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The best constant for an inequality related to the Mathieu series

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Abstract. Let $S(r)$ be the Mathieu series defined by

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2},\quad r\in(0,\infty),$$

which converges to $2\zeta(3)$ as $r\to 0+$. Here ζ is the Riemann zeta function. Let $b\in(0,\infty)$ be a constant. Our aim of this paper is to show that the inequality

$$S(r)<\frac{2\zeta(3)}{br^2+1},\quad r\in(0,\infty)$$

holds if and only if $b\le b^*:=(2\zeta(5))/\zeta(3)$. Hence b^* is the best constant satisfying this inequality. This best constant is related to the Taylor expansion of S at $r=0$.

1. Introduction

In 1890, French mathematician Émile Léonard Mathieu [5] first introduced the series $S(r)$, which is now called the Mathieu series, defined as

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2},\quad r\in(0,\infty). \tag{1.1}$$

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He conjectured that $S(r) < 1/r^2$ for all $r \in (0, \infty)$, and used this inequality in his work on elasticity of solid bodies. However, he could not prove this conjecture.

In 1952, Berg [2] proved this inequality, although his proof was difficult enough. It was Makai [4] who gave in 1957 an elementary proof and obtained the following estimates:

$$\frac{1}{r^2 + \frac{1}{2}} < S(r) < \frac{1}{r^2}, \quad r \in (0, \infty). \tag{1.2}$$

In 1998, Alzer, Brenner and Ruehr [1] showed that the best constants k_1 and k_2 in the two–sided estimations

$$\frac{1}{r^2 + k_1} < S(r) < \frac{1}{r^2 + k_2}, \quad r \in (0, \infty) \tag{1.3}$$

are given by $k_1 = 1/(2\zeta(3))$ and $k_2 = 1/6$, where $\zeta(s)$ is the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in (1, \infty). \tag{1.4}$$

In 2007, Hoorfar and Qi [3] gave the inequality

$$S(r) < \frac{1}{\sqrt{(r^2+1)^2 + 1} - 1}, \quad r \in (0, \infty). \tag{1.5}$$

By a simple calculation, the inequality

$$\frac{1}{\sqrt{(r^2+1)^2 + 1} - 1} < \frac{1}{r^2 + \frac{1}{6}}$$

holds if and only if $0 < r < \sqrt{\frac{23}{12}} = 1.38443\cdots$. Thus inequality (1.5) is better than the right-hand side inequality in (1.3) with $k_2 = 1/6$ only when $0 < r < \sqrt{\frac{23}{12}}$.

Till now, the Mathieu series has attracted increasing interest and attention. However, as far as the authors know, the upper bounds obtained for $S(r)$ never reflect the fact such that $S(r) \to 2\zeta(3) = 2.40411\cdots$ as $r \to 0+$. Indeed, the upper bound in (1.3) with $k_2 = 1/6$ converges to 6 which is greater than $2\zeta(3)$ as $r \to 0+$; the upper bound in (1.5) converges to $\sqrt{2}+1 = 2.41421\cdots$ which is also greater than $2\zeta(3)$ as $r \to 0+$. This is contrary to lower bounds for $S(r)$, since the lower bound in (1.3) with $k_1 = 1/(2\zeta(3))$ converges to $2\zeta(3)$ as $r \to 0+$.

In this paper, we would like to present an inequality for $S(r)$, which reflects the fact such that $S(r) \to 2\zeta(3)$ as $r \to 0+$. For this purpose, for a given constant $b \in (0,\infty)$, we consider the inequality of the form

$$S(r) < \frac{2\zeta(3)}{br^2+1}, \quad r \in \big(0,\infty\big). \tag{1.6}$$

Note that the right–hand side of inequality (1.6) converges to $2\zeta(3)$ as $r \to 0+$. In this sense, inequality (1.6) reflects the fact such that $S(r) \to 2\zeta(3)$ as $r \to 0+$.

Furthermore, the denominator of the right–hand side of (1.6) comes from (1.3). Indeed, the right–hand side of (1.6) can be rewritten as $\dfrac{(2\zeta(3)/b)}{r^2+(1/b)}$ whose denominator has the same form as those of (1.3).

