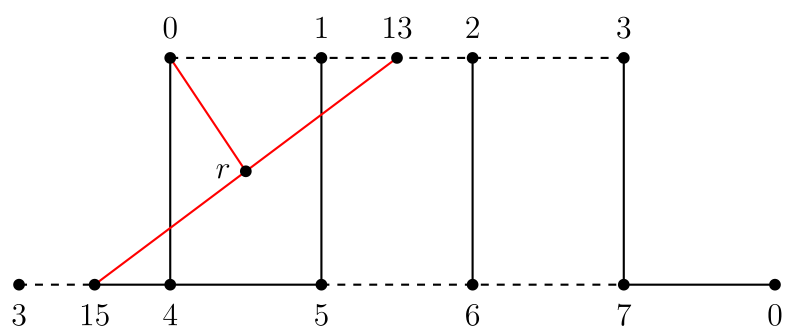

Figure A.34: Coverage provided by the 3-star in Figure 4.23.

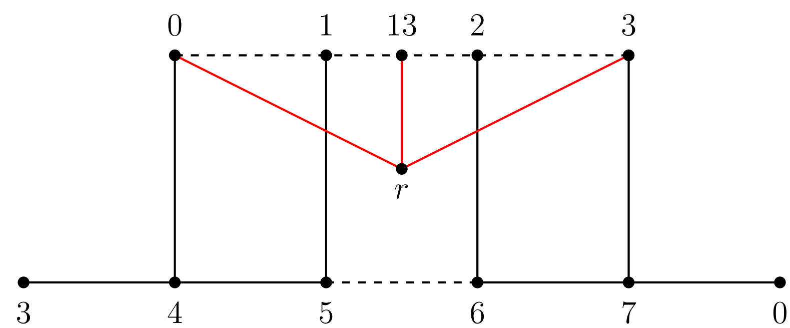

Figure A.35: Coverage provided by the 3-star in Figure 4.26.

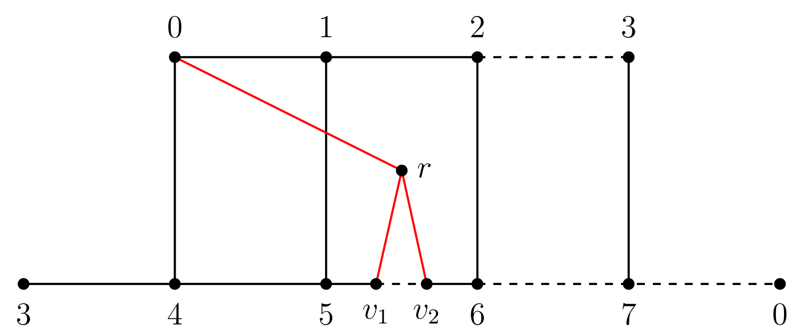

Figure A.36: Coverage provided by the 3-star in Figure 5.5.

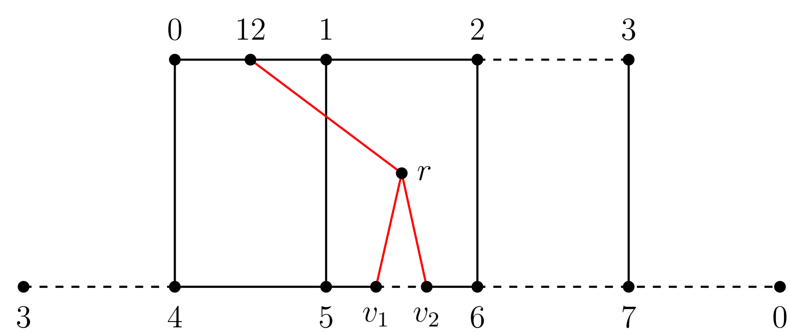

Figure A.37: Coverage provided by the 3-star in Figure 5.6.

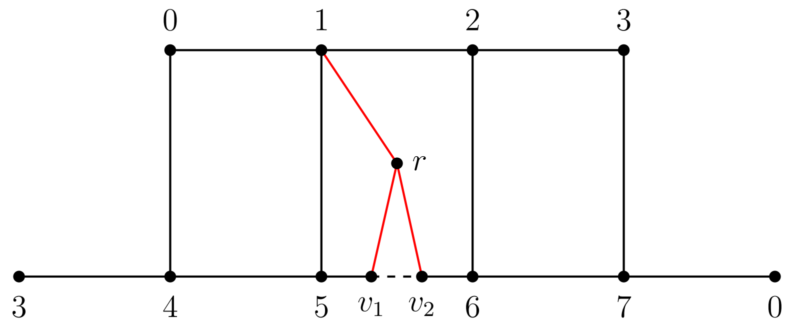

Figure A.38: Minimum coverage provided by the 3-star in Figure 5.7.

