A portfolio stock selection model based on expected utility, entropy and variance

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Abstract

In the context of investment decision-making, the selection of stocks is important for a successful construction of portfolios. In this paper the expected utility, entropy and variance (EU-EV) model is applied for stock selection, which can be used as preselection model for mean-variance portfolio optimization problems. Based on the EU-EV risk, stocks are ranked and the best ranked stocks with lower risk are selected in order to form subsets of stocks, which are then used to construct portfolios. The EU-EV model is applied to the PSI 20 index, to the Euro Stoxx 50 index and to the Nasdaq 100 index. Subsets of selected stocks are analysed and their portfolios' efficiencies are compared with those of the portfolios obtained from the whole set of stocks using the mean-variance model. The results reveal that the EU-EV model is an adequate stock selection model for building up efficient portfolios with a lower number of stocks.

Keywords: decision analysis, portfolio optimization, stock selection, risk analysis, expected utility, entropy and variance model

1. Introduction

In modern portfolio theory introduced by Markowitz (Markowitz (1952), Markowitz (2000)), mean-variance analysis is used in investment decision-making. According to that theory, portfolios, which are weighted combinations of their component stocks, are assessed and assembled by minimizing the risk, expressed by variance, for a given expected return or maximizing the expected return for a given risk. In this sense, the optima portfolios constitute the efficient set of portfolios. Depending on the risk aversion of the investor, a portfolio lying on the efficient frontier can then be chosen to invest in. The risk aversion is related to the level of risk the investor is willing to undertake, where a higher risk level is associated with a higher expected return. The diversification in building up portfolios is a strategy to reduce their risks, since increasing the number of stocks in a portfolio can lead to a decreasing portfolio variance. However, the problem of determining the number of stocks to invest in or the selection of those stocks is not directly addressed with the mean-variance analysis. For example it would be useful to have an insight in advance about the subsets of stocks that are worthwhile to invest in and regarding diversification to have an idea about the number of stocks to construct optimal portfolios. Since the mean-variance model was presented by Markowitz, various other portfolio stock selection methods have been proposed in the literature and few of them attempt to tackle the issue of the particular selection of stocks. The cardinality-constrained portfolio optimization limits the maximum number of stocks for investment, allowing the mean-variance model to select at most the imposed maximum number of stocks. Due to the introduction of a cardinality constraint the portfolio selection problem becomes more complex, implied by the higher combinatorial possibilities (Gao & Li, 2013). Various solution methods have been proposed for cardinality-constrained portfolio optimization, including heuristics or relaxation methods (see e.g. Chang, Meade, Beasley, & Sharaiha (2000), Deng, Lin, & Lo (2012), or Gao & Li (2013) and Leung & Wang (2022) and references therein). Other portfolio stock selection methods deal also with the problem of assessing risk in an adequate way, using other risk or uncertainty measures than variance. Ortobelli, Rachev, Stoyanov, Fabozzi, & Biglova (2005) emphasized that risk cannot be assessed by measuring only the uncertainty and popular uncertainty measures, such as the standard deviation or variance, are not always adequate as a proxy for risk (Rachev, Ortobelli, Stoyanov, Fabozzi, & Biglova, 2018). Even so there are cases where the variance can serve as an index for risk as discussed in Levi (1992). A popular measure used in several models to assess risk is entropy, which has the advantage that it can be computed from nonmetric data and is free from an assumption concerning the underlying distribution. In finance and economic literature one can find entropy models and measures to model an uncertain environment and to obtain optimal economic decisions in modelling portfolio investment risk, e.g the mean-entropy model used in Philippatos & Wilson (1972) and studied in Philippatos & Gressis (1975), the logarithmic expectation entropy model (Yin, 2019), or the adaptive entropy model (Song & Chan, 2020), which incorporates entropy into the mean-variance model. Bera & Park (2008) used a cross-entropy measure for optimal portfolio diversification. Models based on entropy and also on higher order moments are the mean-variance-skewness-entropy model (Usta & Kantar, 2011) and the mean-variance-skewness-kurtosis-entropy model (Aksarayli & Pala, 2018). Mercurio, Wu, & Xie (2020) introduced the return-entropy portfolio optimization problems, where a mean-entropy objective function is used.

Portfolio selection problems were also discussed for fuzzy environments. Recently, in Georgescu & Fono (2019), a possibilistic porfolio choice problem using possibilistic expected utility was presented, where the return of a risky asset is a fuzzy number. Huang (2008b) proposed fuzzy mean-semivariance models for portfolio selection and Li, Zhang, & Xu (2015) developed a fuzzy portfolio selection model with background risk, based on the definitions of the possibilistic return and possibilistic risk. Parra, Terol, & Uria (2001) developed a fuzzy goal programming approach for the portfolio selection problem. A mean-variance-skewness model for portfolio selection considering fuzzy returns was applied in Li, Qin, & Kar (2010). Models based on fuzzy set theory, including entropy, were presented in Jana, Roy, & Mazumder (2009), in Huang (2008a), where mean-entropy models for fuzzy portfolio selection were proposed, and in Qin, Li, & Ji (2009) fuzzy cross-entropy models were considered. Galankashi, Mokhatab Rafiei, & Ghezelbash (2020) developed a fuzzy analytic network process to assess and select portfolios and presented a literature review about portfolio selection models investigating the portfolio selection criteria.

Other portfolio optimization problems are based on uncertainty theory. Mehralizade, Amini, Gildeh, & Ahmadzade (2020) considered the uncertain random portfolio selection problem and a new corresponding risk criterion. In Li, Sun, Aw, & Teo (2019) a new uncertain risk measure for the modelling of investment risk was defined and an uncertain portfolio optimization model was formulated. In the context of uncertainty theory, stock selection models based on the entropy measure are the mean-entropy-skewness models (Bhattacharyya, Chatterjee, & Samarjit, 2013) and the mean-variance-entropy model (Li & Zhang, 2021).

Other extensions of the mean-variance model were proposed in Xia, Liu, Wang, & Lai (2000), where the portfolio selection model is based on an order of expected returns of securities, and in Dai & Wang (2019), where sparse and robust mean-variance portfolio optimization problems were introduced.

Also machine learning methodologies have been employed in the research on stock selection and portfolio optimization. For example, Liu & Yeh (2017) used neural networks to build stock selection decision support systems. Min, Dong, Liu, & Gong (2021) developed hybrid robust portfolio models, introducing a trade-off parameter to adjust the portfolio optimism level and used machine learning algorithms including long short-term memory (LSTM) and eXtreme Gradient Boosting (XGBoost) to evaluate the potential market movements and provide forecasting information to generate the parameter for modeling. Huang (2012) used support vector regression, together with genetic algorithms, to develop a methodology for effective stock selection, where top-ranked stocks are selected to form a portfolio. This can be seen as a preselection method for optimal portfolio construction. In fact, the preselection of stocks is relevant for portfolio optimization. For example, Chang, Yang, & Chang (2009) analysed and solved portfolio optimization problems with different risk measures using genetic algorithms and concluded that portfolios with a smaller number of assets (one third of the total assets) outperform those containing more assets. Several preselection methods, where assets or stocks are preselected before forming optimal portfolios, were considered in the research context of portfolio theory. Wang, Li, Zhang, & Liu (2020) explored the preselection process of assets using a deep LSTM method in order to obtain high-quality inputs for an optimal portfolio formation and proposed a combined method of preselection and mean-variance model. Paiva, Cardoso, Hanaoka, & Duarte (2019) applied the support vector machine (SVM) method, a classifier based on machine learning, to classify assets that reach a certain target of gain and select the best assets to compose optimal investment portfolios with the mean-variance method. Hai & Min (2021) designed a machine-learning based preselection method, using random forests (RF) and SVM, for picking out high-quality risky assets. Lozza, Shalit, & Fabozzi (2013) used a preselection technique to reduce the dimensionality of large scale portfolio problems, where stocks that present the highest Rachev ratio, a reward-risk performance measure depending on the conditional value-at-risk, are chosen before optimizing the portfolio. Qu, Zhou, Xiao, Liang, & Suganthan (2017) proposed two asset preselection procedures for large-scale portfolio optimization that consider return and risk of individual asset and pairwise correlation to remove assets that may not potentially be selected into any portfolio. Chen, Zhang, Mehlawat, & Jia (2021) developed a novel portfolio optimization model based on machine learning, using extreme gradient boosting for preselection of stocks with higher potential returns before employing the mean - variance model. Yang, Feng, & Qiu (2017) proposed the expected utility and entropy (EU-E) model for stock selection. There, the authors show that the efficient portfolios constructed from smaller subsets of stocks selected with their model from the total set of stocks have almost the same efficient frontier than the initial set. Marasović, Kalinić, & Jerković (2021) and Marasović & Kalinić (2019) applied the EU-E model to stock selection for different markets and managed to reduce the number of stocks in a given capital market by 50% without changing majorly the properties of efficient portfolios.

The purpose of this paper is to present a new stock selection model based on expected utility, entropy and variance, the EU-EV model, which can be used as stock preselection model for optimal portfolio construction. The paper is organized as follows. In section 2, the EU-EV risk model is presented and its application to the modelling of stock investment risks and to stock selection is proposed. In section 3, the model is applied to the selection of stock components of the PSI 20 index. The stocks are ranked using the EU-EV risk measure and the best ranked stocks with lowest EU-EV risk are selected. Subsets with different numbers of stocks are formed and optimal portfolios are constructed applying the mean-variance model. The efficient frontiers are compared with the efficient frontier obtained from the whole set of stocks. Furthermore, the results are also compared with the solutions obtained by the cardinality-constrained mean-variance model. In section 4, the EU-EV risk measure and the mean-variance model are applied to the Euro Stoxx 50 index and to the Nadaq 100 index. The numerical experiments in sections 3 and 4 were done in Matlab. The Conclusions are presented in section 4. The appendix contains information about stocks used in section 4.

2. EU-EV stock selection model

The EU-EV model, that will be presented in this section, can be adapted and applied to stock selection for the construction of portfolios. In order to set up the EU-EV model for stock selection, consider a portfolio or a set of stocks $S = \{S_1, \ldots, S_I\}$ and let a_i be the action of selecting stock S_i, $i = 1, \ldots, I$, so that the action space is given by $A = \{a_1, \ldots, a_I\}$. The returns of stock S_i collected over T previous days will be denoted by $r_{i1}, \ldots, r_{iT}$ and the return at day t, where $t = 1, \ldots, T$, from stock S_i is represented by r_{it}. Setting

$$
\begin{aligned}
r_{\min} &= \min_{1 \le i \le I} \{r_{i1}, \ldots, r_{iT}\} \\
r_{\max} &= \max_{1 \le i \le I} \{r_{i1}, \ldots, r_{iT}\},
\end{aligned}
$$

then the interval $[r_{\min}, r_{\max}]$ contains all returns of the stocks S_i, $i = 1, \ldots, I$. In order to determine the frequency distribution of stock returns, the interval $[r_{\min}, r_{\max}]$ is divided into N subintervals, or bins, of equal lengths: $[r_0, r_1)$, $[r_1, r_2)$,…, $[r_{N-1}, r_N]$, where $r_{\min} = r_0 < r_1 < \cdots < r_{N-1} < r_N = r_{\max}$. The subintervals will be denoted by J_n, $n = 1, \ldots, N$. The length of each subinterval is given by

$$\Delta = \frac{r_{\max} - r_{\min}}{N}$$

and the subintervals can be constructed using

$$J_n = \begin{cases} [r_{n-1}, r_{n-1} + \Delta), & n = 1, \ldots, N-1, \\ [r_{n-1}, r_n], & n = N. \end{cases} \tag{1}$$

The relative frequency of the return of stock S_i in the subinterval J_n is given by

$$p_{in} = \frac{|\{r_{it} \in J_n : t = 1, \ldots, T\}|}{T}, \tag{2}$$

where $|\cdot|$ denotes the number of elements in the set. The expected return of stock S_i from the subinterval J_n is estimated by

$$x_{in} = \frac{1}{|\{r_{it} \in J_n : t = 1, \ldots, T\}|} \sum_{\substack{r_{it} \in J_n \\ t=1,\ldots,T}} r_{it}. \tag{3}$$

The EU-EV model was proposed in Brito (2020) for risk decision making. The model depends on entropy and variance as uncertainty risk factors, which are combined with expected utility as preference factor using a trade-off parameter. The model can be seen as an extension of the EU-E decision model introduced in Yang & Qiu (2005), since it uses the variance as additional risk factor.

Let $G(\Theta, A, u)$ be a decision analysis model, where Θ is the state space, A, the action space, and $u = u(X(a, \theta))$ is the utility function, a non-decreasing function, with $X(a, \theta)$ being the outcome corresponding to state $\theta \in \Theta$ when taking the action $a \in A$. For a finite action space with finite state space, consider $A = \{a_1, a_2, \ldots, a_I\}$ and let θ_i be the state corresponding to action a_i for $i = 1, \ldots, I$. Suppose that each state θ_i has N outcomes:

$\theta_i = \{\theta_{i1}, \theta_{i2}, \ldots, \theta_{iN}\}$, the state space being then $\Theta = \{\theta_1, \ldots, \theta_I\}$. Let p_{in}, $i = 1, \ldots, I$, $n = 1, \ldots, N$, denote the distribution law of θ_i, where $\sum_{n=1}^{N} p_{in} = 1$ and $p_{in} \geq 0$. Then $p_{in} = P(\theta = \theta_{in}|a = a_i)$ denotes the probability that state θ_{in} occurs when taking action a_i. The corresponding payoff, when taking action a_i while state θ_{in} occurs, will be denoted by $x_{in} = X(a_i, \theta_{in})$, $i = 1, \ldots, I$, $n = 1, \ldots, N$. The action a_i can be represented in the form

$$a_i = (x_{i1}, p_{i1}; x_{i2}, p_{i2}; \ldots; x_{iN}, p_{iN}). \tag{4}$$

Given an action a_i defined by (4), one can characterize the distribution of its outcomes using a simplified notation, where the outcomes can be represented by a discrete random variable X_i taking values $x_{i1}, \ldots, x_{iN}$ with probabilities $P(X_i = x_{in}) = p_{in}$, $n = 1, \ldots, N$.

The EU-EV risk measure and associated EU-EV risk decision model are presented in Definition 1 and Definition 2, respectively.

Definition 1. *Consider an action space $A = \{a_1, a_2, \ldots, a_I\}$, where the actions*

$$a_i = (x_{i1}, p_{i1}; x_{i2}, p_{i2}; \ldots; x_{iN}, p_{iN}),\ i = 1, \ldots, I,$$

have outcomes x_{in} occurring with probabilities p_{in}, $n = 1, \ldots, N$. The expected utility, entropy and variance (EU-EV) measure of risk for the action $a_i \in A$ is defined by

$$R(a_i) = \frac{\lambda}{2}\left[H(X_i) + \frac{\mathrm{Var}[X_i]}{\max\limits_{a_i \in A}\{\mathrm{Var}[X_i]\}}\right] - (1-\lambda)\frac{\mathbb{E}[u(X_i)]}{\max\limits_{a_i \in A}\{|\mathbb{E}[u(X_i)]|\}}, \tag{5}$$

where λ is a real constant satisfying $0 \leq \lambda \leq 1$, $u(\cdot)$ is the utility function and $H(X_i)$ is the entropy expressed by

$$H(X_i) = -\sum_{n=1}^{N} p_{in} \ln p_{in}. \tag{6}$$

The risk of action a_i, $R(a_i)$, depends on the trade-off parameter λ, which is used to balance the decision maker's expected utility of an action and the uncertainty reflected by the entropy and variance associated with the action. In this risk measure, entropy and variance are combined as arithmetic mean. If $\lambda = 0$, then the risk measure depends only on the expected utility and if $\lambda = 1$ the risk measure uses only the uncertainty factors entropy and variance to assess risk. If $\lambda \in (0, 1)$, then the effect of the expected utility on the risk measure is bigger if λ approaches 0 and if λ approaches 1, the risk measure will be more influenced by the uncertainty than by the expected utility.