Our aim of this paper is to determine completely the range of b satisfying inequality (1.6). Let us define the constant b^* by

$$b^* = \frac{2\zeta(5)}{\zeta(3)} = 1.72525\cdots . \tag{1.7}$$

Our main result is

Theorem 1.1. *Let $b \in (0,\infty)$ be a constant. Inequality* (1.6) *holds if and only if $b \le b^*$.*

This theorem shows that the best constant b satisfying inequality (1.6) is given by b^*. As indicated by Lemma 2.1 and the proof of Theorem 1.1 below, this best constant is related to the Taylor expansion of S at $r = 0$.

We emphasize that inequality (1.6) with $b = b^*$ is important for a small r. Indeed, as shown in Lemma 2.6 below, there exists a constant $r_0 \in (0,\frac{1}{2})$ such that the inequality

$$\frac{1}{\sqrt{(r^2+1)^2+1}-1} < \frac{2\zeta(3)}{b^*r^2+1}$$

holds if and only if $r \in (r_0,\infty)$. Reflecting inequality (1.5), the objective inequality (1.6) with $b = b^*$ has an advantage for only $r \in (0, r_0]$.

The paper is organized as follow: In Section 2, we give some preliminaries. In Section 3, we prove Theorem 1.1. In Section 4, we give a concluding remark to compare the three inequalities (1.3) with $k_2 = 1/6$, (1.5) and (1.6) with $b = b^*$.

2. Preliminaries

In this section, we give some preliminaries to prove Theorem 1.1. First we give the Taylor expansion of S at $r = 0$.

Lemma 2.1.

$$\lim_{r\to 0+} \frac{S(r) - 2\zeta(3)}{r^2} = -4\zeta(5).$$

Proof. This is clear, since

$$\frac{S(r) - 2\zeta(3)}{r^2} = -2\sum_{n=1}^{\infty} \frac{2n^2 + r^2}{n^3(n^2 + r^2)^2} \to -2\sum_{n=1}^{\infty} \frac{2n^2}{n^7} = -4\zeta(5) \quad (r \to 0+).$$

This completes the proof. □

Lemma 2.2. *Let $a \in (0, \infty)$ be a constant. If a function $g : [a, \infty) \to \mathbb{R}$ is strictly convex on $[a, \infty)$, then we have*

$$g(x) < \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} g(t)\,dt, \quad x \in [a + \frac{1}{2}, \infty).$$

Proof. Let $x \in [a + \frac{1}{2}, \infty)$ and $y \in [-\frac{1}{2}, \frac{1}{2}]$. Note that $a \le x + y, x - y$. Since g is strictly convex on $[a, \infty)$ and $x + y \ne x - y$ for $y \ne 0$, we have

$$g(x) = g\big(\frac{1}{2}(x+y) + \frac{1}{2}(x-y)\big) < \frac{1}{2}\,g\big(x+y\big) + \frac{1}{2}\,g(x-y), \quad y \in [-\frac{1}{2}, \frac{1}{2}] \setminus \{0\}.$$

Integrating this inequality with respect to y over $[-\frac{1}{2}, \frac{1}{2}]$, we conclude that

$$g(x) < \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} g\big(x + y\big)\,dy + \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} g(x - y)\,dy = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} g(t)\,dt.$$

This completes the proof. □

Lemma 2.3. *We have*

$$S(r) < \frac{1}{r^2 + \frac{1}{4}}, \quad r \in (0, \frac{1}{2}].$$

Remark 2.4. Milovanović and Pogány [6] showed this inequality for $r \in (0, \frac{\sqrt{3}}{2}]$. In this paper, it is sufficient to show this inequality for $r \in (0, \frac{1}{2}]$. In order to make this paper self–contained, we give a proof to this lemma.

Proof. Fix $r \in (0, \frac{1}{2}]$. Let

$$f(x) = \frac{2x}{(x^2 + r^2)^2}, \quad x \in (0, \infty).$$

Since

$$f''(x) = 24x \, \frac{x^2 - r^2}{(x^2 + r^2)^4}, \quad x \in (0, \infty),$$

we see that f is strictly convex on $[r, \infty)$. Note that $r + \frac{1}{2} \leq 1 \leq n$ for each positive integer n. By Lemma 2.2, we have

$$f(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(t) \, dt.$$

Thus

$$S(r) = \sum_{n=1}^{\infty} f(n) < \int_{\frac{1}{2}}^{\infty} f(t) \, dt = \left[-(t^2 + r^2)^{-1} \right]_{\frac{1}{2}}^{\infty} = \frac{1}{r^2 + \frac{1}{4}}.$$