A.3 4-Stars

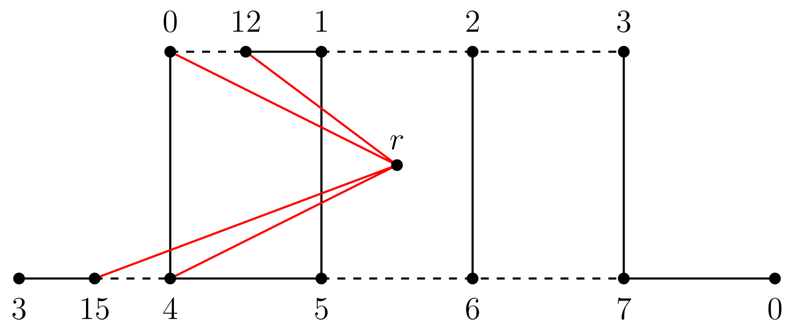

Figure A.39: Coverage provided by the 4-star in Figure 6.4.

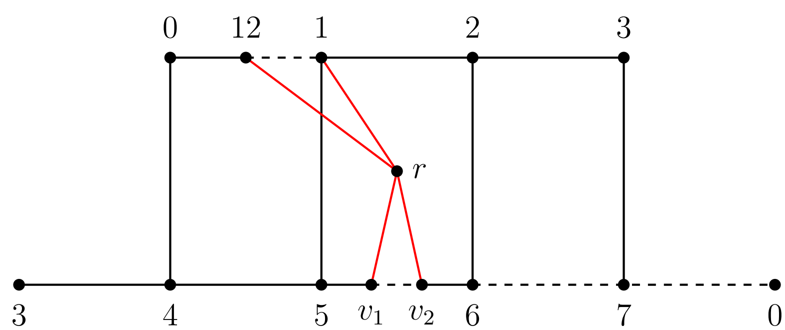

Figure A.40: Coverage provided by the 4-star in Figure 6.8.

A.4 4+-Trees

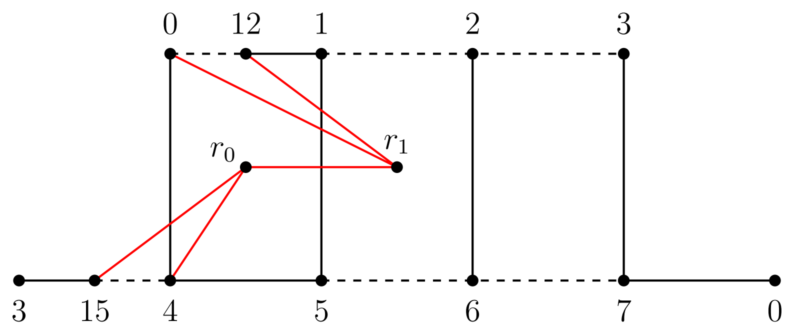

Figure A.41: Coverage provided by Figure 7.17.

A.5 Miscellaneous

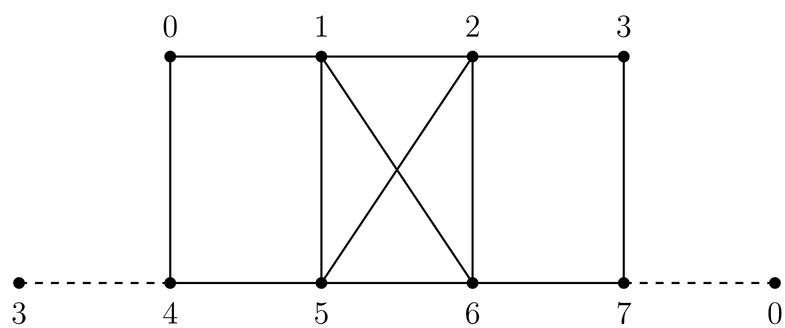

Figure A.42: A V_8 with two opposing diagonals inside of a quad. Covered edges are denoted with dashed lines.