Assuming that there exists a preference relation based on the risk measure denoted by $\succeq$ (also designated as weak preference relation), where $\succ$ denotes the strict preference and $\sim$ the indifference, and that the preference relation is used by a decision-maker in choosing between two actions, where $a_1 \succeq a_2$ means that a_1 is preferred over a_2 ($a_1 \succ a_2$ meaning that a_1 is strictly preferred over a_2 and $a_1 \sim a_2$ that a_1 is indifferent to a_2), the EU-EV decision model is defined as follows.

Definition 2. *Let $G(\Theta, A, u)$ be a decision analysis model.*

1. *Consider two actions $a_1, a_2 \in A$ with EU-EV risk measures $R(a_1)$ and $R(a_2)$. Then:*
   (a) *$a_1 \succ a_2$ if $R(a_1) < R(a_2)$;*
   (b) *$a_1 \succeq a_2$ if $R(a_1) \leq R(a_2)$;*
   (c) *$a_1 \sim a_2$ if $R(a_1) = R(a_2)$.*
2. *Consider various actions, $A = \{a_1, a_2, \ldots, a_I\}$, then they can be ordered using the EU-EV measure of risk. The optimal action is the one with minimum EU-EV risk. In that case, one chooses a_k if*

$$R(a_k) = \min_{a_i \in A} R(a_i).$$

The EU-EV risk measure for stock selection can then be formulated as follows. Consider that a_i is the action of investing in stock S_i. The EU-EV risk measure presented in Definition 1 for investing in stock S_i is then given by (5), with x_{in} and p_{in}, $n = 1, \ldots, N$, defined in (3) and (2), respectively, and $\mathrm{Var}[X_i] = \sum_{n=1}^{N} x_{in}^2 p_{in} - \left(\sum_{n=1}^{N} x_{in} p_{in}\right)^2$, $\mathbb{E}[u(X_i)] = \sum_{n=1}^{N} u(x_{in}) p_{in}$.

Consider two stocks S_1 and S_2. If $R(a_1) < R(a_2)$, then, according to Definition 2, one has $a_1 \succ a_2$, indicating that it is preferable to invest in stock S_1 than in stock S_2. This means that the optimal stock with lowest EU-EV risk is S_1. We can establish a correspondence between the preference in terms of a_i and in terms of S_i and we will write $S_1 \succ S_2$, indicating that stock S_1 is preferred over S_2. In this case a decision-maker would select S_1.

Consider the set of stocks $S = \{S_1, \ldots, S_I\}$. The stock selection problem consists in choosing from the set S a subset with $K \leq I$ stocks with lowest EU-EV risk. For that purpose, the I stocks are ordered according to their EU-EV risk-measure in ascending order and the K stocks with lowest EU-EV risk measure are selected.

3. Application to the selection of stock components of the PSI 20 index

The aim is to apply the EU-EV decision model to the selection of stocks from a stock market in order to investigate if the EU-EV model adequately selects the relevant stocks for an efficient portfolio construction with a reduced number of stocks. For this purpose, in subsection 3.1, subsets of stocks are preselected with the EU-EV model and the mean-variance model is applied to those subsets. The obtained results are compared with those obtained from considering the whole set of stocks. Later, in subsection 3.2, the cardinality-constrained mean-variance model is applied and the results obtained with both methodologies are compared.

In this analysis, the Portuguese Stock market will be considered. The Portuguese Stock Index PSI 20 consists, from January 2019 to December 2020, of 18 component stocks of companies with the largest market capitalization. As initial set the set $S = \{S_1, \dots, S_{18}\}$ containing the 18 stocks from PSI 20 indicated in Table 1 will be considered.

Table 1: Components of PSI 20.

Stock	Company	Stock	Company	Stock	Company
S_1	Altri	S_7	GALP	S_{13}	The Navigator
S_2	BCP	S_8	Ibersol	S_{14}	Pharol
S_3	Corticeira Amorim	S_9	Jerónimo Martins	S_{15}	Ramada
S_4	CTT	S_{10}	Mota-Engil	S_{16}	REN
S_5	Energias de Portugal	S_{11}	Novabase	S_{17}	Semapa
S_6	EDP Renováveis	S_{12}	NOS	S_{18}	Sonae

From each stock, the daily closing prices from January 2019 to December 2020 are collected from Investing.com, yielding a set $\{P_{i0}, \dots, P_{iT}\}$ with $T + 1 = 512$ closing prices for each S_i, $i = 1, \dots, 18$.

The stock returns are calculated using the formula

$$r_{it} = \ln\left(\frac{P_{it}}{P_{i(t-1)}}\right), \tag{7}$$

where $i = 1, \dots, 18$ and $t = 1, \dots, 511$. The set of returns from stock S_i is then given by $\{r_{i1}, \dots, r_{i511}\}$. From all stock returns, one determines the minimum and maximum return, which for the given data are $r_{\min} = -0.258$ and $r_{\max} = 0.28$, so that

$$r_{it} \in [-0.258, 0.28],$$

for $i = 1, \dots, 18$ and $t = 1, \dots, 511$. The interval $[-0.258, 0.28]$ is divided into $N = 15$ subintervals of length $\Delta = 0.036$ according to (1):

$$J_1 = [-0.258, -0.222), \dots, J_{15} = [0.244, 0.28].$$

Consider the action $a_i = (x_{i1}, p_{i1}; \dots; x_{i15}, p_{i15})$ of investing in stock S_i, $i = 1, \dots, 18$, with p_{in} and x_{in}, $n = 1, \dots, 15$, defined in (2) and (3), respectively. In order to calculate the EU-EV risk measure (5) for each stock S_i one needs further to define the utility function. Here the following utility function will be used

$$u(x) = \begin{cases} \ln(1+x), & x \geq 0, \\ -\ln(1-x), & x < 0. \end{cases} \tag{8}$$

This utility function is a S-shaped utility function, which is concave for gains and convex for losses. S-shaped utility functions were proposed in Kahneman & Tversky (1979) to model the behaviour of decision makers towards risk. Now, the EU-EV risk measure (5) can be determined for each stock S_i. As for the trade-off coefficient, in the present study the risk measures are calculated for values of λ from 0 to 1 equally spaced in steps of 0.05, as done in Marasović & Kalinić (2019) with the EU-E risk measure. The stocks are ranked according to their EU-EV risk for all values of λ. Table 2 contains the normalized expected utilities (NEU), the entropies (H) and the normalized variances (NVar) and the EU-EV risks corresponding to the trade-off factors $\lambda = 0, 0.25, 0.5, 0.75, 1$ for all 18 stocks.

Table 2: EU-EV risks for components of PSI 20.

Stock	NEU(a_i)	$H(a_i)$	NVar(a_i)	$R(a_i)$				
				$\lambda=0$	$\lambda=0.25$	$\lambda=0.5$	$\lambda=0.75$	$\lambda=1$
S_1	-0.1190	1.0654	0.6634	0.1190	0.3054	0.4917	0.6781	0.8644
S_2	-0.5714	1.0893	0.7647	0.5714	0.6603	0.7492	0.8381	0.9270
S_3	0.2352	0.8488	0.2959	-0.2352	-0.0333	0.1685	0.3704	0.5723
S_4	-0.2108	1.0164	0.5379	0.2108	0.3524	0.4940	0.6356	0.7772
S_5	0.5238	0.7677	0.3212	-0.5238	-0.2567	0.0103	0.2774	0.5445
S_6	1.0000	0.8002	0.3038	-1.0000	-0.6120	-0.2240	0.1640	0.5520
S_7	-0.4299	1.0078	0.6400	0.4299	0.5285	0.6270	0.7254	0.8239
S_8	-0.1781	1.0080	1.0000	0.1781	0.3846	0.5911	0.7975	1.0040
S_9	0.2770	0.7622	0.2726	-0.2770	-0.0784	0.1202	0.3188	0.5174
S_{10}	-0.4018	1.1948	0.9801	0.4018	0.5732	0.7446	0.9160	1.0874
S_{11}	0.3885	0.9828	0.7835	-0.3885	-0.0706	0.2473	0.5653	0.8832
S_{12}	-0.5714	0.8607	0.3550	0.5714	0.5805	0.5896	0.5988	0.6079
S_{13}	-0.3364	0.9241	0.4209	0.3364	0.4204	0.5045	0.5885	0.6725
S_{14}	-0.2854	1.0450	0.6925	0.2854	0.4312	0.5770	0.7229	0.8687
S_{15}	-0.4312	0.9940	0.7376	0.4312	0.5398	0.6485	0.7571	0.8658
S_{16}	-0.0215	0.5005	0.1313	0.0215	0.0951	0.1687	0.2423	0.3159
S_{17}	-0.3256	0.8978	0.3961	0.3256	0.4059	0.4863	0.5666	0.6470
S_{18}	-0.1788	0.8762	0.3392	0.1788	0.2860	0.3933	0.5005	0.6077

Table 3 contains the ranked stocks for various trade-off factors $\lambda \in [0, 1]$, where the number from 1 to 10 represents the order of a stock, considering the ranking in ascending order, where 1 corresponds to the lowest EU-EV risk and 10 to the highest EU-EV risk.

Table 3: Ranked stocks by EU-EV risk for trade-off factors $\lambda \in [0, 1]$.

λ	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}	S_{17}	S_{18}
0	7	17/18	5	10	2	1	15	8	4	14	3	17/18	13	11	16	6	12	9
0.05	7	18	5	10	2	1	15	9	4	14	3	17	13	11	16	6	12	8
0.1	7	18	5	10	2	1	14	9	4	15	3	17	13	11	16	6	12	8
0.15	7	18	5	9	2	1	14	10	4	15	3	17	13	11	16	6	12	8
0.2	8	18	5	9	2	1	14	10	4	16	3	17	13	12	15	6	11	7
0.25	8	18	5	9	2	1	14	10	3	16	4	17	12	13	15	6	11	7
0.3	8	18	5	9	2	1	14	11	3	17	4	16	12	13	15	6	10	7
0.35	8	18	4	9	2	1	14	12	3	17	5	16	11	13	15	6	10	7
0.4	8	18	4	9	2	1	14	12	3	17	5	15	11	13	16	6	10	7
0.45	8	18	4	9	2	1	14	13	3	17	5	15	11	12	16	6	10	7
0.5	9	18	4	10	2	1	15	14	3	17	6	13	11	12	16	5	8	7
0.55	11	17	5	10	2	1	15	14	3	18	6	12	9	13	16	4	8	7
0.6	11	17	5	10	2	1	14	15	4	18	6	12	9	13	16	3	8	7
0.65	12	17	5	10	2	1	14	16	4	18	6	11	9	13	15	3	8	7
0.7	12	17	5	11	2	1	14	16	4	18	7	10	9	13	15	3	8	6
0.75	12	17	5	11	3	1	14	16	4	18	7	10	9	13	15	2	8	6
0.8	12	17	5	11	3	1	13	16	4	18	10	8	9	14	15	2	7	6
0.85	12	16	5	10	3	2	13	17	4	18	11	8	9	14	15	1	7	6
0.9	13	16	5	10	3	2	12	17	4	18	11	7	9	14	15	1	8	6
0.95	13	16	5	10	4	2	11	17	3	18	12	7	9	14	15	1	8	6
1	12	16	5	10	3	4	11	17	2	18	15	7	9	14	13	1	8	6

3.1. EU-EV stock selection and MV model

Based on the ranking of stocks obtained with the EU-EV model, the best ranked stocks with lowest EU-EV risk will be selected, considering different ranges of the trade-off parameter. Subsets with a half number of stocks and subsets with four stocks will be formed. The mean-variance model will be applied to those subsets in order to construct efficient portfolios. The efficient frontiers will be compared with the efficient frontier of the portfolios obtained from the whole set of stocks. The purpose of this analysis is to investigate if the EU-EV model is able to select the best and relevant stocks for an efficient portfolio construction with a reduced number of stocks. Considering the number of stocks necessary for the optimum portfolio construction, this analysis will also give an insight about the number of stocks to invest in to obtain the efficient portfolios.

3.1.1. EU-EV preselection with 9 stocks

From the set of 18 ranked stocks by EU-EV risk, first, subsets with the half number of stocks will be determined. Considering different values of λ, 9 stocks with lowest EU-EV risk are selected. One can observe from Table 3 that for certain values of λ the subsets with 9 stocks having lowest EU-EV risk are maintained. One obtains the following subsets Q_i, $i = 1, \ldots, 5$, for the values of λ considered in Table 3:

$$
\begin{array}{ll}
Q_1 = \{S_1, S_3, S_5, S_6, S_8, S_9, S_{11}, S_{16}, S_{18}\}, & \lambda = 0, 0.05, 0.1; \\
Q_2 = \{S_1, S_3, S_4, S_5, S_6, S_9, S_{11}, S_{16}, S_{18}\}, & \lambda = 0.15, \ldots, 0.45; \\
Q_3 = \{S_1, S_3, S_5, S_6, S_9, S_{11}, S_{16}, S_{17}, S_{18}\}, & \lambda = 0.5; \\
Q_4 = \{S_3, S_5, S_6, S_9, S_{11}, S_{13}, S_{16}, S_{17}, S_{18}\}, & \lambda = 0.55, \ldots, 0.75; \\
Q_5 = \{S_3, S_5, S_6, S_9, S_{12}, S_{13}, S_{16}, S_{17}, S_{18}\}, & \lambda = 0.8, \ldots, 1.
\end{array}
\tag{9}
$$

Comparing Q_1 and Q_2, the difference is that stock S_8, which belongs to Q_1 is substituted in Q_2 by stock S_4, the other stocks remaining the same in Q_1 and Q_2. Calculating the corresponding risks using (5) and the results in Table 2 one obtains $R(a_4) = 0.5664\lambda + 0.2108$ and $R(a_8) = 0.8259\lambda + 0.1781$. Since the risks are strictly increasing functions of $\lambda \in [0, 1]$, one concludes that there exists a value $\lambda^\star \in [0.1, 0.15]$ such that $R(a_4) > R(a_8)$ for $\lambda < \lambda^\star$ and $R(a_4) < R(a_8)$ for $\lambda > \lambda^\star$. Equating both risk expressions $R(a_4) = R(a_8)$ and solving for λ, it turns ou that $\lambda^\star = 0.1260$. In an analogous way one can determine the other values of λ, which correspond to intersection points of two risk measures implying the replacement of a given stock in one subset by a different stock in the succeding subset due to the change in the inequality of the corresponding risk measures. Proceeding in this way, one obtains the interval ranges $I_\lambda \subset [0, 1]$, listed in Table 4, for which Q_i, $i = 1, \ldots, 5$, contain the stocks given in (9).

Table 4: Sets of 9 stocks with lowest EU-EV risk.

I_λ	Q_i
$[0, 0.1260)$	$Q_1 = \{S_1, S_3, S_5, S_6, S_8, S_9, S_{11}, S_{16}, S_{18}\}$
$[0.1260, 0.4685)$	$Q_2 = \{S_1, S_3, S_4, S_5, S_6, S_9, S_{11}, S_{16}, S_{18}\}$
$[0.4685, 0.5311)$	$Q_3 = \{S_1, S_3, S_5, S_6, S_9, S_{11}, S_{16}, S_{17}, S_{18}\}$
$[0.5311, 0.7771)$	$Q_4 = \{S_3, S_5, S_6, S_9, S_{11}, S_{13}, S_{16}, S_{17}, S_{18}\}$
$[0.7771, 1]$	$Q_5 = \{S_3, S_5, S_6, S_9, S_{12}, S_{13}, S_{16}, S_{17}, S_{18}\}$

One can observe that there are 6 stocks which belong to all subsets Q_i, $i = 1, \ldots, 5$, namely S_3, S_5, S_6, S_9, S_{16}, S_{18}. S_{11} belongs to 4 subsets, S_1 and S_{17} to 3 subsets, S_{13} to 2 subsets and S_4 and S_{12} to one subset.

The set Q_1 corresponds to trade-off coefficients $\lambda \in [0, 0.1260)$, in which case more weight is given to the return's expected utility, almost ingnoring the effect of variance and entropy. In fact, Q_1 is equal to the set obtained with $\lambda = 0$, where only the return's expected utility is taken into account.