This completes the proof. □

Lemma 2.5. *We have*

$$S'(r) < -\frac{8\zeta(5) r}{(1 + r^2)^3}, \quad r \in (0, \infty).$$

Proof. Note that

$$S'(r) = -8r \sum_{n=1}^{\infty} \frac{n}{(n^2 + r^2)^3}, \quad r \in (0, \infty).$$

Since

$$\frac{n}{(n^2+r^2)^3} > \frac{1}{n^5(1+r^2)^3}, \quad n \in \{2,3,4,\cdots\},\ r \in (0,\infty),$$

we have

$$S'(r) = -8r\sum_{n=1}^{\infty}\frac{n}{(n^2+r^2)^3} < -8r\sum_{n=1}^{\infty}\frac{1}{n^5}\frac{1}{(1+r^2)^3} = -\frac{8\zeta(5)r}{(1+r^2)^3}, \quad r \in (0,\infty).$$

This completes the proof. □

In the following, we often use the following estimates:

$$1.20205 < \zeta(3) < 1.20206, \quad 1.03692 < \zeta(5) < 1.03693. \tag{2.1}$$

Lemma 2.6. *There exists a constant $r_0 \in (0, \frac{1}{2})$ such that the inequality*

$$\frac{1}{\sqrt{(r^2+1)^2+1}-1} < \frac{2\zeta(3)}{b^*r^2+1}$$

holds if and only if $r \in (r_0,\infty)$.

Proof. Let $r \in (0,\infty)$. By (1.7), we have

$$\begin{aligned}&\frac{2\zeta(3)}{b^*r^2+1} - \frac{1}{\sqrt{(r^2+1)^2+1}-1}\\ =\ &\frac{2\zeta(3)^2\sqrt{(r^2+1)^2+1}-2\zeta(5)r^2-2\zeta(3)^2-\zeta(3)}{(2\zeta(5)r^2+\zeta(3))(\sqrt{(r^2+1)^2+1}-1)}.\end{aligned}$$

Let $t = r^2 \in (0,\infty)$. We set

$$f(t) = 2\zeta(3)^2\sqrt{(t+1)^2+1} - 2\zeta(5)t - 2\zeta(3)^2 - \zeta(3), \quad t \in (0,\infty).$$

We consider the range of $t \in (0,\infty)$ such that $f(t) > 0$. We see that $f(t) > 0$ if and only if the inequality

$$4\zeta(3)^4[(t+1)^2+1] > [2\zeta(5)t + 2\zeta(3)^2 + \zeta(3)]^2$$

holds. This inequality is equivalent to the one

$$at^2 + bt + c > 0,$$

where a, b, c are the constants given by

$$\begin{aligned} a &= 4(\zeta(3)^4 - \zeta(5)^2), \\ b &= 4\zeta(3)(2\zeta(3)^3 - 2\zeta(5)\zeta(3) - \zeta(5)), \\ c &= \zeta(3)^2(4\zeta(3)^2 - 4\zeta(3) - 1). \end{aligned}$$

By (2.1), we have $\zeta(3)^2 - \zeta(5) > 1.20205^2 - 1.03693 > 0.40799$, so that $a > 0$. Since $0 < \zeta(3) < 1.20206 < \frac{1+\sqrt{2}}{2} =: \gamma$ and γ is a solution of the equation $4s^2 - 4s - 1 = 0$, we see that $c < 0$; (although we can determine the sign of b, it is not necessary in this proof.) Hence, reflecting $t > 0$, we see that the inequality $at^2 + bt + c > 0$ holds if and only if

$$t > \frac{-b + \sqrt{b^2 - 4ac}}{2a} > 0.$$

Letting

$$r_0 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)^{1/2},$$

we conclude that $f(r^2) > 0$ if and only if $r \in (r_0, \infty)$.

Finally, when $r = \frac{1}{2}$, we have $t = \frac{1}{4}$. Note that $\sqrt{41} > 6.40312$. By (2.1), we have

$$\begin{aligned} f\left(\frac{1}{4}\right) &= \left(\frac{\sqrt{41}}{2} - 2\right)\zeta(3)^2 - \frac{\zeta(5)}{2} - \zeta(3) \\ &> \left(\frac{6.40312}{2} - 2\right) \cdot 1.20205^2 - \frac{1.03693}{2} - 1.20206 > 0.01563. \end{aligned}$$

This implies that $r_0 < \frac{1}{2}$. This completes the proof. □

Proposition 2.7. *Let $b \in (0, \zeta(5)]$. Then the inequality*

$$S(r) < \frac{2\zeta(3)}{br^2 + 1}, \quad r \in \left(0, \frac{1}{2}\right]$$

holds.