Considering the ordered set for $\lambda = 0$ with respect to the EU-EV risk, from the lowest to the highest risk, or, in this case with respect to the expected utility, from the highest to the lowest, according to Table 3, the ordered nine stocks are:

$$S_6 \succ S_5 \succ S_{11} \succ S_9 \succ S_3 \succ S_{16} \succ S_1 \succ S_8 \succ S_{18}.$$

Thus, in set Q_1, S_6 has the lowest risk (or the highest expected utility) and S_{18} the highest risk (or the lowest expected utility). Considering the set Q_2, where more weight is given to expected utility than to entropy and variance, since $\lambda \in [0.1260, 0.4685)$, one obtains the following ordering:

$$S_6 \succ S_5 \succ S_9 \succ S_3 \succ S_{11} \succ S_{16} \succ S_{18} \succ S_1 \succ S_4.$$

In this set, S_4, which entered in Q_2 by replacing S_8 in Q_1, has now the highest risk. The set Q_3 corresponds to trade-off coefficients $\lambda \in [0.4685, 0.5311)$, which is equal to the set obtained with $\lambda = 0.5$, where an equal weight is given to variance and entropy, on the one hand, and to expected utility, on the other hand. In this case, where $\lambda = 0.5$, the set has the following ordered stocks (see Table 3):

$$S_6 \succ S_5 \succ S_9 \succ S_3 \succ S_{16} \succ S_{11} \succ S_{18} \succ S_{17} \succ S_1,$$

where now S_1 has the highest EU-EV risk. One can observe that the risk positions of the stocks in Q_2 are maintained in Q_3, except the positions of S_{11} and S_{16}, which interchange, of S_1 and S_4 being replaced by S_{17}. Considering the set Q_4, corresponding to decisions where more weight is given to entropy and variance than to expected utility, with $\lambda \in [0.5311, 0.7771)$, the ordering is the following:

$$S_6 \succ S_5 \succ S_{16} \succ S_9 \succ S_3 \succ S_{11} \succ S_{18} \succ S_{17} \succ S_{13},$$

where S_{13} replaces S_1 of Q_3 in the highest risk position and considering the remaining higher ranked stocks, the only change being that S_{16} passes from the fifth lowest risk position to the third lowest risk position. Considering the set Q_5, obtained for $\lambda \in [0.7771, 1]$, it equals the set corresponding to $\lambda = 1$, where the preferences are taken regarding only uncertainty given by entropy and variance. The ordered stocks with respect to the EU-EV risk of the set for $\lambda = 1$ are:

$$S_{16} \succ S_9 \succ S_5 \succ S_6 \succ S_3 \succ S_{18} \succ S_{12} \succ S_{17} \succ S_{13},$$

where, differently from the previous sets, S_{16} appears now in the lowest risk position, this stock having the lowest uncertainty given by entropy and variance.

Figures 1—3 contain the graphs of the EU-EV risk measures as functions of λ for the stocks of the sets Q_1, Q_2, Q_3, Q_4 and Q_5.

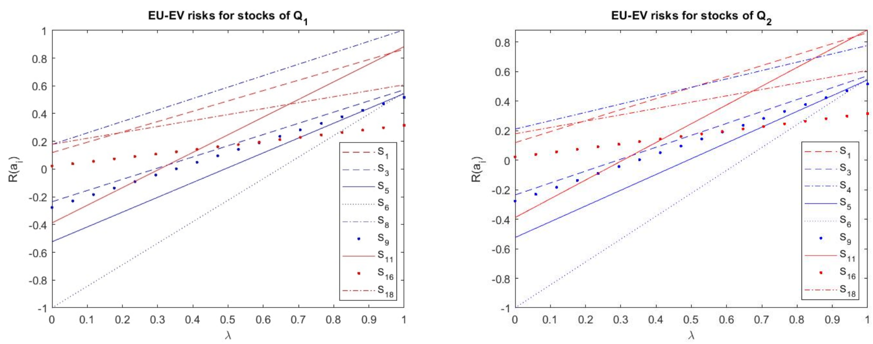

Figure 1: *Left panel*: EU-EV risks for Q_1. *Right panel*: EU-EV risks for Q_2.

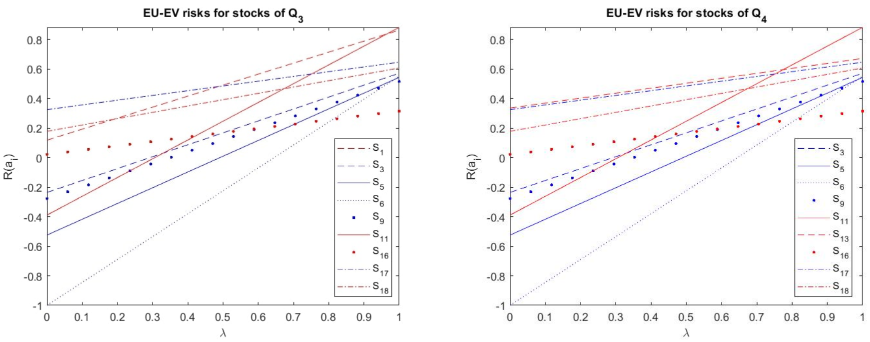

Figure 2: *Left panel*: EU-EV risks for Q_3. *Right panel*: EU-EV risks for Q_4.

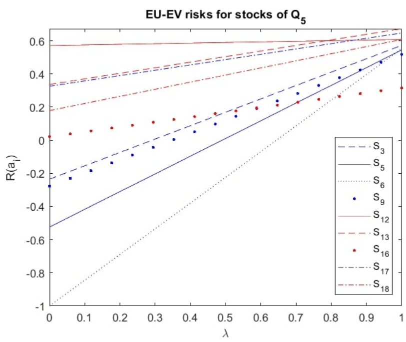

Figure 3: EU-EV risks for Q_5.

One can observe in Figures 1—3 that the risks of all stocks increase with λ, that is, the risks increase with the uncertainty described by variance and entropy. In all three sets, stock S_6 has the lowest EU-EV risk for a range of $\lambda \in [0, 0.8122)$, 0.8122 being the intersection point of the EU-EV risk lines $R(a_6)$ and $R(a_{16})$. For this reason, S_6 was ranked with lowest risk in the sets Q_1, Q_2, Q_3 and Q_4, corresponding to $\lambda < 0.7771$. Considering $\lambda \in (0.8122, 1]$, stock S_{16} reveals to be the optimal one, so that for $\lambda = 1$ this stock appears in the highest ranking position in terms of lowest EU-EV risk. Among the considered stocks, $S_8 \in Q_1$ has the highest EU-EV risk for a wide range of λ, which was classified as the stock with second highest risk for $\lambda = 0$, where it is exceeded by the risk of S_{18}, and it was excluded from the other sets by the ranking. In Q_3 (see left panel of Figure 2), it is S_1 that for $\lambda = 0.5$ has the highest risk, which due to a higher slope, intersects the EU-EV risk of S_{17} at $\lambda = 0.4872$ and exceeds it for higher values. In Q_4 (see right panel of Figure 2), one can see that the high ranked EU-EV risks of S_{13} and S_{17} are very similar. In Figure 3, one can observe that S_{12}, which for low values of λ has the highest EU-EV risk among all considered stocks and was therefore not included in the other sests, has a line with approximately constant slope, which for values of λ higher than 0.8 is overtaken by the risks of S_{17} and S_{13}.

In general, one can conclude that stock S_6 followed by S_5 have the best performance in terms of lowest EU-EV risk for $\lambda \in [0, 0.7771)$. For higher values of λ these stocks remain in low risk positions (S_5 has the third lowest risk and S_6 the fourth lowest risk), where then S_{16} appears as best performing stock (for preferences based on a high weight given to entropy and variance). Notable is that the EU-EV risk of S_6 has the highest slope, it increases from -1 to 0.5520, considering the range of λ, and that the EU-EV risk of S_{12} is almost not affected by varying λ, it increases from 0.5714 to 0.6079. Interesting is also the very similar EU-EV risk performance of the higher risk ranked stocks S_{13} and S_{17}.

Having obtained a preselection of stocks with lowest EU-EV risk for different values of λ presented in Table 4 and analysed the ranking of stocks for different values of λ, now the mean-variance optimization problem will be considered.

3.1.2. MV model for subsets with 9 stocks

The mean-variance model proposed by Markowitz (1952) consists of the following optimization problem. Given the stocks $S_1, \ldots, S_I$, the aim is to determine portfolios, which are weighted combinations of the stocks, that have minimum risk expressed by variance, for a given mean return, denoted by $\bar{\mu}$. The mean-variance portfolio optimization problem can be formulated as follows:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{I} \sum_{j=1}^{I} w_i w_j \sigma_{ij} \\
\text{subject to} \quad & \sum_{i=1}^{I} w_i \mu_i = \bar{\mu} \\
& \sum_{i=1}^{I} w_i = 1 \\
& 0 \leq w_i \leq 1,\ i = 1, \ldots, I,
\end{aligned}
$$

where w_i are the portfolio weights, σ_{ij} is the covariance between the stocks S_i and S_j given by $\sigma_{ij} = \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])]$, and μ_i is the expected return of stock S_i given by $\mu_i = \mathbb{E}[X_i]$.

The expected return of the portfolio is the weighted average of its expected stock returns and by the constraint $\sum_{i=1}^{I} w_i \mu_i = \bar{\mu}$ it is equal to a specified mean return $\bar{\mu}$. The risk of the portfolio is given by the variance and can be expressed by the standard deviation. Efficient portfolios are the optimal portfolios, which have for a given expected return the minimum risk (or, for a given risk the highest expected return). The set of all efficient portfolios defines the efficient frontier.

Applying the mean-variance optimization problem, the efficient frontiers of the sets $Q_1, \ldots, Q_5$ (see Table 4) obtained with the EU-EV model are compared with the efficient frontier of the set S consisting of all 18 stocks of PSI 20. Figures 4-6 contain the graphs of the efficient frontiers of Q_i, $i = 1, \ldots, 5$, and of the set S.

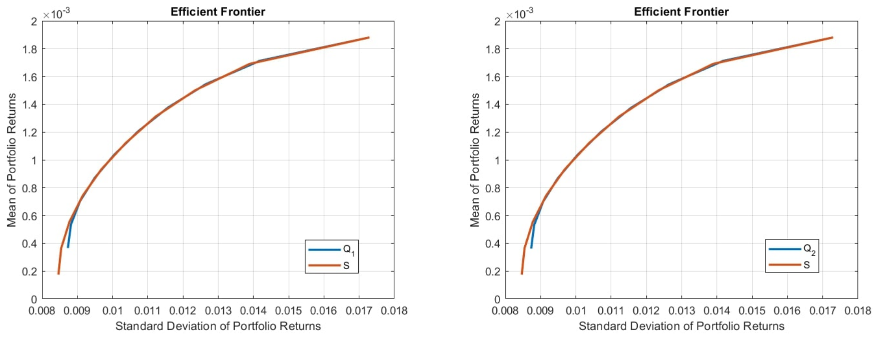

Figure 4: *Left panel*: Efficient frontier for Q_1. *Right panel*: Efficient frontier for Q_2.

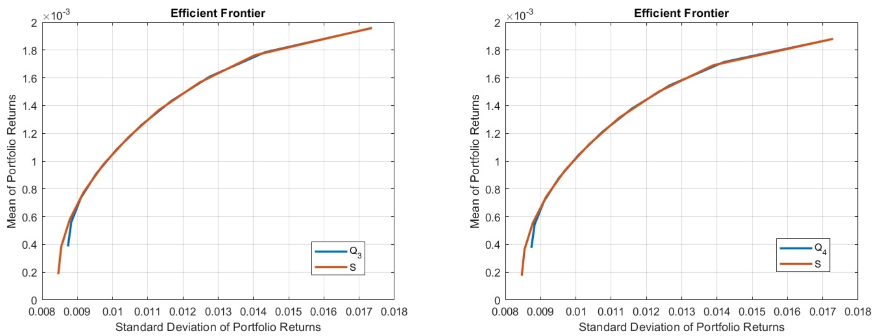

Figure 5: *Left panel*: Efficient frontier for Q_3. *Right panel*: Efficient frontier for Q_4.

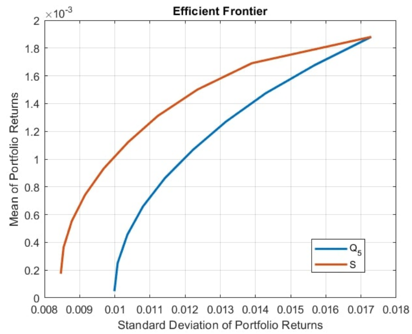

Figure 6: Efficient frontier for Q_5.

One can observe that the efficient frontiers of Q_1, Q_2, Q_3 and Q_4 seem to be very similar or almost equal and these are approximately equal to the efficient frontier of S, the differences being notable for standard deviations in the interval $[0.008, 0.009]$. This means that the performance of the sets $Q_1, \dots, Q_4$ of 9 stocks corresponding to trade-off factors $\lambda \in [0, 0.7771)$ is similar to the performance of S obtained with all 18 stocks, if the aim is to minimize risk expressed by variance (or standard deviation) for a given expected return of the portfolio. Therefore, instead of investing in stocks contained in the set S of 18 stocks, one could invest in stocks belonging to $Q_1, \dots, Q_4$, which only contain 9 stocks, in order to obtain for a given risk (standard deviation) almost the same expected return. In this sense, the subsets of stocks $Q_1, \dots, Q_4$ approximate well S.

As for the set Q_5, which was obtained for λ close to 1 and therefore privileging stocks with lower uncertainty and almost ignoring expected utility, it performs less well than S. Due to the high weight given to uncertainty, which depends on variance, and the very low importance assigned to expected utility in the EU-EV risk, the expected return of the selected stocks may be lower and therefore the efficiency decays with respect to those of S.

Considering the sets $Q_1, \dots, Q_4$, weights of selected stocks of 10 efficient portfolios are determined and the results are compared. The risks (standard deviations) and returns of those portfolios are indicated in Figures 7 and 8. The weights of their stock components are listed in Tables 5–8. Table 9 contains the weights of stocks of 10 efficient portfolios of the set of all stocks S (also plotted in Figures 7 and 8).

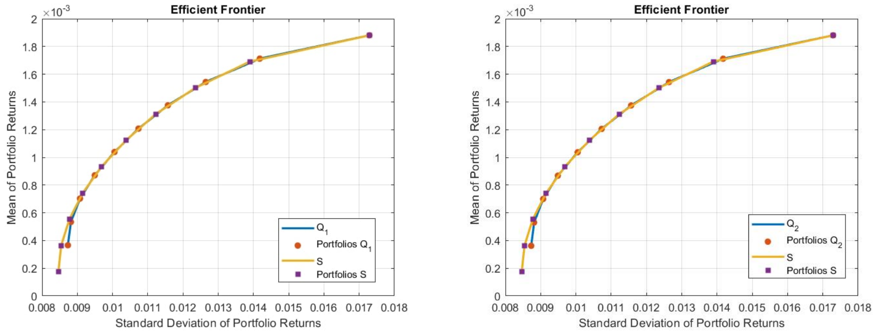

Figure 7: *Left panel*: Efficient frontier with 10 portfolios for Q_1 and S. *Right panel*: Efficient frontier with 10 portfolios for Q_2 and S.

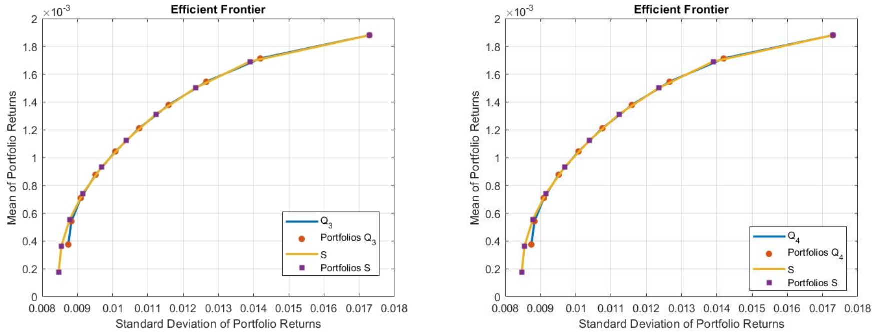

Figure 8: *Left panel*: Efficient frontier with 10 portfolios for Q_3 and S. *Right panel*: Efficient frontier with 10 portfolios for Q_4 and S.