Proof. Let $b \in (0, \zeta(5)]$. By Lemmas 2.3 and 2.5, we have, for $r \in (0, \frac{1}{2})$,

$$\begin{aligned}\frac{d}{dr}\big[(br^2+1)S(r)\big] &= 2brS(r) + (br^2+1)S'(r) \\ &< 2br\,\frac{1}{r^2+\frac{1}{4}} + (br^2+1)\big(-\frac{8\zeta(5)r}{(1+r^2)^3}\big) \\ &= \frac{8r}{(4r^2+1)(1+r^2)^3}\big[\,b(1+r^2)^3 - \zeta(5)(br^2+1)(4r^2+1)\big] \\ &= \frac{8r}{(4r^2+1)(1+r^2)^3}\,g(r),\end{aligned}$$

where

$$g(r) := br^6 + b(3-4\zeta(5))r^4 + [3b - \zeta(5)(b+4)]r^2 + b - \zeta(5), \quad r \in \mathbb{R}.$$

We show that $g(r) \le 0$ for $r \in [0, \frac{1}{2}]$. Then, since

$$\frac{d}{dr}\big[(br^2+1)S(r)\big] < 0, \quad r \in (0, \frac{1}{2}),$$

we conclude that $(br^2+1)S(r) < S(0+) = 2\zeta(3)$ for $r \in (0, \frac{1}{2}]$.

It remains to show that $g(r) \le 0$ for $r \in [0, \frac{1}{2}]$. Set

$$h(t) := bt^3 + b(3-4\zeta(5))t^2 + [3b - \zeta(5)(b+4)]t + b - \zeta(5), \quad t \in \mathbb{R}.$$

Then

$$\begin{aligned}h'(t) &= 3bt^2 + 2b(3-4\zeta(5))t + 3b - \zeta(5)(b+4) \\ &= 3b\big(t - \frac{4\zeta(5)-3}{3}\big)^2 - 3b\big(\frac{4\zeta(5)-3}{3}\big)^2 + 3b - \zeta(5)(b+4).\end{aligned}$$

By (2.1), note that

$$\frac{4\zeta(5)-3}{3} > 0.38256 > \frac{1}{4}.$$

Thus, $h'(t) \le h'(0)$ for $t \in [0, \frac{1}{4}]$. Since

$$\begin{aligned}h'(0) &= 3b - \zeta(5)(b+4) = b(3-\zeta(5)) - 4\zeta(5) \\ &\le \zeta(5)(3-\zeta(5)) - 4\zeta(5) = -\zeta(5)(1+\zeta(5)) < 0,\end{aligned}$$

we have $h'(t) < 0$ for $t \in [0, \frac{1}{4}]$, and h is strictly decreasing on $[0, \frac{1}{4}]$. Therefore,

$$h(t) \le h(0) = b - \zeta(5) \le 0.$$

This implies that $g(r) \le 0$ for $r \in [0, \frac{1}{2}]$. This completes the proof. □

Recall that b^* is the constant of (1.7). By (2.1), we have

$$b^* < \frac{2 \cdot 1.03693}{1.20205} < 1.72527. \tag{2.2}$$

Thus

$$\frac{\zeta(5)}{8} b^* + \frac{\zeta(5)}{2} = \frac{\zeta(5)}{8}(b^* + 4) < \frac{1.03693}{8} \cdot 5.72527 < 0.74209.$$

Since

$$\frac{125}{64}\zeta(3) > 2.34775,$$

we have

$$\frac{125}{64}\zeta(3) - \frac{\zeta(5)}{8} b^* - \frac{\zeta(5)}{2} > 0.$$

Hence,

$$\eta(a) := \frac{\zeta(5)(\frac{a}{2} + 2)}{\frac{125}{64}\zeta(3) - \frac{\zeta(5)}{8} a - \frac{\zeta(5)}{2}} > 0, \quad a \in (0, b^*]. \tag{2.3}$$

This inequality is important in the following proposition.