Table 5: Weights of components of efficient portfolios (EP) of Q_1.

EP	Weights									Risk	Return
	S_6	S_5	S_{11}	S_9	S_3	S_{16}	S_1	S_8	S_{18}		
P_1	0.0659	0	0.1774	0.1666	0.1654	0.4147	0	0.0101	0	0.0087	0.0004
P_2	0.1459	0	0.1882	0.1579	0.1631	0.3441	0	0.0008	0	0.0088	0.0005
P_3	0.2292	0	0.1992	0.1477	0.1605	0.2635	0	0	0	0.0091	0.0007
P_4	0.3128	0	0.2102	0.1373	0.1579	0.1819	0	0	0	0.0095	0.0009
P_5	0.3963	0	0.2212	0.1269	0.1553	0.1003	0	0	0	0.0101	0.0010
P_6	0.4799	0	0.2322	0.1164	0.1527	0.0188	0	0	0	0.0107	0.0012
P_7	0.5855	0	0.2405	0.0702	0.1038	0	0	0	0	0.0116	0.0014
P_8	0.6976	0	0.2480	0.0133	0.0410	0	0	0	0	0.0126	0.0015
P_9	0.8248	0	0.1652	0	0	0	0	0	0	0.0142	0.0017
P_{10}	1	0	0	0	0	0	0	0	0	0.0173	0.0019

Table 6: Weights of components of efficient portfolios (EP) of Q_2.

EP	Weights									Risk	Return
	S_6	S_5	S_9	S_3	S_{11}	S_{16}	S_{18}	S_1	S_4		
P_1	0.0651	0	0.1635	0.1617	0.1778	0.4156	0	0	0.0162	0.0087	0.0004
P_2	0.1439	0	0.1583	0.1631	0.1880	0.3467	0	0	0	0.0088	0.0005
P_3	0.2277	0	0.1479	0.1605	0.1990	0.2649	0	0	0	0.0091	0.0007
P_4	0.3115	0	0.1374	0.1579	0.2100	0.1831	0	0	0	0.0095	0.0009
P_5	0.3953	0	0.1270	0.1553	0.2211	0.1014	0	0	0	0.0100	0.0010
P_6	0.4790	0	0.1166	0.1527	0.2321	0.0196	0	0	0	0.0107	0.0012
P_7	0.5846	0	0.0707	0.1043	0.2405	0	0	0	0	0.0116	0.0014
P_8	0.6970	0	0.0136	0.0414	0.2480	0	0	0	0	0.0126	0.0015
P_9	0.8344	0	0	0	0.1656	0	0	0	0	0.0142	0.0017
P_{10}	1	0	0	0	0	0	0	0	0	0.0173	0.0019

Table 7: Weights of components of efficient portfolios (EP) of Q_3.

EP	Weights									Risk	Return
	S_6	S_5	S_9	S_3	S_{16}	S_{11}	S_{18}	S_{17}	S_1		
P_1	0.0666	0	0.1679	0.1655	0.4222	0.1778	0	0	0	0.0087	0.0004
P_2	0.1496	0	0.1576	0.1630	0.3411	0.1887	0	0	0	0.0088	0.0005
P_3	0.2327	0	0.1472	0.1604	0.2600	0.1996	0	0	0	0.0091	0.0007
P_4	0.3158	0	0.1369	0.1578	0.1790	0.2106	0	0	0	0.0095	0.0009
P_5	0.3988	0	0.1265	0.1552	0.0979	0.2215	0	0	0	0.0101	0.0010
P_6	0.4819	0	0.1162	0.1526	0.0168	0.2325	0	0	0	0.0108	0.0012
P_7	0.5875	0	0.0692	0.1027	0	0.2407	0	0	0	0.0116	0.0014
P_8	0.6990	0	0.0126	0.0403	0	0.2481	0	0	0	0.0127	0.0015
P_9	0.8358	0	0	0	0	0.1642	0	0	0	0.0142	0.0017
P_{10}	1	0	0	0	0	0	0	0	0	0.0173	0.0019

Table 8: Weights of components of efficient portfolios (EP) of Q_4.

EP	Weights									Risk	Return
	S_6	S_5	S_{16}	S_9	S_3	S_{11}	S_{18}	S_{17}	S_{13}		
P_1	0.0666	0	0.4222	0.1679	0.1655	0.1778	0	0	0	0.0087	0.0004
P_2	0.1496	0	0.3411	0.1576	0.1630	0.1887	0	0	0	0.0088	0.0005
P_3	0.2327	0	0.2600	0.1472	0.1604	0.1996	0	0	0	0.0091	0.0007
P_4	0.3158	0	0.1790	0.1369	0.1578	0.2106	0	0	0	0.0095	0.0009
P_5	0.3988	0	0.0979	0.1265	0.1552	0.2215	0	0	0	0.0101	0.0010
P_6	0.4819	0	0.0168	0.1162	0.1526	0.2325	0	0	0	0.0108	0.0012
P_7	0.5875	0	0	0.0692	0.1027	0.2407	0	0	0	0.0116	0.0014
P_8	0.6990	0	0	0.0126	0.0403	0.2481	0	0	0	0.0127	0.0015
P_9	0.8358	0	0	0	0	0.1642	0	0	0	0.0142	0.0017
P_{10}	1	0	0	0	0	0	0	0	0	0.0173	0.0019

Table 9: Weights of components of efficient portfolios (EP) of S.

EP	Weights								Risk	Return
	S_3	S_6	S_9	S_{11}	S_{12}	S_{14}	S_{15}	S_{16}		
P_1	0.1415	0.0549	0.1461	0.1603	0.0776	0.0055	0.0594	0.3546	0.0085	0.0002
P_2	0.1479	0.1206	0.1467	0.1734	0.0508	0	0.0446	0.3162	0.0085	0.0004
P_3	0.1533	0.1869	0.1461	0.1865	0.0220	0	0.0298	0.2755	0.0088	0.0006
P_4	0.1575	0.2569	0.1435	0.2001	0	0	0.0135	0.2285	0.0092	0.0007
P_5	0.1569	0.3435	0.1334	0.2142	0	0	0	0.1519	0.0097	0.0009
P_6	0.1540	0.4376	0.1217	0.2266	0	0	0	0.0600	0.0104	0.0011
P_7	0.1276	0.5429	0.0918	0.2377	0	0	0	0	0.0112	0.0013
P_8	0.0569	0.6693	0.0277	0.2461	0	0	0	0	0.0124	0.0015
P_9	0	0.8140	0	0.1860	0	0	0	0	0.0139	0.0017
P_{10}	0	1	0	0	0	0	0	0	0.0173	0.0019

From Tables 5–8, one can observe that the stocks S_6, S_{11}, S_3, S_9, S_{16} are privileged in the efficient portfolio construction from Q_1, Q_2, Q_3 and Q_4, where in Q_1 the additional stock S_{18} enters with very low weights for portfolios with low risks and in Q_2 the additional stock S_4. The risks and returns for the efficient portfolios P_1-P_{10} obtained for Q_1, Q_2, Q_3 and Q_4 are almost equal, only the weights of the corresponding stocks differ slightly. It is notable that the high returns are achieved with higher weights of S_6 in the first position for Q_1, Q_2, Q_3 and Q_4, followed by weights of S_{11} and then also by weights of S_3 and S_9, however these representing a very small contribution when compared with S_6, or also with S_{11}. In all cases, stock S_{16} plays an important role, followed by stock S_{11}, in constructing efficient portfolios with low risks, since their weights are higher than those of other stocks in the portfolios P_1 and P_2.

The results in Table 9 indicate that the efficient portfolio construction based on the whole set of stocks S privileges stocks S_6, S_{11}, S_3, S_9, S_{16}, S_{15}, S_{12} and S_{14}, where as in the efficient portfolio construction for Q_1, Q_2, Q_3 and Q_4, the stock S_6 with highest weights is again the most relevant one for obtaining high returns, followed by S_{11} and by S_3 and S_9 with very low weights. One can observe that the efficient portfolios with higher returns constructed with stocks from S are very similar to those obtained with stocks from Q_1, Q_2, Q_3 and Q_4. Concerning efficient portfolios with low risks, here also stock S_{16} contributes with the highest weight, followed by stock S_{11}. The risks and returns obtained from S are similar to those obtained from Q_1, Q_2, Q_3 and Q_4, it is only worth to mention the slight difference that S allows for portfolios with lower risks (0.0085 instead of 0.0087) as one can also observe in Figures 7 and 8.

The results show that the efficient portfolios can be constructed with a smaller subset of stocks, namely a number of 5 or 6 stocks from Q_1, Q_2, Q_3 and Q_4, containing 9 stocks, than the number of 8 stocks from S, containing 18 stocks, since the risk and return values are approximately equal. Another conclusion is that the sets Q_1, Q_2, Q_3 and Q_4, which consist of 9 of the 18 stocks of S, contain the most relevant stocks for the efficient portfolio construction, these being S_6, S_{11}, S_3 and S_9, considering their presence and weights in the efficient porfolios. And these 4 stocks play also the most relevant role in the efficient portfolios obtained from S. These results suggest to construct efficient portfolios from smaller subsets of S, in particular to analyse the construction with sets of 4 stocks, which will be considered in the following subsection.

In order to confirm that the EU-EV model is in fact useful for the selection of stocks for efficient portfolio construction, before carrying out the cardinality-constrained portfolio optimization (in subsection 3.2), a comparison with randomly picked stock sets, in number equal to the number of stocks in Q_i, $i = 1, \ldots, 5$, will be performed. The mean-variance optimization problem will be applied to the new sets and the efficient frontiers will be confronted with the efficient frontier of S. Additionally, the specific stock set Q_6 containing the stocks that were mostly left out by the EU-EV selection for Q_1, Q_2, Q_3, Q_4 and Q_5 will also be used in this analysis:

$$Q_6 = \{S_2, S_4, S_7, S_8, S_{10}, S_{12}, S_{13}, S_{14}, S_{15}\}.$$

The following randomly picked stock sets will be considered:

$$\begin{aligned}
Q_7 &= \{S_3, S_5, S_6, S_7, S_8, S_{11}, S_{14}, S_{16}, S_{17}\},\\
Q_8 &= \{S_3, S_4, S_{10}, S_{11}, S_{12}, S_{14}, S_{15}, S_{16}, S_{17}\},\\
Q_9 &= \{S_2, S_4, S_5, S_6, S_9, S_{10}, S_{11}, S_{15}, S_{18}\},\\
Q_{10} &= \{S_1, S_2, S_4, S_6, S_8, S_{14}, S_{15}, S_{16}, S_{18}\}.
\end{aligned}$$

Applying the mean-variance optimization problem to the set Q_6, that consists of stocks classified with the EU-EV model as having the highest risk, one obtains the efficient frontier in Figure 9, containing also the efficient frontier of the whole stock set S. The set Q_6 has in fact a very poor performance, all mean return values are negative.

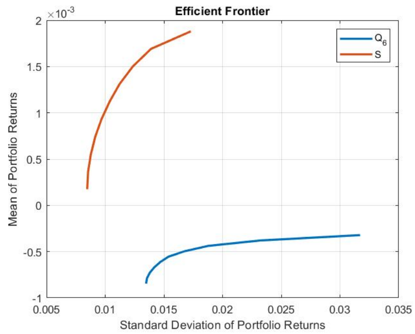

Figure 9: Efficient frontier for Q_6.

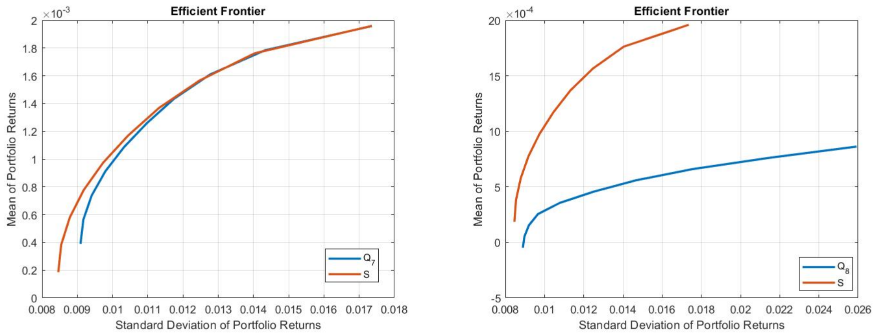

Figure 10: *Left panel*: Efficient frontier for Q_7. *Right panel*: Efficient frontier for Q_8.

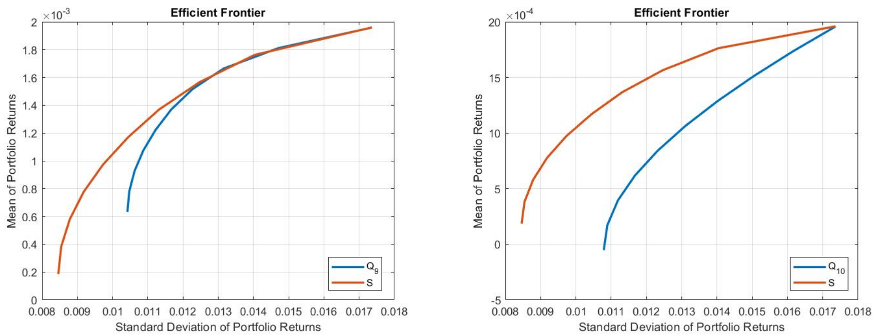

Figure 11: *Left panel*: Efficient frontier for Q_9. *Right panel*: Efficient frontier for Q_{10}.

Comparing the efficient frontiers of Q_7, Q_8, Q_9 and Q_{10} with those of S (see Figures 10 and 11), one can observe that with Q_8 and Q_{10} it is impossible to achieve the return levels of S in the given range (the highest return level associated with highest risk obtained with Q_{10} being an exception). The efficient frontiers of Q_7 and Q_9 partially approache the efficient frontier of S, namely for standard deviations greater than 0.012, where Q_7 yields a better approximation. However, comparing the performance of these stocks, in particular that of Q_7, with those of $Q_1 - Q_4$ (see Figures 4 and 5), it becomes evident that for lower risk ranges $Q_1 - Q_4$ lead to higher returns, so that these sets obtained with the EU-EV model outperform Q_7 (and $Q_8 - Q_{10}$). Note that the set Q_7 contains the stocks S_3, S_5, S_6, S_{11} and S_{16}, which are common to $Q_1 - Q_4$, and these stocks occupy in those sets the lower EU-EV risk ranking positions. For this reason, Q_7 performs better than Q_6 and $Q_8 - Q_{10}$.

3.1.3. EU-EV preselection with 4 stocks

In the following paragraphs, subsets of 4 stocks with lowest EU-EV risk will be formed and analysed, analogously to the previous case. Considering the results in Table 3, one obtains the sets of 4 stocks with lowest EU-EV risk presented in Table 10 corresponding to specific interval ranges I_λ of λ.

Table 10: Sets of 4 stocks with lowest EU-EV risk.

I_λ	Q_i
$[0, 0.3303)$	$Q_1 = \{S_5, S_6, S_9, S_{11}\}$
$[0.3303, 0.5001)$	$Q_2 = \{S_3, S_5, S_6, S_9\}$
$[0.5001, 1]$	$Q_3 = \{S_5, S_6, S_9, S_{16}\}$

The set Q_1 equals the set obtained for $\lambda = 0$. The stocks in this set maximize therefore the return's expected utility, the entropy and variance being more irrelevant. The ordered stocks for $\lambda = 0$, with respect to the EU-EV risk, are

$$S_6 \succ S_5 \succ S_{11} \succ S_9.$$

The set Q_2 equals the set for $\lambda = 0.5$. The stocks in this set are those which perform best, when the assessment is characterized by weighting equally the return's expected utility, on the one hand, and the entropy and variance, on the other hand. The ordered stocks for $\lambda = 0.5$, with respect to the EU-EV risk, are

$$S_6 \succ S_5 \succ S_9 \succ S_3.$$

The set Q_3 equals the set obtained for $\lambda = 1$. The stocks in this set minimize therefore the return's entropy and variance, ignoring the return's expected utility. The ordered stocks for $\lambda = 1$, with respect to the EU-EV risk, are

$$S_{16} \succ S_9 \succ S_5 \succ S_6.$$

Figures 12 and 13 contain the graphs of the EU-EV risks as functions of λ for the stocks of sets Q_1, Q_2 and Q_3.