Proposition 2.8. *Let $a \in (0, b^*]$. Assume that the inequality*

$$S(r) < \frac{2\zeta(3)}{ar^2 + 1}, \quad r \in (0, \frac{1}{2}]$$

holds. Then, for a constant b with

$$0 < b \leq \min\Big\{ b^*, \eta(a) \Big\}, \tag{2.4}$$

the inequality

$$S(r) < \frac{2\zeta(3)}{br^2 + 1}, \quad r \in (0, \frac{1}{2}]$$

holds.

Proof. Let b be a constant with (2.4). By Lemma 2.5, we have, for $r \in (0, \frac{1}{2})$,

$$\begin{aligned} \frac{d}{dr}\big[(br^2 + 1)S(r)\big] &= 2brS(r) + (br^2 + 1)S'(r) \\ &< 2br \, \frac{2\zeta(3)}{ar^2 + 1} + (br^2 + 1)\big(-\frac{8\zeta(5)r}{(1 + r^2)^3}\big) \\ &= \frac{4r}{(ar^2 + 1)(1 + r^2)^3}\big[b\zeta(3)(1 + r^2)^3 - 2\zeta(5)(ar^2 + 1)(br^2 + 1)\big] \\ &= \frac{4r}{(ar^2 + 1)(1 + r^2)^3} \, g(r), \end{aligned}$$

where

$$g(r) := b\zeta(3)r^6+b(3\zeta(3)-2a\zeta(5))r^4+[3b\zeta(3)-2\zeta(5)(a+b)]r^2+b\zeta(3)-2\zeta(5)$$

for $r \in \mathbb{R}$. We show that $g(r) \leq 0$ for $r \in [0, \frac{1}{2}]$. Then, since

$$\frac{d}{dr}\big[(br^2+1)S(r)\big] < 0, \quad r \in (0, \frac{1}{2}),$$

we conclude that $(br^2+1)S(r) < S(0+) = 2\zeta(3)$ for $r \in (0, \frac{1}{2}]$.

It remains to show that $g(r) \leq 0$ for $r \in [0, \frac{1}{2}]$. Set

$$h(t) := b\zeta(3)t^3+b(3\zeta(3)-2a\zeta(5))t^2+[3b\zeta(3)-2\zeta(5)(a+b)]t+b\zeta(3)-2\zeta(5)$$

for $t \in \mathbb{R}$. Then

$$h''(t) = 6b\zeta(3)t + 2b(3\zeta(3) - 2a\zeta(5)).$$

Since $a \leq b^*$, we note by (2.1) and (2.2) that

$$3\zeta(3) - 2a\zeta(5) \geq 3\zeta(3) - 2b^*\zeta(5) > 3 \cdot 1.20205 - 2 \cdot 1.72527 \cdot 1.03693 > 0.$$

Thus h is convex on $[0, \frac{1}{4}]$. Hence,

$$h(t) \leq \max\big\{h(0), h(\frac{1}{4})\big\}, \quad t \in [0, \frac{1}{4}].$$

Note that

$$\begin{aligned} h(0) &= b\zeta(3) - 2\zeta(5) \leq 0, \\ h(\frac{1}{4}) &= b\big(\frac{125}{64}\zeta(3) - \frac{\zeta(5)}{8}a - \frac{\zeta(5)}{2}\big) - \zeta(5)(\frac{a}{2}+2) \\ &= \big(\frac{125}{64}\zeta(3) - \frac{\zeta(5)}{8}a - \frac{\zeta(5)}{2}\big)\,(b - \eta(a)) \leq 0 \end{aligned}$$

by (2.3) and (2.4). Hence, $h(t) \leq 0$ for $t \in [0, \frac{1}{4}]$. This implies that $g(r) \leq 0$ for $r \in [0, \frac{1}{2}]$. This completes the proof. □

3. Proof of Theorem 1.1

Proof of Theorem 1.1. First, we show only if part. Let $b \in (0, \infty)$ be a constant. Assume that inequality (1.6) holds. From inequality (1.6), we have

$$S(r) - 2\zeta(3) < \frac{2\zeta(3)}{br^2+1} - 2\zeta(3) = -\frac{2\zeta(3)br^2}{br^2+1}, \quad r \in (0, \infty).$$

Thus we have

$$\frac{S(r)-2\zeta(3)}{r^2} < -\frac{2\zeta(3)b}{br^2+1} \quad r \in (0,\infty).$$

Letting $r \to 0+$ and using Lemma 2.1, we have

$$-4\zeta(5) \leq -2\zeta(3)b.$$

Thus we conclude that $b \leq (2\zeta(5))/\zeta(3) = b^*$. Hence we have shown only if part.