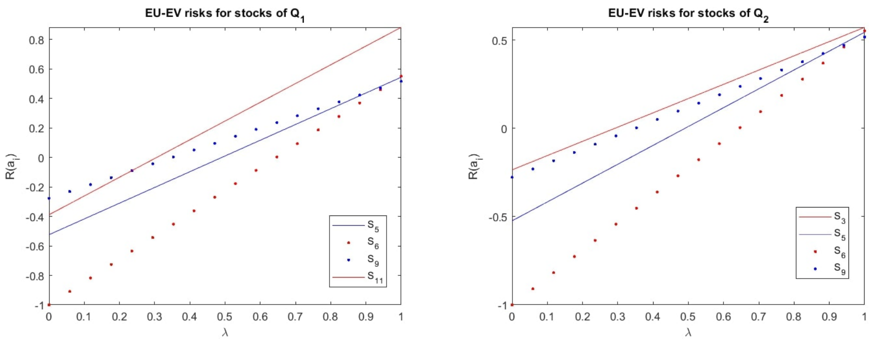

Figure 12: *Left panel*: EU-EV risks for Q_1. *Right panel*: EU-EV risks for Q_2.

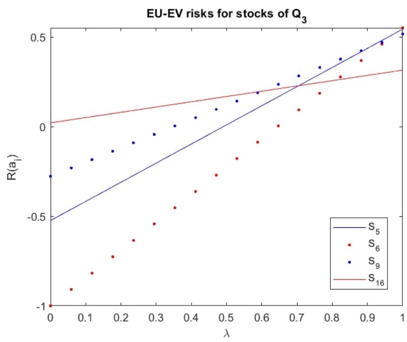

Figure 13: EU-EV risks for Q_3.

The sets Q_1, Q_2 and Q_3 have in common the stocks S_5, S_6 and S_9. One can observe in Figures 12 and 13 that, in set Q_1, the additional stock S_{11} has a low EU-EV risk for low values of λ, however it increases with a higher slope when compared with the EU-EV risks of S_3, included in Q_2, and S_{16}, included in Q_3. On the contrary, the EU-EV risk of S_{16} due to its lower slope is responsible for a better performance in terms of lower EU-EV risk of set Q_3 for higher values of λ. Considering set Q_2, one can observe that the evolution of the EU-EV risk with λ of the additional stock S_3 is very similar to that of S_9, the slopes of the corresponding lines being almost equal and the difference between both risks being small. In all sets, S_6 has the lowest risk for a wider range of λ, except for higher values, where in Q_3 it is more evident that S_{16} performs better for higher values of λ, this being consistent with the ranking obtained previously for the cases $\lambda = 0$, $\lambda = 0.5$ and $\lambda = 1$.

3.1.4. MV model for subsets with 4 stocks

The mean-variance optimization problem will now be applied to the sets consisting of 4 stocks and their efficiencies will be compared with those of the whole set S through a graphical analysis and an analysis of particular efficient portfolios. The efficient frontiers of the sets Q_1, Q_2, Q_3 and of the set containing all stocks S are presented in Figure 14.

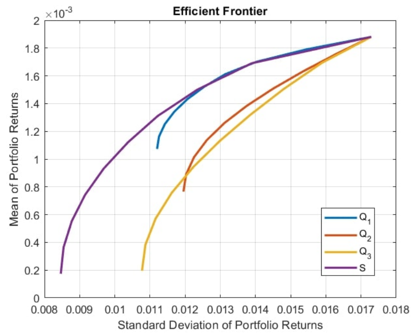

Figure 14: Efficient frontiers for Q_1, Q_2, Q_3 and S.

Comparing the efficient frontiers corresponding the three subsets of 4 stocks Q_1, Q_2, Q_3 and the initial set of 18 stocks S, one can observe that the efficient frontier of portfolios constructed from stocks of set Q_1, consisting of the best ranked stocks for $\lambda \in [0, 0.3303)$, approximates the efficient frontier of portfolios from S better than the efficient frontiers corresponding to Q_2 and Q_3. However, the approximation is only achieved for standard deviations greater than 0.011, since it is not possible to attain lower risks with Q_1. Also, one can observe that for standard deviations greater than 0.012, the efficient frontier of Q_1 approximates well the efficient frontier of S. This means that the optimal portfolios constructed with stocks of Q_1 yield approximately the same expected returns as the optimal portfolios constructed with stocks from S for given standard deviations greater than 0.012. The efficient frontiers of Q_2 and Q_3 lie notably below those of Q_1 and S. These results indicate that the higher weight given to the expected utility in the EU-EV risk, through values of λ close to 0, is relevant for the sets of four stocks to attain the optimal portfolios with higher mean returns. Instead, an approximate equal weight given to entropy and variance on the one hand and expected utility on the other hand, as in the formation of set Q_2, and a higher weight attributed to entropy and variance, as in the formation of set Q_3, will decrease the mean returns of the portfolios

One concludes that the 4 stocks of set Q_1 are sufficient to obtain the highest expected returns for risks higher than 0.012. Therefore, one could pay attention to these stocks instead of considering the set of all 18 stocks, if the aim is to achieve the highest expected returns, although coupled with a higher degree of risk.

Considering the set Q_1, weights of 10 efficient portfolios constructed with stocks of Q_1 are listed in Table 11, together with the corresponding risks and returns. The risks (standard deviations) and returns of those portfolios are plotted on the efficient frontier in Figure 15 together with the efficient frontier and risks and returns of 10 efficient portfolios of the set of all stocks S. The weights of the 10 efficient portfolios indicated on the efficient frontier of S in Figure 15 are listed in Table 9.

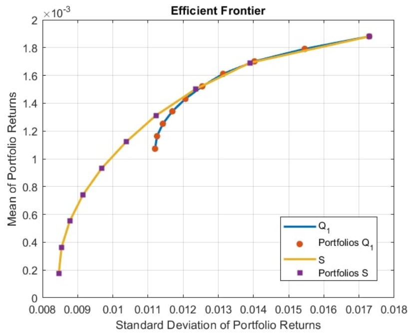

Figure 15: Efficient frontier with 10 portfolios for Q_1 and S.

Table 11: Weights of components of efficient portfolios (EP) of Q_1.

EP	Weights				Risk	Return
	S_6	S_5	S_{11}	S_9		
P_1	0.2695	0.1683	0.2603	0.3019	0.0112	0.0011
P_2	0.3526	0.1245	0.2600	0.2630	0.0113	0.0012
P_3	0.4357	0.0807	0.2596	0.2241	0.0114	0.0013
P_4	0.5187	0.0369	0.2592	0.1852	0.0117	0.0013
P_5	0.5996	0	0.2586	0.1418	0.0121	0.0014
P_6	0.6688	0	0.2563	0.0749	0.0125	0.0015
P_7	0.7380	0	0.2540	0.0079	0.0131	0.0016
P_8	0.8239	0	0.1761	0	0.0140	0.0017
P_9	0.9119	0	0.0881	0	0.0155	0.0018
P_{10}	1	0	0	0	0.0173	0.0019

One can observe in Table 11 that the efficient portfolios with higher returns can be formed with stocks S_6 and S_{11} of Q_1, since only these contribute with their weights to the highest returns, where S_6 enters with higher weights in the portfolio construction. Efficient porfolios with lower risks are constructed with all four stocks, where S_9 contributes with the highest weight to achieve the lowest risk. Comparing these results with those of the efficient portfolios obtained from S, which are listed in Table 9, one can observe that also for S, the stocks S_6 and S_{11} lead to the highest returns. The difference is that in S, S_3 and S_9 contribute next to the intermediate returns, followed then by S_{16}, whereas in Q_1 the stock S_9 is included, however with low weights, to obtain portfolios with intermediate returns. The 4 stocks in Q_1 are not sufficient to achieve lower risks associated with lower returns. For that purpose, more stocks should be included, diversifying more the portfolios. In fact, in S (see Table 9) the lowest risk portfolios on the efficient frontier are constructed by including more stocks.

3.2. Comparison with the cardinality-constrained MV model

In this subsection, the results obtained in the previous subsection will be compared with the results obtained by adding a cardinality constraint to the MV model. The maximum number of stocks in the portfolio optimization problem will first be limited to 9 stocks and then to 4 stocks.

The mean-variance portfolio optimization problem with cardinality constraint can be formulated as follows:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{I} \sum_{j=1}^{I} w_i w_j \sigma_{ij} \\
\text{subject to} \quad & \sum_{i=1}^{I} w_i \mu_i = \bar{\mu} \\
& \sum_{i=1}^{I} w_i = 1 \\
& 0 \leq w_i \leq 1,\ i = 1, \ldots, I, \\
& \sum_{i=1}^{I} \delta(w_i) \leq K,
\end{aligned}
$$

where now the last condition is introduced to limit the maximum number of stocks to K, with $K \leq I$, and $\delta(w_i) = 0$ if $w_i = 0$ and $\delta(w_i) = 1$ if $w_i \neq 0$, indicating thus if stock S_i will be included in the portfolio.

3.2.1. MV model with a maximum number of 9 stocks

Applying the mean-variance optimization problem with cardinality contraint, considering $K = 9$, one obtains the efficient frontier in Figure 16, where U denotes the set of portfolios formed with the additional constraint. Figure 16 also contains the efficient frontier of S, however both efficient frontiers seem to coincide. Table 12 contains the weights of 10 efficient constrained portfolios, plotted in Figure 16.

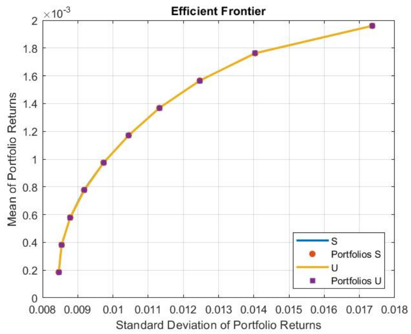

Figure 16: Efficient frontier with 10 portfolios for U and S.

Table 12: Weights of components of efficient portfolios (EP) of U.

EP	Weights								Risk	Return
	S_3	S_6	S_9	S_{11}	S_{12}	S_{14}	S_{15}	S_{16}		
P_1	0.1413	0.0548	0.1459	0.1603	0.0773	0.0065	0.0595	0.3543	0.0085	0.0002
P_2	0.1479	0.1204	0.1467	0.1733	0.0508	0	0.0446	0.3162	0.0085	0.0004
P_3	0.1568	0.1745	0.1525	0.1850	0	0	0.0348	0.2965	0.0088	0.0006
P_4	0.1599	0.2492	0.1452	0.2018	0	0	0	0.2439	0.0092	0.0007
P_5	0.1569	0.3434	0.1335	0.2142	0	0	0	0.1520	0.0097	0.0009
P_6	0.1540	0.4375	0.1217	0.2266	0	0	0	0.0601	0.0104	0.0011
P_7	0.1276	0.5428	0.0919	0.2377	0	0	0	0	0.0112	0.0013
P_8	0.0570	0.6692	0.0277	0.2461	0	0	0	0	0.0124	0.0015
P_9	0	0.8139	0	0.1861	0	0	0	0	0.0139	0.0017
P_{10}	0	1	0	0	0	0	0	0	0.0173	0.0019

Comparing the results of Table 12 with those of Table 9, where no cardinality constraint was imposed, one concludes that the results are very similar, the same eight stocks are used to build up the efficient portfolios, when the maximum number of stocks in the portfolio is $K = 9$.

Comparing these results with those of subsection 3.1.2, where the mean-variance model was applied to the 9 stocks, preselected with the EU-EV model, one concludes that the cardinality-constrained mean-variance model represents in this case only an improvement for approaching the efficient frontier of S for low standard deviations in the interval $[0.008, 0.009]$, associated with the lowest expected returns, as one can observe in Figure 17. Note that in Figure 17, the efficient frontiers of Q_i, $i = 1, \dots, 4$ coincide (or almost coincide, c.f. subsection 3.1.2), as well as the efficient frontiers of S and U. One can conclude that applying the EU-EV model with trade-off factors $\lambda \in [0, 0.7771)$ together with the mean-variance model or using the cardinality-constrained mean-variance model will lead approximately to the same efficient frontier for standard deviations greater than 0.009.

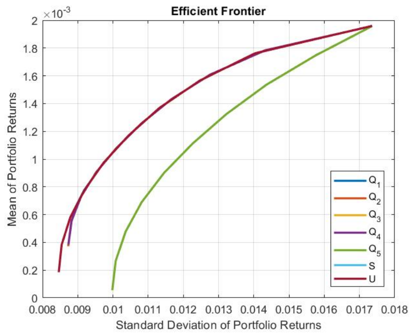

Figure 17: Efficient frontier with 10 portfolios for Q_i, $i = 1, 2, 3$, U and S.

3.2.2. MV model with a maximum number of 4 stocks

Considering now the cardinality-constrained mean-variance optimization problem with $K = 4$, one obtains the efficient frontier in Figure 18, where U denotes the corresponding set of portfolios formed with the imposed constraint. Comparing the efficient frontier with those of S, one can observe that both efficient frontiers seem to coincide for standard deviations greater than 0.0105 (or expected returns greater than 1.2). For lower standard deviations, the cardinality-constrained portfolios have lower expected returns than the unconstrained portfolios, the deviation being small. Table 13 contains the weights of 10 efficient constrained portfolios, plotted in Figure 18.

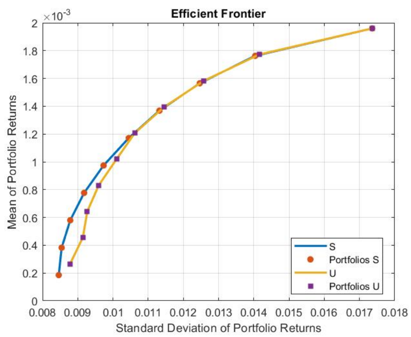

Figure 18: Efficient frontier with 10 portfolios for U and S.

Table 13: Weights of components of efficient portfolios (EP) of U.

EP	Weights					Risk	Return
	S_3	S_6	S_9	S_{11}	S_{16}		
P_1	0.1713	0	0.1887	0.1749	0.4651	0.0088	0.0003
P_2	0.1946	0.1426	0	0.1966	0.4662	0.0091	0.0004
P_3	0.1899	0.2274	0	0.2072	0.3755	0.0093	0.0006
P_4	0.1851	0.3140	0	0.2180	0.2829	0.0096	0.0008
P_5	0.1802	0.4007	0	0.2288	0.1903	0.0101	0.0010
P_6	0.1847	0.4409	0.1436	0.2308	0	0.0106	0.0012
P_7	0.1174	0.5611	0.0826	0.2389	0	0.0114	0.0013
P_8	0.0503	0.6815	0.0214	0.2469	0	0.0125	0.0015
P_9	0	0.8229	0	0.1771	0	0.0140	0.0017
P_{10}	0	1	0	0	0	0.0137	0.0019

Comparing the results of Table 13 with those of Table 11, where the cardinality constraint was ignored in the mean-variance model and the preselection of 4 stocks with the EU-EV model was applied with $\lambda \in [0, 0.3303)$, one can see that in both methods the stock S_6 followed by S_{11} are the most relevant stocks for obtaining efficient portfolios. Also stock S_9 contributes to the efficient portfolio construction with both methods. The difference is that with the EU-EV model, stock S_5 was selected as fourth contributing stock, whereas with the cardinality-constrained model the stocks S_3 and S_{16} are the third and fourth most relevant stocks for the efficient portfolios. The stock S_{16} is important for constructing efficient portfolios with lower risks, since as one can see, S_{16} contributes with higher weights to the construction of those portfolios. This explains the better performance of U when compared with those of Q_1 for lower standard deviations (see Figure 19).

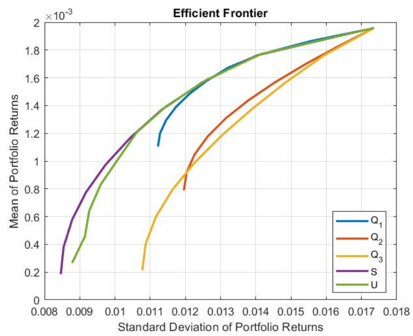

Figure 19: Efficient frontier with 10 portfolios for Q_i, $i = 1, 2, 3$, U and S.

In this case, both methods have in common that the efficient frontier of S is better approximated for middle and higher standard deviations in the given range, associated with higher returns, than for lower standard deviations. However, considering also lower standard deviations, then one can conclude that the cardinality-constrained mean-variance model permits obtaining portfolios with efficiency close to the efficiency of S, whereas with the preselection of 4 stocks with the EU-EV model it is not possible to construct portfolios with standard deviations approximately lower than 0.011.

3.2.3. General remarks

The mathematical advantage of the cardinality-constrained mean-variance model is that all combinatorial possibilities of sets with the limited number of stocks are directly taken into account in the optimization process. However, from the practical or numerical point of view, the cardinality-constrained portfolio optimization is very complex. Although, the solution methods for this optimization problem, including exact, relaxation and heuristic methods have evolved in determining optimal solutions, some computational problems were referred in the literature (see e.g. Gao & Li (2013)), such as failure in computing optimal solutions for higher numbers of stocks, compromised quality of the solutions and failure in guaranteeing the optimality of solutions. In the here considered problem, where the total number of stocks is small, the cardinality-constrained portfolio optimization determines effectively optimal stock sets, whose efficiency is very close to that of the whole stock set (the approximation being better for a maximum number of 9 stocks than for a maximum number of 4 stocks).

Considering the combined EU-EV preselection and mean-variance model, one advantage is the computational simplification, achieved by applying the mean-variance model to a specific set with a reduced number of stocks, obtained from a simple ranking procedure of individual stocks with respect to the EU-EV risk. Another advantage is that the preselection with the EU-EV risk measure gives a priori an insight about the individual characterization and individual performance of stocks in terms of variance, entropy and expected utility for different values of the trade-off factor. Also of interest is that this methodology permits in this way assessing risk in a different way taking into account more different aspects of risk, since the use variance alone as risk measure in the traditional mean-variance model may limit the risk assessment (as also pointed out in the literature). The EU-EV preselection and mean-variance model permit thus obtaining both and individual stock assessment and then the implied collective assessment of stocks. The disadvantage of this process is that the ranking with the EU-EV risk ignores the linear combination of weighted stocks and therefore the collective behaviour of stocks, so that optimal combinations may not be captured by leaving out certain stocks when applying the mean-variance model. However, the results of the application show that optimal solutions with efficiency very close to that of the whole set of stocks were obtained with this method, when reducing the number of stocks to the half. With sets of four stocks, i.e. with 20%

of the total number of stocks, the efficiency is achieved only for medium and high risks associated with medium and high expected returns.

4. Application to the selection of stock components of Euro Stoxx 50 index and Nasdaq 100 index

In this section, the EU-EV model and the mean-variance model will be applied to the stock market index Euro Stoxx 50, which consists of 50 component stocks of companies from different European countries, and to the Nasdaq 100 index, consisting of 102 component stocks. The data were collected from January 2019 to December 2020 from Yahoo Finance.

4.1. Euro Stoxx 50 index

In the following analysis, 502 daily closing prices, from January 2019 to December 2020, of 48 stocks of the Euro Stoxx 50 index, listed in Table A.14 in the appendix, will be considered and the EU-EV model will be applied to select subsets with 24 stocks having lowest EU-EV risk. Then, the mean-variance model will be applied to those subsets and their efficient frontiers will be compared with the efficient frontier of the whole set of stocks.

Following the methodology explained in detail in the previous section, the EU-EV risks are calculated for the 48 stocks and the stocks are ordered with respect to the EU-EV risk considering the trade-off factors $\lambda = 0, 0.25, 0.5, 0.75, 1$. Altogether there are 13 different subsets of stocks that can be formed for the equally spaced values of λ, ranging from 0 to 1 in intervals of 0.05. However, due to the similarity, only five subsets corresponding to the mentioned trade-off factors are considered for the respresentation of the efficient frontiers in order to analyse the influence of the change in λ and for the comparison with the efficient frontier obtained from the set of all 48 stocks. The best ranked 24 stocks are selected to form the subsets designated by $Q_{\lambda=0}$, $Q_{\lambda=0.25}$,..., $Q_{\lambda=1}$. The stocks belonging to these subsets are listed in Table A.15 in the appendix. Applying the mean-variance model to these subsets consisting of 24 stocks and to the set of all 48 stocks, designated by S, one obtains the efficient frontiers presented in Figures 20-22, where Figure 22 (right panel) contains all efficient frontiers of the considered subsets.

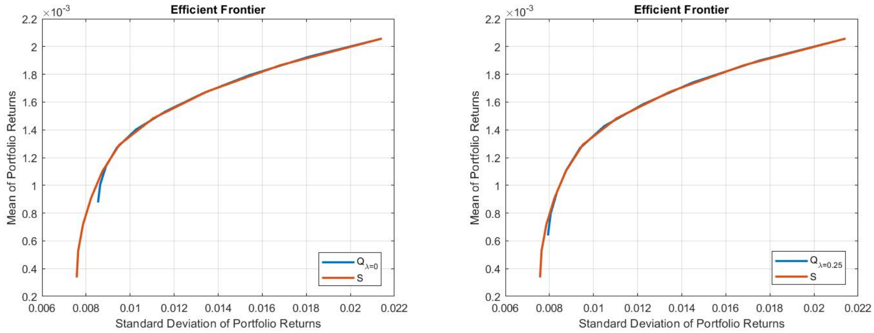

Figure 20: *Left panel*: Efficient frontier for $Q_{\lambda=0}$. *Right panel*: Efficient frontier for $Q_{\lambda=0.25}$.

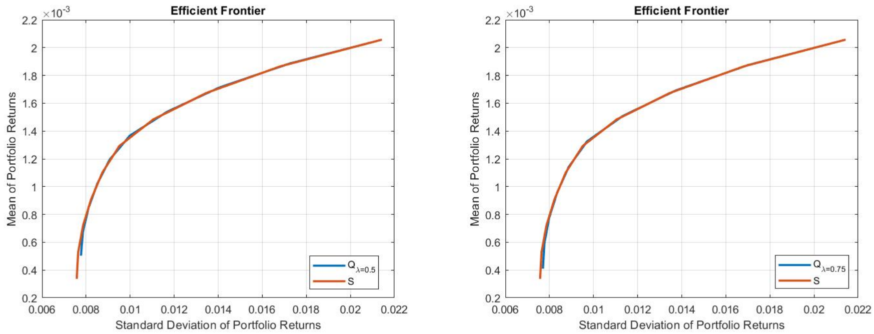

Figure 21: *Left panel*: Efficient frontier for $Q_{\lambda=0.5}$. *Right panel*: Efficient frontier for $Q_{\lambda=0.75}$.

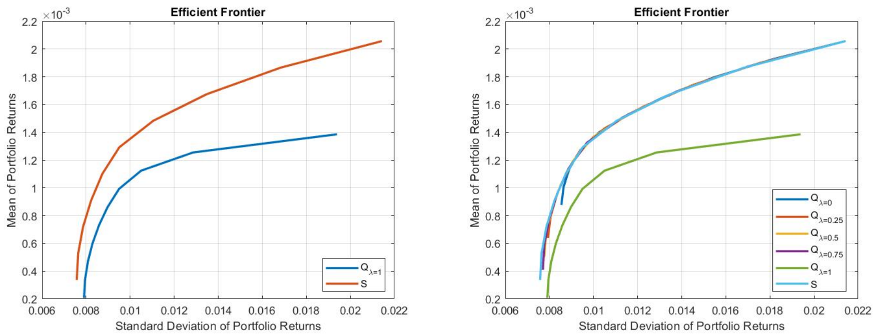

Figure 22: *Left panel*: Efficient frontier for $Q_{\lambda=1}$. *Right panel*: Efficient frontier for $Q_{\lambda=0}$, $Q_{\lambda=0.25}$, $Q_{\lambda=0.5}$, $Q_{\lambda=0.75}$, $Q_{\lambda=1}$ and S.

One can observe that the subsets with intermediate values of λ: $\lambda = 0.25$,

$\lambda = 0.5$ and $\lambda = 0.75$, lead to efficient frontiers that are approximately equal to the efficient frontier of S, the approximation with $\lambda = 0.75$ being the best. For $\lambda = 0$, the efficient frontier is also close to the efficient frontier of S, however for a smaller range of risk values, namely for standard deviations greater than 0.009. The set formed with $\lambda = 1$ presents a poor performance, since its efficient frontier lies entirely below those of S. Due to the fact that in this case the stocks are assessed using only the risk factors entropy and variance, the objective of achieving the high portfolio returns of S fails (see Figure 22 on the left).

One concludes with this experiment that the EU-EV model is able to select the stocks that are relevant for the efficient portfolio construction with the half number of stocks, reducing the initial set of 48 stocks to 24 stocks.

4.2. Nasdaq 100 index

In the following analysis, 504 daily closing prices, from January 2019 to December 2020, of 96 component stocks of the Nadaq 100 index, listed in Table A.16 in the appendix, will be considered. In this case the EU-EV model is used to form subsets with 48 stocks and the mean-variance model is then applied in order to compare the subsets' efficient frontiers with the efficient frontier of the entire set with 96 stocks.

The EU-EV risks are computed for the 96 stocks and the stocks are ordered with respect to the EU-EV risk for the trade-off factors $\lambda = 0$, 0.1,..., 0.9, 1. Here, 17 different subsets of stocks can be formed for λ ranging from 0 to 1 in intervals of 0.05. However, for the illlustrations of the efficient frontiers only 11 subsets for the mentioned trade-off factors will be used, due to the same reasons noted in the previous analysis. Then, the best ranked 48 stocks are selected to form the subsets designated by $Q_{\lambda=0}$, $Q_{\lambda=0.1}$,..., $Q_{\lambda=0.9}$, $Q_{\lambda=1}$, whose corresponding stocks are listed in Table A.17 in the appendix. Applying the mean-variance model to these subsets and to the set of all 96 stocks, S, one obtains the efficient frontiers presented in Figures 23-28, where Figure 28 (right panel) contains all efficient frontiers of the considered subsets.

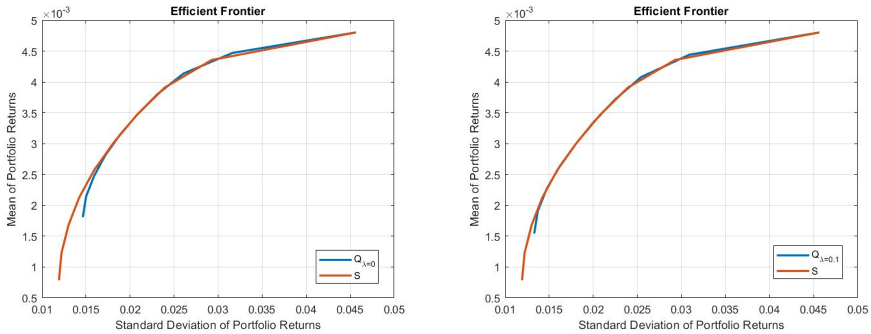

Figure 23: *Left panel*: Efficient frontier for $Q_{\lambda=0}$. *Right panel*: Efficient frontier for $Q_{\lambda=0.1}$.

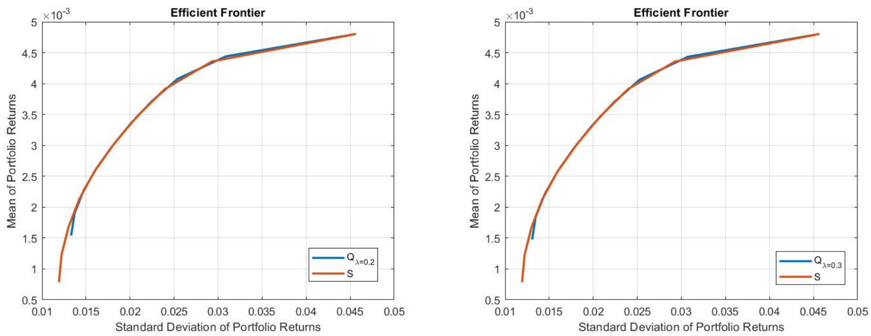

Figure 24: *Left panel*: Efficient frontier for $Q_{\lambda=0.2}$. *Right panel*: Efficient frontier for $Q_{\lambda=0.3}$.

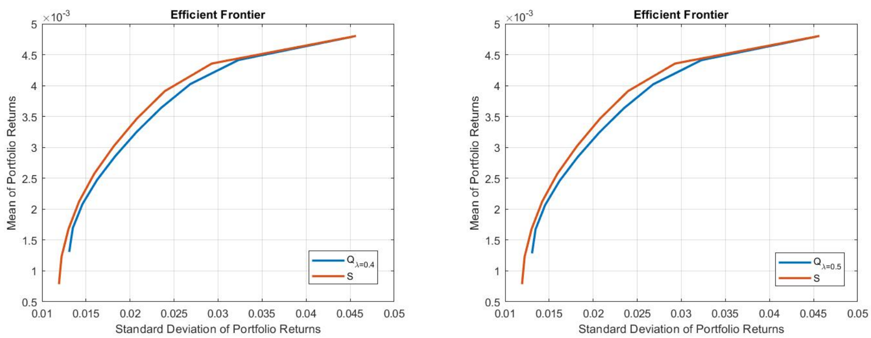

Figure 25: *Left panel*: Efficient frontier for $Q_{\lambda=0.4}$. *Right panel*: Efficient frontier for $Q_{\lambda=0.5}$.

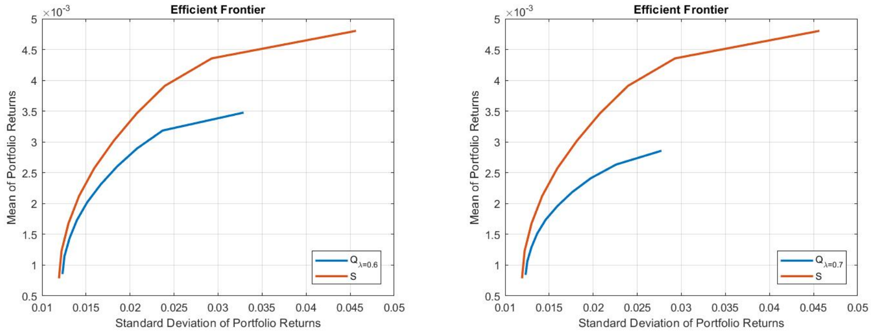

Figure 26: *Left panel*: Efficient frontier for $Q_{\lambda=0.6}$. *Right panel*: Efficient frontier for $Q_{\lambda=0.7}$.

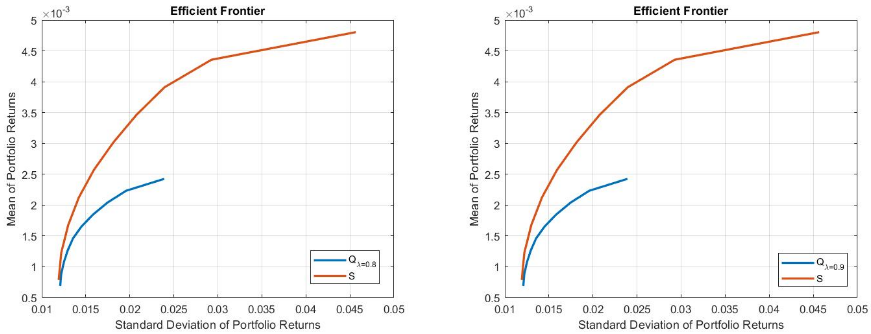

Figure 27: *Left panel*: Efficient frontier for $Q_{\lambda=0.8}$. *Right panel*: Efficient frontier for $Q_{\lambda=0.9}$.