Next we prove if part. By (2.1), we have

$$\frac{125}{64}\zeta(3) < 2.34778, \quad \frac{\zeta(5)}{2} > 0.51846. \tag{3.1}$$

On the other hand, by (2.1) and Proposition 2.7, we see that the assumption of Proposition 2.8 is fulfilled for $a = 1.03692$. For this a, we have, by (2.1),

$$\zeta(5)(\frac{a}{2}+2) > 2.61144, \quad \frac{\zeta(5)}{8}a > 0.13440.$$

Thus, by (3.1), we obtain

$$\eta(a) = \frac{\zeta(5)(\frac{a}{2}+2)}{\frac{125}{64}\zeta(3) - \frac{\zeta(5)}{8}a - \frac{\zeta(5)}{2}} > \frac{2.61144}{2.34778-0.13440-0.51846} > 1.54074.$$

Since $1.54074 < b^*$ by (2.2), Proposition 2.8 implies that the inequality

$$S(r) < \frac{2\zeta(3)}{br^2+1}, \quad r \in (0, \frac{1}{2}]$$

holds for $b \in (0, 1.54074]$.

Now, let $\hat{a} = 1.54074$. For this $\hat{a}$, we have, by (2.1),

$$\zeta(5)(\frac{\hat{a}}{2}+2) > 2.87265, \quad \frac{\zeta(5)}{8}\hat{a} > 0.19970.$$

Thus, by (3.1), we obtain

$$\eta(\hat{a}) = \frac{\zeta(5)(\frac{\hat{a}}{2}+2)}{\frac{125}{64}\zeta(3) - \frac{\zeta(5)}{8}\hat{a} - \frac{\zeta(5)}{2}} > \frac{2.87265}{2.34778-0.19970-0.51846} > 1.76277.$$

Since $1.76277 > b^*$ by (2.2), Proposition 2.8 implies that the inequality

$$S(r) < \frac{2\zeta(3)}{br^2+1}, \quad r \in (0, \frac{1}{2}]$$

holds for $b \in (0, b^*]$. Hence, reflecting Lemma 2.6 and inequality (1.5), we have shown if part. This completes the proof. □

4. A concluding remark

In this section, we give a concluding remark for the three inequalities (1.3) with $k_2 = 1/6$, (1.5) and (1.6) with $b = b^*$. Let us define the three functions f, g, h on $(0, \infty)$ as follows:

$$\begin{aligned} f(r) &= \frac{1}{r^2+\frac{1}{6}}, \\ g(r) &= \frac{1}{\sqrt{(r^2+1)^2+1}-1}, \\ h(r) &= \frac{2\zeta(3)}{b^* r^2+1}. \end{aligned}$$

Note that these functions are different from ones used in Section 2. Then, we have considered the following inequalities:

$$S(r) < f(r), \tag{4.1}$$

$$S(r) < g(r), \tag{4.2}$$

$$S(r) < h(r). \tag{4.3}$$

Inequalities (4.1), (4.2) and (4.3) appear, respectively, in (1.3) with $k_2 = 1/6$, (1.5) and (1.6) with $b = b^*$. Dividing the range of $r \in (0, \infty)$, we give a table to show which inequality is best among (4.1), (4.2) and (4.3). Let r_0 be the constant of Lemma 2.6, and

$$r_1 = \sqrt{\frac{\zeta(3)(3-\zeta(3))}{6(\zeta(3)^2-\zeta(5))}} = 0.93958\cdots, \quad r_2 = \sqrt{\frac{23}{12}}.$$

Note that $0 < r_0 < r_1 < r_2$ and that

$$\begin{cases} r_0 \text{ is a unique solution of the equation } g(r) = h(r), \\ r_1 \text{ is a unique solution of the equation } f(r) = h(r), \\ r_2 \text{ is a unique solution of the equation } f(r) = g(r). \end{cases}$$

Then we have the following table:

The range of r	Inequality	The best one among (4.1), (4.2) and (4.3)
$0 < r < r_0$	$h(r) < g(r) < f(r)$	(4.3)
$r_0 < r < r_1$	$g(r) < h(r) < f(r)$	(4.2)
$r_1 < r < r_2$	$g(r) < f(r) < h(r)$	(4.2)
$r_2 < r$	$f(r) < g(r) < h(r)$	(4.1)

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