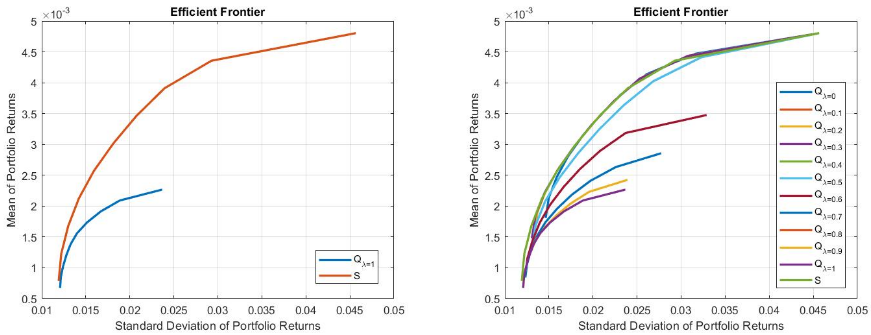

Figure 28: *Left panel*: Efficient frontier for $Q_{\lambda=1}$. *Right panel*: Efficient frontier for $Q_{\lambda=0}$, $Q_{\lambda=0.1}$,..., $Q_{\lambda=0.9}$, $Q_{\lambda=1}$ and S.

Observing Figures 23-28, one can conclude that the efficient frontiers of the subsets corresponding to $\lambda = 0.1, 0.2, 0.3$ are very similar and approximately equal to the efficient frontier of S (ignoring the very low standard deviation values, below 0.013), the best approximation being obtained for $\lambda = 0.3$. Considering $\lambda = 0$, the efficient frontier of the corresponding subset also approximates well the efficient frontier of S, however for values of standard deviations greater than 0.017. Regarding the subsets corresponding to $\lambda = 0.4, 0.5$, a good approximation to the efficient frontier of S is achieved for higher risks (standard deviations greater than 0.033) associated with higher expected returns (see Figure 25); for lower risks the efficient frontier deviates slightly from those of S. As for λ greater than 0.6, the results reveal that the efficiencies of the subsets' frontiers decrease with increasing λ, the deviation from the efficient frontier of S being highest when λ approaches 1. Thus, when more weight is given to entropy and variance in the EU-EV risk measure and ignoring the expected utility term when $\lambda = 1$, it becomes evident in the results that the selected stocks lead to portfolios with lower expected returns and that the efficient frontier with minimal portfolio returns corresponds to $\lambda = 1$.

The conclusion of this experiment, where an initial set of 96 stocks was reduced to subsets with 48 stocks having lowest EU-EV risk, is that the EU-EV model effectively selects the most relevant stocks for an efficient portfolio construction and that the performance of the subsets is more similar to the performance of S when more weight is given to the expected utility term in the EU-EV risk, that is for trade-off factors less than 0.5.

5. Conclusions

In this paper, a new stock selection model, the EU-EV model, was presented, which can be used as preselection model for the construction of optimal mean-variance portfolios. The EU-EV risk model depends on entropy and variance and on expected utility, which are combined through a trade-off parameter. The selection is based on minimizing the EU-EV risk by reducing the uncertainty given by variance and entropy and increasing the expected utility. The proposed methodology consists in ranking the stocks of a given set according to their EU-EV risk and in selecting the best ranked stocks with lowest EU-EV risk with the purpose to form subsets with a lower number of stocks than the initial whole set of stocks. The mean-variance model is then applied to the subsets containing the preselected stocks in order to construct efficient portfolios.

The methodology was applied to the PSI 20 index containing 18 stocks, with data from January 2019 to December 2020. Subsets with a half number of stocks and subsets with four stocks, preselected with the EU-EV risk model, were built and the mean-variance model was applied to those subsets of stocks. The efficient frontiers of the subsets were compared with the efficient frontier of the total set of stocks. The results showed that the efficient frontiers of portfolios with a half number of stocks, preselected with the EU-EV model, excluding higher values of the trade-off parameter, are approximately equal to the efficient frontier obtained with the whole set of stocks. Considering the subsets of four stocks, the efficient frontier of the whole set is well approximated, however only for medium and higher risks coupled to medium and higher returns, respectively, and for low values of the trade-off parameter. However, in order to decrease risk and to have portfolios with lower risks coupled to lower returns, which can be preferred by risk-averse investors, one should construct portfolios from a larger set of stocks. Four stocks were not sufficient to capture returns with lower risks, however the portfolios constructed from sets of nine stocks revealed that it is possible to achieve lower risks. A comparison with the cardinality-constrained mean-variance model, where the maximum number of stocks was limited to 9 stocks and to 4 stocks, revealed that, considering sets with 4 stocks, the cardinality-constrained mean-variance model permitted also obtaining efficient portfolios with lower risks. However, considering sets with the half number of stocks, it was possible to obtain approximately equal efficient portfolios with both methodologies. One concludes that a reduction of the number of stocks to the half using the EU-EV risk model as stock preselection model, with low and intermediate values of the trade-off parameter, leads to an approximate equal performance in the mean-variance optimization problem regarding the efficient portfolio construction. If the aim is to concentrate on efficient portfolios with higher risks coupled to higher returns, regarding for example risk-tolerant investors, one could reduce the number of stocks more by applying the EU-EV stock preselection model with very low values of the trade-off parameter and construct the efficient portfolios with the reduced number of stocks.

In order to further test the applicability of the proposed methodology to reduce the number of stocks to the half for an efficient portfolio construction, the EU-EV risk model and the mean-variance model were applied to the Euro Stoxx 50 index, containing 48 stocks with data from January 2019 to December 2020, and to the Nasdaq 100 index, containing 96 stocks with data in the same time range. The results indicated also in these cases that the reduction of the number of stocks to the half using the EU-EV risk model with low and intermediate values of the trade-off parameter leads to an efficient portfolio construction in the mean-variance optimization problem, with an approximately equal performance to those obtained with the entire initial sets of stocks.

One can conclude that with the EU-EV risk model it is possible to capture the optimal stocks that play the most relevant role in the efficient mean-variance portfolio construction. The EU-EV risk model can therefore be used as stock preselection model for the formation of efficient portfolios with a reduced number of stocks.

Acknowledgments

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Appendix A. Components of Euro Stoxx 50 index and Nasdaq 100 index and selected sets

Table A.14: Components of Euro Stoxx 50.

Stock	Ticker	Stock	Ticker	Stock	Ticker	Stock	Ticker
S_1	AI.PA	S_{13}	BNP.PA	S_{25}	IBE.MC	S_{37}	RWE.DE
S_2	ALV.DE	S_{14}	CA.PA	S_{26}	ITX.MC	S_{38}	SAN.PA
S_3	ABI.BR	S_{15}	SGO.PA	S_{27}	INGA.AS	S_{39}	SAP.DE
S_4	MT.AS	S_{16}	CRH.L	S_{28}	ISP.MI	S_{40}	SU.PA
S_5	ASML.AS	S_{17}	BN.PA	S_{29}	PHIA.AS	S_{41}	SIE.DE
S_6	G.MI	S_{18}	DBK.DE	S_{30}	OR.PA	S_{42}	GLE.PA
S_7	CS.PA	S_{19}	DTE.DE	S_{31}	MC.PA	S_{43}	TEF.MC
S_8	BBVA.MC	S_{20}	ENEL.MI	S_{32}	MBG.DE	S_{44}	TTE.PA
S_9	SAN.MC	S_{21}	ENGI.PA	S_{33}	MUV2.DE	S_{45}	UCG.MI
S_{10}	BAS.DE	S_{22}	ENI.MI	S_{34}	NOKIA.HE	S_{46}	DG.PA
S_{11}	BAYN.DE	S_{23}	EOAN.DE	S_{35}	ORA.PA	S_{47}	VIV.PA
S_{12}	BMW.DE	S_{24}	EL.PA	S_{36}	REP.MC	S_{48}	VOW.DE

Table A.15: Sets of 24 stocks from Euro Stoxx 50 with lowest EU-EV risk.

$Q_{\lambda=0}$	$Q_{\lambda=0.25}$	$Q_{\lambda=0.5}$	$Q_{\lambda=0.75}$	$Q_{\lambda=1}$
S_1	S_1	S_1	S_1	S_1
S_2	S_2	S_2	S_2	S_2
S_4	S_5	S_5	S_5	S_6
S_5	S_7	S_6	S_6	S_7
S_{10}	S_{15}	S_7	S_7	S_{14}
S_{13}	S_{16}	S_{15}	S_{16}	S_{17}
S_{15}	S_{18}	S_{16}	S_{17}	S_{19}
S_{16}	S_{19}	S_{19}	S_{19}	S_{20}
S_{18}	S_{20}	S_{20}	S_{20}	S_{21}
S_{20}	S_{23}	S_{23}	S_{21}	S_{23}
S_{24}	S_{24}	S_{24}	S_{23}	S_{24}
S_{25}	S_{25}	S_{25}	S_{24}	S_{25}
S_{26}	S_{26}	S_{26}	S_{25}	S_{26}
S_{29}	S_{29}	S_{29}	S_{29}	S_{29}
S_{30}	S_{30}	S_{30}	S_{30}	S_{30}
S_{31}	S_{31}	S_{31}	S_{31}	S_{31}
S_{33}	S_{33}	S_{33}	S_{33}	S_{33}
S_{37}	S_{37}	S_{37}	S_{35}	S_{35}
S_{39}	S_{39}	S_{38}	S_{37}	S_{37}
S_{40}	S_{40}	S_{39}	S_{38}	S_{38}
S_{41}	S_{41}	S_{40}	S_{39}	S_{39}
S_{46}	S_{46}	S_{41}	S_{40}	S_{40}
S_{47}	S_{47}	S_{47}	S_{41}	S_{41}
S_{48}	S_{48}	S_{48}	S_{47}	S_{47}

Table A.16: Components of Nasdaq 100.

Stock	Ticker	Stock	Ticker	Stock	Ticker	Stock	Ticker
S_1	AAPL	S_{25}	ADP	S_{49}	CDNS	S_{73}	ILMN
S_2	MSFT	S_{26}	SBUX	S_{50}	ORLY	S_{74}	ROST
S_3	AMZN	S_{27}	MDLZ	S_{51}	KHC	S_{75}	DLTR
S_4	TSLA	S_{28}	GILD	S_{52}	PAYX	S_{76}	WBA
S_5	GOOG	S_{29}	AMAT	S_{53}	ADSK	S_{77}	BIDU
S_6	GOOGL	S_{30}	ADI	S_{54}	EXC	S_{78}	PCAR
S_7	META	S_{31}	BKNG	S_{55}	MELI	S_{79}	VRSK
S_8	NVDA	S_{32}	ISRG	S_{56}	NXPI	S_{80}	FAST
S_9	PEP	S_{33}	VRTX	S_{57}	CTAS	S_{81}	IDXX
S_{10}	COST	S_{34}	CHTR	S_{58}	XEL	S_{82}	BIIB
S_{11}	AVGO	S_{35}	CSX	S_{59}	ASML	S_{83}	SGEN
S_{12}	CSCO	S_{36}	FISV	S_{60}	FTNT	S_{84}	CPRT
S_{13}	TMUS	S_{37}	MU	S_{61}	MRVL	S_{85}	EBAY
S_{14}	ADBE	S_{38}	REGN	S_{62}	LULU	S_{86}	SIRI
S_{15}	CMCSA	S_{39}	ATVI	S_{63}	MCHP	S_{87}	ANSS
S_{16}	TXN	S_{40}	LRCX	S_{64}	EA	S_{88}	ZS
S_{17}	QCOM	S_{41}	MRNA	S_{65}	AZN	S_{89}	VRSN
S_{18}	AMD	S_{42}	PANW	S_{66}	TEAM	S_{90}	ALGN
S_{19}	AMGN	S_{43}	KDP	S_{67}	CTSH	S_{91}	NTES
S_{20}	INTC	S_{44}	AEP	S_{68}	DXCM	S_{92}	SWKS
S_{21}	HON	S_{45}	SNPS	S_{69}	PDD	S_{93}	MTCH
S_{22}	INTU	S_{46}	MAR	S_{70}	WDAY	S_{94}	SPLK
S_{23}	PYPL	S_{47}	KLAC	S_{71}	JD	S_{95}	DOCU
S_{24}	NFLX	S_{48}	MNST	S_{72}	ODFL	S_{96}	OKTA

Table A.17: Sets of 48 stocks from Nasdaq 100 with lowest EU-EV risk.

$Q_{\lambda=0}$	$Q_{\lambda=0.1}$	$Q_{\lambda=0.2}$	$Q_{\lambda=0.3}$	$Q_{\lambda=0.4}$	$Q_{\lambda=0.5}$	$Q_{\lambda=0.6}$	$Q_{\lambda=0.7}$	$Q_{\lambda=0.8}$	$Q_{\lambda=0.9}$	$Q_{\lambda=1}$
S_{1}	S_{1}	S_{1}	S_{1}	S_{1}	S_{1}	S_{1}	S_{1}	S_{1}	S_{1}	S_{2}
S_{2}	S_{2}	S_{2}	S_{2}	S_{2}	S_{2}	S_{2}	S_{2}	S_{2}	S_{2}	S_{3}
S_{3}	S_{3}	S_{3}	S_{3}	S_{3}	S_{3}	S_{3}	S_{3}	S_{3}	S_{3}	S_{5}
S_{4}	S_{4}	S_{4}	S_{4}	S_{4}	S_{4}	S_{5}	S_{5}	S_{5}	S_{5}	S_{6}
S_{7}	S_{7}	S_{7}	S_{7}	S_{5}	S_{5}	S_{6}	S_{6}	S_{6}	S_{6}	S_{9}
S_{8}	S_{8}	S_{8}	S_{8}	S_{6}	S_{6}	S_{7}	S_{7}	S_{7}	S_{9}	S_{10}
S_{13}	S_{10}	S_{10}	S_{10}	S_{7}	S_{7}	S_{9}	S_{9}	S_{9}	S_{10}	S_{12}
S_{14}	S_{13}	S_{13}	S_{13}	S_{8}	S_{8}	S_{10}	S_{10}	S_{10}	S_{12}	S_{13}
S_{17}	S_{14}	S_{14}	S_{14}	S_{9}	S_{9}	S_{13}	S_{13}	S_{12}	S_{13}	S_{14}
S_{18}	S_{17}	S_{17}	S_{17}	S_{10}	S_{10}	S_{14}	S_{14}	S_{13}	S_{14}	S_{15}
S_{22}	S_{18}	S_{18}	S_{18}	S_{13}	S_{13}	S_{15}	S_{15}	S_{14}	S_{15}	S_{16}
S_{23}	S_{22}	S_{22}	S_{22}	S_{14}	S_{14}	S_{16}	S_{16}	S_{15}	S_{16}	S_{19}
S_{24}	S_{23}	S_{23}	S_{23}	S_{17}	S_{15}	S_{21}	S_{19}	S_{16}	S_{19}	S_{21}
S_{29}	S_{29}	S_{29}	S_{29}	S_{18}	S_{18}	S_{22}	S_{21}	S_{19}	S_{21}	S_{22}
S_{34}	S_{34}	S_{34}	S_{34}	S_{22}	S_{22}	S_{23}	S_{22}	S_{21}	S_{22}	S_{25}
S_{37}	S_{37}	S_{39}	S_{39}	S_{23}	S_{23}	S_{25}	S_{23}	S_{22}	S_{25}	S_{26}
S_{39}	S_{39}	S_{40}	S_{40}	S_{27}	S_{26}	S_{26}	S_{25}	S_{23}	S_{26}	S_{27}
S_{40}	S_{40}	S_{41}	S_{41}	S_{29}	S_{27}	S_{27}	S_{26}	S_{25}	S_{27}	S_{28}
S_{41}	S_{41}	S_{42}	S_{42}	S_{34}	S_{34}	S_{34}	S_{27}	S_{26}	S_{28}	S_{32}
S_{42}	S_{42}	S_{45}	S_{45}	S_{39}	S_{39}	S_{39}	S_{32}	S_{27}	S_{32}	S_{34}
S_{45}	S_{45}	S_{47}	S_{47}	S_{40}	S_{40}	S_{42}	S_{34}	S_{28}	S_{34}	S_{35}
S_{47}	S_{47}	S_{48}	S_{48}	S_{42}	S_{42}	S_{43}	S_{35}	S_{32}	S_{35}	S_{36}
S_{49}	S_{49}	S_{49}	S_{49}	S_{45}	S_{45}	S_{44}	S_{39}	S_{34}	S_{36}	S_{42}
S_{53}	S_{53}	S_{53}	S_{53}	S_{47}	S_{47}	S_{45}	S_{42}	S_{35}	S_{42}	S_{43}
S_{55}	S_{55}	S_{55}	S_{55}	S_{48}	S_{48}	S_{48}	S_{43}	S_{39}	S_{43}	S_{44}
S_{56}	S_{56}	S_{57}	S_{57}	S_{49}	S_{49}	S_{49}	S_{44}	S_{42}	S_{44}	S_{45}
S_{57}	S_{57}	S_{59}	S_{59}	S_{53}	S_{53}	S_{50}	S_{45}	S_{43}	S_{45}	S_{48}
S_{59}	S_{59}	S_{60}	S_{60}	S_{55}	S_{55}	S_{52}	S_{48}	S_{44}	S_{48}	S_{49}
S_{60}	S_{60}	S_{61}	S_{61}	S_{57}	S_{57}	S_{55}	S_{49}	S_{45}	S_{49}	S_{50}
S_{61}	S_{61}	S_{62}	S_{62}	S_{59}	S_{58}	S_{57}	S_{50}	S_{48}	S_{50}	S_{51}
S_{62}	S_{62}	S_{66}	S_{66}	S_{61}	S_{59}	S_{58}	S_{52}	S_{49}	S_{52}	S_{52}
S_{63}	S_{66}	S_{68}	S_{68}	S_{62}	S_{61}	S_{59}	S_{53}	S_{50}	S_{54}	S_{54}
S_{66}	S_{68}	S_{69}	S_{69}	S_{66}	S_{62}	S_{62}	S_{57}	S_{52}	S_{57}	S_{57}
S_{68}	S_{69}	S_{71}	S_{71}	S_{68}	S_{66}	S_{64}	S_{58}	S_{57}	S_{58}	S_{58}
S_{69}	S_{71}	S_{72}	S_{72}	S_{69}	S_{69}	S_{65}	S_{59}	S_{58}	S_{59}	S_{64}
S_{71}	S_{72}	S_{79}	S_{79}	S_{71}	S_{71}	S_{71}	S_{62}	S_{59}	S_{64}	S_{65}
S_{72}	S_{79}	S_{80}	S_{80}	S_{72}	S_{72}	S_{72}	S_{64}	S_{64}	S_{65}	S_{67}
S_{81}	S_{81}	S_{81}	S_{81}	S_{79}	S_{79}	S_{78}	S_{65}	S_{65}	S_{67}	S_{72}
S_{83}	S_{83}	S_{83}	S_{83}	S_{80}	S_{80}	S_{79}	S_{71}	S_{72}	S_{72}	S_{75}
S_{84}	S_{84}	S_{84}	S_{84}	S_{81}	S_{81}	S_{80}	S_{72}	S_{78}	S_{78}	S_{78}
S_{87}	S_{87}	S_{87}	S_{85}	S_{83}	S_{83}	S_{81}	S_{78}	S_{79}	S_{79}	S_{79}
S_{88}	S_{88}	S_{88}	S_{87}	S_{84}	S_{84}	S_{83}	S_{79}	S_{80}	S_{80}	S_{80}
S_{90}	S_{90}	S_{90}	S_{88}	S_{85}	S_{85}	S_{84}	S_{80}	S_{81}	S_{81}	S_{81}
S_{91}	S_{91}	S_{91}	S_{90}	S_{87}	S_{87}	S_{85}	S_{81}	S_{84}	S_{84}	S_{84}
S_{92}	S_{92}	S_{92}	S_{92}	S_{88}	S_{88}	S_{87}	S_{84}	S_{85}	S_{85}	S_{85}
S_{93}	S_{93}	S_{93}	S_{93}	S_{93}	S_{93}	S_{89}	S_{85}	S_{86}	S_{86}	S_{86}
S_{95}	S_{95}	S_{95}	S_{95}	S_{95}	S_{95}	S_{95}	S_{87}	S_{87}	S_{87}	S_{87}
S_{96}	S_{96}	S_{96}	S_{96}	S_{96}	S_{96}	S_{96}	S_{89}	S_{89}	S_{89}	S_{89}

References

Aksarayli, M., & Pala O. (2018). A polynomial goal programming model for portfolio optimization based on entropy and higher moments. *Expert Systems with Applications*, 94, 185-192. https://doi.org/10.1016/j.eswa.2017.10.056.

Bhattacharyya, R., Chatterjee, A., & Samarjit, K. (2013). Uncertainty theory based multiple objective mean-entropy-skewness stock portfolio selection model with transactions costs. *Journal of Uncertainty Analysis and Applications*, 16, 1-17. https://doi.org/10.1186/2195-5468-1-16.

Bera, A.K., & Park, S.Y. (2008). Optimal portfolio diversification using the maximum entropy principle. *Econometric Reviews*, 27, 484-512. https://doi.org/10.1080/07474930801960394

Brito, I. (2020). A decision model based on expected utility, entropy and variance. *Applied Mathematics and Computation*, 379, Article 125285. https://doi.org/10.1016/j.amc.2020.125285.

Chang, T.-J., Meade, N., Beasley, J.E., & Sharaiha, Y.M. (2000). Heuristics for cardinality-constrained portfolio optimisation. *Computers & Operations Research*, 27, *13*, 1271-1302. https://doi.org/10.1016/S0305-0548(99)00074-X.

Chang, T.-J., Yang, S.-C., & Chang, K.-J. (2009). Portfolio optimization problems in different risk measures using genetic algorithm. *Expert Systems with Applications*, 36, *7*, 10529-10537. https://doi.org/10.1016/j.eswa.2009.02.062.

Chen, W., Zhang, H., Mehlawat, M.K. &, Jia, L. (2021). Mean-variance portfolio optimization using machine learning-based stock price prediction. *Applied Soft Computing*, 100, Article 106943. https://doi.org/10.1016/j.asoc.2020.106943.

Dai, Z., & Wang, F. (2019). Sparse and robust mean-variance portfolio optimization problems. *Physica A: Statistical Mechanics and its Applications*, 523, 1371-1378. https://doi.org/10.1016/j.physa.2019.04.151.

Deng, G.F., Lin, W.-T., & Lo, C.-C. (2012). Markowitz-based portfolio selection with cardinality constraints using improved particle swarm optimization. *Expert Systems with Applications*, 39, *4*, 4558-4566. https://doi.org/10.1016/j.eswa.2011.09.129.

Galankashi, R. M., Mokhatab Rafiei, F., & Ghezelbash, M. (2020). Portfolio selection: a fuzzy-ANP approach. *Financial Innovation*, 6, Article 17. https://doi.org/10.1186/s40854-020-00175-4.

Gao, J., & Li, D. (2013). Optimal Cardinality Constrained Porfolio Selection. *Operations Research*, 61, 745-761. https://doi.org/10.1287/opre.2013.1170.

Georgescu, I., & Fono, L.A. (2019). A Portfolio Choice Problem in the Framework of Expected Utility Operators. *Mathematics*, 7, *8*, Article 669. https://doi.org/10.3390/math7080669.

Hai, T., & Min, L. (2021). Hybrid Robust Portfolio Selection Model Using Machine Learning-based Preselection. *Engineering Letters*, 29, 1626-1635.

Huang, C.F. (2012). A hybrid stock selection model using genetic algorithms and support vector regression. *Applied Soft Computing*, 12, 807-818. https://doi.org/10.1016/j.asoc.2011.10.009.

Huang, X. (2008a). Mean-entropy models for fuzzy portfolio selection. *IEEE Transactions on Fuzzy Systems*, 16, 1096-1101. https://doi.org/10.1109/TFUZZ.2008.924200.

Huang, X. (2008b). Mean-semivariance models for fuzzy portfolio selection. *Journal of Computational and Applied Mathematics*, 217, *1*, 1-8. https://doi.org/10.1016/j.cam.2007.06.009.

Jana, P., Roy, T.K., & Mazumder, S.K. (2009). Multi-objective possibilistic model for portfolio selection with transaction cost. *Journal of Computational and Applied Mathematics*, 228, *1*, 188-196. https://doi.org/10.1016/j.cam.2008.09.008.

Kahneman, D., & Tversky, A. (1979). Prospect theory: An analysis of decisions under risk. *Econometrica*, 47, 263-290. https://doi.org/10.2307/1914185.

Leung, M.-F., & Wang, J. (2022). Cardinality-constrained portfolio selection based on collaborative neurodynamic optimization. *Neural Networks*, 145, 68-79. https://doi.org/10.1016/j.neunet.2021.10.007.

Levi, H. (1992). Stochastic dominance and expected utility: survey and analysis. *Management Science*, 38, *4*, 555-593. https://www.jstor.org/stable/2632436.

Li, B., Sun, Y., Aw, G., & Teo, K.L. (2019). Uncertain portfolio optimization problem under a minimax risk measure. *Applied Mathematical Modelling*, 76, 274-281. https://doi.org/10.1016/j.apm.2019.06.019.

Li, B., & Zhang, R. (2021). A new mean-variance-entropy model for uncertain portfolio optimization with liquidity and diversification. *Chaos, Solitons & Fractals*, 146, Article 110842. https://doi.org/10.1016/j.chaos.2021.110842.

Li, T., Zhang, W., & Xu, W. (2015). A fuzzy portfolio selection model with background risk. *Applied Mathematics and Computation*, 256, 505-513. https://doi.org/10.1016/j.amc.2015.01.007.

Li, X., Qin, Z., & Kar, K. (2010). Mean-variance-skewness model for portfolio selection with fuzzy returns. *European Journal of Operational Research*, 202, *1*, 239-247. https://doi.org/10.1016/j.ejor.2009.05.003.

Liu, Y.C., & Yeh, I.C. (2017). Using mixture design and neural networks to build stock selection decision support systems. *Neural Computing and Applications*, 28, 521-535. https://doi.org/10.1007/s00521-015-2090-x.

Lozza, S.O., Shalit, H., & Fabozzi, F.J. (2013). Portfolio selection problems consistent with given preference orderings. *International Journal of Theoretical and Applied Finance*, 16, Article 1350029. https://doi.org/10.1142/S0219024913500295.

Marasović, B., Kalinić, T., & Jerković, I. (2021). Determining Expected Utility and Entropy Ratio in the Expected Utility-Entropy Decision Model for Stock Selection Depending on Capital Market Development. In Škrinjari'c, T., Čižmešija, M., & Christiansen, B. (Ed.), *Recent Applications of Financial Risk Modelling and Portfolio Management* (pp. 1-21). IGI Global. https://doi.org/10.4018/978-1-7998-5083-0.ch001.

Marasović, B., & Kalinić, T. (2019). Is the expected utility-entropy decision model for stock selection applicable on emerging capital markets? *Economic and social development: Book of Proceedings*, 39, 225-235.

Markowitz, H. (1952). Portfolio Selection. *The Journal of Finance*, 7, *1*, 77-91. https://doi.org/10.2307/2975974.

Markowitz, H. (2000). *Mean-Variance Analysis in Portfolio Choice and Capital Markets.* Wiley.

Mehralizade, R., Amini, M., Gildeh, B.S., & Ahmadzade, H. (2020). Uncertain random portfolio selection based on risk curve. *Soft Computing*, 24, 13331-13345. https://doi.org/10.1007/s00500-020-04751-9.

Mercurio, P.J., Wu, Y., & Xie, H. (2020). An Entropy-Based Approach to Portfolio Optimization. *Entropy*, 22, *3*, Article 332. https://doi.org/10.3390/e22030332.

Min, L., Dong, J., Liu, J., & Gong, X. (2021). Robust mean-risk portfolio optimization using machine learning-based trade-off parameter. *Applied Soft Computing*, 113, Article 107948. https://doi.org/10.1016/j.asoc.2021.107948.

Ortobelli, S., Rachev, S., Stoyanov, S., Fabozzi, F., & Biglova, A. (2005). The proper use of risk measures in portfolio theory. *International Journal of Theoretical and Applied Finance*, 8, 1107-1133. https://doi.org/10.1142/S0219024905003402.

Paiva, F.P., Cardoso, R.T.M., Hanaoka, G.P., & Duarte, W.M. (2019). Decision-making for financial trading: A fusion approach of machine learning and portfolio selection. *Expert Systems with Applications*, 115, 635-655. https://doi.org/10.1016/j.eswa.2018.08.003.

Parra, M.A., Terol, A.B., & Uria, M.V.R. (2001). A fuzzy goal programming approach to portfolio selection. *European Journal of Operational Research*, 133, 287-297. https://doi.org/10.1016/S0377-2217(00)00298-8.

Philippatos, G., & Gressis, N. (1975). Conditions of equivalence among E-V, SSD and E-H portfolio selection criteria: The case of uniform, normal and lognormal distributions. *Management Science*, 21, 617-625. https://www.jstor.org/stable/2629595.

Philippatos, G., & Wilson, C. (1972). Entropy, market risk and the selection of efficient portfolios. *Applied Economics*, 4, 209-220. https://doi.org/10.1080/00036847200000017.

Qin, Z., Li, X., & Ji, X. (2009). Portfolio selection based on fuzzy cross-entropy. *Journal of Computational and Applied Mathematics*, 228, 188-196. https://doi.org/10.1016/j.cam.2008.09.010.

Qu, B.Y., Zhou, Q., Xiao, J.M., Liang, J.J., & Suganthan, P.N. (2017). Large-Scale Portfolio Optimization Using Multiobjective Evolutionary Algorithms and Preselection Methods. *Mathematical Problems in Engineering*, 2017, Article 4197914. https://doi.org/10.1155/2017/4197914.

Rachev, S., Ortobelli, S., Stoyanov, S., Fabozzi, F., & Biglova, A. (2008). Desirable properties of an ideal risk measure in portfolio theory. *International Journal of Theoretical and Applied Finance*, 11, 19-54. https://doi.org/10.1142/S0219024908004713.

Song, R., & Chan, Y. (2020). A New Adaptive Entropy Portfolio Selection Model. *Entropy*, 22, *9*, Article 951. https://doi.org/10.3390/e22090951.

Usta, I., & Kantar, Y.M. (2011). Mean-Variance-Skewness-Entropy Measures: A Multi-Objective Approach for Portfolio Selection. *Entropy*, 13, 117-133. https://doi.org/10.3390/e13010117.

Wang, W., Li, W., Zhang, N., & Liu, K. (2020). Portfolio formation with preselection using deep learning from long-term financial data. *Expert Systems with Applications*, 143, Article 113042. https://doi.org/10.1016/j.eswa.2019.113042.

Xia, Y., Liu, B., Wang, S., & Lai,K.K. (2000). A model for portfolio selection with order of expected returns. *Computers and Operations Research*, 27, 409-422. https://doi.org/10.1016/S0305-0548(99)00059-3.

Yang, J., & Qiu, W. (2005). A measure of risk and a decision-making model based on expected utility and entropy. *European Journal of Operational Reserach*, 164, 792-799. https://doi.org/10.1016/j.ejor.2004.01.031.

Yang, J., Feng, Y. Qiu, W. (2017). Stock Selection for Portfolios Using Expected Utility-Entropy Decision Model. *Entropy*, 19, Article 508. https://doi.org/10.3390/e19100508.

Yin, D. (2019). Investment decision based on entropy theory. *Modern Economy*, 10, 1211-1228. https://doi.org/10.4236/me.2019.104083.